COINCIDENCE INVARIANTS AND HIGHER REIDEMEISTER TRACES

KATE PONTO

Abstract. The Lefschetz number and fixed point index can be thought of as two different descriptions of the same invariant. The Lefschetz number is algebraic and defined using homology. The index is defined more directly from the topology and is a stable homotopy class. Both the Lefschetz number and index admit generalizations to coincidences and the comparison of these invariants retains its central role. In this paper we show that the identification of the Lefschetz number and index using formal properties of the symmetric monoidal trace extends to coincidence invariants. This perspective on the coincidence index and Lefschetz number also suggests difficulties for generalizations to a coincidence Reidemeister trace.

Introduction

A **coincidence point** for a pair of maps $f, g\colon M \to N$ is a point x of M such that $f(x) = g(x)$. Coincidence points are a natural generalization of fixed points and there is a corresponding generalization of the Lefschetz fixed point theorem.

Theorem A. *[12] Suppose M and N are closed, smooth, $\mathbb{Q}$-orientable manifolds of the same dimension and $f, g\colon M \to N$ are continuous maps. If f and g have no coincidence points then the Lefschetz number of f and g*

$$L(f,g) \coloneqq \sum_i (-1)^i \mathrm{tr} \left(\begin{array}{ccccc} H_i(M;\mathbb{Q}) & \xrightarrow{f_*} & H_i(N;\mathbb{Q}) & & H_i(M;\mathbb{Q}) \\ & & {\scriptstyle -\cap[N]}\uparrow & & {\scriptstyle -\cap[M]}\uparrow \\ & & H^{\dim(N)-i}(N;\mathbb{Q}) & \xrightarrow{g^*} & H^{\dim(N)-i}(M;\mathbb{Q}) \end{array} \right)$$

is zero.

The vertical maps above are the Poincaré duality isomorphism and they play in essential role in the definition of $L(f,g)$. The main result of this note is to give a simple proof of the following generalization.

Theorem B. *Suppose M and N are closed, smooth manifolds and*

$$\theta\colon T\nu_{\triangle\subset N\times N} \wedge K \to L \wedge M_+$$

Date: July 14, 2014.
719 Patterson Office Tower, Department of Mathematics, University of Kentucky, Lexington, KY, kate.ponto@uky.edu
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is a stable map for spaces (or spectra) K and L. If continuous maps $f,g\colon M\to N$ have no coincidence points then the Lefschetz number of f and g relative to θ

$$L_\theta(f,g) \coloneqq \sum_i (-1)^i \mathrm{tr}\left(H_i(M;\mathbb{Q}) \xrightarrow{(f\times g)_*} H_i(T\nu_{\triangle\subset N\times N};\mathbb{Q}) \xrightarrow{\theta_*} H_i(M;\mathbb{Q}) \right)$$

is zero.

While the formulation of this result using the map θ is nonstandard, Theorem A and the generalizations in [18, 19] follow from this result.

The proofs here use duality and trace in symmetric monoidal categories [3, 15]. This allows for short, conceptual proofs that are very similar to the corresponding proof of the Lefschetz fixed point theorem [3] and Reidemeister trace [14].

Remark. In this paper we focus on closed smooth manifolds. Many of the results could also be stated in terms of compact ENRs (or finite CW complexes) by replacing normal bundles by mapping cylinders.

1. Lefschetz Numbers

Following [4–6, 8–11, 18, 19] we start from the observation that the coincidence points of maps $f,g\colon M\to N$ are the intersection of the diagonal in N with the image of the product

$$f\times g\colon M\to N\times N.$$

If we use $\nu_{\triangle\subset N\times N}$ to denote the normal bundle of the diagonal in $N\times N$ and $T\nu_{\triangle\subset N\times N}$ to denote the Thom space, for coincidence free maps the composite

$$M \xrightarrow{f\times g} N\times N \longrightarrow T\nu_{\triangle\subset N\times N}$$

where the second map is the Thom collapse will be homotopic to the constant map at the collapse point. We denote this composite by $f\times g$ since context will make the meaning unambiguous.

To define the invariants described in the introduction and prove comparison results we need some additional structure. In this paper we encode that structure using a stable map

$$\theta\colon T\nu_{\triangle\subset N\times N}\wedge K\to L\wedge M_+.$$

If f and g have no coincidences the composite

$$M_+\wedge K \xrightarrow{(f\times g)\wedge \mathrm{id}_K} T\nu_{\triangle\subset N\times N}\wedge K \xrightarrow{\theta} L\wedge M_+$$

will be homotopically trivial.

Example 1.1. Let k_* be a homology theory and suppose M and N are k_*-orientable. If k is the spectrum associated to k_* there are Thom isomorphisms [13, 20.5.8]

$$k\wedge T\nu_M \cong k\wedge \Sigma^{p-m}M_+ \quad \text{and} \quad k\wedge T\nu_{\triangle\subset N\times N}\cong k\wedge \Sigma^n N_+$$

where ν_M is the normal bundle of an embedding of M in $\mathbb{R}^p$ for some integer p, m is the dimension of M, and n is the dimension of N. These induce the familiar homology isomorphisms

$$\tilde{k}_*(T\nu_M)\cong \tilde{k}_*(\Sigma^{p-m}M_+) \quad \text{and} \quad \tilde{k}_*(T\nu_{\triangle\subset N\times N})\cong \tilde{k}_*(\Sigma^n N_+)$$

and define a map

$$\begin{array}{ccccc} \theta\colon k\wedge S^{p-m}\wedge T\nu_{\triangle\subset N\times N} & & & & k\wedge S^{p-2m+n}\wedge M_+ \\ \downarrow\sim & & & & \sim\uparrow \\ k\wedge S^{p-m+n}\wedge N_+ & \longrightarrow & k\wedge S^{p-m+n} & \longrightarrow & k\wedge S^{-m+n}\wedge T\nu_M \end{array}$$

where the first horizontal map is the projection map for N and the second is the Thom collapse for an embedding of M in $\mathbb{R}^p$.

Example 1.2. Classes $\alpha\in k_a(M_+)$ and $\beta\in k^b(T\nu_{\triangle\subset N\times N})$ are associated to stable maps $\alpha\colon S^a\to M_+\wedge k$ and $\beta\colon T\nu_{\triangle\subset N\times N}\wedge S^{-b}\to k$. If k is multiplicative define a map θ by

$$\theta\colon S^{a-b}\wedge T\nu_{\triangle\subset N\times N}\xrightarrow{\beta} k\wedge S^a\xrightarrow{\alpha} k\wedge k\wedge M_+\longrightarrow k\wedge M_+\ .$$

This example corresponds to the results in [18, 19].

We now generalize the proof of the Lefschetz fixed point theorem from [3].

Definition 1.3. The **coincidence index of f and g relative to θ** is the symmetric monoidal trace of the composite

$$K\wedge M_+\xrightarrow{\mathrm{id}\wedge(f\times g)} K\wedge T\nu_{\triangle\subset N\times N}\xrightarrow{\theta} M_+\wedge L.$$

By definition the index is the composite

$$\begin{array}{ccccc} K\wedge S^p & & & & L\wedge S^p \\ \downarrow{\scriptstyle \mathrm{id}\wedge\eta} & & & & \uparrow{\scriptstyle \epsilon\wedge\mathrm{id}} \\ K\wedge M_+\wedge T\nu_M & \xrightarrow{\mathrm{id}\wedge(f\times g)} & K\wedge T\nu_{\triangle\subset N\times N}\wedge T\nu_M & \xrightarrow{\theta} & M_+\wedge L\wedge T\nu_M \end{array}$$

where η is the coevaluation for the dual pair $(M_+, T\nu_M)$ and ϵ is the evaluation [3, 15]. This trace is an example of the twisted traces in [15] that generalizes the trace in [3]. The homotopy class of the index is clearly trivial if f and g have no coincidences or are homotopic to maps without coincidences.

There is also a corresponding *Lefschetz number*. It is the symmetric monoidal trace of the composite

$$H_*(K)\otimes H_*(M)\xrightarrow{\mathrm{id}\otimes(f\times g)_*} H_*(K)\otimes H_*(T\nu_{\triangle\subset N\times N})\xrightarrow{\theta_*} H_*(L)\otimes H_*(M)$$

where $H_*(-)$ is rational homology or any other homology theory with a Künneth isomorphism. This trace is a generalization of the usual trace from linear algebra defined using the diagonal entries of a matrix determined by a basis

Functoriality of the symmetric monoidal trace [3, 15] implies the following result.

Theorem B. *The map induced on homology by the coincidence index of f and g relative to θ is the same as the Lefschetz number of $H_*(f)$ and $H_*(g)$ relative to $H_*(\theta)$.*

This theorem is the coincidence generalization of the familiar Lefschetz-Hopf result that compares topologically and algebraically defined invariants.

If M and N are k_*-orientable the k_*-**Lefschetz number** of f and g, denoted $L_{k_*}(f,g)$, is

$$\sum_i (-1)^i \mathrm{tr} \left(\begin{array}{ccccc} \tilde{k}_i(M_+) & \xrightarrow{f_*} & \tilde{k}_i(N_+) & & \tilde{k}_{(q-n)-(p-m)}(S^0)\otimes \tilde{k}_i(M_+) \\ & & \downarrow \cong & & \cong \uparrow \\ & & \tilde{k}_{i+q-n}(T\nu_N) & \xrightarrow{(Dg)_*} & \tilde{k}_{i+q-n}(T\nu_M) \end{array} \right)$$

Theorem 1.4. *If k_* has a Künneth isomorphism and M and N are closed smooth k-orientable manifolds the stable homotopy class of the k_*-index is the same as $L_{k_*}(f,g)$.*

Since Poincaré duality is the composite of Spanier-Whitehead duality and the Thom isomorphism $L_{H_*(-,\mathbb{Q})}(f,g)$ agrees with the invariant in Theorem A.

Proof. The k_*-**index** of f and g is the symmetric monoidal trace of the composite

$$k\wedge S^m \wedge M_+ \xrightarrow{\mathrm{id}\wedge \mathrm{id}\wedge (f\times g)} k\wedge S^m\wedge T\nu_{\triangle\subset N\times N} \xrightarrow{\theta} k\wedge S^n \wedge M_+.$$

This is a map $k_*(S^m)\to k_*(S^n)$. A diagram chase shows the trace of the composite

$$\tilde{k}_*(S^m\wedge M_+) \xrightarrow{k_*(f\times g)} \tilde{k}_*(T\nu_{\triangle\subset N\times N}) \xrightarrow{k_*(\theta)} \tilde{k}_*(S^n\wedge M_+)$$

is the trace of

$$\begin{array}{ccccc} \tilde{k}_i(M_+) & \xrightarrow{f_*} & \tilde{k}_i(N_+) & & \tilde{k}_{(q-n)-(p-m)}(S^0)\otimes \tilde{k}_i(M_+) \\ & & \downarrow \cong & & \cong \uparrow \\ & & \tilde{k}_{i+q-n}(T\nu_N) & \xrightarrow{(Dg)_*} & \tilde{k}_{i+q-n}(T\nu_M) \end{array}$$

If M_+ is dualizable and k_* satisfies a Künneth isomorphism then $\tilde{k}_*(M_+)$ is also dualizable. The result then follows from the independence of the symmetric monoidal trace from the choice of dual pair [3, 15]. □

This theorem followed Example 1.1. The same approach gives an analogous statement for Example 1.2. In this case we define the **Lefschetz number** relative to α and β, $L_{k_*,\alpha,\beta}(f,g)$, to be

$$\sum_i (-1)^i \mathrm{tr} \left(\begin{array}{ccccc} \tilde{k}_i(M_+) & \xrightarrow{f_*\times g_*} & \tilde{k}_i(T\nu_{\triangle\subset N\times N}) & & \\ & & \downarrow \beta & & \\ & & \tilde{k}_i(S^b) & \xrightarrow{\alpha} & \tilde{k}_0(S^{b-a})\otimes \tilde{k}_i(M_+) \end{array} \right)$$

Functoriality again gives a comparison of topologically and algebraically defined invariants.

Theorem 1.5. *If k_* has a Künneth isomorphism and M and N are closed smooth manifolds the k_*-index relative to $\alpha\in k_a(M_+)$ and $\beta \in k^b(T\nu_{\triangle\subset N\times N})$ is the same as $L_{k_*,\alpha,\beta}(f,g)$.*

Note that [15, 4.4 and 5.5] imply $L_{k_*,\alpha,\beta}(f,g)$ is the composite

$$\tilde{k}_i(S^b) \xrightarrow{\alpha} \tilde{k}_0(S^{b-a}) \otimes \tilde{k}_*(M_+) \xrightarrow{f_* \times g_*} \tilde{k}_0(S^{b-a}) \otimes \tilde{k}_*(T\nu_{\triangle \subset N \times N}) \xrightarrow{\beta} \tilde{k}_i(S^{2b-a})$$

Remark 1.6. The top composite in the commutative diagram below is the k_* index and the composite along the bottom is the coincidence index from [19]

$$\begin{array}{ccccccccc}
S^p \xrightarrow{\eta} & M_+ \wedge T\nu_M & \xrightarrow{(f\times g)\wedge 1} & T\nu_\triangle \wedge T\nu_M & \xrightarrow{\beta \wedge 1} & k \wedge S^b \wedge T\nu_M & \xrightarrow{\alpha\wedge 1} & k \wedge S^{b-a} \wedge M \wedge T\nu_M \\
\searrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \epsilon \\
& \mathrm{Hom}(M,M) & \xrightarrow{(f\times g)_*} & \mathrm{Hom}(M,T\nu_\triangle) & \xrightarrow{\beta_*} & \mathrm{Hom}(M,S^b\wedge k) & \longrightarrow & S^{b-a+p}\wedge k
\end{array}$$

With this comparison Theorem 1.5 recovers results from [18, 19].

If N is orientable we can choose β in the statement above to be the fundamental class of N and define a map $g^\alpha\colon N \to M$ by

$$\begin{array}{ccccccc}
S^{q-n+p}\wedge N & & & & & & M \wedge S^{q-2a} \\
\downarrow \sim & & & & & & \uparrow \hat{\alpha} \\
S^p \wedge T\nu_N & \xrightarrow{\mathrm{id}\wedge\eta} & T\nu_N \wedge M \wedge T\nu_M & \xrightarrow{\mathrm{id}\wedge g \wedge \mathrm{id}} & T\nu_N \wedge N \wedge T\nu_M & \xrightarrow{\mathrm{id}\wedge\epsilon} & S^q \wedge T\nu_M
\end{array}$$

where $\hat{\alpha}$ is the adjoint of the map $T\nu_M \wedge T\nu_M \xrightarrow{\alpha\wedge\alpha} S^{-2a}\wedge k \wedge k \longrightarrow S^{-2a}\wedge k$. The composite across the bottom is the dual of g. A diagram chase shows that

$$L_{k_*,\alpha,[N]}(f,g) = \sum_i (-1)^i \mathrm{tr}\left(\tilde{k}_i(M_+) \xrightarrow{f_*} \tilde{k}_i(N_+) \xrightarrow{(g^\alpha)_*} \tilde{k}_{i+q-2a}(M_+) \right)$$

where $[N]$ is the fundamental class of N. This and the theorem above imply the following corollary.

Corollary 1.7. *Suppose k_* is a homology theory, $\alpha \in k_a M$ and N is k_*-orientable. If f and g are coincidence free then $L((g^\alpha)_* f_*) = 0$.*

2. Generalizations

We finish by considering two generalizations of the Lefschetz fixed point theorem for coincidences - a similar result for intersections and the generalization to Reidemeister traces. The approach here generalizes to the first but does not appear to generalize to the second.

The failure of this approach to generalize to the Reidemeister trace is very suggestive. If there are algebraic generalizations of the Reidemeister trace to coincidences similar to the original definition in [7] we would expect to see an easy comparison of this invariant and a topological invariant using functoriality as in the previous section. Instead we have a very natural description of the Reidemeister trace for fixed points in terms of the categorical trace while the generalization to coincidences suggested by [1, 8] is fundamentally incompatible with the trace.

We start with intersections.

2.1. **Intersections.** Let Q be a submanifold of a manifold P and $f\colon M \to P$ be a continuous map. If the image of f is disjoint from Q the composite of f with the Thom collapse for the normal bundle of Q in P

$$M \xrightarrow{f} P \longrightarrow T\nu_{Q\subset P}$$

is trivial. It is homotopically trivial if f is homotopic to a map g whose image is disjoint from Q. In general the converse is not true, see [8] for a refinement that gives a necessary and sufficient condition.

As in the previous section a stable map $\theta\colon K \wedge T\nu_{Q\subset N} \to L \wedge M_+$ defines both an index and Lefschetz number.

Definition 2.1. The **intersection index of f and Q relative to θ** is the symmetric monoidal trace of the composite

$$K \wedge M_+ \xrightarrow{\mathrm{id}\wedge f} K \wedge T\nu_{Q\subset N} \xrightarrow{\theta} L \wedge M_+.$$

The **Lefschetz number** is the symmetric monoidal trace of the composite

$$H_*(K) \otimes H_*(M) \xrightarrow{\mathrm{id}\otimes f_*} H_*(K) \otimes H_*(T\nu_{Q\subset N}) \xrightarrow{\theta_*} H_*(L) \otimes H_*(M)$$

where H_* is rational homology.

With these definitions Theorem B and its proof generalize immediately.

Theorem 2.2. *The map induced on homology by the coincidence index of f and Q relative to θ is the same as the Lefschetz number of f and Q relative to $H_*(\theta)$.*

The other examples in the previous section generalize similarly.

2.2. **Reidemeister trace.** Now we consider corresponding generalizations to the Nielsen number and Reidemeister trace. There is a coincidence Nielsen number [20] but our interest here is the Reidemeister trace and so we are looking for a trace description. This requires that we consider parametrized spaces.

A **parametrized space** over a space B is a space E along with maps $\sigma\colon B \to E$ and $p\colon E \to B$ such that $p\circ\sigma$ is the identity map of B. A map of parametrized spaces commutes with both p and σ. For notation and terminology we will follow [17] which builds on [13].

Parametrized spaces are the 1-cells in a bicategory Ex. The 0-cells are topological spaces and a 1-cell from A to B is a parametrized space over $A\times B$. The two cells are fiberwise stable homotopy classes of maps. The bicategorical composition is defined using the pullback along the diagonal followed by a quotient map. In [14,16] duality and trace in symmetric monoidal categories are generalized to bicategories.

For a topological space M $S^0_M \coloneqq M \amalg M$. For a continuous map $g\colon M \to N$ let

$${}_gN \coloneqq \{(x,\gamma) \in M \times N^I | g(x) = \gamma(0)\} \amalg (M \times N).$$

This is a space over $M \times N$ using the map $(x,\gamma) \mapsto (x,\gamma(1))$. The space N_g is similar. These are referred to as *base change objects* and a continuous map $f\colon M \to M$ induces a fiberwise map $S_M \to S_M \odot M_f$.

Theorem 2.3. *The bicategorical trace of $S_M \to S_M \odot M_f$ is the Reidemeister trace of f.*

There are two ways to approach Theorem 2.3. We can think of the invariants defined in [8] as the definition of the Reidemeister trace and then the identification we require can be found in [1]. Alternatively, we can use a more classical description of the Reidemeister trace in terms of fixed point indices and fixed point classes and apply techniques from [14]. We give a proof using the second approach here.

Proof. The universal cover $\tilde{M} \to M$ is classified by a map $\phi\colon M \to B\pi_1(M)$ and so the $\pi_1(M)$-space $\tilde{M}$ is equivalent to the pullback $M \times_{B\pi_1(M)} E\pi_1(M)$. Using the notation above we can rewrite this as

$$\tilde{M}_+ \cong S^0_M \odot {}_\phi B(\pi_1(M)) \odot \widehat{(E\pi_1(M),\rho)}_+$$

where $\widehat{(E\pi_1(M),\rho)}_+$ is the parametrized space $E\pi_1(M) \amalg B\pi_1(M) \to B\pi_1(M)$ regarded as a space over $B\pi_1(M) \times *$. The space $E\pi_1 M$ also has an action of $\pi_1 M$ that commutes with the quotient map.

If M is a closed smooth manifold or compact ENR S^0_M is dualizable [13, 18.5.1]. The object ${}_\phi B\pi_1(M)$ is dualizable [13, 17.3.1]. For $\widehat{(E\pi_1(M),\rho)}_+$ we do not have a dual pair in a bicategory, but we do have a map

$$\triangle_! S^0_{B\pi_1(M)} \to \widehat{(E\pi_1(M),\rho)}_+ \wedge_{\pi_1(M)} (E\pi_1(M),\rho)_+$$

over $B\pi_1(M) \times B\pi_1(M)$ and a $\pi_1(M) \times \pi_1(M)$-equivariant map

$$(E\pi_1(M),\rho)_+ \odot \widehat{(E\pi_1(M),\rho)}_+ \to \pi_1(M)_+$$

which make the usual triangle diagrams for a dual pair commute. The first map is defined by lifting any path in $B\pi_1(M)$ to $E\pi_1(M)$ and then evaluating at the end points. The quotient by $\pi_1(M)$ implies this will be independent of choices. For the second map two points in the same fiber are taken to the group element that transforms one to the other.

If $\hat{f}\colon B\pi_1(M) \to B\pi_1(M)$ is the map induced by f the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ B\pi_1(M) & \xrightarrow{\hat{f}} & B\pi_1(M) \end{array}$$

defines a map $S_f \odot {}_\phi B\pi_1(M) \to {}_\phi B\pi_1(M) \odot B\pi_1(M)_{B(\hat{f})}$ [17, 3.3]. If $\pi_1(M)_{f_*}$ is the $\pi_1(M) - \pi_1(M)$ set $\pi_1(M)$ where the right action is via f_* we can define a map

$$B\pi_1(M)_{B(\hat{f})} \odot (E\pi_1(M),\rho)_+ \to (E\pi_1(M),\rho)_+ \wedge_{\pi_1(M)} \pi_1(M)_{f_*}$$

by $((\gamma, x), e) \mapsto \tilde{\gamma}(0)$ where $\tilde{\gamma}$ is a lift of γ to a path ending at $\hat{f}(e)$. This is a map over $B\pi_1(M)$ and equivariant with respect to the right action of $\pi_1(M)$.

Using the identification $\tilde{M}_+ \cong S^0_M \odot {}_\phi B(\pi_1(M)) \odot (E\pi_1(M),\rho)_+$ the composite

$$\begin{aligned} S^0_M \odot {}_\phi B\pi_1(M) \odot \widehat{(E\pi_1(M),\rho)}_+ &\xrightarrow{f\odot \mathrm{id}} S^0_M \odot M_f \odot {}_\phi B\pi_1(M) \odot \widehat{(E\pi_1(M),\rho)}_+ \\ &\longrightarrow S^0_M \odot {}_\phi B\pi_1(M) \odot B\pi_1(M)_{B(\hat{f})} \odot \widehat{(E\pi_1(M),\rho)}_+ \\ &\longrightarrow S^0_M \odot {}_\phi B\pi_1(M) \odot \widehat{(E\pi_1(M),\rho)}_+ \wedge \pi_1(M)_{f_*}. \end{aligned}$$

is the map $\tilde{f}\colon \tilde{M} \to \tilde{M} \odot (\pi_1 M)_{\hat{f}}$ induced by f. Then the trace of $\tilde{f}\colon \tilde{M} \to \tilde{M} \odot (\pi_1 M)_{\hat{f}}$, which is identified with the more classical descriptions of the Reidemeister trace in [14], can also be written as the composite [16, 7.5] [17, 5.2]

$$S^0 \xrightarrow{\mathrm{tr}(f)} \langle\langle M_f \rangle\rangle \to \langle\langle B\pi_1(M)_{B(\hat{f})} \rangle\rangle \to \langle\langle \pi_1(M)_{\hat{f}} \rangle\rangle.$$

The composite of the second and third maps takes a twisted loop in M to its associated fixed point class. □

We now attempt to mimic this description for coincidences. If Q is a submanifold of P $S^{\nu_{Q\subset P}}$ is the fiberwise one point compactification of the normal bundle of this embedding. This is a parametrized space over Q where the map is induced by the projection map for the bundle. The map $Q \to S^{\nu_{Q\subset P}}$ is the section at infinity.

Corresponding to the classical Thom collapse there is a fiberwise homotopy Pontryagin-Thom collapse for $\triangle$ in $N \times N$

$$\psi\colon S^0_{N\times N} \to S^{\nu_{\triangle\subset N\times N}} \odot {}_{i_\triangle}(N \times N)$$

defined in [1, §6] and [2, II.12]. Composing the fiberwise map induced by f and g $S^0_M \to S^0_{N\times N} \odot (N \times N)_{f\times g}$ with the homotopy Pontryagin-Thom collapse we have a map

$$S^0_M \to S^{\nu_{\triangle\subset N\times N}} \odot {}_{i_\triangle}(N \times N) \otimes (N \times N)_{f\times g}. \tag{2.4}$$

Further, this is precisely the invariant that detects intersections.

Theorem 2.5. *[8, Theorem 3.4] If $\dim(M) + 3 \leq 2\dim(N)$ the fiberwise stable homotopy class of 2.4 is trivial if and only if there is are maps $f', g'\colon M \to N$, homotopic to f and g, such that f' and g' have no coincidences.*

Then the natural composite to consider for the coincidence Reidemeister trace is

$$S^0 \to S^0_M \odot S^{\nu_M} \to S^{\nu_{\triangle\subset N\times N}} \odot {}_{i_\triangle}(N \times N) \odot (N \times N)_{f\times g} \odot S^{\nu_M}$$
$$\xrightarrow{?} S^0_M \odot {}_f N \odot N_g \odot S^{\nu_M} \to \langle\langle \Lambda^{f,g} N \rangle\rangle$$

where the second to last map would need to be a generalization of the Thom isomorphisms for M and N. To use the approach of the first section we need to rewrite the last two maps using the evaluation for the dual pair (S^0_M, S^{ν_M}). At this point we encounter the major difference between duality in monoidal categories and in bicategories - duality in symmetric monoidal categories is symmetric but it is sided in a bicategory. There is no adjunction that will allow us to introduce the the evaluation as we did in the previous section. This is a major obstruction to defining generalizations of the Reidemeister trace like those in [7] for coincidences and suggests that a very different approach may be needed.

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