

Nondeterministic State Complexity of Positional Addition

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Consider nondeterministic finite automata recognizing base- k positional notation of numbers. Assume that numbers are read starting from their least significant digits. It is proved that if two sets of numbers S and T are represented by nondeterministic automata of m and n states, respectively, then their sum $\{s + t \mid s \in S, t \in T\}$ is represented by a nondeterministic automaton with $2mn + 2m + 2n + 1$ states. Moreover, this number of states is necessary in the worst case for all $k \geq 9$.

1 Introduction

Descriptional complexity of operations on regular languages with respect to their representation by finite automata and regular expressions is among the common topics of automata theory. With respect to deterministic finite automata (DFAs), and using the number of states as a complexity measure, the *state complexity* of basic operations on languages was determined by Maslov [11] in 1970. In particular, such results as “if languages K and L are recognized by DFAs of m and n states, respectively, then the language KL requires a DFA with up to $(2m - 1)2^{n-1}$ states” originate from that paper.

Over the last two decades, similar results were obtained for nondeterministic finite automata (NFAs). In particular, Birget [2] has shown that the complement of a language recognized by an n -state NFA may require an NFA with as many as 2^n states, and this result was later improved by Jirásková [9] who reduced the alphabet for the witness language from $\{a, b, c, d\}$ to $\{a, b\}$. The systematic study of *nondeterministic state complexity*, that is, state complexity with respect to NFAs, of different operations was started by Holzer and Kutrib [6], who obtained, in particular, the precise results for union, intersection and concatenation. More recently Jirásková and Okhotin [10] determined the nondeterministic state complexity of cyclic shift, Gruber and Holzer [4] established precise results for scattered substrings and scattered superstrings, Domaratzki and Okhotin [3] studied k -th power of a language, L^k , while Han, K. Salomaa and Wood [5] considered the standard operations on NFAs in the context of prefix-free languages.

The present paper continues this study by investigating another operation, which has recently been used by Jež and Okhotin [7, 8] in the study of language equations. This is the operation of addition of strings in base- k positional notation. Let $\Sigma_k = \{0, 1, \dots, k - 1\}$ with $k \geq 2$ be an alphabet of digits. Then a string $a_{\ell-1} \dots a_0 \in \Sigma_k^*$ represents a number $(a_{\ell-1} \dots a_0)_k = \sum_{i=0}^{\ell-1} a_i \cdot k^i$, and there is a correspondence between natural numbers and strings in $\Sigma_k^* \setminus 0\Sigma_k^*$. For two strings $u, v \in \Sigma_k^* \setminus 0\Sigma_k^*$, their sum can be defined as $w = u \boxplus v$ as the unique string $w \in \Sigma_k^* \setminus 0\Sigma_k^*$, for which $(w)_k = (u)_k + (v)_k$. The operation extends to languages as follows: for all $K, L \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$, $K \boxplus L = \{u \boxplus v \mid u \in K, v \in L\}$.

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This operation preserves regularity, and proving that can be regarded as an exercise in automata theory. The paper begins with a solution to this exercise, given in Section 2. For convenience, it is assumed that automata read a notation of a number starting from its least significant digit; to put it formally, a slightly different operation is studied: $K \boxplus^R L = (K^R \boxplus L^R)^R$. This variant seems to be more natural in the context of automata, and furthermore, since the nondeterministic state complexity of reversal is $n + 1$, the complexity of these two operations is almost the same.

The straightforward construction of an automaton recognizing the language $L(A) \boxplus^R L(B)$ for an m -state NFA A and an n -state NFA B yields an NFA with $2mn + 2m + 2n + 1$ states. The purpose of this paper is to show that this construction is in fact optimal, and there are witness languages, for which exactly this number of states is required. This is established in Section 3, where worst-case automata are presented for $m, n \geq 1$ with $m + n \geq 3$. The case of $m = n = 1$ requires a special treatment, and it is proved that the NFA recognizing a positional sum of two one-state automata requires 6 states in the worst case.

2 Constructing an NFA for $K \boxplus^R L$

A *nondeterministic finite automaton* (NFA) is a quintuple $A = (Q, \Sigma, \delta, q_0, F)$, in which Q is a finite set of states, Σ is a finite input alphabet, $\delta : Q \times \Sigma \rightarrow 2^Q$ is the (nondeterministic) transition function, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of accepting states. An NFA is called a *deterministic finite automaton* (DFA) if $|\delta(q, a)| = 1$ for all q and a , and it is a *partial DFA* if $|\delta(q, a)| \leq 1$. The transition function can be naturally extended to the domain $Q \times \Sigma^*$. The language recognized by the NFA A , denoted $L(A)$, is the set $\{w \in \Sigma^* \mid \delta(q_0, w) \cap F \neq \emptyset\}$.

Throughout this paper, the letters in an alphabet of size k are always considered as digits in base- k notation, and the alphabet is $\Sigma_k = \{0, 1, \dots, k-1\}$. With such an alphabet fixed, the *nondeterministic state complexity of positional addition* of NFAs is defined as a function $f_k : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, where $f_k(m, n)$ is the least number of states in an NFA sufficient to represent $L(A) \boxplus^R L(B)$ for every m -state NFA A and n -state NFA B with $L(A), L(B) \subseteq \Sigma_k^* \setminus 0\Sigma_k^*$. The following lemma, besides formally establishing that regular languages are closed under addition in positional notation, gives an upper bound on this function.

Lemma 1. *Let A and B be NFAs over $\Sigma_k = \{0, 1, \dots, k-1\}$ with m and n states, respectively. Let $L(A) \cap 0\Sigma_k^* = L(B) \cap 0\Sigma_k^* = \emptyset$. Then there exists a $(2mn + 2m + 2n + 1)$ -state NFA over Σ_k for the language $L(A) \boxplus^R L(B)$.*

Proof: Let $A = (P, \Sigma_k, \delta_A, p_0, F_A)$ and $B = (Q, \Sigma_k, \delta_B, q_0, F_B)$. The new NFA C has a set of states split into four groups: $\hat{Q} = Q^{AB} \cup Q^A \cup Q^B \cup \{q_{acc}\}$, where

$$\begin{aligned} Q^{AB} &= P \times Q \times \{0, 1\}, \\ Q^A &= \{A\} \times P \times \{0, 1\}, \\ Q^B &= \{B\} \times Q \times \{0, 1\}. \end{aligned}$$

(I) Each state $(p, q, c) \in Q^{AB}$ corresponds to A in state p , B in state q and carry digit $c \in \{0, 1\}$. In particular, the state $(p_0, q_0, 0)$ is the initial state of this NFA. State (p, q, c) represents the case shown in Figure 1(left). A string of digits $dddd$ has been read, and C has guessed its representation as a sum of two strings of digits, $aaaa \boxplus^R bbbb$, where A goes to p by $aaaa$ and B goes to q by $bbbb$. If $c = 1$, then $aaaa \boxplus^R bbbb = 1dddd$.

The transitions from one state of this kind to another are defined as follows. Suppose A reads a digit a and goes from p to p' , while B may go from q to q' by a digit b . Then, taking the carry digit c into account, the sum may contain a digit $a + b + c$ or $a + b + c - k$ in this position depending on whether $a + b + c < k$ or not, and also the carry should be adjusted accordingly. Thus C has a transition from (p, q, c) to $(p', q', 0)$ by $a + b + c$ if $a + b + c < k$, or a transition to $(p', q', 1)$ by $a + b + c - k$ if $a + b + c \geq k$. This procedure continues until the string of digits recognized by A or by B finishes. Then C enters a state of one of the following two groups.

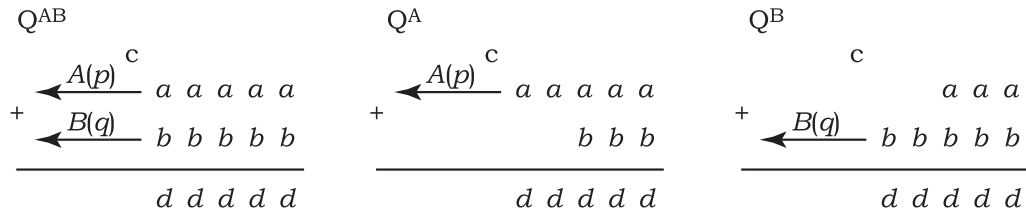


Figure 1: Transitions out of $(p, q, 0) \in Q^{AB}$ in the constructed NFA.

(II) If the automaton B is no longer running (that is, the notation of the second number has ended), while A still produces some digits, this case is implemented in states $(A, p, c) \in Q^A$, where p is a state of A and c is a carry. This case is illustrated in Figure 1(middle). The NFA C reaches this group of states as follows. For every state $(p, q, c) \in Q^{AB}$, such that q is an accepting state of B , the string recognized by B can be pronounced finished. Suppose that A may go from p to p' by a digit a . Then the sum may contain a digit $a + c$ or $a + c - k$. This case is represented by a transition of C from (p, q, c) to $(A, p', 0)$ by $a + c$ if $a + c < k$, or to $(A, p', 1)$ by $a + c - k$ if $a + c \geq k$. Once C enters the subset Q^A , it can continue reading the number as follows. For every state (A, p, c) , if A may go from p to p' by a digit a , then there is a transition from (A, p, c) to $(A, p', 0)$ by $(a + c)$ if $a + c < k$, or to $(A, p', 1)$ by $(a + c - k)$ if $a + c \geq k$.

(III) Symmetrically, there is a group of states (B, q, c) , which correspond to the case when the number read by A has ended. For each state $(p, q, c) \in Q^{AB}$ with $p \in F_A$, for every digit b and for every state q' , such that B has a transition from q to q' by b , the new automaton C has a transition from (p, q, c) to $(B, q', 0)$ by $b + c$ if $b + c < k$, or to $(B, q', 1)$ by $b + c - k$ if $b + c \geq k$. Second, for every state (B, q, c) , if B may go from q to q' by a digit b , then C has a transition from (B, q, c) to $(B, q', 0)$ by $(b + c)$ if $b + c < k$, or to $(B, q', 1)$ by $(b + c - k)$ if $b + c \geq k$.

(IV) q_{acc} is a special accepting state with no outgoing transitions. This state is needed when the strings of digits recognized by A and B have already finished, but the carry digit remains, and thus an extra input symbol has to be read. The automaton C reaches this state by reading the digit 1 under the following conditions: for all $p \in F_A$ and $q \in F_B$, there are transitions by 1 from $(p, q, 1)$, from $(A, p, 1)$ and from $(B, q, 1)$ to q_{acc} .

The other accepting states are all states of the form $(p, q, 0)$, $(A, p, 0)$ and $(B, q, 0)$, with $p \in F_A$ and $q \in F_B$.

This completes the construction. The general form of transitions from a state $(p, q, c) \in Q^{AB}$ is illustrated in Figure 2, separately for $c = 0$ and $c = 1$. \square

Example 5. Consider the regular language $L = \{w \in \Sigma^* \mid \text{the number of } a\text{'s in } w \text{ is a multiple of } n\}$. The set of pairs of strings $\{(a, a^{n-1}), (a^2, a^{n-2}), \dots, (a^n, \varepsilon)\}$ is a fooling set for the language L because for every i and j in $\{1, 2, \dots, n\}$,

(F1) $a^i a^{n-i} = a^n$, and the string a^n is in the language L , and

(F2) if $1 \leq i < j \leq n$, then $a^i a^{n-j} = a^{n-(j-i)}$, and the string $a^{n-(j-i)}$ is not in the language L since $0 < n - (j - i) < n$.

Hence by Lemma 4, every NFA for the language L needs at least n states. \diamond

Lemma 6. Let $\Sigma_k = \{0, 1, \dots, k-1\}$ be an alphabet with $k \geq 9$. Let $m \geq 1$ and $n \geq 2$, and consider the partial DFAs A_m and B_n over Σ_k given in Figure 3. Then every NFA for $L(A_m) \boxplus^R L(B_n)$ has at least $2mn + 2m + 2n + 1$ states.

Proof: In plain words, $L(A_m)$ represents all numbers with their base- k notation using only digits 1, 2 and $k-1$, with the number of 1s equal to $m-1$ modulo m . Similarly, the base- k notation of all numbers in $L(B_n)$ uses only digits 1, 3 and $k-1$, and the total number of 1s and $(k-1)$ s should be $n-1$ modulo n .

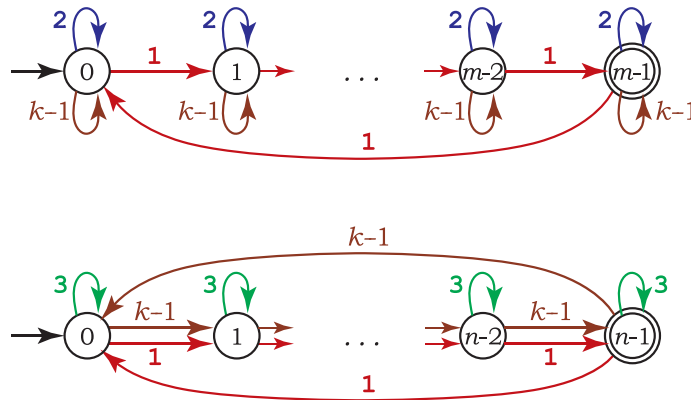


Figure 3: The nondeterministic finite automata A_m and B_n over $\Sigma_k = \{0, 1, \dots, k-1\}$ with $k \geq 9$.

Let the set of states of A_m be $P = \{0, \dots, m-1\}$ and let the states of B_n be $Q = \{0, \dots, n-1\}$. Let $L = L(A_m) \boxplus^R L(B_n)$, and let us construct a $(2mn + 2m + 2n + 1)$ -state NFA

$$M = (Q^{AB} \cup Q^A \cup Q^B \cup \{q_{acc}\}, \Sigma_k, \delta, q_0, F)$$

for the language L as in Lemma 1. The initial state of M is $q_0 = (0, 0, 0)$. The full set of transitions is omitted due to space constraints; the reader can reconstruct it according to Lemma 1. The below incomplete list represents all information about M used later in the proof:

- Each state $(i, j, 0)$ goes to itself by 5; to state $(i, j+1, 0)$ by 3; to state $(i+1, j, 0)$ by 4, and to state $(i, j+1, 1)$ by $k-2$. Each state $(m-1, j, 0)$ also goes to state $(B, j, 0)$ by 3.
- Each state $(i, j, 1)$ goes to state $(i, j, 0)$ by 6. Each state $(i, n-1, 1)$ also goes to state $(A, i, 1)$ by 0, and each state $(m-1, j, 1)$ also goes to state $(B, j+1, 1)$ by 0.
- Each state $(A, i, 1)$ goes to itself by 0; to state $(A, i, 0)$ by 3; and to state $(A, i+1, 0)$ by 2.
- Each state $(A, i, 0)$ goes to itself by 2 and $k-1$; and to $(A, i+1, 0)$ by 1.

- Each state $(B, j, 1)$ goes to state $(B, j + 1, 1)$ by 0; to state $(B, j, 0)$ by 4; and to state $(B, j + 1, 0)$ by 2.
- Each state $(B, j, 0)$ goes to itself by 3; and to $(B, j + 1, 0)$ by 1 and $k - 1$.
- State $(A, m - 1, 1)$ goes to state q_{acc} by 1.

Notice that in states (A, i, c) and (B, j, c) , transitions by 5 and by 6 are not defined, and no transitions are defined in state q_{acc} . There are four accepting states: $(m - n, n - 1, 0)$, $(A, m - 1, 0)$, $(B, n - 1, 0)$ and q_{acc} . Transitions from $(i, j, 0)$ and $(i, j, 1)$ are illustrated in Figure 4, where transitions not used in the proof are shown in grey.

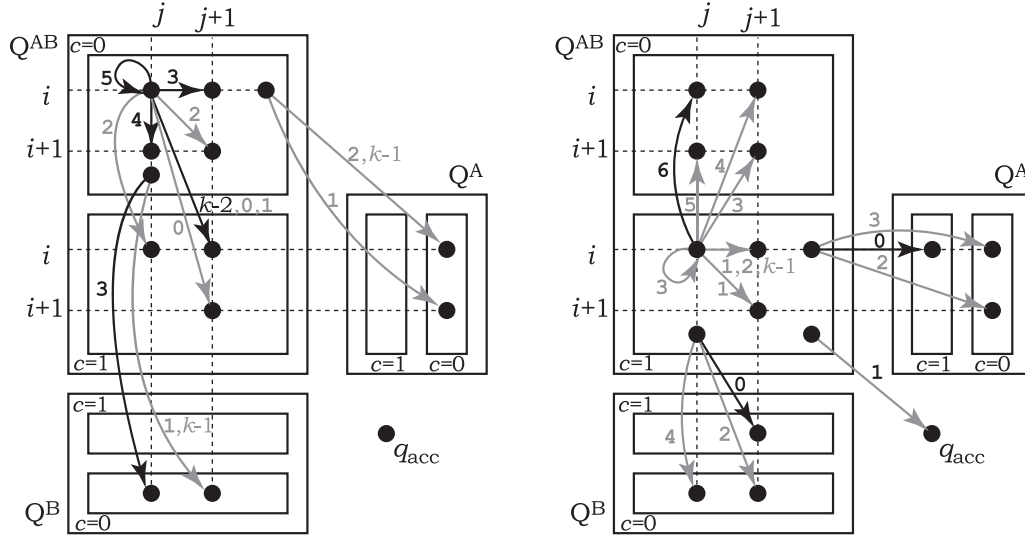


Figure 4: NFA M : transitions out of states $(i, j, 0)$ and $(i, j, 1)$.

Our goal is to show that every NFA for the language L requires at least $2mn + 2m + 2n + 1$ states. We prove this by describing a fooling set for the language L of size $2mn + 2m + 2n + 1$. Consider the following sets of pairs of strings, in which the difference $j - 1$ is modulo n (that is, $j - 1 = n - 1$ for $j = 0$):

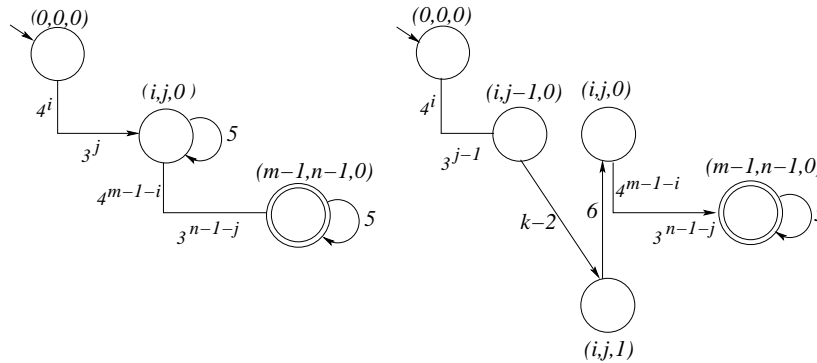
$$\begin{aligned} \mathcal{A} &= \{(4^i 3^j, 54^{m-1-i} 3^{n-1-j} 5) \mid i = 0, 1, \dots, m-1, j = 0, 1, \dots, n-1\}, \\ \mathcal{B} &= \{(4^i 3^{j-1} (k-2), 64^{m-1-i} 3^{n-1-j} 5) \mid i = 0, 1, \dots, m-1, j = 0, 1, \dots, n-1\}, \\ \mathcal{C} &= \{(4^i 3^{n-2} (k-2) 0, 31^{m-1-i} 22) \mid i = 0, 1, \dots, m-1\} \cup \\ &\quad \{(4^i 3^{n-2} (k-2) 03, 1^{m-1-i} 22) \mid i = 0, 1, \dots, m-1\}, \\ \mathcal{D} &= \{(4^{m-1} 3^{n-1} (k-2) 00^j, 0^{n-1-j} 41^{n-1} 33) \mid j = 0, 1, \dots, n-1\} \cup \\ &\quad \{(4^{m-1} 3^{n-1} (k-2) 0^n 41^j, 1^{n-1-j} 33) \mid j = 0, 1, \dots, n-1\}. \end{aligned}$$

Let $\mathcal{F} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. Let us show that the set \mathcal{F} is a fooling set for L , that is,

- (F1) for each pair (x, y) in \mathcal{F} , the string xy is in L ;
- (F2) if (x, y) and (u, v) are two different pairs in \mathcal{F} , then $xv \notin L$ or $uy \notin L$.

We prove the statement (F1) by examination of each pair:

- If (x, y) is a pair in \mathcal{A} , then $xy = 4^i 3^j 5 4^{m-1-i} 3^{n-1-j} 5$. The initial state $(0, 0, 0)$ of M goes to state $(i, j, 0)$ by $4^i 3^j$, which goes to itself by 5, and then to the accepting state $(m-1, n-1, 0)$ by $4^{m-1-i} 3^{n-1-j} 5$. Thus xy is accepted by M , and so is in L . This case is illustrated in Figure 5, left.
- If (x, y) is a pair in \mathcal{B} , then $xy = 4^i 3^{j-1} (k-2) 6 4^{m-1-i} 3^{n-1-j} 5$. State $(0, 0, 0)$ goes to state $(i, j-1, 0)$ by $4^i 3^{j-1}$, which goes to state $(i, j, 1)$ by $k-2$. State $(i, j, 1)$ goes to state $(i, j, 0)$ by 6, and then to the accepting state $(m-1, n-1, 0)$ by $4^{m-1-i} 3^{n-1-j} 5$, which is shown in Figure 5, right.

Figure 5: A pair in \mathcal{A} and a pair in \mathcal{B} .

- If (x, y) is a pair in \mathcal{C} , then $xy = 4^i 3^{n-2} (k-2) 0 3 1^{m-1-i} 2 2$. State $(0, 0, 0)$ goes to state $(i, n-1, 1)$ by $4^i 3^{n-2} (k-2)$, which goes to state $(A, i, 1)$ by 0, and then to state $(A, i, 0)$ by 3, and to the accepting state $(A, m-1, 0)$ by $1^{m-1-i} 2 2$. This computation path is presented in Figure 6, left.
- If (x, y) is a pair in \mathcal{D} , then $xy = 4^{m-1} 3^{n-1} (k-2) 0^n 4 1^{n-1} 3 3$. State $(0, 0, 0)$ goes to $(m-1, 0, 1)$ by $4^{m-1} 3^{n-1} (k-2)$, which goes to state $(B, 1, 1)$ by 0, and then to state $(B, 0, 1)$ by 0^{n-1} , and to state $(B, 0, 0)$ by 4, and to the accepting state $(B, n-1, 0)$ by $1^{n-1} 3 3$, as shown in Figure 6, right.

Thus in all four cases, the string xy is accepted by the NFA M , and so is in the language L . This proves (F1). To prove (F2) let us consider the following seven cases:

- If (x, y) and (u, v) are two different pairs in \mathcal{A} , then

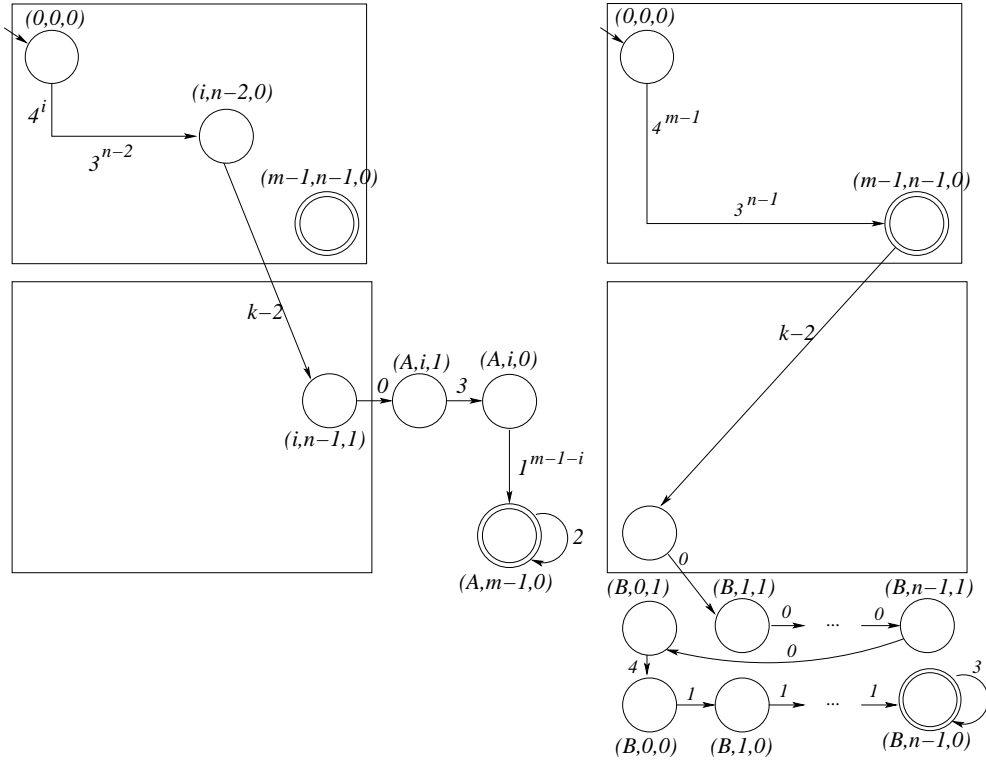
$$(x, y) = (4^i 3^j, 5 4^{m-1-i} 3^{n-1-j} 5) \text{ and } (u, v) = (4^r 3^s, 5 4^{m-1-r} 3^{n-1-s} 5),$$

where $(i, j) \neq (r, s)$. Consider the string $xv = 4^i 3^j 5 4^{m-1-r} 3^{n-1-s} 5$. Since the digit 5 cannot be read in any state $(B, p, 0)$, after reading xv , the NFA M may only be in state

$$(m-1-r+i, n-1-s+j, 0).$$

This state is rejecting if $i \neq r$ or $j \neq s$. So the string xv is not in L .

- If (x, y) is a pair in \mathcal{A} and (u, v) is a pair in \mathcal{B} , then $x = 4^i 3^j$ and $v = 6w$ for a string w . After reading x , the NFA M is either in state $(i, j, 0)$ or in a state $(B, p, 0)$. In these states, transitions by 6 are not defined. Thus the string xv is rejected by M , and so is not in L .

Figure 6: A pair in \mathcal{C} and a pair in \mathcal{D} .

- If (x, y) is a pair in $\mathcal{A} \cup \mathcal{B}$, and (u, v) is a pair in $\mathcal{C} \cup \mathcal{D}$, then $y = 5w$ or $y = 6w$ for a string w . Let us show that the string uy is not in L . Notice that after reading the string u , the NFA M is either in a state (A, p, c) or in a state (B, q, c) . In these states, no transitions by 5 and by 6 are defined. Therefore, the string uy is not in L .
- If (x, y) and (u, v) are two different pairs in \mathcal{B} , then $(x, y) = (4^i 3^{j-1} (k-2), 64^{m-1-i} 3^{n-1-j} 5)$ and $(u, v) = (4^r 3^{s-1} (k-2), 64^{m-1-r} 3^{n-1-s} 5)$, where $(i, j) \neq (r, s)$. After reading x , the nfa M may only be in state $(i, j, 1)$; notice that transitions by $k-2$ are not defined in states $(B, q, 0)$. State $(i, j, 1)$ goes to state $(i, j, 0)$ by 6. From this state, by reading $4^{m-1-r} 3^{n-1-s} 5$, the NFA may only reach the rejecting state $(m-1-r+i, n-1-s+j, 0)$. Hence the string xv is not in L .
- If (x, y) and (u, v) are two different pairs in \mathcal{C} , then we have three subcases:
 - $(x, y) = (4^i 3^{n-2} (k-2) 0, 31^{m-1-i} 22)$ and $(u, v) = (4^r 3^{n-2} (k-2) 0, 31^{m-1-r} 22)$, where $0 \leq i < r \leq m-1$.
After reading x , the NFA M is in state $(A, i, 1)$, which goes to state $(A, i, 0)$ by 3, and then to rejecting state $(A, m-1-r+i, 0)$ by $1^{m-1-r} 22$. Thus xv is not in L .
 - $(x, y) = (4^i 3^{n-2} (k-2) 03, 1^{m-1-i} 22)$ and $(u, v) = (4^r 3^{n-2} (k-2) 03, 1^{m-1-r} 22)$, where $0 \leq i < r \leq m-1$.
After reading x , the NFA is in state $(A, i, 0)$, which goes to rejecting state $(A, m-1-r+i, 0)$ by $1^{m-1-r} 22$. Thus xv is not in L .
 - $(x, y) = (4^i 3^{n-2} (k-2) 0, 31^{m-1-i} 22)$ and $(u, v) = (4^r 3^{n-2} (k-2) 03, 1^{m-1-r} 22)$.

After reading u , the NFA may only be in state $(A, r, 0)$, where it cannot read symbol 3. Thus uy is not in L .

- If (x, y) is a pair in \mathcal{C} , and (u, v) is a pair in \mathcal{D} , then $y = w22$ for a string w . Consider the string uy . After reading u , the NFA may only be in a state from Q^B (notice that $n \geq 2$). By reading w , it either hangs, or remains in Q^B , and then cannot read 22. Therefore, uy is not in L .
- If (x, y) and (u, v) are two different pairs in \mathcal{D} , then there are three subcases again:
 - $(x, y) = (4^{m-1}3^{n-1}(k-2)00^j, 0^{n-1-j}41^{n-1}33)$ and $(u, v) = (4^{m-1}3^{n-1}(k-2)00^s, 0^{n-1-s}41^{n-1}33)$, where $0 \leq j < s \leq n-1$. Since $n \geq 2$, state $(m-1, 0, 1)$ only goes to state $(B, 1, 1)$ by 0. After reading x , the NFA is in state $(B, j+1, 1)$, which goes to rejecting state $(B, n-1-s+j, 0)$ by $0^{n-1-s}41^{n-1}33$. Thus xv is not in L .
 - $(x, y) = (4^{m-1}3^{n-1}(k-2)0^n41^j, 1^{n-1-j}33)$ and $(u, v) = (4^{m-1}3^{n-1}(k-2)0^n41^s, 1^{n-1-s}33)$, where $0 \leq j < s \leq n-1$. After reading x , the NFA is in state $(B, j, 0)$, which goes to rejecting state $(B, n-1-s+j, 0)$ by $1^{n-1-s}33$. Thus xv is not in L .
 - $(x, y) = (4^{m-1}3^{n-1}(k-2)00^j, 0^{n-1-j}41^{n-1}33)$ and $(u, v) = (4^{m-1}3^{n-1}(k-2)0^n41^s, 1^{n-1-s}33)$. After reading x , the NFA M is in state $(B, j+1, 1)$, where it can read neither 1 nor 3. Thus xv is not in L .

We have shown (F2), which means that the set \mathcal{F} is a fooling set for the language L . Consider one more pair $(4^{m-1}3^{n-2}(k-2)01, \varepsilon)$. The NFA M may only be in the accepting state q_{acc} after reading the string $4^{m-1}3^{n-2}(k-2)01$. Since in this state no transitions are defined, and the second part of each pair in \mathcal{F} is nonempty, the set

$$\mathcal{F} \cup \{(4^{m-1}3^{n-2}(k-2)01, \varepsilon)\}$$

is a fooling set for the language L of size $2mn + 2m + 2n + 1$. This means that every NFA for the language L requires at least $2mn + 2m + 2n + 1$ states. \square

The above lower bound is not applicable in the case of a pair of one-state automata. In fact, in this special case the complexity of this operation is lower. While Lemma 1 gives an upper bound of 7 states for this case, 6 states are actually sufficient.

Lemma 7. *Let A and B be two 1-state NFAs over an alphabet Σ_k . Then the language $L(A) \boxplus^R L(B)$ is representable by an NFA with 6 states.*

Proof: Note that these 1-state NFAs must be partial DFAs. Following the notation of Lemma 6, let 0 denote the state in the NFA A , as well as the state in the NFA B . If NFA A has no transition on $k-1$, then state $(A, 0, 1)$ cannot be reached; similarly for NFA B and state $(B, 0, 1)$. If both A and B have transitions by $k-1$, then states $(A, 0, 1)$ and $(B, 0, 1)$ can be merged into a state q_{01} , which goes by 0 to itself, by a symbol $a+1$ to state $(A, 0, 0)$ if the NFA A has a transition by a , by a symbol $b+1$ to state $(B, 0, 0)$ if the NFA B has a transition by b , for all a, b in $\Sigma_k \setminus \{k-1\}$. \square

The next lemma establishes a matching lower bound of 6 states.

Lemma 8. *Let $\Sigma_k = \{0, 1, \dots, k-1\}$ be an alphabet with $k \geq 9$, and consider 1-state partial DFAs A and B over Σ_k which accept languages $\{2, k-1\}^*$ and $\{3, k-1\}^*$, respectively. Then every NFA for $L(A) \boxplus^R L(B)$ has at least 6 states.*

Proof: Let $L = L(A) \boxplus^R L(B)$. Let the state in the NFA A as well as the state in the NFA B be denoted by 0. Consider a six-state NFA for the language L defined in Lemma 7, with the states $(0,0,0)$, $(0,0,1)$, q_{01} , $(A,0,0)$, $(B,0,0)$ and q_{acc} . The transitions of this automaton are shown in Figure 7. Let

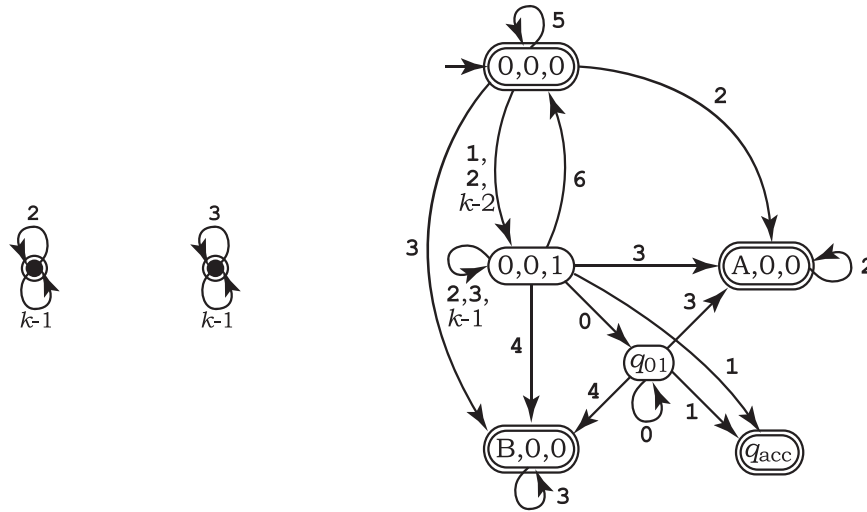


Figure 7: The 1-state NFAs A and B , and the 6-state NFA for $L(A) \boxplus^R L(B)$.

$$\mathcal{A} = \{(\varepsilon, 5), (k-2, 6), ((k-2)0, 32), ((k-2)03, 2), ((k-2)04, 3), ((k-2)01, \varepsilon)\},$$

and let us show that this set is a fooling set for the language L . Since the strings 5, $(k-2)6$, $(k-2)032$, $(k-2)043$, and $(k-2)01$ are accepted by the NFA, the statement (F1) holds for \mathcal{A} . On the other hand, the following strings are not accepted by this NFA: the string 6, any string starting with $(k-2)0$ and ending with 5 or with 6, the strings $(k-2)033$, $(k-2)0432$, $(k-2)042$, and any string $(k-2)01w$ with $w \neq \varepsilon$. This means that the statement (F2) also holds for \mathcal{A} . Hence \mathcal{A} is a fooling set for the language L , and so every NFA for this language needs at least 6 states. \square

Putting together all the above lemmata, the following result is obtained.

Theorem 9. *For every $k \geq 9$, the nondeterministic state complexity of positional addition is given by the function*

$$f_k(m, n) = \begin{cases} 6, & \text{if } m = n = 1, \\ 2mn + 2m + 2n + 1, & \text{if } m + n \geq 3. \end{cases}$$

An obvious question left open in this paper is the state complexity of positional addition with respect to deterministic finite automata. A straightforward upper bound is given by $2^{2mn+2m+2n+1}$, though calculations show that for small values of k, m, n this bound is not reached. Though the exact values of this complexity function might involve too difficult combinatorics, determining its asymptotics is an interesting problem, which is proposed for future study.

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