On the Strength of Ramsey's Theorem

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May 31, 1995

1 Introduction

In this paper we study the logical strength of Ramsey's Theorem (1930), especially of Ramsey's Theorem for partitions of pairs into two pieces.

Definition 1.1 For $X \subseteq \mathbb{N}$, let $[X]^n$ denote the size n subsets of X. Suppose that n and m are positive integers and F is a function from $[\mathbb{N}]^n$ to $\{0, \ldots, m-1\}$. We say that $H \subseteq \mathbb{N}$ is *homogeneous* for F if F is constant on $[H]^n$.

Theorem 1.2 (Ramsey) *For all positive integers n and m, if F maps $[\mathbb{N}]^n$ to $\{0, \ldots, m-1\}$ then there is an infinite set H such that H is homogeneous for F.*

If we fix n and m, we represent the above conclusion as $\mathbb{N} \rightarrow [\mathbb{N}]^n_m$.

Theorem 1.2 has a curiously noneffective proof and has been a fruitful example for mathematical logicians.

1.1 Recursion theoretic analysis

Jockusch (1972) showed that the noneffective methods in the proof of Theorem 1.2 cannot be eliminated.

Theorem 1.3 (Jockusch)

- *There is a recursive partition of $[\mathbb{N}]^3$ into 2 pieces such that $0'$ is recursive in any infinite homogeneous set.*
- *There is a recursive partition of $[\mathbb{N}]^2$ into 2 pieces with no infinite homogeneous set recursive in $0'$.*

Theorem 1.3 gives a good recursion theoretic understanding of Ramsey's theorem except for the case of partitions of $[\mathbb{N}]^2$. Jockusch posed the following question.

Question 1.4 (Jockusch) Is there a recursive partition of $[\mathbb{N}]^2$ into 2 pieces such that $0'$ is recursive in any infinite homogeneous set?

Seetapun answered Jockusch's question negatively. We present the proof of Seetapun's theorem in Section 2. We also give Seetapun's application showing that there are no nontrivial bi-introreducible subsets of $\mathbb{N}$.

1.2 Fragments of second order arithmetic

In Section 3, we analyze Ramsey's Theorem as a formal statement within second order arithmetic. To review, $P^- + I\mathbf{\Sigma}^0_1$ states the algebraic properties of addition and multiplication and the scheme that every set that is defined by a $\mathbf{\Sigma}^0_1$ formula, contains 0 and is closed under the successor function contains every natural number. Primarily, the second order systems which will concern us are RCA_0, $P^- + I\mathbf{\Sigma}^0_1$ with the scheme for recursive comprehension; WKL_0, RCA_0 with the statement that every infinite binary tree has an infinite path; and ACA_0, RCA_0 with the scheme for arithmetic comprehension. A detailed discussion of these systems can be found in (Friedman 1975).

Jockusch's theorem can be recast in terms of fragments of arithmetic:

$$RCA_0 + \mathbb{N} \rightarrow [\mathbb{N}]^3_2 \vdash ACA_0; \tag{1}$$

$$WKL_0 \not\vdash \mathbb{N} \rightarrow [\mathbb{N}]^2_2. \tag{2}$$

In (1), one notes that Jockusch's proof can be formalized in RCA_0. In (2), one must observe that there is a standard model of WKL_0 in which every set is Δ^0_2. We will say more about obtaining such a model Section 2. Then one can conclude from Theorem 1.3 that this model fails to satisfy $\mathbb{N} \rightarrow [\mathbb{N}]^2_2$.

Jockusch's question translates to the following, which was known as the 3-2 question.

Question 1.5 Does $RCA_0 + \mathbb{N} \rightarrow [\mathbb{N}]^2_2 \vdash ACA_0$?

We will show that a negative answer to Question 1.5 follows from Seetapun's solution to Jockusch's original question.

In response to Seetapun's results, Simpson asked whether $\mathbb{N} \to [\mathbb{N}]^2_2$ is conservative over RCA_0 for Π^1_1 sentences. Such is the case for WKL_0 by a theorem of Harrington. We will prove Slaman's theorem that there is a Π^0_4 sentence which is provable from $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2$ but which is not provable in RCA_0, and hence not provable in WKL_0.

2 Analysis by recursion theoretic complexity.

In this section, we prove Seetapun's theorem and answer Question 1.4. The proof that we give is due to Jockusch, which is an improved version of Seetapun's original proof.

Theorem 2.1 (Seetapun) *Fix a real Z and a partition $F : [\mathbb{N}]^2 \to \{0, 1\}$ such that F is recursive in Z. Let $C_1, C_2, \ldots$ be a countable list of reals such that for each i, $C_i \not\leq_T Z$. Then F has an infinite homogeneous set H such that for each i, $C_i \not\leq_T H$.*

2.1 Notation

We regard Turing functionals as recursively enumerable sets of axioms. In what follows, all strings will be increasing sequences in $\mathbb{N}^{<\mathbb{N}}$. In the course of the argument below, we constantly need to refer to the range of strings $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we write this as $rng(\sigma)$. Also if $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and X is a real, $\sigma \subset X$ means σ is an initial segment of X.

Definition 2.2 If ? is a Turing functional, Z is a real and $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we define

$$? \upharpoonright \sigma \oplus Z = \{\tau \mid \tau = \alpha \oplus \beta \wedge \beta \subseteq Z \wedge (\sigma \frown \alpha) \oplus \beta \in ?\}.$$

We note $? \upharpoonright \sigma \oplus Z$ is recursively enumerable in Z.

Thus $? \upharpoonright \sigma \oplus Z$ is the set of extensions of σ that together with Z force ? to converge.

2.2 Scott sets

Definition 2.3 (Scott) Fix a real Z. A Scott set $\mathcal{S}$ containing Z is defined as follows.

- $\mathcal{S}$ is a set of reals which form an ideal under Turing reducibility and recursive join.

- $Z \in \mathcal{S}$.
- If T is a Y-recursive Y-recursively bounded infinite tree and $Y \in \mathcal{S}$ then there is a path $f \in [T]$ with $f \in \mathcal{S}$.

Given a real Z, we expand the language of arithmetic by adding a unary predicate U and we add to the axioms of PA axioms for the predicate: $n \in U$ if $n \in Z$ and $n \notin U$ if $n \notin Z$. We call the resulting system PA^Z. By a relativization of a theorem of Scott, have that the reals recursively coded in a nonstandard model of PA^Z form a Scott set containing Z.

Now we may obtain maximal consistent extensions of any recursive extension of PA^Z as paths in Z-recursive binary branching trees and thus the following theorem comes into play.

Theorem 2.4 (Jockusch-Soare) *If Z is a real and C_i is a countable list of reals with each $C_i \not\leq_T Z$ then any Z-recursive binary branching tree has a path f with $C_i \not\leq f \oplus Z$.*

Noting that the above observations yield, by the Henkin construction, models recursive in paths of an appropriate Z-recursive binary branching tree and using the lemma, we obtain.

Lemma 2.5 *If Z is a real and C_i is a countable list of reals with each $C_i \not\leq_T Z$ then there is a real S and a Scott set $\mathcal{S}$ containing Z whose elements are uniformly recursive in S and for each i, $C_i \not\leq_T S$.*

We note we may also build a Scott set containing Z by iteratively applying Theorem 2.4 and then finding an upper bound on the Scott set which avoids computing any of the C_i's.

2.3 Forcing over Scott sets.

In what follows fix a real Z and a partition of pairs F recursive in Z. We will be forcing over Scott sets containing this real Z. All notions related to a partition refer to F. We will say $\{x, y\}$ is red or blue to mean that $F(x, y)$ is equal to 0 or 1, respectively.

Definition 2.6 Suppose $\langle a_0, \dots, a_k\rangle$ is a sequence of numbers and each a_i is designated red or blue. A number x is *acceptable* for the sequence if $\{a_i, x\}$ has the same color as a_i. Similarly, if $\sigma_0, \dots, \sigma_k$ are sequences as above then x is acceptable for $\langle\sigma_0, \dots, \sigma_k\rangle$ if it is acceptable for each σ_i.

Definition 2.7 If ? is a Turing functional and X is a set of numbers, a *red split* $\langle \sigma, \tau \rangle$ of ? in X consists of two axioms σ, τ in ? with $rng(\sigma), rng(\tau) \subset X$ such that $rng(\sigma)$ and $rng(\tau)$ are finite red homogeneous sets (in the sense of F) which force different values of ?.

We note we have an analogous definition of blue split or of a split relative to Z. Note that to say that ? $\restriction \sigma \oplus Z$ does not blue split in X or does not red-split in X is $\Pi^0_1(X \oplus Z)$

Definition 2.8 Let $\mathcal{S}$ be a Scott set containing Z. We define $\mathbb{P}_{\mathcal{S}}$ to be the collection of all triples $\langle \rho_R, \rho_B, X \rangle$ such that

1. $\rho_R, \rho_B \in \mathbb{N}^{<\mathbb{N}}$
2. $rng(\rho_R)$ is a finite red homogeneous set and $rng(\rho_B)$ is a finite blue homogeneous set.
3. $X \in \mathcal{S}$
4. X is an infinite set of acceptable numbers for ρ_R and ρ_B where each member of ρ_R and ρ_B is designated the obvious color.
5. $max(rng(\rho_R) \cup rng(\rho_B)) < min(X)$

Definition 2.9 If $p, q \in \mathbb{P}_{\mathcal{S}}$ with $p = \langle \rho_R, \rho_B, X \rangle$ and $q = \langle \rho'_R, \rho'_B, X' \rangle$ then $q \leq p$ if

1. $\rho_R \subseteq \rho'_R$, $\rho_B \subseteq \rho'_B$;
2. $rng(\rho'_R) - rng(\rho_R) \subset X$, $rng(\rho'_B) - rng(\rho_B) \subset X$, and $X' \subseteq X$.

If G is generic over $\mathbb{P}_{\mathcal{S}}$ then we have generic homogeneous sets ρ^G_R and ρ^G_B.

This next lemma allows us to force the generic homogeneous sets through any segment which has infinitely many acceptable numbers.

Lemma 2.10 *If σ is a string and $\langle \rho_R, \rho_B, X \rangle$ is a condition with $rng(\sigma) \subset X$ and $rng(\sigma)$ is a finite red homogeneous set, then $\langle \rho_R \frown \sigma, \rho_B, X^* \rangle$ is a condition, where X^* is the set of acceptable numbers in X for $rng(\sigma)$ when each element of σ is designated red.*

Proof: Now $rng(\rho_R \frown \sigma)$ is a finite red homogeneous set since $rng(\sigma)$ is such a set and is contained in a set of acceptable numbers for $rng(\rho_R)$. Also, X^* is an infinite collection of acceptable numbers for $rng(\rho_R \frown \sigma) \cup rng(\rho_B)$ since $X^* \subseteq X$ and each element of X^* is acceptable for $rng(\sigma)$. Thus it suffices to show $X^* \in \mathcal{S}$. To see this, X^* is recursive in $X \oplus Z$ and the finite sets $rng(\rho_R \frown \sigma)$ and $rng(\rho_B)$. But $X \oplus Z \in \mathcal{S}$ which implies $X^* \in \mathcal{S}$. ■

Lemma 2.11 *If $\langle \rho_R, \rho_B, X\rangle \Vdash \rho_R^G$ is finite, then there is a blue homogeneous set in $\mathcal{S}$.*

Proof: By moving to a stronger condition if necessary, we may assume $\langle \rho_R, \rho_B, X\rangle \Vdash \rho_R^G$ is bounded by b. Set X^* to be those numbers in X which are bigger than b. Clearly $X^* \leq_T X$ and there is no number in X^* which is colored red with infinitely many numbers in X^* for otherwise as above we may concatenate this number to ρ_R to obtain a contradiction.

We may now define a blue homogeneous set recursively in $X^* \oplus Z$. Pick a number $n_1 \in X^*$ and wait for the least number $n_2 \in X^*$ which is colored blue with n_1. If we wait forever, it is easy to see there are infinitely many numbers colored red with n_1. Inductively, we may suppose we have picked a finite homogeneous set $n_1, \ldots, n_k$. To define n_{k+1}, we wait for the least number in X^* colored blue with every $n_1, \ldots, n_k$. If no such number appears, one of the numbers n_i, $i \leq k$ is colored red with infinitely many numbers in X^*. ■

Lemma 2.12 *Let Φ be a Turing functional and $p = \langle \rho_R, \rho_B, X\rangle$ be a condition. If $p \Vdash \Phi^{\rho_R^G \oplus Z}$ is total and $\Phi \restriction \rho_R \oplus Z$ does not red-split on X, then $p \Vdash \Phi^{\rho_R^G \oplus Z} \leq_T X \oplus Z$*

Proof: Since $p \Vdash \Phi^{\rho_R^G \oplus Z}$ is total, we may expect to see an axiom in $\Phi \restriction \rho_R \oplus Z$ whose range is a finite homogeneous subset of X. The value of $\Phi \restriction \rho_R \oplus Z$ forced by this axiom must be the value forced by the axiom which applies to the generic set for otherwise we have a red-split. ■

Lemma 2.13 *Fix a real C and suppose $\langle \rho_R, \rho_B, X\rangle \Vdash\ ?^{\rho_R^G \oplus Z} = C$, then every red split of $?^{\rho_R \oplus Z}$ in X has finitely many acceptable numbers in X.*

Proof: Suppose not and we have a red-split $\langle \sigma, \tau\rangle$ of $?^{\rho_R \oplus Z}$ in X with infinitely many acceptable numbers in X. We now note $\langle \rho_R \frown \sigma, \rho_B, X^*\rangle$ is a condition (the elements of X^* are the acceptable numbers for $\rho_R \frown \sigma$ in X) and $\langle \rho_R \frown \tau, \rho_B, X^{**}\rangle$ is a condition (the elements of X^{**} are the acceptable numbers for $\rho_R \frown \tau$ in X). These two conditions are below $\langle \rho_R, \rho_B, X\rangle$ and force incompatible values of $?^{\rho_R \oplus Z}$. This is a contradiction since $\langle \rho_R, \rho_B, X\rangle \Vdash\ ?^{\rho_R^G \oplus Z} = C$. ■

Lemma 2.14 *Let ? and Φ be Turing functionals. Fix reals C and D. If $p \Vdash ?^{\rho_R^G \oplus Z} = C \wedge \Phi^{\rho_B^G \oplus Z} = D$, then $p \Vdash C \in \mathcal{S} \vee D \in \mathcal{S}$.*

Proof: Fix any condition $q \leq p$, we show there is $r \leq q$ such that $r \Vdash C \in \mathcal{S}$ or $r \Vdash D \in \mathcal{S}$.

Suppose $q = \langle \rho_R, \rho_B, Y \rangle$. We define the sequence $\langle \sigma_1, \tau_1 \rangle, \langle \sigma_2, \tau_2 \rangle, \ldots$ recursively in Y. Assume $\langle \sigma_1, \tau_1 \rangle, \ldots, \langle \sigma_n, \tau_n \rangle$ are defined such that

1. For $1 \leq i < n$, $max(rng(\sigma_i) \cup rng(\tau_i)) < min(rng(\sigma_{i+1}) \cup rng(\tau_{i+1}))$.

2. For $1 \leq i \leq n$, $\langle \sigma_i, \tau_i \rangle$ red-split $? \restriction \rho_R \oplus Z$.

Search recursively in $Y \oplus Z$ for the least axioms σ and τ which red-split $? \restriction \rho_R \oplus Z$ in Y and $max(rng(\sigma_n) \cup rng(\tau_n)) < min(rng(\sigma) \cup rng(\tau))$.

If no such axioms are found, let $b = max(rng(\sigma_n) \cup rng(\tau_n))$. Set Y^* to be those numbers in Y which are bigger than b. Y^* is recursive in Y so $Y^* \in \mathcal{S}$. We now see $\langle \rho_R, \rho_B, Y^* \rangle \Vdash ?^{\rho_R^G \oplus Z}$ is total and $? \restriction \rho_R \oplus Z$ does not red-split in Y^*. Applying Lemma 2.12 now yields the result.

Thus we may assume we have an infinite sequence $\langle \sigma_1, \tau_1 \rangle, \langle \sigma_2, \tau_2 \rangle, \ldots$. We now let $T = \{\alpha \in \mathbb{N}^{<\mathbb{N}} \mid \alpha(n) \in rng(\sigma_n) \cup rng(\tau_n)\}$. T is a $Y \oplus Z$-recursive, $Y \oplus Z$-recursively bounded finitely branching tree.

Set $U = \{\beta \mid \Phi^{\rho_B \oplus Z}$ does not blue-split along β in $lh(\beta)$ steps$\}$. U is a Z-recursive tree. We now distinguish two cases.

In case one, we suppose first $T \cap U$ is finite with bound l and obtain a contradiction. We may suppose each of the red-splits $\langle \sigma_i, \tau_i \rangle$ has finitely many acceptable numbers in any subset of Y, for otherwise Lemma 2.10 yields a contradiction. We now show there is a node γ of length l in T such that if we designate each member of γ blue, γ has infinitely many acceptable numbers in Y. We do this by induction. Since there are finitely many acceptable numbers for $\langle \sigma_1, \tau_1 \rangle$ there is a number $k_1 \in rng(\sigma_1) \cup rng(\tau_1)$ such that there are infinitely many numbers in Y colored blue with k_1. By induction, suppose we have $\gamma_j = \langle k_1, k_2, \ldots, k_j \rangle$ such that there is an infinite set $Y_j \subset Y$ of acceptable numbers for γ_j where we designate each member of γ_j blue. There are only be finitely many elements of Y and hence of Y_{j+1} which are acceptable for $\langle \sigma_{j+1}, \tau_{j+1} \rangle$ when they are designated red. Now we observe there is a number $k_{j+1} \in rng(\sigma_{j+1}) \cup rng(\tau_{j+1})$ such that there are infinitely many numbers in Y_j colored blue with k_{j+1}. We now set $\gamma_{j+1} = \langle k_1, k_2, \ldots, k_{j+1} \rangle$. The string γ_l is a node of length l for which there are infinitely many acceptable numbers in Y. Now since the height of $T \cap U$ is less than l, we see $\Phi \restriction \rho_B \oplus Z$ blue-splits along γ. Thus we have a blue split with infinitely many acceptable numbers in Y. This is the desired contradiction (to Lemma 2.13 and the assumption that $\langle \rho_R, \rho_B, X\rangle \Vdash \Phi^{\rho_B^G \oplus Z} = D$).

In case two, we suppose $T \cap U$ is infinite. Now $T \cap U$ is a $Y \oplus Z$-recursive, $Y \oplus Z$-recursively bounded finitely branching tree. Thus there is a path $Y' \in \mathcal{S}$. We now have $\langle \rho_R, \rho_B, Y'\rangle \Vdash \Phi^{\rho_B^G \oplus Z}$ is total and $\Phi \upharpoonright \rho_B \oplus Z$ does not blue-split in Y'. Now we apply Lemma 2.12. ■

In fact, the meet of the two generic homogeneous sets is contained in $\mathcal{S}$.

Lemma 2.15 *Let ? and Φ be a Turing functionals. If $p \Vdash ?^{\rho_R^G \oplus Z} = \Phi^{\rho_B^G \oplus Z}$, then $p \Vdash ?^{\rho_R^G \oplus Z} = \Phi^{\rho_B^G \oplus Z} \in \mathcal{S}$*

Proof: Fix any condition $q \leq p$, we show there is $r \leq q$ such that $r \Vdash ?^{\rho_R^G \oplus Z} = \Phi^{\rho_B^G \oplus Z} \in \mathcal{S}$. Suppose $q = \langle \rho_R, \rho_B, Y\rangle$. It suffices to derive a contradiction in the case where we have a red-split or a blue-split with infinitely many acceptable numbers in Y. Let us suppose we have a red-split $\langle \sigma, \tau\rangle$ of $? \upharpoonright \rho_R \oplus Z$ with infinitely many acceptable numbers in Y. Consider the condition $q' = \langle \rho_R, \rho_B, Y^*\rangle$ where Y^* is the set of acceptable numbers for $\langle \sigma, \tau\rangle$ in Y. Now $q' \leq q$ so we may find an axiom α in $\Phi \upharpoonright \rho_B \oplus Z$ with infinitely many acceptable numbers in Y^*. Set Y^{**} to be the acceptable numbers for α in Y^*. We may now choose the axiom β of the red split which forces a value of $? \upharpoonright \rho_R \oplus Z$ different from the value of $\Phi^{\rho_B \oplus Z}$ forced by α. The condition $q'' = \langle \rho_R \frown \beta, \rho_B \frown \alpha, Y^{**}\rangle$ is below q and $q'' \Vdash ?^{\rho_R^G \oplus Z} \neq \Phi^{\rho_B^G \oplus Z}$. ■

2.4 Bi-introreducible sets

The first author thanks C. G. Jockusch, Jr. for pointing out the following fact which enabled an argument of his to be combined with Theorem 2.1 to yield Theorem 2.19.

Definition 2.16 A set X is *bi-introreducible* if for every infinite set Y, if Y is a subset of X or of the complement of X then $Y \geq_T X$.

Lemma 2.17 *If C is a nonrecursive set, then there is a set A such that $C \not\leq_T A$ and $C \leq_T A'$.*

Proof: Given C, we construct A as in the proof of the Friedberg Jump Inversion Theorem (1957). We alternate the following steps: we decide facts about the jump of A to diagonalize against computing C; we code atomic facts about C. ■

Lemma 2.18 (Jockusch) *Suppose $C \leq_T A'$ then there is a partition of pairs recursive in A such that any infinite homogeneous set is a subset of C or a subset of its complement.*

Proof: Since $C \leq_T A'$, C is an A-recursive limit of A recursive predicates. Let $C(x)[y]$ equal 0 if during th yth stage in A's approximation to C it appears that x is not an element of C. Let $C(x)[y]$ equal 1, otherwise. Consider the partition F given by

$$F(x,y) = \begin{cases} 0, & \text{if } C(x)[y] = C(y)[y]; \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that H is an infinite set which is homogeneous for F.

Suppose that x_0 and x_1 are in H, $x_0 \notin C$ and $x_1 \in C$. Fix y_0 so that for every y greater than or equal to y_0, $C(x_0)[y] = 0$ and $C(x_1)[y] = 1$. Now let y be an element of H such that y is greater than y_0. But then $C(x_0)[y] \neq C(x_1)[y]$ and H is not homogeneous. Thus, either H is contained in C or is contained in the complement of C. ∎

Theorem 2.19 (Seetapun) *The only bi-introreducible sets are the recursive sets.*

Proof: Suppose that C is not recursive. By Lemma 2.17, fix A so that C is recursive in A' but not recursive in A. By Lemma 2.18, fix F so that $F : [\mathbb{N}]^2 \to 2$, F is recursive in A, and any infinite set which is homogeneous for F is either a subset of C or a subset of the complement of C. By Theorem 2.1, fix H so that H is homogeneous for F and $H \not\geq_T C$. Then H is a counterexample to C's being bi-introreducible. ∎

3 Analysis by axiomatic strength.

3.1 Second order consequences of $\mathbb{N} \to [\mathbb{N}]^2_2$

We begin by showing that Ramsey's theorem for pairs is a relatively weak subtheory of second order arithmetic. It does not imply the arithmetic comprehension axiom.

Theorem 3.1 (Seetapun) *$WKL_0 + \mathbb{N} \to [\mathbb{N}]^2_2$ does not prove ACA_0.*

Proof: By recursion, we construct a set of reals $\mathcal{S}$ such that $\mathcal{S}$ is a Scott set; for each $X \in \mathcal{S}$, if $F : [\mathbb{N}]^2 \to 2$ is a recursive in X then there is an infinite set H such that H is homogeneous for F and $H \in \mathcal{S}$; and $0'$ is not an element of $\mathcal{S}$. We begin with the collection of recursive sets and let S_1 be a recursive real. At step n, we consider a partition $F : [\mathbb{N}]^2 \to 2$ which is recursive in some element of $\mathcal{S}_n$. By Theorem 2.1, there is an infinite set H such that H is homogeneous for F and $H \oplus S_n \not\geq_T 0'$. By Lemma 2.5, let S_{n+1} compute a Scott set $\mathcal{S}_{n+1}$ such that $H \oplus S_n \in \mathcal{S}_{n+1}$ and $S_{n+1} \not\geq_T 0'$. We let $\mathcal{S}$ be the union of the $\mathcal{S}_n$. We arrange our recursion so that for every X in $\mathcal{S}$ and F recursive in X as above, there is a step n such that we add an infinite homogeneous set for F to $\mathcal{S}$ during step n. ■

3.2 Conservation

Definition 3.2 If T_1 and T_2 are two theories and ? is a set of formulas then T_2 is *? -conservative over* T_1 if whenever $\varphi \in$? and $T_2 \vdash \varphi$ then $T_1 \vdash \varphi$.

In the analysis of WKL_0, Harrington showed that if $\mathfrak{N}$ is a countable model of RCA_0 then there is a second order model $\mathfrak{M}$ such that

- The numbers of $\mathfrak{M}$ are exactly those in $\mathfrak{N}$;
- $\mathfrak{M} \models WKL_0$.

That is, $\mathfrak{M}$ is obtained from $\mathfrak{N}$ by adjoining additional sets of numbers. The following theorem results.

Theorem 3.3 (Harrington)

- *WKL_0 is Π^1_1-conservative over RCA_0.*
- *For all n, WKL_0 is Π^0_n-conservative over $P^- + I\Sigma_1$.*

Proof: For the first claim in Harrington's theorem, suppose that φ is a Π^1_1 sentence and φ fails in some model of RCA_0. Then let $\mathfrak{N}$ be a countable model of RCA_0 in which φ fails and let $X_1, \ldots, X_n$ be sets in $\mathfrak{N}$ such that $\mathfrak{N}$ satisfies the arithmetic sentence about $X_1, \ldots, X_n$ which makes them a counterexample to φ. Now, if $\mathfrak{M}$ is an extension of $\mathfrak{N}$ obtained by adding new sets but not new natural numbers to $\mathfrak{N}$ then $X_1, \ldots, X_n$ will still satisfy the arithmetic statement that makes them a counterexample to φ even when that statement is interpreted in $\mathfrak{M}$. In short, the meaning of the arithmetic functions and relations, the relation ϵ and the arithmetic quantifiers is absolute between $\mathfrak{N}$ and $\mathfrak{M}$. Now, if $\mathfrak{M}$ is the model of WKL_0 produced by Theorem 3.3 then $\mathfrak{M}$ shows that φ is not a consequence of WKL_0.

The second claim follows from the first and the observation that if $\mathfrak{N}_0$ is a model of $P^- + I\Sigma_1$ then the second order model $\mathfrak{N}$ obtained by adding the sets which are recursively definable in $\mathfrak{N}_0$ is a model of RCA_0. ∎

Harrington produced $\mathfrak{M}$ from $\mathfrak{N}$ by iterating the forcing of the Jockusch and Soare Theorem 2.4 over $\mathfrak{N}$ to add paths through recursively bounded trees. In the proof of Theorem 2.4, one uses a forcing construction to add define a path through a recursive binary tree and control its Turing jump. Harrington showed that this forcing preserves $I\Sigma_1$.

Upon hearing of Seetapun's Theorem 2.1, Simpson raised the question of whether Seetapun's forcing could be adapted similarly.

However, there immediate difference between the two situations. Suppose that $F : [\mathbb{N}]^2 \to 2$ is a recursive partition and H is the F-homogeneous set obtained in the proof of Theorem 2.1. For any recursive function f and any condition $p = \langle \rho_R, \rho_B, X \rangle$ if X^* is the subset of X chosen so that for each n the nth elements of $rng(\rho_R) \cup X^*$ and of $rng(\rho_B) \cup X^*$ are greater than $f(n)$ then $\langle \rho_R, \rho_B, X^* \rangle$ is a condition extending p. Consequently, the function enumerating the elements of H in increasing order eventually dominates every recursive function. By a theorem of Martin (1966), H must be high.

This apparent obstruction to adapting Seetapun's forcing is insurmountable. Slaman showed that Ramsey's theorem for pairs has first order consequences beyond $P^- + I\Sigma_1$, as we shall see in Theorem 3.6.

We begin with a well known lemma.

Lemma 3.4 *There is a model $\mathfrak{N}$ such that*

- $\mathfrak{N} \models P^- + I\Sigma_1$.
- *There is an projection π of $\mathfrak{N}$ into its standard part such that π is a recursive limit in $\mathfrak{N}$.*

Proof: Let $\mathfrak{N}^*$ be a model of first order Peano Arithmetic. We define $\mathfrak{N}$ so that for every $\mathbf{\Sigma}_1$ unary formula φ with parameters from $\mathfrak{N}$, the least solution to φ in $\mathfrak{N}^*$ is an element of $\mathfrak{N}$.

We proceed by recursion. Suppose that $a_0, \ldots, a_n$ have been determined to lie in $\mathfrak{N}$ and that a_0 is not standard. Let φ_{n+1} be the $n+1$st unary $\mathbf{\Sigma}_1$ formula in the parameters $a_0, \ldots, a_n$. If $\mathfrak{N}^* \models (\forall x)\neg\varphi$ then let a_{n+1} equal a_0. Otherwise, let a_{n+1} be the least element a of $\mathfrak{N}^*$ such that $\mathfrak{N}^* \models \varphi(a)$. There is such an a since $\mathfrak{N}^*$ is a model of Peano Arithmetic. We organize our construction so that for every Σ_1 formula $\varphi(x, y_0, \ldots, y_k)$ and every $a_{i_0}, \ldots, a_{i_k}$ there is an n such that $\varphi(x, a_{i_0}, \ldots, a_{i_k})$ is equal to φ_{n+1}.

Note that by closing $\mathfrak{N}$ under the operation of adding the least solutions to $\mathbf{\Sigma}_1$ predicates we have ensured that $\mathfrak{N}$ is a Σ_1 substructure of $\mathfrak{N}^*$. But then the least solution to a $\mathbf{\Sigma}_1$ predicate with parameters from $\mathfrak{N}$ is the same whether computed in $\mathfrak{N}$ or in $\mathfrak{N}^*$. Thus, $\mathfrak{N}$ is a model of $P^- + I\Sigma_1$.

Now, in $\mathfrak{N}$ we can approximate the above construction. By recursion, let $a_{n+1}[s]$ be our approximation to a_{n+1} during stage s. First, we define $\varphi_n[s]$ to be the $\mathbf{\Sigma}_1$ formula which would be used in the above recursion should $a_0, \ldots, a_n$ equal $a_0[s], \ldots, a_n[s]$. Define $a_n[s]$ to be the least a less than or equal to s such that a is a solution to $\varphi_{n+1}[s]$ and the witnesses to its existential quantifiers are all less than s, if there is such an a; define $a_{n+1}[s]$ to be a_0, otherwise.

As $\mathfrak{N}$ is a model of $I\Sigma_1$, for each s, this recursion is well defined in $\mathfrak{N}$. For each standard n, once s is so large that for each m less than or equal to $n+1$ s bounds a_m and, if necessary, the witnesses needed to verify its existential property then $a_{n+1}[s]$ is equal to a_{n+1}. Of course the approximation need not reach a limit a_n when n is not standard. ■

Lemma 3.5 *Let $\mathfrak{N}$ be the model of Lemma 3.4. Then there is a recursive predicate F such that $\mathfrak{N}$ is a model of the following propositions.*

1. *F is a total function mapping the pairs of numbers to $\{0, 1\}$.*
2. *There is an a in $\mathfrak{N}$ such that for all h, if h is (the code for) a finite set with a many elements and h is homogeneous for F then there is a y such that for all $z > y$, $h \cup \{z\}$ is not homogeneous for F.*

Proof: Let a be a nonstandard element of $\mathfrak{N}$. Let π be the projection of $\mathfrak{N}$ described in Lemma 3.4.

We define F by recursion. During stage $s+1$, we define $F(x, s+1)$ for each x less than or equal to s as follows. If $s+1$ is less than or equal to $a+1$ then set $F(x, s+1)$ equal to 0. Otherwise, let $h_0[s+1], \ldots, h_a[s+1]$ be first a many sets of cardinality a all of whose elements are less than $s+1$ in the stage $s+1$ approximation to the ordering of the range of π. That is, we order the domain of the stage $s+1$ approximation to π by saying that x comes before y if $\pi(x)$ is approximated to be less than $\pi(y)$ during stage $s+1$. Define F so that for each i less than or equal to a, $h_i \cup \{s+1\}$ is not F homogeneous. This may be accomplished by recursion on i: choose an element x_i from h_i so that $F(x_i, s+1)$ is not defined, possible since the recursion has taken less than a steps and h_i has a many elements; define

$F(x_i, s+1)$ different from the value of F on the first two elements of h_i. Now, define $F(x, s+1)$ to be 0 for each x for which the previous recursion did not decide the value of $F(x, s+1)$. By $I\Sigma_1$ in $\mathfrak{N}$, $F(x, y)$ is defined for all $x < y$ in $\mathfrak{N}$.

For every set h with a many elements there is a standard n and a t such that for all $s+1 > t$ h is the nth element of the domain of the approximation to π during stage $s+1$. Then for every $s+1$ greater than t, $h \cup \{s+1\}$ is not homogeneous for F. ■

Theorem 3.6 (Slaman) *There is a Π^0_4 statement φ such that $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2 \vdash \varphi$ and $RCA_0 \not\vdash \varphi$.*

Proof: Suppose that $\mathfrak{M}$ is a model of $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2$. Suppose that F is a recursive partition of pairs into two pieces in $\mathfrak{M}$. For each a in $\mathfrak{M}$, the first a many elements of an infinite homogeneous set H for F would have infinitely many one point homogeneous extensions, namely those given by the larger elements of H. Thus, we may conclude that $\mathfrak{M}$ does not satisfy item 2 of Lemma 3.5. Counting the quantifiers, $\mathfrak{M}$ must satisfy the Π^0_4 statement which is the negation of Item 2.

Now, since $\mathfrak{N}$ does not satisfy this Π^0_4 statement it cannot be provable from RCA_0. ■

3.3 The cardinality scheme

Definition 3.7 We let ? be a set of formulas and define the *cardinality scheme* C? for ?. If $\varphi(x, y) \in$? then the universal closure of the following formula is in C?: *If $\varphi(x, y)$ defines an injective function then its range is unbounded.*

Let C be the $\bigcup_{n \in \mathbb{N}} C\Sigma_n$.

Remark 3.8 Our proof of Theorem 3.6 shows that $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2$ proves C? for ? the set of formulas which define functions as a recursive limit.

Slaman gave an examples of models of P^- with an additional unary predicate U which were models of $I\Sigma_k(U) + C(U)$ but not models of $PA(U)$. Slaman posed the question, answered by Kaye (1994a) with the following theorem, of whether the same theorem is true when the extra predicate is removed.

Theorem 3.9 (Kaye) *For each k there is a model of $P^- + B\Sigma_k + C$ which is not a model of $I\Sigma_k$.*

In fact, Kaye has uncovered a great deal of information on models of C and its variants. See also (Kaye 1994b).

4 Questions and further remarks

4.1 A recursion theoretic question.

A particular case of Theorem 2.1 states that there is no recursive partition of pairs such that every infinite homogeneous set computes $0'$. However, the forcing to produce the example homogeneous which avoid the cone above $0'$ produces high sets. We observed that any notion of forcing which produces low generic sets is likely to lead to a conservation theorem, as in Theorem 3.3. For another example, Brown and Simpson (1993) proved that the Baire Category Theorem (suitably stated as BCT-$\mathbf{\Pi}^0_1$) is Π^1_1 conservative over RCA_0. Their proof rests on showng that Cohen forcing preserves $I\mathbf{\Sigma}_1$. Of course, Cohen's forcing with finite conditions is well known to produce sets G whose Turing jump is well behaved.

Question 4.1 Does there exist an n such that every $F : [\mathbb{N}]^2 \to 2$ has an infinite homogeneous set H such that $H^{(n)}$ is recursive in $F^{(n)}$? Here $H^{(n)}$ and $F^{(n)}$ refer to the nth iterates of the Turing jump applied to H and F, respectively.

One would expect that an affirmative answer to Question 4.1 would lead to a Π^1_1 conservation theorem over $RCA_0 + I\mathbf{\Sigma}^0_n$, for that n which appears in the affirmative answer to the question.

4.2 Fragments of arithmetic.

Theorem 3.1 gives the impression that the principal $\mathbb{N} \to [\mathbb{N}]^2_2$ produces a relatively weak fragment of second order arithmetic. However, a curious restriction appears in its proof. Seetapun's notion of forcing to construct homogeneous sets requires that the conditions be drawn from a Scott set. To iterate this forcing and produce a model of $\mathbb{N} \to [\mathbb{N}]^2_2$, one must also iterate the forcing to produce a model of WKL_0.

Question 4.2 (Seetapun) Does $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2 \vdash WKL_0$?

Question 4.3 (Slaman) Characterize the set of first order consequences of $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2$.

- Does $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2$ prove PA? Does $RCA_0 + \mathbb{N} \to [\mathbb{N}]^2_2$ prove C?

- Is there an n such that $RCA_0 + \mathbb{N} \rightarrow [\mathbb{N}]^2_2$ is conservative over $P^- + I\Sigma_n$ for sentences in first order arithmetic?

In Figure 1, we display the known relationships between the subsystems of ACA_0 introduced by Friedman; $BCT\text{-}\mathbf{\Pi}^0_1$, an equivalent to the version of the Baire Category Theorem studied by Brown and Simpson; Ramsey's Theorem for pairs, as studied here; and Ramsey's Theorem for partitions of pairs into finitely many pieces. The calculations involving $\mathbb{N} \rightarrow [\mathbb{N}]^2_{<\mathbb{N}}$ may be found in (Mytilinaios and Slaman 1994). Solid arrows indicate implication; dashed arrows indicate that whether implication holds is not known; and dotted arrows indicate going from a second order theory to the set of its first order consequences.

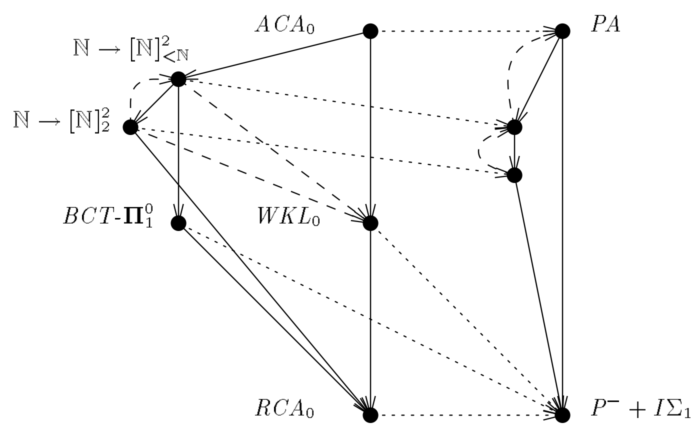

Figure 1: Subsystems of ACA_0 and their first order parts.

The picture one obtains is that the ordering by direct provability of subsystems of analysis is complicated, even for these few natural examples. In addition to the questions that we raised above concerning the unknown features of this ordering, we wonder whether there is a clearer way to organize these systems. Perhaps the only workable answer is to adopt the ordering by relative consistency, as has been adopted in axiomatic set theory.

Question 4.4
- Does $RCA_0 + \mathbb{N} \rightarrow [\mathbb{N}]^2_2$ prove the consistency of $P^- + I\Sigma_1$?
- What is the consistence strength of $RCA_0 + \mathbb{N} \rightarrow [\mathbb{N}]^2_2$.

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