QUANTITATIVE UNIFORM IN TIME CHAOS PROPAGATION FOR BOLTZMANN COLLISION PROCESSES

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Abstract. This paper is devoted to the study of mean-field limit for systems of indistinguables particles undergoing collision processes. As formulated by Kac [22] this limit is based on the *chaos propagation*, and we (1) prove and quantify this property for Boltzmann collision processes with unbounded collision rates (hard spheres or long-range interactions), (2) prove and quantify this property *uniformly in time*. This yields the first chaos propagation result for the spatially homogeneous Boltzmann equation for true (without cut-off) Maxwell molecules whose "Master equation" shares similarities with the one of a Lévy process and the first *quantitative* chaos propagation result for the spatially homogeneous Boltzmann equation for hard spheres (improvement of the convergence result of Sznitman [40]). Moreover our chaos propagation results are the first uniform in time ones for Boltzmann collision processes (to our knowledge), which partly answers the important question raised by Kac of relating the long-time behavior of a particle system with the one of its mean-field limit, and we provide as a surprising application a new proof of the well-known result of gaussian limit of rescaled marginals of uniform measure on the N-dimensional sphere as N goes to infinity (more applications will be provided in a forthcoming work). Our results are based on a new method which reduces the question of chaos propagation to the one of proving a purely functional estimate on some generator operators (*consistency estimate*) together with fine stability estimates on the flow of the limiting non-linear equation (*stability estimates*).

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Keywords: mean-field limit; quantitative; uniform in time; jump process; collision process; Boltzmann equation; Maxwell molecules; non cutoff; hard spheres.

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Contents

1. Introduction and main results

1.1. **The Boltzmann equation.** The Boltzmann equation (Cf. [10] and [11]) describes the behavior of a dilute gas when the only interactions taken into account are binary collisions. It writes

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) \tag{1.1}$$

where $Q = Q(f,f)$ is the bilinear *Boltzmann collision operator* acting only on the velocity variable.

In the case when the distribution function is assumed to be independent on the position x, we obtain the so-called *spatially homogeneous Boltzmann equation*, which reads

$$\frac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v), \qquad v \in \mathbb{R}^d, \quad t \geq 0, \tag{1.2}$$

where $d \geq 2$ is the dimension.

Let us now focus on the collision operator Q. It is defined by the bilinear symmetrized form

$$Q(g,f)(v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v-v_*|, \cos\theta) \left(g'_* f' + g' f'_* - g_* f - g f_* \right) dv_* \, d\sigma, \tag{1.3}$$

where we have used the shorthands $f = f(v)$, $f' = f(v')$, $g_* = g(v_*)$ and $g'_* = g(v'_*)$. Moreover, v' and v'_* are parametrized by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2} \sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2} \sigma, \qquad \sigma \in \mathbb{S}^{d-1}. \tag{1.4}$$

Finally, $\theta \in [0,\pi]$ is the deviation angle between $v'-v'_*$ and $v-v_*$ defined by $\cos\theta = \sigma \cdot \hat{u}$, $u = v - v_*$, $\hat{u} = u/|u|$, and B is the Boltzmann collision kernel determined by physics (related to the cross-section $\Sigma(v-v_*,\sigma)$ by the formula $B = |v-v_*| \, \Sigma$).

Boltzmann's collision operator has the fundamental properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^d} Q(f,f) \, \phi(v) \, dv = 0, \qquad \phi(v) = 1, v, |v|^2, \tag{1.5}$$

and satisfying the so-called Boltzmann's H theorem

We shall consider collision kernels $B = \Gamma(|v-v_*|) \, b(\cos\theta)$ (with Γ, b nonnegative functions). Typical physical interesting kernels are in dimension 3:

- **(HS)**: the hard spheres collision kernel $B(|v-v_*|, \cos\theta) = \mathrm{cst}\, |v-v_*|$;
- collision kernels deriving from interaction potentials $V(r) = \mathrm{cst}\, r^{-s}$, $s > 2$: $\Gamma(z) = |z|^\gamma$ with $\gamma = (s-4)/s$, and b is L^1 apart from $\theta \sim 0$, where

  $b(\cos\theta) \sim_{\theta\sim 0} C_b \, \theta^{-(d-1)-\nu}$ with $\nu = 2/s$ (see [10]) including in particular:

  (tMM): the true Maxwell molecules collision kernel when $\gamma = 0$ and $\nu = 1/2$;
- **(GMM)**: Grad's cutoff Maxwell molecules when $B = 1$.

1.2. **Deriving the Boltzmann equation from many-particle systems.** The question of deriving the Boltzmann equation from particles systems (interacting *via* Newton's laws) is a famous problem, related to the so-called 6-th Hilbert problem mentionned by Hilbert at the International Congress of Mathematics at Paris in 1900.

At least at the formal level, the correct limiting procedure has been identified by Grad [18] in the late fourties (see also [9] for mathematical formulation of the open question): it is now called the *Boltzmann-Grad* or *low density* limit. However the original question of Hilbert remains largely open, in spite of a striking breakthrough due to Lanford [25], who proved the limit for short times. The tremendous difficulty underlying this limit is the *irreversibility* of the Boltzmann equation, whereas the particle system interacting *via* Newton's laws is a reversible Hamiltonian system.

In 1954-1955, Kac [22] proposed a simplified problem in order to make mathematical progress on the question: start from the Markov process corresponding to collisions only, and try to prove the limit towards the *spatially homogeneous* Boltzmann equation. Going back to the idea of Boltzmann of "stosszahlansatz" (molecular chaos), he formulated the by now standard notion of *chaos propagation*.

Let us first define the key notion of *chaoticity* for a sequence $(f^N)_{N\geq 1}$ of probabilities on E^N, where E is some given Polish space (and we will take $E = \mathbb{R}^d$ in the applications): roughly speaking it means that

$$f^N \sim f^{\otimes N} \quad \text{when} \quad N \to \infty$$

for some given one-particle probability f on E. It was clear since Boltzmann that in the case when the joint probability density f^N of the N-particles system is tensorized into N copies $f^{\otimes N}$ of a 1-particle probability density, the latter would satisfy the limiting Boltzmann equation. Then Kac made the key remark that although in general coupling between a finite number of particles prevents any possibility of propagation of the "tensorization" property, the weaker property of chaoticity can be propagated (hopefully!) in the correct scaling limit. The application example of [22] was a simplified one-dimensional collision model inspired from the spatially homogeneous Boltzmann equation.

The framework set by Kac is our starting point. Let us emphasize that the limit performed in this setting is different from the Boltzmann-Grad limit. It is in fact a *mean-field limit*. This limiting procedure is most well-known for deriving Vlasov-like equations. In a companion paper [34] we shall develop systematically our new functional approach for Vlasov equations, McKean-Vlasov equations, and granular gases Boltzmann collision models.

1.3. **Goals, existing results and method.** Our goal in this paper is to prove (and set up a general robust method for proving) chaos propagation with *quantitative rate* in terms of the number of particles N and of the final time of observation T. Let us explain briefly what it means. The original formulation of Kac [22] of chaoticity is: a sequence $f^N \in P_{sym}(E^N)$ of symmetric probabilities on E^N is f-chaotic, for a given probability $f \in P(E)$, if for any $\ell \in \mathbb{N}^*$ and any $\varphi \in C_b(E)^{\otimes \ell}$ there holds

$$\lim_{N\to\infty} \left\langle f^N, \varphi \otimes \mathbf{1}^{N-\ell} \right\rangle = \left\langle f^{\otimes \ell}, \varphi \right\rangle$$

which amounts to the weak convergence of any marginals (see also [6] for another stronger notion of "entropic" chaoticity). Here we will deal with *quantified chaoticity*, in the sense that we measure precisely the rate of convergence in the above limit. Namely, we say that f^N is f-chaotic with rate $\varepsilon(N)$, where $\varepsilon(N) \to 0$ when $N \to \infty$ (typically $\varepsilon(N) = N^{-r}$,

$r > 0$), if for some normed space of *smooth functions* $\mathcal{F} \subset C_b(E)$ (to be precised) and for any $\ell \in \mathbb{N}^*$ there exists $K_\ell \in (0, \infty)$ such that for any $\varphi \in \mathcal{F}^{\otimes \ell}$, $\|\varphi\|_{\mathcal{F}} \le 1$, there holds

$$\left| \left\langle \Pi_\ell \left[f^N \right] - f^{\otimes \ell}, \varphi \right\rangle \right| \le K_\ell \, \varepsilon(N), \tag{1.6}$$

where $\Pi_\ell \left[f^N \right]$ stands for the ℓ-th marginal of f^N.

Now, considering a sequence of densities of a N-particles system $f^N \in C([0,\infty); P_{sym}(E^N))$ and a 1-particle density of the expected mean field limit $f \in C([0,\infty); P(E))$, we say that there is *propagation of chaos* on some time interval $[0,T]$ if the f_0-chaoticity of the initial family f_0^N implies the f_t-chaoticity of the family f_t^N for any time $t \in [0,T]$.

Moreover one can roughly classify the different questions around chaos propagation into the following layers (in parenthesis, the corresponding probabilistic interpretation for the empirical measure, see below):

(1) Proof of propagation in time of the convergence in the chaoticity definition (propagation of the law of large numbers along time).
(2) Same result with a rate $\varepsilon(N)$ as above (estimates of the rates in the law of large numbers, estimates on the size of fluctuations around the deterministic limit along time).
(3) Proof of propagation in time of the convergence to a "universal behavior" around the deterministic limit (central limit theorem). See for instance [31, 38] for related results.
(4) Proof of propagation in time of bounds of exponential type on the "rare" events far from chaoticity (large deviation estimates).
(5) Estimations (2)–(4) can be made uniformly on intervals $[0,T)$ with T finite or $T = +\infty$.

For Boltzmann collision processes, Kac [22]-[23] has proved the point (1) in the case of his baby one-dimensional model. The key point in his analysis is a clever combinatorial use of a semi-explicit form of the solution (Wild sums). It was generalized by McKean [32] to the Boltzmann collision operator but only for "Maxwell molecules with cutoff", *i.e.*, roughly when the collision kernel B above is constant. In this case the combinatorial argument of Kac can be extended. Kac raised in [22] the question of proving chaos propagation in the case of hard spheres and more generally unbounded collision kernels, although his method seemed impossible to extend (no semi-explicit combinatorial formula of the solution exists in this case).

In the seventies, Grünbaum [20] then proposed in a very compact and abstract paper another method for dealing with hard spheres, based on the Trotter-Kato formula for semigroups and a clever functional framework (partially remindful of the tools used for mean-field limit for McKean-Vlasov equations). Unfortunately this paper was incomplete for two reasons: (1) It was based on two "unproved assumptions on the Boltzmann flow" (page 328): (a) existence and uniqueness for measure solutions and (b) a smoothness assumption. Assumption (a) was indeed recently proved in [17] using Wasserstein metrics techniques and in [14] adapting the classical DiBlasio trick [13], but concerning assumption (b), although it was inspired by cutoff maxwell molecules (for which it is true), it fails for hard spheres (cf. the counterexample built by Lu and Wennberg in [29]) and is somehow "too rough" in this case. (2) A key part in the proof in this paper is the expansion of the "H_f" function, which is an a clever idea of Grünbaum (and the starting point for our idea of developing an abstract differential calculus in order to control fluctuations) — however it is again too rough and is adapted for cutoff Maxwell molecules but not hard spheres.

A completely different approach was undertaken by Sznitman in the eighties [41] (see also Tanaka [43]). Starting from the observation that Grünbaum's proof was incomplete, he gave a full proof of chaos propagation for hard spheres. His work was based on: (1) a new uniqueness result for measures for the hard spheres Boltzmann equation (based on a probabilistic reasoning on an enlarged space of "trajectories"); (2) an idea already present in Grünbaum's approach: reduce by a combinatorial argument on symmetric probabilities the question of chaos propagation to a law of large numbers on measures; (3) a new compactness result at the level of the empirical measures; (4) the identification of the limit by an "abstract test function" construction showing that the (infinite particle) system has trajectories included in the chaotic ones. Hence the method of Sznitman proves convergence but does not provide any rate for chaoticity. Let us also emphasize that Graham and Méléard in [19] have obtained a rate of convergence (of order $1/\sqrt{N}$) on any bounded finite interval of the N-particles system to the deterministic Boltzmann dynamic in the case of Maxwell molecules under Grad's cut-off hypothesis, and that Fournier and Méléard in [15, 16] have obtained the convergence of the Monte-Carlo approximation (with numerical cutoff) of the Boltzmann equation for true Maxwell molecules with a rate of convergence (depending on the numerical cutoff and on the number N of particles).

Our starting point was Grünbaum's paper [20]. Our original goal was to construct a general and robust method able to deal with mixture of jump and diffusion processes, as it occurs for granular gases (see for this point the companion paper [34]). It turns out that it lead us to develop a new theory, inspiring from more recent tools such as the course of Lions on "Mean-field games" at Collège de France, and the master courses of Méléard [33] and Villani [45] on mean-field limits. One of the byproduct of our paper is that we make fully rigorous the original intuition of Grünbaum in order to prove chaos propagation for the Boltzmann velocities jump process associated to hard spheres contact interactions.

As Grünbaum [20] we shall use a duality argument. We introduce S^N_t the semigroup associated to the flow of the N-particle system and T^N_t its *"dual" semigroup*. We also introduce S^{NL}_t the (nonlinear) semigroup associated to the meanfield dynamic (the exponent "NL" recalling that the limit semigroup is nonlinear in the most physics interesting cases) as well as T^∞_t the associated (linear) *"pushforward" semigroup* (see below for the definition). Then we will prove the above kind of convergence on the linear semigroups T^N_t and T^∞_t.

The first step consists in defining a common functional framework in which the N-particles dynamic and the limit dynamic make sense so that we can compare them. Hence we work at the level of the "full" limit space $P(P(E))$ (see below). Then we shall identify the regularity required in order to prove the "consistency estimate" between the generators G^N and G^∞ of the dual semigroups T^N_t and T^∞_t, and then prove a corresponding "stability estimate" at the level of the limiting semigroup S^{NL}_t. The latter crucial step shall lead us to introduce an abstract differential calculus for functions acting on measures endowed with various metrics.

In terms of existing open questions, this paper solves two related problems. First it proves quantitative rates for chaos propagation for hard spheres and for (non cutoff) Maxwell molecules. These two results can be seen as two advances into proving chaos propagation for collision processes with unbounded kernels (in the two physically relevant "orthogonal" directions: unboundedness is either due to growth at large velocities, or to grazing collisions). Second we provide the first *uniform in time* chaos propagation results (moreover quantitative), which answers partly the question raised by Kac of relating the long-time behavior of the N-particle system with the one of its mean-field limit.

Finally the general method that we provide for solving these problems is, we hope, interesting by itself for several reasons: (1) it is fully quantitative, (2) it is highly flexible in terms of the functional spaces used in the proof, (3) it requires a minimal amount of informations on the N-particles systems but more stability information on the limiting PDE (we intentionally presented the assumption as for the proof of the convergence of a numerical scheme, which was our "methodological model"), (4) the "differential stability" conditions that are required on the limiting PDE seem (to our knowledge) new, at least at the level of Boltzmann or more generally transport equations.

1.4. Main results.

Without waiting for the full abstract framework, let us give a slightly fuzzy version of our main results, gathered in a single theorem. The full definitions of all the objects considered shall be given in the forthcoming sections, together with fully rigorous statements (see Theorems 3.27, 4.1 and 5.1).

Theorem 1.1. *Let S_t^{NL} denotes the semigroup of the spatially homogeneous Boltzmann equation (acting on $P(\mathbb{R}^d)$) and S_t^N, $N \geq 1$ denotes the semigroup of the N-particle system satisfying a collision Markov process. Then for any (one particle) initial datum $f_0 \in P(\mathbb{R}^d)$ with compact support and the corresponding tensorized N particle initial data $f_0^{\otimes N}$, we have*

(i) *In the case where S_t^{NL} and S_t^N correspond to the* **hard spheres collision kernels (HS)***, we have for any $\ell \in \mathbb{N}^*$ and any $N \geq 2\ell$:*

$$\sup_{t\in(0,\infty)} \sup_{\varphi\in W^{1,\infty}(\mathbb{R}^d)^{\otimes\ell},\, \|\varphi\|_{W^{1,\infty}(\mathbb{R}^d)^{\otimes\ell}}\leq 1} \left\langle \Pi_\ell \left[S_t^N \left(f_0^{\otimes N} \right) \right] - S_t^{NL}(f_0)^{\otimes \ell}, \varphi \right\rangle \leq \ell\, \varepsilon(N)$$

(ii) *In the case where S_t^∞ and S_t^N correspond to the* **true (or cutoff) Maxwell molecules (tMM)-(GMM)***, we have for any $\ell \in \mathbb{N}^*$, any $N \geq 2\ell$ and any $\delta \in (0, 2/d)$:*

$$\sup_{t\in(0,\infty)} \sup_{\varphi\in \mathcal{F}^{\otimes\ell},\, \|\varphi\|_{\mathcal{F}^{\otimes\ell}}\leq 1} \left\langle \Pi_\ell \left[S_t^N \left(f_0^{\otimes N} \right) \right] - S_t^{NL}(f_0)^{\otimes \ell}, \varphi \right\rangle \leq \ell^2 \, \frac{C_\delta}{N^{\frac{2}{d}-\delta}}$$

for some constant $C_\eta \in (0,\infty)$ which may blow up when $\eta \to 0$ and where

$$\mathcal{F} := \left\{ \varphi : \mathbb{R}^d \to \mathbb{R};\ \|\varphi\|_{\mathcal{F}} := \int_{\mathbb{R}^d} (1+|\xi|^4)\, |\hat\varphi(\xi)|\, d\xi < \infty \right\},$$

$\hat\varphi$ standing for the Fourier transform of φ.

(iii) *In the case where S_t^∞ and S_t^N correspond to the* **cutoff Maxwell molecules (GMM)***, we have for any $\ell \in \mathbb{N}^*$, any $N \geq 2\ell$, any $s \in (d/2, d/2+1)$ and any $T \in (0,\infty)$:*

$$\sup_{t\in(0,T)} \sup_{\varphi\in H^s(\mathbb{R}^d)^{\otimes\ell},\, \|\varphi\|_{H^s(\mathbb{R}^d)^{\otimes\ell}}\leq 1} \left\langle \Pi_\ell \left[S_t^N \left(f_0^{\otimes N} \right) \right] - S_t^{NL}(f_0)^{\otimes \ell}, \varphi \right\rangle \leq \ell^2 \, \frac{C_{s,T}}{N^{\frac{1}{2}}}$$

for some constant $C_{s,T} \in (0,\infty)$ which may blow up when $s \to d/2$ or $s \to d/2+1$ or when $T \to \infty$.

Remarks 1.2.

- *To be more precise the constants depend on the the initial datum f_0 through polynomial moments bounds for **(GMM)** and **(tMM)**, and exponential moment bound for **(HS)**. However some one needs also a bound on the support of the energy of the N-particle empirical measure. When the N-particle initial data are simply tensor products of the 1-particle initial datum, the simplest sufficient condition is the compact support of f_0. It could be relaxed at the price of an* *additional error term. This restriction can also be relaxed by considering N-particle initial datum conditioned to the sphere $S^{Nd-1}(\sqrt{N})$ of constant energy as in* [22]*: these extensions are considered in Section 6.*

- *Note that the two first estimates (i) and (ii) are* global in time. *This is an important qualitative improvement over previous chaos propagation results for Boltzmann collision processes.*
- *In the third estimate (iii), the rate of convergence $\mathcal{O}(1/N^{1/2})$ is optimal in the case of Maxwellian kernels, as predicted by the law of large numbers.*

1.5. **Some open questions and extensions.**

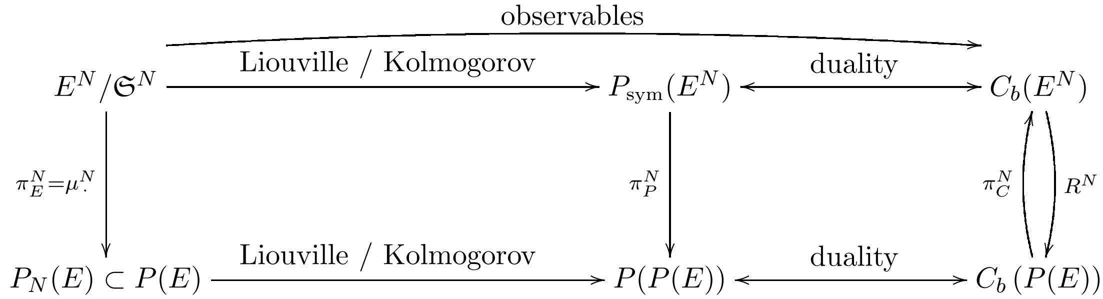
- What about larger classes of initial data, say with only and as few as possible polynomial moments?
- What about optimizing the rate? Is the rate $N^{-1/2}$ always optimal and how to obtain in general when no Hilbert structure seems available for the estimates (such as for the hard spheres case **(HS)**)?
- What about more general Boltzmann models? A work is in progress for applying the method to inelastic hard spheres and inelastic Maxwell molecules Boltzmann equation with thermal or stochastic baths. Another issue is the true (without cut-off) Boltzmann equation for hard or soft potential: only assumption **(A4)** (see below) remains to be proved in that case. The question of proving uniform in time chaos propagation also remains open in this case.
- What about a central limit theorem and/or a large deviation result with the help of our setting? A natural guess would be that a higher-order "differential stability" (see below) is required on the limit system, and this point should be clarified.
- Finally let us mention that works are in progress to apply our method to Vlasov and McKean-Vlasov equations.

1.6. **Plan of the paper.** In Section 2 we set the abstract functional framework together with the general assumption and in Section 3 we state and prove the abstract Theorem 3.27. In Section 4 we apply this method to the (true) Maxwell molecules: we show how to choose metrics so that the general assumptions can be proved (Theorem 4.1). In Section 5 we apply the method to hard spheres molecules and quantitative chaos propagation in Theorem 5.1. Finally in Section 6 we address several important extensions and applications: we relaxe the assumption of compactly supported initial datum by considering N-particle initial data conditioned on the energy sphere as suggested by Kac, and as a consequence of our uniform in time chaos propagation result we study the chaoticity of the steady state. Finally we conclude with some remarks and computations on the BBGKY hierarchy and its link with our work.

2. The abstract setting

In this section we shall state and prove the key abstract result. This will motivate the introduction of a general functional framework.

2.1. **The general functional framework of the duality approach.** Let us set the framework. Here is a diagram which sums up the duality approach (norms and duality brackets shall be precised in Subsections 2.3):



In this diagram:

- E denotes a Polish space.
- $\mathfrak{S}^N$ denotes the N-permutation group.
- $P_{\rm sym}(E^N)$ denotes the set of symmetric probabilities on E^N: For a given permutation $\sigma \in \mathfrak{S}^N$, a vector $V = (v_1, ..., v_N) \in E^N$, a function $\phi \in C_b(E^N)$ and a probability $\rho^N \in P(E^N)$ we successively define $V_\sigma = (v_{\sigma(1)}, ..., v_{\sigma(N)}) \in E^N$, $\phi_\sigma \in C_b(E^N)$ by setting $\phi_\sigma(V) = \phi(V_\sigma)$ and $\rho^N_\sigma \in P(E^N)$ by setting $\langle \rho^N_\sigma, \phi\rangle = \langle \rho^N, \phi_\sigma\rangle$. We say that a probability ρ^N on E^N is symmetric or invariant under permutations if $\rho^N_\sigma = \rho^N$ for any permutation $\sigma \in \mathfrak{S}^N$.
- For any $V \in E^N$ the probability measure μ^N_V denotes the *empirical measure*:
$$\mu^N_V = \frac{1}{N}\sum_{i=1}^N \delta_{v_i}, \quad V = (v_1, \dots, v_N)$$
where δ_{v_i} denotes the Dirac mass on E at point v_i.
- $P_N(E)$ denotes the subset $\{\mu^N_V,\, V \in E^N\}$ of $P(E)$.
- $P(P(E))$ denotes the set of probabilities on the polish space $P(E)$.
- $C_b\left(P(E)\right)$ denotes the space of continuous and bounded functions on $P(E)$, the latter space being endowed with the weak or strong topologies (see Subsection 2.3).
- The arrow pointing from $E^N/\mathfrak{S}^N$ to $P_N(E)$ denotes the map π^N_E defined by
$$\forall\, V \in E^N/\mathfrak{S}^N, \quad \pi^N_E(V) := \mu^N_V.$$
- The arrow pointing from $C_b(P(E))$ to $C_b(E^N)$ denotes the following map π^N_C
$$\forall\, \Phi \in C_b\left(P(E)\right), \ \forall\, V \in E^N, \ \left(\pi^N_C \Phi\right)(V) := \Phi\left(\mu^N_V\right).$$
- The counter arrow pointing from $C_b(E^N)$ to $C_b(P(E))$ denotes the transformation R^N defined by:
$$\forall\, \phi \in C_b(E^N), \ \forall\, \rho \in P(E), \quad R^N[\phi](\rho) := \left\langle \rho^{\otimes N}, \phi\right\rangle.$$
In the sequel we shall sometimes use the shorthand notation R^ℓ_ϕ instead of $R^\ell[\phi]$ for any $\ell \in \mathbb{N}^*$ and $\phi \in C_b(E^\ell)$.
- The arrow pointing from $P_{\rm sym}(E^N)$ to $P(P(E))$ denotes the following transformation: consider a symmetric probability $\rho^N \in P_{\rm sym}(E^N)$ on E^N, we define the probability on probability $\pi^N_P \rho^N \in P(P(E))$ by setting
$$\forall\, \varphi \in \Phi \in C_b(P(E)), \quad \left\langle \pi^N_P \rho^N, \Phi\right\rangle = \left\langle \rho^N, \pi^N_C \Phi\right\rangle = \left\langle \rho^N, \Phi\left(\mu^N_V\right)\right\rangle,$$
where the first bracket means $\langle \cdot,\cdot\rangle_{P(P(E)),C_b(P(E))}$ and the second bracket means $\langle\cdot,\cdot\rangle_{P(E^N),C_b(E^N)}$.

- The arrows pointing from the first column to the second one denote the procedure of the writing either of the *Liouville* transport equation associated with the set of ODEs of a particle system (as for mean-field limits for Vlasov equations), or the writing of the *Kolmogorov* equation associated with a stochastic Markov process of a particles system (e.g. jump or diffusion processes).
- Finally the dual spaces of the spaces of probabilities on the phase space can be interpreted as the spaces of observables on the original systems. We shall discuss this point later.

Remark 2.1. *Consider a random variable V on E^N with law $\rho^N \in P(E^N)$. Then it is often denoted by μ^N_V the random variable on $P(E)$ with law $\pi^N_P \rho^N \in P(P(E))$. Our notation is slightly less compact and intuitive, but at the same time more accurate.*

Remark 2.2. *Our functional framework shall be applied to weighted probability spaces rather than directly in $P(E)$. More precisely, for a given weight function $m : E \to \mathbb{R}_+$ we shall use (subsets) of the weighted probability space*

$$\{\rho \in P(E);\ M_m(\rho) := \langle \rho, m\rangle < \infty\} \tag{2.1}$$

as our core functional space. Typical examples are $m(v) := \tilde m(dist_E(v, v_0))$, for some fixed $v_0 \in E$ and $\tilde m(z) = z^k$ or $\tilde m(z) = e^{a\,z^k}$, $a, k > 0$. We shall sometimes abuse notation by writing M_k for M_m when $\tilde m(z) = z^k$ in the above example.

2.2. **The evolution semigroups.** Let us introduce the mathematical objects living in these spaces, for any $N \geq 1$.

Step 1. Consider a process $(\mathcal{V}^N_t)$ on E^N which describes the trajectories of the particles (Lagrangian viewpoint). The evolution can correspond to stochastic ODEs (Markov process), or deterministic ODEs (deterministic Hamiltonian flow). We make the fundamental assumption that this flow commutes with permutations: for any $\sigma \in \mathfrak{S}^N$, the solution at time t starting from $\left(\mathcal{V}^N_0\right)_\sigma$ is $\left(\mathcal{V}^N_t\right)_\sigma$. This reflects mathematically the fact that particles are indistinguishable.

Step 2. One naturally derives from this flow on E^N a corresponding semigroup S^N_t acting on $P_{\mathrm{sym}}(E^N)$ for the presence density of particles in the phase space E^N. This corresponds to a linear evolution equation

$$\partial_t f^N = A^N f^N, \qquad f^N \in P_{\mathrm{sym}}(E^N), \tag{2.2}$$

which can be interpreted as the forward Kolmogorov equation on the law in case of a Markov process at the particle level, or the Liouville equation on the probability density in case of an Hamiltonian process at the particle level. As a consequence of the previous assumption that the flow $(\mathcal{V}^N_t)$ commutes with permutation, we have that S^N_t acts on $P_{\mathrm{sym}}(E^N)$. In other words, if the law f^N_0 of $\mathcal{V}^N_0$ belongs to $P_{\mathrm{sym}}(E^N)$, then for any times the law f^N_t of $\mathcal{V}^N_t$ also belongs to $P_{\mathrm{sym}}(E^N)$.

Step 3. We then define the *dual* semigroup T^N_t of S^N_t acting on functions $\phi \in C_b(E^N)$ by

$$\forall\, f^N \in P(E^N),\ \phi \in C_b(E^N), \quad \left\langle f^N, T^N_t(\phi)\right\rangle := \left\langle S^N_t(f^N), \phi\right\rangle$$

and we denote its generator by G^N, which corresponds to the following *linear* evolution equation:

$$\partial_t \phi = G^N(\phi), \qquad \phi \in C_b(E^N). \tag{2.3}$$

This is the semigroup of the *observables* on the evolution system $(\mathcal{V}^N_t)$ on E^N.

Let us state precisely our assumptions on the N-particles dynamics. Here and below, for a given 1-particle weight function $m : E \to \mathbb{R}_+$, we define the N-particle weight function

$$\forall\, V = (v_1, ..., v_N) \in E^N, \qquad M^N_m(V) := \frac{1}{N} \sum_{i=1}^N m(v_i) = \langle \mu^N_V, m\rangle = M_m(\mu^N_V). \tag{2.4}$$

Again, we shall sometimes abuse notation by writing M^N_k and M_k instead of M^N_m and M_k with $\tilde m(z) = z^k$ in the example of Remark 2.2. In particular, when $E = \mathbb{R}^d$ endowed with the euclidian structure, we have

$$\forall\, V = (v_1, ..., v_N) \in \mathbb{R}^{dN}, \qquad M^N_k(V) := \frac{1}{N} \sum_{i=1}^N |v_i|^k = \langle \mu^N_V, |\cdot|^k\rangle = M_k(\mu^N_V). \tag{2.5}$$

(A1) On the N-particle system. Together with the fact that G^N and T^N_t are well defined and satisfy the symmetry condition introduced in Step 2 above, we assume that the following moments conditions hold:

(i) *Energy bounds*: There exists a weight function m_e and a constant $\mathcal{E} \in (0, \infty)$ such that

$$\operatorname{supp} f^N_t \subset \mathbb{E}_N := \left\{V \in E^N;\ M^N_{m_e}(V) \le \mathcal{E}\right\}. \tag{2.6}$$

(ii) *Integral moment bound*: There exists a weight function m_1, a time $T \in (0, \infty]$ and a constant $C^N_{T,m_1} \in (0, \infty)$, possibly depending on T, and m_1, m_e and $\mathcal{E}$, but not on the number of particles N, such that

$$\sup_{0 \le t < T} \left\langle f^N_t, M^N_{m_1}\right\rangle \le C^N_{T,m_1}. \tag{2.7}$$

(iii) *Support moment bound at initial time*: There exists a weight function m_3 and a constant $C^N_{0,m_3} \in (0, +\infty)$, possibly depending on the number of particles N and on $\mathcal{E}$, such that

$$\operatorname{supp} f^N_0 \subset \left\{V \in E^N; M^N_{m_3}(V) \le C^N_{0,m_3}\right\}. \tag{2.8}$$

Remark 2.3. *Note that these assumptions on the N-particle system are very weak, and do not require any precise knowledge of the N-particle dynamics.*

Step 4. Consider a (possibly nonlinear) semigroup S^{NL}_t acting on $P(E)$ associated with the limit (possibly nonlinear) kinetic equation: for any $\rho \in P(E)$, $S^{NL}_t(\rho) := f_t$ where $f_t \in C(\mathbb{R}_+, P(E))$ is the solution to

$$\partial_t f_t = Q(f_t), \quad f_0 = \rho. \tag{2.9}$$

Step 5. Then we consider its *pushforward* semigroup T^∞_t acting on $C_b(P(E))$ defined by:

$$\forall\, \rho \in P(E),\ \Phi \in C_b(P(E)), \quad T^\infty_t[\Phi](\rho) := \Phi\left(S^{NL}_t(\rho)\right).$$

Note carefully that T^∞_t is always *linear* as a function of Φ (although of course be careful that $T^\infty_t[\Phi](\rho)$ is not linear as a function of ρ). We denote its generator by G^∞, which corresponds to the following *linear* evolution equation on $C_b(P(E))$:

$$\partial_t \Phi = G^\infty(\Phi). \tag{2.10}$$

Remark 2.4. *The semigroup T_t^∞ can be interpreted physically as the semigroup of the evolution of* observables *of the nonlinear equation* (2.9)*:*
Given a nonlinear ODE $V' = F(V)$ on $\mathbb{R}^d$, one can define (at least formally) the linear *Liouville transport PDE*

$$\partial_t \rho + \nabla_v \cdot (F\, \rho) = 0,$$

where $\rho = \rho_t(v)$ is a time-dependent probability density. When the trajectories $(V_t(v))$ of the ODE are properly defined, the solution of the associated transport equation is given by $\rho_t(v) = V_{-t}^*(\rho_0)$, that is the pullback of the initial measure ρ_0 by V_{-t} (for smooth functions, this amounts to $\rho_t(v) = \rho_0(V_{-t}(v))$). Now, instead of the Liouville viewpoint, one can adopt the viewpoint of observables*, that is functions depending on the position of the system in the phase space (e.g. energy, momentum, etc.) For some observable function Φ_0 defined on $\mathbb{R}^d$, the evolution of the value of this observable along the trajectory is given by $\phi_t(v) = \phi_0(V_t(v))$. In other words we have $\phi_t = V_t^* \phi_0 = \phi_0 \circ V_t$. Then ϕ_t is solution to the following* dual *linear PDE*

$$\partial_t \phi - F \cdot \nabla_v \phi = 0,$$

and it satisfies

$$\langle \phi_t, \rho_0 \rangle = \langle V_t^* \phi_0, \rho_0 \rangle = \langle \phi_0, V_{-t}^* \rho_0 \rangle = \langle \phi_0, \rho_t \rangle.$$

Now let us consider a nonlinear evolution system $V' = Q(V)$ in an asbtract space H. Then (keeping in mind the analogy with ODE/PDE above) we see that one can formally naturally define two linear evolution systems on the larger functional spaces $P(H)$ and $C_b(H)$: first the transport equation (trajectories level)

$$\partial_t \pi + \nabla \cdot (Q(v)\, \pi) = 0, \qquad \pi \in P(H)$$

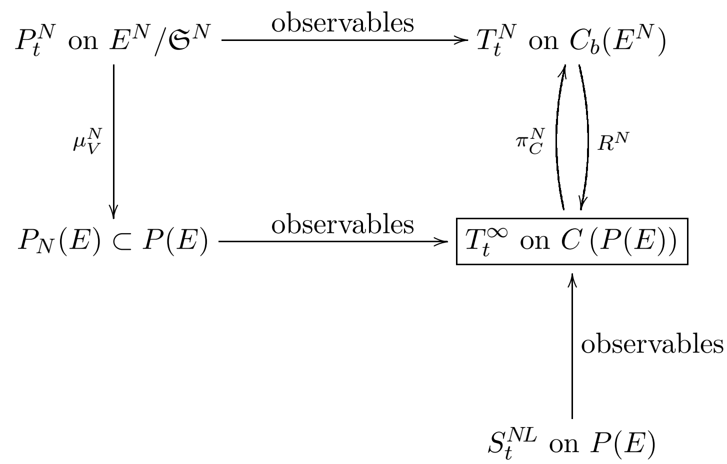
(where the second term of this equation has to be properly defined) and second the dual *equation for the evolution of observables*

$$\partial_t \Phi - Q(v) \cdot \nabla \Phi = 0, \qquad \Phi \in C_b(H).$$

Taking $H = P(E)$, this provides an intuition for our functional construction, and also for the formula of the generator G^∞ below (compare the previous equations with formula (2.17)*) and the need for developping a differential calculus in $P(E)$. Be careful that when $H = P(E)$, the word "trajectories" refers to trajectories* in the space of probabilites $P(E)$ *(*i.e.*, solutions to the nonlinear equation* (2.9)*), and not trajectories of a particle in E.*

Another important point to notice is that for a dissipative equation *at the level of H, one cannot define reverse "characteristics" (the flow in H is not defined backwards), and therefore for the nonlinear Boltzmann equation, only the equation for the observables can be defined in terms of trajectories.*

Summing up we obtain the following picture for the semigroups:



Hence we see that the key point of our construction is that, through the evolution of *observables* one can "interface" the two evolution systems (the nonlinear limit equation and the N-particles system) *via* the applications π_C^N and R^N. From now on we shall denote $\pi^N = \pi_C^N$.

2.3. **The metric issue.** $P(E)$ is our fundamental space, where we shall compare (through their observables) the marginals of the N-particle density f_t^N and the marginals of the chaotic ∞-particle dynamic $f_t^{\otimes\infty}$. Let us make precise the topological and metric structure used on $P(E)$. At the topological level there are two canonical choices (which determine two different sets $C(P(E))$): (1) the strong topology (associated to the total variation norm that we denote by $\|\cdot\|_{M^1}$) and (2) the weak topology (*i.e.*, the trace on $P(E)$ of the weak topology $M^1(E)$, the space of Radon measures on E with finite mass, induced by $C_b(E)$).

The set $C_b(P(E))$ depends on the choice of the topology on $P(E)$. In the sequel, we will denote $C_b(P(E), w)$ the space of continuous and bounded functions on $P(E)$ *endowed with the weak topology*, and $C_b(P(E), TV)$ the similar space on $P(E)$ *endowed with the total variation norm*. It is clear that $C_b(P(E), w) \subset C_b(P(E), TV)$.

The supremum norm $\|\Phi\|_{L^\infty(P(E))}$ *does not* depend on the choice of topology on $P(E)$, and induces a Banach topology on the space $C_b(P(E))$. The transformations π^N and R^N satisfy:

$$\left\|\pi^N \Phi\right\|_{L^\infty(E^N)} \le \|\Phi\|_{L^\infty(P(E))} \quad\text{and}\quad \|R^N[\phi]\|_{L^\infty(P(E))} \le \|\phi\|_{L^\infty(E^N)}. \tag{2.11}$$

The transformation π^N is well defined from $C_b(P(E), w)$ to $C_b(E^N)$, but in general, it does not map $C_b(P(E), TV)$ into $C_b(E^N)$ since $V \in E^N \mapsto \mu_V^N \in (P(E), TV)$ is not continuous.

In the other way round, the transformation R^N is well defined from $C_b(E^N)$ to $C_b(P(E), w)$, and therefore also from $C_b(E^N)$ to $C_b(P(E), TV)$: for any $\phi \in C_b(E^N)$ and for any sequence $f_k \rightharpoonup f$ weakly, we have $f_k^{\otimes N} \rightharpoonup f^{\otimes N}$ weakly, and then $R^N[\phi](f_k) \to R^N[\phi](f)$.

The different possible metric structures inducing the weak topology are not seen at the level of $C_b(P(E), w)$. However any Hölder or C^k-like space will strongly depend on this choice, as we shall see.

Definition 2.5. *For a given weight functions $m_\mathcal{G} : E \to \mathbb{R}_+$ and some constant $\mathcal{E} \in (0,\infty)$, we define the subspaces of probabilities:*

$$P_\mathcal{G} := \{f \in P(E);\ \langle f, m_\mathcal{G}\rangle < \infty,\ \langle f, m_e\rangle \le \mathcal{E}\}, \tag{2.12}$$

where m_e is introduced in **(A1)**. *For a given constraint function $\mathbf{m}_\mathcal{G} : E \to \mathbb{R}^D$ such that the components $\mathbf{m}_\mathcal{G}$ are controlled by $m_\mathcal{G}$, we also define the corresponding constrained subsets*

$$P_{\mathcal{G},\mathbf{r}} := \{f \in P_\mathcal{G};\ \langle f, \mathbf{m}_\mathcal{G}\rangle = \mathbf{r}\}, \quad \mathbf{r} \in \mathbb{R}^D,$$

the corresponding bounded subsets for $a > 0$

$$\mathcal{B}P_{\mathcal{G},a} := \{f \in P_G G);\ \langle f, m_\mathcal{G}\rangle < a\}, \quad P_{\mathcal{G},\mathbf{r},a} := \{f \in \mathcal{B}P_{\mathcal{G},a};\ \langle f, \mathbf{m}_\mathcal{G}\rangle = \mathbf{r}\},$$

and the corresponding vectorial space of "increments"

$$\mathcal{I}P_\mathcal{G} := \left\{f_2 - f_1;\ \exists\, \mathbf{r} \in \mathbb{R}^D \ s.t.\ f_1, f_2 \in P_{\mathcal{G},\mathbf{r}}\right\}$$

and $\mathcal{I}P_{\mathcal{G},a}$ as expected. Now, we shall encounter two situations:

- *Either $dist_\mathcal{G}$ denotes a distance defined on the whole space $P_\mathcal{G}(E)$, and thus on $P_{\mathcal{G},\mathbf{r}}$ for any $\mathbf{r} \in \mathbb{R}^D$.*
- *Either there is a vectorial space $\mathcal{G} \supset \mathcal{I}P_\mathcal{G}$ endowed with a norm $\|\cdot\|_\mathcal{G}$ such that we can define a distance $dist_\mathcal{G}$ on $P_{\mathcal{G},\mathbf{r}}$ for any $\mathbf{r} \in \mathbb{R}^D$ by setting*

$$\forall\, f,\, g \in P_{\mathcal{G},\mathbf{r}}, \qquad dist_\mathcal{G}(f,g) := \|g - f\|_\mathcal{G}.$$

Finally, we say that two metrics d_0 and d_1 on $P_\mathcal{G}$ are topologically uniformly equivalent on bounded sets if there exists $\kappa \in (0,\infty)$ and for any $a \in (0,\infty)$ there exists $C_a \in (0,\infty)$ such that

$$\forall\, f,\, g \in \mathcal{B}P_{\mathcal{G},a} \quad d_0(f,g) \le C_a\, [d_1(f,g)]^\kappa, \quad d_1(f,g) \le C_a\, [d_0(f,g)]^\kappa.$$

If d_0 and d_1 are resulting from some normed spaces $\mathcal{G}_0$ and $\mathcal{G}_1$, we abusively say that $\mathcal{G}_0$ and $\mathcal{G}_1$ are topologically uniformly equivalent (on bounded sets).

Example 2.6. *The choice $m_\mathcal{G} := 1$, $\mathbf{m}_\mathcal{G} := 0$, $\|\cdot\|_\mathcal{G} := \|\cdot\|_{M^1}$ recovers $P_\mathcal{G}(E) = P(E)$. More generally on can choose $m_{\mathcal{G}_k}(v) := dist_E(v,v_0)^k$, $\mathbf{m}_{\mathcal{G}_k} := 0$, $\|\cdot\|_{\mathcal{G}_k} := \|\cdot\, dist_E(v,v_0)^k\|_{M^1}$. For $k_1 > k_2, k_3 > 0$, the spaces $P_{\mathcal{G}_{k_2}}$ and $P_{\mathcal{G}_{k_3}}$ are topologically uniformly equivalent on bounded sets of $P_{\mathcal{G}_{k_1}}$.*

Example 2.7. *There are many distances on $P(E)$ which induce the weak topology, see for instance* [37]. *In section 2.7 below, we will present some of them which have a practical interest for us, and which are all topologically uniformly equivalent on "bounded sets" of $P(E)$, when the bounded sets are defined thanks to a convenient (strong enough) weight function.*

2.4. **Differential calculus for functions of probability measures.** We start with a purely metric definition in the case of usual Hölder regularity.

Definition 2.8. *For some metric spaces $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$, some weight function $\Lambda : \tilde{\mathcal{G}}_1 \mapsto \mathbb{R}_+^*$ and some $\eta \in (0,1]$, we denote by $C^{0,\eta}_\Lambda(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2)$ the weighted space of functions from $\tilde{\mathcal{G}}_1$ to $\tilde{\mathcal{G}}_2$ with η-Hölder regularity, that is the functions $\mathcal{S} : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_2$ such that there exists a constant $C > 0$ so that*

$$\forall\, f_1,\, f_2 \in \tilde{\mathcal{G}}_1 \qquad \mathrm{dist}_{\mathcal{G}_2}(\mathcal{S}(f_1), \mathcal{S}(f_2)) \le C\, \Lambda(f_1,f_2)\, \mathrm{dist}_{\mathcal{G}_1}(f_1,f_2)^\eta, \tag{2.13}$$

with $\Lambda(f_1,f_2) := \max\{\Lambda(f_1), \Lambda(f_2)\}$ and $\mathrm{dist}_{\mathcal{G}_k}$ denotes the metric of $\tilde{\mathcal{G}}_k$ (the tilde sign in the notation of the distance has been removed in order to present unified notation with the *next definition). We define the semi-norm* $[\cdot]_{C^{0,\eta}_\Lambda(\tilde{\mathcal G}_1,\tilde{\mathcal G}_2)}$ *in* $C^{0,\eta}_\Lambda(\tilde{\mathcal G}_1,\tilde{\mathcal G}_2)$ *as the infimum of the constants* $C>0$ *such that* (2.13) *holds.*

Second we define a first order differential calculus, for which we require a norm structure on the functional spaces.

Definition 2.9. *For some normed spaces* $\mathcal G_1$ *and* $\mathcal G_2$*, some metric sets* $\tilde{\mathcal G}_1$ *and* $\tilde{\mathcal G}_2$ *such that* $\tilde{\mathcal G}_i-\tilde{\mathcal G}_i\subset\mathcal G_i$*, some weight function* $\Lambda:\tilde{\mathcal G}_1\mapsto[1,\infty)$ *and some* $\eta\in(0,1]$*, we denote* $C^{1,\eta}_\Lambda(\tilde{\mathcal G}_1,\mathcal G_1;\tilde{\mathcal G}_2,\mathcal G_2)$ *(or simply* $C^{1,\eta}_\Lambda(\tilde{\mathcal G}_1;\tilde{\mathcal G}_2)$*), the space of continuously differentiable functions from* $\tilde{\mathcal G}_1$ *to* $\tilde{\mathcal G}_2$*, whose derivative satisfies some weighted* η*-Hölder regularity. More explicitely, these are the continuous functions* $\mathcal S:\tilde{\mathcal G}_1\to\tilde{\mathcal G}_2$ *such that there exists a continuous fonction* $D\mathcal S:\tilde{\mathcal G}_1\to\mathcal B(\mathcal G_1,\mathcal G_2)$ *(where* $\mathcal B(\mathcal G_1,\mathcal G_2)$ *denotes the space of bounded linear applications from* $\mathcal G_1$ *to* $\mathcal G_2$ *endowed with the usual norm operator), and some constants* $C_i>0$*,* $i=1,\,2,\,3$*, so that for any* $f_1,\,f_2\in\tilde{\mathcal G}_1$*:*

$$\|\mathcal S(f_2)-\mathcal S(f_1)\|_{\mathcal G_2}\ \le\ C_1\,\Lambda(f_1,f_2)\,\|f_2-f_1\|_{\mathcal G_1} \tag{2.14}$$

$$\|\langle D\mathcal S[f_1],f_2-f_1\rangle\|_{\mathcal G_2}\ \le\ C_2\,\Lambda(f_1,f_2)\,\|f_2-f_1\|_{\mathcal G_1} \tag{2.15}$$

$$\|\mathcal S(f_2)-\mathcal S(f_1)-\langle D\mathcal S[f_1],f_2-f_1\rangle\|_{\mathcal G_2}\ \le\ C_3\,\Lambda(f_1,f_2)\,\|f_2-f_1\|^{1+\eta}_{\mathcal G_1}. \tag{2.16}$$

Notations 2.10. *For* $\mathcal S\in C^{1,\eta}_\Lambda$*, we define* $C^{\mathcal S}_i$*,* $i=1,\,2,\,3$*, as the infimum of the constants* $C_i>0$ *such that* (2.14) *(resp.* (2.15)*,* (2.16)*) holds. We then denote*

$$[\mathcal S]_{C^{0,1}_\Lambda}:=C^{\mathcal S}_1,\quad [\mathcal S]_{C^{1,0}_\Lambda}:=C^{\mathcal S}_2,\quad [\mathcal S]_{C^{1,\eta}_\Lambda}:=C^{\mathcal S}_3,\quad \|\mathcal S\|_{C^{1,\eta}_\Lambda}:=C^{\mathcal S}_1+C^{\mathcal S}_2+C^{\mathcal S}_3,$$

and we will remove the subscript Λ *when* $\Lambda\equiv1$*.*

Remark 2.11. *In the sequel, we shall apply this differential calculus with some suitable subspaces* $\tilde{\mathcal G}_i\subset P(E)$*. This choice of subspaces is crucial in order to make rigorous the intuition of Grünbaum* [20] *(see the — unjustified — expansion of* H_f *in* [20]*). It is worth emphasizing that our differential calculus is based on the idea of considering* $P(E)$ *(or subsets of* $P(E)$*) as "plunged sub-manifolds" of some larger normed spaces* $\mathcal G_i$*. Our approach thus differs from the approach of P.-L. Lions recently developed in his course at Collège de France* [26] *or the one developed by L. Ambrosio et al in order to deal with gradient flows in probability measures spaces, see for instance* [2]*. In the sequel we develop a differential calculus in probability measures spaces into a simple and robust framework, well suited to deal with the different objects we have to manipulate (*1*-particle semigroup, polynomial, generators, ...). And the main innovation from our work is the use of this differential calculus to state some subtle "differential" stability conditions on the limiting semigroup. Roughly speaking the latter estimates measure how this limiting semigroup handles fluctuations departing from chaoticity. They are the corner stone of our analysis. Surprisingly they seem new, at least for Boltzmann type equations.*

This differential calculus behaves well for composition in the sense that for any given $\mathcal U\in C^{1,\eta}_{\Lambda_{\mathcal U}}(\tilde{\mathcal G}_1,\tilde{\mathcal G}_2)$ and $\mathcal V\in C^{1,\eta}_{\Lambda_{\mathcal V}}(\tilde{\mathcal G}_2,\tilde{\mathcal G}_3)$ there holds $\mathcal S:=\mathcal V\circ\mathcal U\in C^{1,\eta}_{\Lambda_{\mathcal S}}(\tilde{\mathcal G}_1,\tilde{\mathcal G}_3)$ for some appropriate weight function $\Lambda_{\mathcal S}$. We conclude the section by stating a precise result well adapted to our applications. The proof is straightforward by writing and compounding the expansions of $\mathcal U$ and $\mathcal V$ provided by Definition 2.9 and we then skip it.

Lemma 2.12. *For any given* $\mathcal U\in C^{1,\eta}_\Lambda(\tilde{\mathcal G}_1,\tilde{\mathcal G}_2)$ *and* $\mathcal V\in C^{1,\eta}(\tilde{\mathcal G}_2,\tilde{\mathcal G}_3)$ *there holds* $\mathcal S:=\mathcal V\circ\mathcal U\in C^{1,\eta}_{\Lambda^{1+\eta}}(\tilde{\mathcal G}_1,\tilde{\mathcal G}_3)$ *and* $D\mathcal S[f]=D\mathcal V[\mathcal U(f)]\circ D\mathcal U[f]$*. More precisely, there holds*

$$[\mathcal S]_{C^{0,1}_\Lambda}\le[\mathcal V]_{C^{0,1}}\,[\mathcal U]_{C^{0,1}_\Lambda},\quad [\mathcal S]_{C^{1,0}_\Lambda}\le[\mathcal V]_{C^{1,0}}\,[\mathcal U]_{C^{1,0}_\Lambda}$$

and

$$[\mathcal{S}]_{C^{1,\eta}_{\Lambda^{1+\eta}}} \le [\mathcal{V}]_{C^{1,0}} \, [\mathcal{U}]_{C^{1,\eta}_\Lambda} + [\mathcal{V}]_{C^{1,\eta}} \, [\mathcal{U}]^{1+\eta}_{C^{0,1}_\Lambda}.$$

When further $\mathcal{V} \in C^{1,1}(\tilde{\mathcal{G}}_2, \tilde{\mathcal{G}}_3)$*, we also have* $\mathcal{S} := \mathcal{V} \circ \mathcal{U} \in C^{1,\eta}_{\Lambda^2}(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_3)$ *with*

$$[\mathcal{S}]_{C^{1,\eta}_{\Lambda^2}} \le [\mathcal{V}]_{C^{1,0}} \, [\mathcal{U}]_{C^{1,\eta}_\Lambda} + [\mathcal{V}]_{C^{1,1}} \, [\mathcal{U}]^2_{C^{0,(1+\eta)/2}_\Lambda}.$$

2.5. **The pushforward generator.** As a first example of application of this differential calculus, let us compute the generator of the pushforward limiting semigroup. Assume the following:

(A2) Existence of the generator of the pushforward semigroup. For some normed space $\mathcal{G}_1$ and some probability space $P_{\mathcal{G}_1}(E)$ (defined as above) associated to a weight function m_1 and constraint function $\mathbf{m_1}$, and endowed with the metric induced from $\mathcal{G}_1$, for some $\delta \in (0,1]$ and some $\bar{a} \in (0,\infty)$ we have for any $a \in (\bar{a},\infty)$:
- (i) For any $t \in [0,+\infty)$, $S^{NL}_t : \mathcal{B}P_{\mathcal{G}_1,a} \to \mathcal{B}P_{\mathcal{G}_1,a}$ is continuous (for the metric $\text{dist}_{\mathcal{G}_1}$), uniformly in time.
- (ii) The application Q is δ-Hölder continuous from $\mathcal{B}P_{\mathcal{G}_1,a}$ into $\mathcal{G}_1$.
- (iii) For any $f \in \mathcal{B}P_{\mathcal{G}_1,a}$, for some $\tau > 0$ the application $[0,\tau) \to \mathcal{B}P_{\mathcal{G}_1,a}$, $t \mapsto \mathcal{S}^{NL}_t(f)$ is $C^{1,\delta}([0,\tau); \mathcal{B}P_{\mathcal{G}_1,a})$, with $\mathcal{S}(f)'(0) = Q(f)$.

Lemma 2.13. *Under assumption* **(A2)** *the generator* G^∞ *of the pushforward semigroup* T^∞_t *exists as an unbounded linear operator on* $C(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R})$ *for any* $a \in (\bar{a},\infty)$ *with domain including* $C^{1,\delta}(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R})$*, and on this domain it is defined by the formula*

$$\forall\, \Phi \in C^{1,\delta}(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R}),\ \forall\, f \in \mathcal{B}P_{\mathcal{G}_1,a}(E), \quad (G^\infty \Phi)(f) := \langle D\Phi[f], Q(f) \rangle. \tag{2.17}$$

Proof of Lemma 2.13. We split the proof in several steps.

Step 1. First (T^∞_t) is a continuous semigroup on $C(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R})$: consider $\Phi \in C(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R})$ and some sequence (f_n) of $\mathcal{B}P_{\mathcal{G}_1,a}(E)$ such that $\text{dist}_{\mathcal{G}_1}(f_n, f) \to 0$, then thanks to **(A2)**-(i) we deduce $(T^\infty_t \Phi)(f_n) = \Phi(S^{NL}_t(f_n)) \to \Phi(S^{NL}_t(f)) = (T^\infty_t \Phi)(f)$, and by composition $T^\infty_t \Phi \in C(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R})$.

Next, we have

$$\|T^\infty_t\| = \sup_{\|\Phi\| \le 1} \|T^\infty_t \Phi\| = \sup_{\|\Phi\| \le 1} \sup_{f \in \mathcal{B}P_{\mathcal{G}_1,a}} |\Phi(S^{NL}_t(f))| \le 1, \qquad \|\Phi\| = \sup_{g \in \mathcal{B}P_{\mathcal{G}_1,a}} |\Phi(g)|.$$

Now, from **(A2)**-(iii) there exists a modulus of continuity ω such that $\|S^{NL}_t f - f\|_{\mathcal{G}_1} \le \omega(t) \to 0$ as $t \to 0$ for any $f \in \mathcal{B}P_{\mathcal{G}_1,a}$ (e.g. $\omega(t) = t^\alpha$, $\alpha \in (0,1)$), which implies

$$\forall\, \Phi \in C(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R}), \qquad \|T^\infty_t \Phi - \Phi\| = \sup_{f \in \mathcal{B}P_{\mathcal{G}_1,a}} |\Phi(S^{NL}_t(f)) - \Phi(f)| \to 0.$$

We therefore deduce from Hille-Yosida's Theorem that (T^∞_t) has a closed generator G^∞ with dense domain included in $C(\mathcal{B}P_{\mathcal{G}_1,a}(E);\mathbb{R})$.

Step 2. Let us define $\tilde{G}^\infty \Phi$ by

$$\forall\, \Phi \in C^{1,0}(\mathcal{B}P_{\mathcal{G}_1,a};\mathbb{R}),\ \forall\, f \in \mathcal{B}P_{\mathcal{G}_1,a}, \quad (\tilde{G}^\infty \Phi)(f) := \langle D\Phi[f], Q(f) \rangle.$$

The right-hand side quantity is well defined since $D\Phi(f) \in \mathcal{B}(\mathcal{G}_1,\mathbb{R}) = \mathcal{G}_1'$ and $Q(f) \in \mathcal{G}_1$. Moreover, since both $f \mapsto D\Phi[f]$ and $f \mapsto Q(f)$ are continuous we have $\tilde{G}^\infty \Phi \in C(\mathcal{B}P_{\mathcal{G}_1,a};\mathbb{R})$.

Step 3. Consider $\Phi \in C^{1,\delta}(\mathcal{B}P_{\mathcal{G}_{1},a};\mathbb{R})$. By the composition rule of Lemma 2.12, for any fixed $f \in P_{\mathcal{G}_1}(E)$, $t \mapsto T^\infty_t \Phi(f) = \Phi \circ S^{NL}_t(f)$ is $C^{1,\delta}([0,\eta);\mathbb{R})$ and

$$\begin{aligned} \frac{d}{dt}(T^\infty_t \Phi)(f)\big|_{t=0} &:= \frac{d}{dt}\big(\Phi \circ \mathcal{S}(f)(t)\big)\big|_{t=0} \\ &= \left\langle D\Phi(\mathcal{S}(f))\big|_{t=0}, \frac{d}{dt}\mathcal{S}(f)\big|_{t=0} \right\rangle \\ &= \langle D\Phi[f], Q(f)\rangle = \left(G^\infty \Phi\right)(f), \end{aligned}$$

which precisely means that any such Φ belongs to domain(G^∞) and that (2.17) holds. $\square$

2.6. **Compatibility of the π^N and R^ℓ transformations.** Our transformations π^N and R^ℓ behave nicely for the sup norm on $C_b(P(E), TV)$, see (2.11). More generally we shall consider "duality pairs" of metric spaces:

Definition 2.14. *We say that a pair $(\mathcal{F},\mathcal{G})$ of normed vectorial spaces are "in duality" if*

$$\forall\, f \in \mathcal{G},\ \forall\, \phi \in \mathcal{F} \qquad |\langle f, \phi\rangle| \le \|f\|_{\mathcal{G}}\, \|\phi\|_{\mathcal{F}} \tag{2.18}$$

where $\|\cdot\|_{\mathcal{F}}$ denotes the norm on $\mathcal{F}$ and $\|\cdot\|_{\mathcal{G}}$ denotes the norm on $\mathcal{G}$.

The "compatibility" of the transformation R^ℓ for any such pair follows from the multilinearity: if $\mathcal{F}$ and $\mathcal{G}$ are in duality, $\mathcal{F} \subset C_b(E)$ and $P_{\mathcal{G}}$ is endowed with the metric associated to $\|\cdot\|_{\mathcal{G}}$, then for any $\varphi = \varphi_1 \times \cdots \times \varphi_\ell \in \mathcal{F}^{\otimes \ell}$, the polynomial function R^ℓ_φ is of class $C^{1,\eta}(P_{\mathcal{G}},\mathbb{R})$ for any $\eta \in (0,1]$. Indeed, defining for $f_\alpha \in P_{\mathcal{G}_1}$, $\alpha = 1,2$,

$$\mathcal{G} \to \mathbb{R}, \quad h \mapsto DR^\ell_\varphi[f_1](h) := \sum_{i=1}^{\ell} \left(\prod_{j\ne i} \int_E \varphi\, df_1 \right) \langle \varphi_i, h\rangle,$$

we have

$$R^\ell_\varphi(f_2) - R^\ell_\varphi(f_1) = \sum_{i=1}^{\ell} \left(\prod_{1\le k<i} \int_E \varphi_k\, df_2 \right) \langle \varphi_i, f_2 - f_1\rangle \left(\prod_{i<k\le \ell} \int_E \varphi_k\, df_1 \right),$$

and

$$\begin{aligned} &R^\ell_\varphi(f_2) - R^\ell_\varphi(f_1) - DR^\ell_\varphi[f_1](f_2 - f_1) = \\ &= \sum_{1\le j<i\le \ell} \left(\prod_{1\le k<j} \int_E \varphi_k df_2 \right) \langle \varphi_j, f_2 - f_1\rangle \left(\prod_{j<k<i} \int_E \varphi_k df_1 \right) \langle \varphi_i, f_2 - f_1\rangle \Big(\prod_{i<k\le \ell} \int_E \varphi_k df_1 \Big). \end{aligned}$$

We deduce then for instance $R^\ell_\varphi \in C^{1,1}(\mathcal{G};\mathbb{R})$ since

$$|R^\ell_\varphi(f_2) - R^\ell_\varphi(f_1)| \le \ell\, \|\varphi\|_{\mathcal{F}\otimes(L^\infty)^{\ell-1}}\, \|f_2 - f_1\|_{\mathcal{G}}, \quad |DR^\ell_\varphi[f_1](h)| \le \ell\, \|\varphi\|_{\mathcal{F}\otimes(L^\infty)^{\ell-1}}\, \|h\|_{\mathcal{G}},$$

$$|R^\ell_\varphi(f_2) - R^\ell_\varphi(f_1) - DR^\ell_\varphi[f_1](f_2 - f_1)| \le \frac{\ell(\ell-1)}{2} \|\varphi\|_{\mathcal{F}^2\otimes(L^\infty)^{\ell-2}}\, \|f_2 - f_1\|^2_{\mathcal{G}}, \tag{2.19}$$

where we have defined

$$\|\varphi\|_{\mathcal{F}^k\otimes(L^\infty)^{\ell-k}} := \max_{i_1,\dots,i_k \text{ distincts in } [|1,\ell|]} \|\varphi_{i_1}\|_{\mathcal{F}} \dots \|\varphi_{i_k}\|_{\mathcal{F}} \prod_{j\ne(i_1,\dots,i_k)} \|\varphi_j\|_{L^\infty(E)}.$$

Remarks 2.15.
- *It is easily seen in this computation that the tensorial structure of φ is not necessary. In fact it is likely that this assumption could be relaxed all along our proof. We do not pursue this line of research.*

- *The assumption $\mathcal{F} \subset C_b(E)$ could also be relaxed. For instance, when $\mathcal{F} := \mathrm{Lip}_0(E)$ is the space of Lipschitz function which vanishes in some fixed point $x_0 \in E$, $\mathcal{G}$ is its dual space, and $P_{\mathcal{G}} := \{f \in P_1(E);\ \langle f, \mathrm{dist}_E(\cdot, x_0)\rangle \le a\}$ for some fixed $a > 0$, we have $R^\ell_\varphi \in C^{1,1}(P_1(E);\mathbb{R})$ with*

$$[R^\ell_\varphi]_{C^{0,1}} \le \ell\, a^{\ell-1}\, \|\varphi\|_{\mathcal{F}^{\otimes \ell}}, \quad [R^\ell_\varphi]_{C^{1,1}} \le \frac{\ell(\ell-1)}{2}\, a^{\ell-1}\, \|\varphi\|_{\mathcal{F}^{\otimes \ell}},$$

or equivalently $R^\ell_\varphi \in C^{1,1}_\Lambda(P_1(E);\mathbb{R})$ with $\Lambda(f) := \|f\|^{\ell-1}_{M^1_1}$.

In the other way round, for the projection π^N it is clear that if the empirical measure $V \mapsto \mu^N_V$ belongs to $C^{k,\eta}(E^N, \mathcal{G})$ for some norm space $\mathcal{G} \in M^1(E)$, then by composition one has

$$\left\|\pi^N(\Phi)\right\|_{C^{k,\eta}(E^N;\mathbb{R})} \le C_\pi\, \|\Phi\|_{C^{k,\eta}(\mathcal{G})}. \tag{2.20}$$

However the regularity of the empirical measure depends on the metric $\mathcal{G}$.

Example 2.16. *In the case $\mathcal{F} = (C_b(E), L^\infty)$ and $\mathcal{G} = (M^1(E), TV)$,* (2.20) *is trivial with $k = \eta = 0$.*

Example 2.17. *When $\mathcal{F} = Lip_0(E)$ (Lipschitz function vanishing at some given point v_0) endowed with the norm $\|\phi\|_{Lip}$ and $P_{\mathcal{G}}(E)$ (constructed in Example 2.19) is endowed with the Wasserstein distance W_1 with linear cost, one has* (2.20) *with $k = 0$, $\eta = 1$:*

$$\left|\Phi\left(\mu^N_X\right) - \Phi\left(\mu^N_Y\right)\right| \le \|\Phi\|_{C^{0,1}(P_{\mathcal{G}})}\, W_1\left(\mu^N_X, \hat\mu^N_Y\right) \le \|\Phi\|_{C^{0,1}(P_{\mathcal{G}})}\, \|X - Y\|_{\ell^1},$$

where we use (2.22), which proves that

$$\|\pi^N(\Phi)\|_{C^{0,1}(E^N)} \le \|\Phi\|_{C^{0,1}(P_{\mathcal{G}})},$$

when E^N is endowed with the ℓ^1 distance defined in (2.22).

2.7. **Examples of distances on measures.** Let us list some well-known distances on $P(\mathbb{R}^d)$ (or on subsets of $P(\mathbb{R}^d)$) useful for the sequel. These distances are all topologically equivalent to the weak topology $\sigma(P(E), C_b(E))$ (on the sets $\mathcal{B}P_{k,a}(E)$ for k large enough and for any $a \in (0,\infty)$) and they are all uniformly topologically equivalent (see [44, 8] and section 2.8).

Example 2.18 (Dual-Hölder (or Zolotarev's) distances)**.** *Denote by $dist_E$ a distance on E and let us fix $v_0 \in E$ (e.g. $v_0 = 0$ when $E = \mathbb{R}^d$ in the sequel). Denote by $C^{0,s}_0(E)$, $s \in (0,1)$ (resp. $Lip_0(E)$) the set of s-Hölder functions (resp. Lipschitz functions) on E vanishing at one arbitrary point $v_0 \in E$ endowed with the norm*

$$[\varphi]_s := \sup_{x,y\in E} \frac{|\varphi(y) - \varphi(x)|}{\mathrm{dist}_E(x,y)^s}, s \in (0,1], \qquad [\varphi]_{\mathrm{Lip}} := [\varphi]_1.$$

We then define the dual norm: take $m_{\mathcal{G}} := 1$, $\mathbf{m}_{\mathcal{G}} := 0$ and $P_{\mathcal{G}}(E)$ endowed with

$$\forall\, f, g \in P_{\mathcal{G}}, \qquad [g - f]^*_s := \sup_{\varphi \in C^{0,s}_0(E)} \frac{\langle g - f, \varphi\rangle}{[\varphi]_s}. \tag{2.21}$$

Example 2.19 (Wasserstein distances)**.** *For $q \in (0,\infty)$, define W_q on*

$$P_{\mathcal{G}}(E) = P_q(E) := \{f \in P(E);\ \langle f, dist(\cdot, v_0)^q\rangle < \infty\}$$

by

$$\forall\, f, g \in P_q(E), \quad W_q(f,g) := \inf_{\Pi \in \Pi(f,g)} \int_{E\times E} dist_E(x,y)^q\, \Pi(dx, dy),$$

where $\Pi(f,g)$ denote the set of probability measures $\Pi \in P(E\times E)$ with marginals f and g ($\Pi(A,E) = f(A)$, $\Pi(E,A) = g(A)$ for any Borel set $A \subset E$). Note that for $V, Y \in E^N$ and any $q \in [1,\infty)$, one has

$$W_q\left(\mu^N_V, \mu^N_Y\right) = d_{\ell^q(E^N/\mathfrak{S}_N)}(V,Y) := \min_{\sigma\in\mathfrak{S}_N} \left(\frac{1}{N} \sum_{i=1}^N dist_E(v_i, y_{\sigma(i)})^q \right)^{1/q}, \tag{2.22}$$

and that

$$\forall\, f,\, g \in P_1(E), \quad W_1(f,g) = [f-g]^*_1 = \sup_{\phi\in Lip_0(E)} \langle f-g, \phi\rangle\,. \tag{2.23}$$

We refer to [46] and the references therein for more details on the Wasserstein distances.

Example 2.20 (Fourier-based norms). *For $E = \mathbb{R}^d$, $m_{\mathcal{G}_1} := |v|$, $\mathbf{m}_{\mathcal{G}_1} := 0$, let us define*

$$\forall\, f \in \mathcal{T}P_{\mathcal{G}_1}, \quad \|f\|_{\mathcal{G}_1} = |f|_s := \sup_{\xi\in\mathbb{R}^d} \frac{|\hat f(\xi)|}{|\xi|^s}, \quad s \in (0,1].$$

Similarly, for $E = \mathbb{R}^d$, $m_{\mathcal{G}_2} := |v|^2$, $\mathbf{m}_{\mathcal{G}_2} := v$, we define

$$\forall\, f \in \mathcal{T}P_{\mathcal{G}_2}, \quad \|f\|_{\mathcal{G}_2} = |f|_s := \sup_{\xi\in\mathbb{R}^d} \frac{|\hat f(\xi)|}{|\xi|^s}, \quad s \in (1,2].$$

Example 2.21 (More Fourier-based norms). *More generally, for $E = \mathbb{R}^d$, $k \in \mathbb{N}^*$, we define $m_{\mathcal{G}} := |v|^k$, $\mathbf{m}_{\mathcal{G}} := (v^j)_{j\in\mathbb{N}^d,\,|j|\le k-1}$ where for $j = (j_1,...,j_d) \in \mathbb{N}^d$ we set $v^j = (v_1^{j_1},...,v_d^{j_d})$ and $|j| = j_1 + ... + j_d$, and*

$$\forall\, f \in \mathcal{T}P_{\mathcal{G}}, \quad \|f\|_{\mathcal{G}} = |f|_s := \sup_{\xi\in\mathbb{R}^d} \frac{|\hat f(\xi)|}{|\xi|^s}, \quad s \in (0,k].$$

In fact, we may extend the above norm to $M^1_k(\mathbb{R}^d)$ in the following way. We first define for $f \in M^1_{k-1}(\mathbb{R}^d)$ and $j \in \mathbb{N}^d$, $|j| \le k-1$,

$$M_j[f] := \int_{\mathbb{R}^d} v^j\, f(dv).$$

For a fixed (once for all) function χ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ such that $\chi \equiv 1$ on the set $\{v \in \mathbb{R}^d, |v| \le 1\}$, so that in particular $\int_{\mathbb{R}^d} \mathcal{F}^{-1}(\chi)(v)\,dv = \chi(0) = 1$, we define $\mathcal{M}_k[f]$ be its Fourier transform

$$\hat{\mathcal{M}}_k[f](\xi) := \chi(\xi) \left(\sum_{|j|\le k-1} M_j[f]\, \frac{\xi^j}{j!}\, i^{|j|} \right),$$

which is some smooth version of the Taylor expansion of $\hat f$ at $\xi = 0$. Then we may define the seminorms

$$|f|_k := \sup_{\xi\in\mathbb{R}^d} \left(|\xi|^{-k} \left| \hat f(\xi) - \hat{\mathcal{M}}_k[f](\xi) \right| \right)$$

and

$$|||f|||_k := |f|_k + \sum_{j\in\mathbb{N}^d,\,|j|\le k-1} |M_j[f]|.$$

Example 2.22 (Negative Sobolev norms)**.** *For any* $s \in (d/2, d/2+1/2)$ *take* $E = \mathbb{R}^d$, $m_{\mathcal{G}_1} := |v|$, $\mathbf{m}_{\mathcal{G}_1} := 0$ *and*

$$\forall\, f \in \mathcal{T}P_{\mathcal{G}_1}, \quad \|f\|_{\mathcal{G}_1} = \|f\|_{\dot H^{-s}(\mathbb{R}^d)} := \left\| \frac{\hat f(\xi)}{|\xi|^s} \right\|_{L^2}.$$

Similarly, for any $s \in [d/2+1/2, d/2+1)$ *take* $E = \mathbb{R}^d$, $m_{\mathcal{G}_2} := |v|^2$, $\mathbf{m}_{\mathcal{G}_2} := v$ *and*

$$\forall\, f \in \mathcal{T}P_{\mathcal{G}_2}, \quad \|f\|_{\mathcal{G}_2} = \|f\|_{\dot H^{-s}(\mathbb{R}^d)} := \left\| \frac{\hat f(\xi)}{|\xi|^s} \right\|_{L^2}.$$

2.8. Comparison of distances when $E = \mathbb{R}^d$.

Lemma 2.23. *Let* $f, g \in P(\mathbb{R}^d)$*, then*

$$\forall\, q, k \in (1,\infty) \quad W_1(f,g) \le W_q(f,g) \le M_{k+1}^{1-\alpha}\, W_1(f,g)^\alpha, \tag{2.24}$$

with $\alpha := 1 - (q-1)/k$,

$$\forall\, s \in (0,1], \quad |f-g|_s \le W_s(f,g) \le W_1^s(f,g), \tag{2.25}$$

$$\forall\, s \in (d/2, d/2+1), \quad \|f-g\|_{\dot H^{-s}} \le C\, |f-g|_1^{2s-d}, \quad C = C(d,s) > 0, \tag{2.26}$$

$$\forall\, s > 0,\ k > 0 \quad [f-g]_1^* \le C\, M_{k+1}^{\alpha_1}\, |f-g|_s^{\gamma_1}, \quad C = C(d,s,k) > 0, \tag{2.27}$$

with

$$\alpha_1 := \frac{d}{d+k+k(d+s-1)}, \quad \gamma_1 := \frac{k}{d+k+k(d+s-1)},$$

$$\forall\, s \ge 1,\ k > 0, \quad [f-g]_1^* \le C\, M_{k+1}^{\alpha_2}\, \|f-g\|_{\dot H^{-s}}^{\gamma_2}, \quad C = C(d,s,k) > 0, \tag{2.28}$$

with

$$\alpha_2 := \frac{d/2}{d/2+k+k(s-1)}, \quad \gamma_2 := \frac{k}{d/2+k+k(s-1)}$$

and

$$M_k := \max\left\{ \int_{\mathbb{R}^d} (1+|x|^k)\, f(dx)\,;\ \int_{\mathbb{R}^d} (1+|x|^k)\, g(dx) \right\}.$$

Proof of Lemma 2.23. We split the proof in several steps. For the proof of (2.24) we refer to [44, 8].

Proof of (2.25). Let $\pi \in \Pi(f,g)$. We write

$$\begin{aligned} |\hat f(\xi) - \hat g(\xi)| &= \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} (e^{-i\,v\cdot\xi} - e^{-i\,w\cdot\xi})\, \pi(dv,dw) \right| \\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |e^{-i\,v\cdot\xi} - e^{-i\,w\cdot\xi}|\, \pi(dv,dw) \\ &\le C_s \int_{\mathbb{R}^d\times\mathbb{R}^d} |v-w|^s\, |\xi|^s\, \pi(dv,dw), \end{aligned}$$

which yields (2.25) by taking the supremum in $\xi \in \mathbb{R}^d$ and the infimum in $\pi \in \Pi(f,g)$.

Proof of (2.26). Consider $R > 0$ and the ball $B_R = \{x \in \mathbb{R}^d \; ; \; |x| \le R\}$, and write

$$\begin{aligned} \|f-g\|^2_{\dot H^{-s}} &= \int_{B_R} \frac{|\hat f(\xi) - \hat g(\xi)|^2}{|\xi|^{2s}} \, d\xi + \int_{B_R^c} \frac{|\hat f(\xi) - \hat g(\xi)|^2}{|\xi|^{2s}} \, d\xi \\ &\le |f-g|_1^2 \int_{B_R} \frac{d\xi}{|\xi|^{2(s-1)}} + 4 \int_{B_R^c} \frac{d\xi}{|\xi|^{2s}} \\ &\le C(d)\, R^{d-2(s-1)} \, |f-g|_1^2 + 4\, R^{d-2s}. \end{aligned}$$

Then (2.26) follows by choosing (the optimal) $R := |f-g|_1^{-1}$.

Proof of (2.27). We introduce a truncation function $\chi_R(x) = \chi(x/R)$, $R > 0$, where $\chi \in C^\infty(\mathbb{R}^d)$, $[\chi]_1 \le 1$, $0 \le \chi \le 1$, $\chi \equiv 1$ on $B(0,1)$, $\operatorname{supp} \chi \subset B(0,2)$, and a mollifer function $\omega_\varepsilon(x) = \varepsilon^{-d}\, \omega(x/\varepsilon)$, $\varepsilon > 0$ where for instance $\omega(x) = (2\pi)^{-d/2} \exp(-|x|^2/2)$ (and thus $\hat\omega_\varepsilon(\xi) = \hat\omega(\varepsilon\,\xi) = \exp(-\varepsilon^2\,|\xi|^2/2)$). Fix $\varphi \in W^{1,\infty}(\mathbb{R}^d)$ such that $[\varphi]_1 \le 1$, $\varphi(0) = 0$, define $\varphi_R := \varphi\,\chi_R$, $\varphi_{R,\varepsilon} = \varphi_R * \omega_\varepsilon$ and write

$$\int \varphi\,(df - dg) = \int \varphi_{R,\varepsilon}\,(df-dg) + \int (\varphi_R - \varphi_{R,\varepsilon})\,(df-dg) + \int (\varphi - \varphi_R)\,(df-dg).$$

For the last term, we have

$$\begin{aligned} \left| \int (\varphi_R - \varphi)\,(df - dg) \right| &\le \int (1-\chi_R)\,\varphi\,(df+dg) \\ &\le \int_{B_R^c} [\varphi]_1 \, \frac{|x|^{k+1}}{R^k}\,(df+dg) \le \frac{M_{k+1}[f+g]}{R^k}, \end{aligned} \tag{2.29}$$

where $M_{k+1}[f+g]$ denotes the $(k+1)$-th moment of $f+g$. In order to deal with the second term, we observe that

$$|\nabla\varphi_R| \le \chi(x/R) + |\varphi|\,|\nabla(\chi_R)| \le \chi(x/R) + \frac{|x|}{R}\,|\nabla\chi|(x/R),$$

so that for any $q \in [1,\infty]$ there holds $\|\nabla\varphi_R\|_{L^q} \le C\,R^{d/q}$, for some constant depending only on χ, d. Next, using that

$$\|\varphi_R - \varphi_{R,\varepsilon}\|_\infty \le \|\nabla\varphi_R\|_\infty \int_{\mathbb{R}^d} \omega_\varepsilon(x)\,|x|\,dx \le C\,\varepsilon,$$

we find

$$\left| \int (\varphi_R - \varphi_{R,\varepsilon})\,(df - dg) \right| \le C\,\varepsilon. \tag{2.30}$$

For the first term, using Parseval's identity,

$$\begin{aligned} \left| \int \varphi_{R,\varepsilon}\,(f-g) \right| &= \frac{1}{2\pi} \left| \int \hat\varphi_R\,\hat\omega_\varepsilon\,\overline{(\hat f - \hat g)}\,d\xi \right| \\ &\le \frac{1}{2\pi}\,\|\nabla\varphi_R\|_{L^1} \left\| \frac{\hat f - \hat g}{|\xi|^s} \right\|_{L^\infty} \int |\xi|^{s-1} \exp(-\varepsilon^2\,|\xi|^2/2)\,d\xi \\ &\le C\,R^d \left(\int (1+|y|)\,\chi(y)\,dy \right) |f-g|_s\,\varepsilon^{-(d+s-1)} \left(\int |z|^{s-1}\,e^{-\frac{|z|^2}{2}}\,dz \right) \\ &\le C\,R^d\,\varepsilon^{-(d+s-1)}\,|f-g|_s. \end{aligned} \tag{2.31}$$

Gathering (2.29), (2.30) and (2.31), we get

$$[f-g]_1^* \le C\,\left(\varepsilon + \frac{M_{k+1}[f+g]}{R^k} + R^d\,\varepsilon^{-(d+s-1)}\,|f-g|_s\right).$$

This yields (2.27) by optimizing the parameters ε and R.

Step 4. Proof of (2.28). We start with the same decomposition as before:

$$\int \varphi\,(df-dg) = \int \varphi_{R,\varepsilon}\,(df-dg) + \int (\varphi_R - \varphi_{R,\varepsilon})\,(df-dg) + \int (\varphi - \varphi_R)\,(df-dg).$$

The first term is controled by

$$\left|\int \varphi_{R,\varepsilon}\,(df-dg)\right| = \left|\int \hat\varphi_{R,\varepsilon}\,|\xi|^s\,\frac{(\hat f - \hat g)}{|\xi|^s}\right| \le \|\varphi_{R,\varepsilon}\|_{\dot H^s}\,\|f-g\|_{\dot H^{-s}}$$

with

$$\begin{aligned}\|\varphi_{R,\varepsilon}\|_{\dot H^s} &= \left(\int |\xi|^2\,|\widehat{\varphi\,\chi_R}|^2\,|\xi|^{2(s-1)}\,|\hat\omega_\varepsilon|^2\,d\xi\right)^{1/2}\\ &\le \|\nabla(\varphi\,\chi_R)\|_{L^2}\,\||\xi|^{s-1}\,\hat\omega_\varepsilon(\xi)\|_{L^\infty}\\ &\le \|\nabla(\varphi\,\chi_R)\|_{L^2}\,\varepsilon^{1-s}\||z|^{s-1}\,\hat\omega(z)\|_{L^\infty} \le C\,R^{d/2}\,\varepsilon^{-(s-1)}. \qquad (2.32)\end{aligned}$$

The second term and the last term are controled as before by $C\,\varepsilon$ and $M_{k+1}\,R^{-k}$ respectively. Summing we obtain

$$[f-g]_1^* \le C\,\left(\varepsilon + \frac{M_{k+1}[f+g]}{R^k} + R^{d/2}\,\varepsilon^{-(s-1)}\,|f-g|_{\dot H^{-s}}\right).$$

This yields (2.28) by optimizing the parameters ε and R. □

2.9. **On the law of large numbers for measures.**

2.9.1. *Remark on the meaning of the $\mathcal{W}$ function.* For a given function $D : P(E)\times P(E) \to \mathbb{R}_+$ continuous for the weak topology, and such that $D(f,g) = 0$ if and only $f = g$ (D stands for a distance on $P(E)$ or a function of a distance), we define

$$\mathcal{W}_D^N(f) := \int_{E^N} D\left(\mu_V^N; f\right)\, f^{\otimes N}(dV).$$

On the one hand, from the definition of $\pi_P^N f^{\otimes N}$ in section 2.1, we have

$$\mathcal{W}_D^N(f) = \mathcal{W}_D^N\left(f^{\otimes N}; f\right) = \mathcal{W}_D^\infty\left(\pi_P^N f^{\otimes N}; f\right),$$

with

$$\begin{aligned}&\forall\, f^N \in P(E^N) \qquad \mathcal{W}_D^N(f^N; f) := \int_{E^N} D(\mu_V^N, f)\, f^N(dV)\\ &\forall\, \pi \in P(P(E)) \qquad \mathcal{W}_D^\infty(\pi; f) := \int_{P(E)} D(\rho, f)\,\pi(d\rho).\end{aligned}$$

2.9.2. *Comparison of $\mathcal{W}_D^N$ functions for different distances D.* Let us begin with an elementary result.

Lemma 2.24. *If $D_1 \le C\, D_2^r$ for some constants $C > 0$ and $r < 1$, we have for some constant $C' = C'(C,r)$ and for any $f \in P(E)$*

$$\mathcal{W}_{D_1}^N(f) \le C' \left(\mathcal{W}_{D_2}^N(f)\right)^r. \tag{2.33}$$

Proof of Lemma 2.24. For any $\varepsilon > 0$, we have from Young's inequality

$$\begin{aligned} \mathcal{W}_{D_1}^N(f) &\le C \int_{E^N} D_2\left(\mu_V^N; f\right)^r f^{\otimes N}(dV) \\ &\le C \int_{\mathbb{R}^{Nd}} \left[(1-r)\,\varepsilon^{r/(1-r)} + \frac{r}{\varepsilon} D_2\left(\mu_V^N; f\right)\right] f^{\otimes N}(dV) \\ &\le C(r) \left[\varepsilon^{r/(1-r)} + \frac{\mathcal{W}_{D_2}^N(f)}{\varepsilon}\right]. \end{aligned}$$

This yields (2.33) by optimizing $\varepsilon > 0$. □

Lemma 2.25. *We have the following rates for the $\mathcal{W}$ function:*

- *For any $f \in P_2(\mathbb{R}^d)$, any $s \in (d/2, d/2+1)$ and any $N \ge 1$ there holds*

$$\mathcal{W}_{\|\cdot\|^2_{\dot H^{-s}}}^N(f) = \int_{\mathbb{R}^{Nd}} \left\|\mu_V^N - f\right\|^2_{\dot H^{-s}} f^{\otimes N}(dV) \le \mathrm{Cst}(d, M_2)\, N^{-1}. \tag{2.34}$$

- *For any $\eta > 0$ there exists $k \ge 1$ such that for any $f \in P_k(\mathbb{R}^d)$ and any $N \ge 1$ there holds*

$$\mathcal{W}_{W_1}^N(f) \le \mathrm{Cst}(\eta, k, M_k)\, N^{-1/(d+\eta)}. \tag{2.35}$$

- *For any $\eta > 0$ there exists $k \ge 2$ such that for any $f \in P_k(\mathbb{R}^d)$ and any $N \ge 1$ there holds*

$$\mathcal{W}_{W_2^2}^N(f) \le \mathrm{Cst}(\eta, k, M_k)\, N^{-1/(d+\eta)}. \tag{2.36}$$

Remark 2.26. *Estimate* (2.36) *has to be compared with the following classical estimate established in* [37]*: for any $f \in P_{d+5}(\mathbb{R}^d)$ and any $N \ge 1$ there holds*

$$\mathcal{W}_{W_2^2}^N(f) \le \mathrm{Cst}(d, M_{d+5})\, N^{-\frac{2}{d+4}}. \tag{2.37}$$

It is worth mentioning that (2.36) *improve* (2.37) *when $d \le 3$ and k is large enough (so that $\eta < 2 - d/2$).*

Proof of Lemma 2.25. We split the proof into two steps.

Proof of (2.34). Let us fix $f \in P_2(\mathbb{R}^d)$. First, writing

$$\left(\hat\mu_V^N - \hat f\right)(\xi) = \frac{1}{N} \sum_{j=1}^N \left(e^{-i\,v_j\cdot\xi} - \hat f(\xi)\right),$$

we have

$$\begin{aligned} \mathcal{W}_{\|\cdot\|^2_{\dot H^{-s}}}^N(f) &= \int_{\mathbb{R}^{Nd}} \left(\int_{\mathbb{R}^d} \frac{\left|\hat\mu_V^N - \hat f\right|^2}{|\xi|^{2\,s}}\, d\xi\right) f^{\otimes N}(dV) \\ &= \frac{1}{N^2} \sum_{j_1,j_2=1}^N \int_{\mathbb{R}^{(N+1)d}} \frac{\left(e^{-i\,v_{j_1}\cdot\xi} - \hat f(\xi)\right)\overline{\left(e^{-i\,v_{j_2}\cdot\xi} - \hat f(\xi)\right)}}{|\xi|^{2\,s}}\, d\xi\, f^{\otimes N}(dV). \end{aligned}$$

We use

$$\int_{\mathbb{R}^d} (e^{-i\,v_j\cdot\xi} - \hat f(\xi))\, f(dv_j) = 0, \quad j = 1,\dots,d,$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} \left| e^{-i\,v\cdot\xi} - \hat f(\xi)\right|^2 f(dv) &= \int_{\mathbb{R}^d} \left[1 - e^{-i\,v\cdot\xi}\,\overline{\hat f(\xi)} - e^{i\,v\cdot\xi}\,\hat f(\xi) + |\hat f(\xi)|^2\right] f(dv) \\ &= 1 - |\hat f(\xi)|^2,\end{aligned}$$

to deduce

$$\begin{aligned}\mathcal{W}^N_{\|\cdot\|^2_{\dot H^{-s}}}(f) &= \frac{1}{N^2}\sum_{j=1}^N \int_{\mathbb{R}^{(N+1)d}} \frac{\left| e^{-i\,v_j\cdot\xi} - \hat f(\xi)\right|^2}{|\xi|^{2\,s}}\, d\xi\, f^{\otimes N}(dV) \\ &= \frac{1}{N}\int_{\mathbb{R}^{2\,d}} \frac{\left| e^{-i\,v\cdot\xi} - \hat f(\xi)\right|^2}{|\xi|^{2\,s}}\, d\xi\, f(dv) \\ &= \frac{1}{N}\int_{\mathbb{R}^d} \frac{(1 - |\hat f(\xi)|^2)}{|\xi|^{2\,s}}\, d\xi.\end{aligned}$$

Finally, observing that $\hat f(\xi) = 1 + i\,\langle f, v\rangle\cdot\xi + \mathcal{O}(M_2\,|\xi|^2)$, and therefore

$$\begin{aligned}|\hat f(\xi)|^2 &= \left(1 + i\,\langle f, v\rangle\cdot\xi + \mathcal{O}(M_2\,|\xi|^2)\right)\left(1 - i\,\langle f, v\rangle\cdot\xi + \overline{\mathcal{O}(M_2\,|\xi|^2)}\right) \\ &= 1 + \mathcal{O}\left(M_2\,|\xi|^2\right),\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{W}^N_{\|\cdot\|^2_{\dot H^{-s}}}(f) &= \frac{1}{N}\left(\int_{|\xi|\le 1} \frac{(1 - |\hat f(\xi)|^2)}{|\xi|^{2\,s}}\, d\xi + \int_{|\xi|\ge 1} \frac{(1 - |\hat f(\xi)|^2)}{|\xi|^{2\,s}}\, d\xi\right) \\ &= \frac{1}{N}\left(\int_{|\xi|\le 1} \frac{M_2}{|\xi|^{2\,(s-1)}}\, d\xi + \int_{|\xi|\ge 1} \frac{1}{|\xi|^{2\,s}}\, d\xi\right),\end{aligned}$$

from which (2.34) follows.

Proof of (2.35) *and* (2.36). By gathering (2.34), (2.28) in Lemma 2.24 and Lemma 2.23, we straightforwardly deduce

$$\begin{aligned}\mathcal{W}^N_{W_1}(f) &= \int_{R^{Nd}} [\mu^N_V - f]^*_1\, f^{\otimes N}(dV) \\ &\le C_{k,s,d}(M_{k+1}) \int_{R^{Nd}} \left(\|\mu^N_V - f\|^2_{\dot H^{-s}}\right)^{\gamma_2/2} f^{\otimes N}(dV) \\ &\le C_{k,s,d}(M_{k+1})\, N^{-\gamma_2/2},\end{aligned}$$

from which we deduce (2.35) because in the limit case $k = \infty$ we have $\gamma_2/2 = 1/(2s)$ and we may choose s as close from $d/2$ as we wish.

Proof of (2.36). Estimate (2.36) follows from (2.35) with the help of Lemma 2.23 and (2.24). □

3. The abstract theorem

3.1. Assumptions for the abstract theorem.

Assume that

- **(A1)** and **(A2)** hold, so that in particular the semigroups S_t^N, T_t^N, S_t^{NL} and T_t^∞ are well defined as well as the generators G^N and G^∞.

(A3) Convergence of the generators. In the probability metrized set $P_{\mathcal{G}_1}$ introduced in **(A2)** (associated to the weight function $m_{\mathcal{G}_1}$ and constraint function $\mathbf{m}_{\mathcal{G}_1}$) we define

$$\mathbf{R}_{\mathcal{G}_1} := \{\mathbf{r} \in \mathbb{R}^D;\ \exists\, f \in P_{\mathcal{G}_1} \text{ s.t. } \mathbf{m}_{\mathcal{G}_1}(f) = \mathbf{r}\}.$$

Then for the weight function $\Lambda_1(f) = \langle f, m_1 \rangle$ and for some function $\varepsilon_2(N)$ going to 0 as N goes to infinity, we assume that the generators G^N and G^∞ satisfy

$$\forall\, \Phi \in \bigcap_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} C^{1,\eta}_{\Lambda_1}(P_{\mathcal{G}_1,\mathbf{r}}; \mathbb{R})$$

$$\left\| \left(M^N_{m_1}\right)^{-1} \left(G^N \pi_N - \pi_N G^\infty\right) \Phi \right\|_{L^\infty(\mathbb{E}_N)} \le \varepsilon_2(N) \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} [\Phi]_{C^{1,\eta}_{\Lambda_1}(P_{\mathcal{G}_1,\mathbf{r}})}, \tag{3.38}$$

where $M^N_{m_1}$ was defined in (2.4).

(A4) Differential stability of the limiting semigroup. We assume that the flow S_t^{NL} is $C^{1,\eta}_{\Lambda_2}(P_{\mathcal{G}_1,\mathbf{r}}, P_{\mathcal{G}_2})$ for any $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$ in the sense that there exists $C_T^\infty > 0$ such that

$$\sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^T \left([S_t^{NL}]_{C^{1,\eta}_{\Lambda_2}(P_{\mathcal{G}_1,\mathbf{r}}, P_{\mathcal{G}_2})} + [S_t^{NL}]^{1+\eta'}_{C^{0,\eta''}_{\Lambda_2}(P_{\mathcal{G}_1,\mathbf{r}}, P_{\mathcal{G}_2})} \right) dt \le C_T^\infty, \tag{3.39}$$

where $\eta \in (0,1)$ is the same as in **(A3)**, $(\eta', \eta'') = (\eta, 1)$ or $(\eta', \eta'') = (1, (1+\eta)/2)$, $\Lambda_2 = \Lambda_1^{1/(1+\eta')}$ and $P_{\mathcal{G}_2} = \{\rho,\, \rho \in P_{\mathcal{G}_1}\}$ but it is endowed with the norm associated to a normed space $\mathcal{G}_2 \supset \mathcal{G}_1$.

(A5) Weak stability of the limiting semigroup. We assume that, for some probabilistic space $P_{\mathcal{G}_3}(E)$ (associated to a weight function $m_{\mathcal{G}_3}$, a constraint function $\mathbf{m}_{\mathcal{G}_3}$ and some metric structure $\mathrm{dist}_{\mathcal{G}_3}$) and that for any $a, T > 0$ there exists a concave and continuous function $\Theta_{a,T} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Theta_{a,T}(0) = 0$, we have

$$\forall\, f_1, f_2 \in \mathcal{B}P_{\mathcal{G}_3,a}(E)$$

$$\sup_{[0,T)} \mathrm{dist}_{\mathcal{G}_3}\left(S_t^{NL}(f_1), S_t^{NL}(f_2)\right) \le \Theta_{a,T}\left(\mathrm{dist}_{\mathcal{G}_3}(f_1, f_2)\right). \tag{3.40}$$

3.2. Statement of the result.

Theorem 3.27 (Fluctuation estimate)**.** *Consider a family of N-particle initial conditions $f_0^N \in P_{sym}(E^N)$, $N \ge 1$, and the associated solution $f_t^N = S_t^N f_0^N$. Consider a 1-particle initial condition $f_0 \in P(E)$ and the associated solution $f_t = S_t^{NL} f_0$. Assume that* **(A1)-(A2)-(A3)-(A4)-(A5)** *hold for some spaces $P_{\mathcal{G}_k}$, $\mathcal{G}_k$ and $\mathcal{F}_k$, $k = 1,2,3$ with $\mathcal{F}_k \subset C_b(E)$, and where $\mathcal{F}_k$ and $\mathcal{G}_k$ are in duality.*

Then there is an explicit absolute constant $C \in (0,\infty)$ such that for any $N, \ell \in \mathbb{N}^*$, with $N \ge 2\ell$, and for any

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$$

we have

$$\sup_{[0,T)} \left| \left\langle \left(S^N_t(f^N_0) - \left(S^{NL}_t(f_0)\right)^{\otimes N} \right), \varphi \right\rangle \right| \tag{3.41}$$
$$\leq C \left[\ell^2 \, \frac{\|\varphi\|_\infty}{N} + C^N_{T,m_1} \, C^\infty_T \, \varepsilon_2(N) \, \ell^2 \, \|\varphi\|_{\mathcal{F}^2_2 \otimes (L^\infty)^{\ell-2}} \right.$$
$$\left. + \ell \, \|\varphi\|_{\mathcal{F}_3 \otimes (L^\infty)^{\ell-1}} \, \Theta_{C^N_{0,m_3},T} \left(\mathcal{W}_{\mathrm{dist}_{\mathcal{G}_3}}(\pi^N_P f^N_0, \delta_{f_0}) \right) \right],$$

where $\mathcal{W}_{\mathrm{dist}_{\mathcal{G}_3}}$ *stands for the Monge-Kantorovich distance in* $P(P_{\mathcal{G}_3}(E))$*, see example 2.19, which means for such particular probabilities*

$$\mathcal{W}_{\mathrm{dist}_{\mathcal{G}_3}}(\pi^N_P f^N_0, \delta_{f_0}) = \int_{E^N} \mathrm{dist}_{\mathcal{G}_3}(\mu^N_V, f_0) \, f^N_0(dV). \tag{3.42}$$

Remark 3.28. *1) Our goal here is to treat the* N*-particles system as a perturbation (in a very degenerated sense) of the limiting problem, and to minimize assumptions on the many-particle systems in order to avoid complications of many dimensions dynamics.*

2) In the applications the worst decay rate in the right-hand side of (3.41) *is always the last one, which deals with the chaoticity of the initial data.*

3) It is worth mentioning that in the case when $f^N_0 = f_0^{\otimes N}$ *we have*

$$\mathcal{W}_{\mathrm{dist}_{\mathcal{G}_3}}(\pi^N_P f^N_0, \delta_{f_0}) = \mathcal{W}^N_{\mathrm{dist}_{\mathcal{G}_3}}(f_0)$$

and the decay rate is obtained thanks to the law of large numbers for measures presented in section 2.9. For more general initial datum we refer to section 6.

3.3. **Proof of Theorems 3.27.** For a given function $\varphi \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$, we break up the term to be estimated into four parts:

$$\left| \left\langle \left(S^N_t(f^N_0) - (S^\infty_t(f_0))^{\otimes N} \right), \varphi \otimes 1^{\otimes N-\ell} \right\rangle \right| \leq$$
$$\leq \left| \left\langle S^N_t(f^N_0), \varphi \otimes 1^{\otimes N-\ell} \right\rangle - \left\langle S^N_t(f^N_0), R^\ell_\varphi \circ \mu^N_V \right\rangle \right| \qquad (=: \mathcal{T}_1)$$
$$+ \left| \left\langle f^N_0, T^N_t(R^\ell_\varphi \circ \mu^N_V) \right\rangle - \left\langle f^N_0, (T^\infty_t R^\ell_\varphi) \circ \mu^N_V) \right\rangle \right| \qquad (=: \mathcal{T}_2)$$
$$+ \left| \left\langle f^N_0, (T^\infty_t R^\ell_\varphi) \circ \mu^N_V) \right\rangle - \left\langle (S^\infty_t(f_0))^{\otimes \ell}, \varphi \right\rangle \right| \qquad (=: \mathcal{T}_3).$$

We deal separately with each part step by step:

- $\mathcal{T}_1$ is controled by a purely combinatorial arguments introduced in [20]. In some sense it is the price we have to pay when we use the injection π^N;
- $\mathcal{T}_2$ is controled thanks to the consistency estimate **(A3)** on the generators, the differential stability assumption **(A4)** on the limiting semigroup and the moments propagation **(A1)**;
- $\mathcal{T}_3$ is controled in terms of the chaoticity of the initial data thanks to the weak stability assumption **(A5)** on the limiting semigroup and **(A1)**-(iii).

Step 1: Estimate of the first term $\mathcal{T}_1$. Let us prove that for any $t \geq 0$ and any $N \geq 2\ell$ there holds

$$\mathcal{T}_1 := \left| \left\langle S^N_t(f^N_0), \varphi \otimes 1^{\otimes N-\ell} \right\rangle - \left\langle S^N_t(f^N_0), R^\ell_\varphi \circ \mu^N_V \right\rangle \right| \leq \frac{2\,\ell^2 \, \|\varphi\|_{L^\infty(E^\ell)}}{N}. \tag{3.43}$$

Since $S^N_t(f^N_0)$ is a symmetric probability measure, estimate (3.43) is a direct consequence of the following lemma

Lemma 3.29. *For any $\varphi \in C_b(E^\ell)$ we have*

$$\forall\, N \geq 2\ell, \qquad \left| \left(\varphi \otimes \mathbf{1}^{\otimes N-\ell}\right)_{sym} - \pi_N R^\ell_\varphi \right| \leq \frac{2\,\ell^2\, \|\varphi\|_{L^\infty(E^\ell)}}{N} \tag{3.44}$$

where for a function $\phi \in C_b(E^N)$, we define its symmetrized version ϕ_{sym} as:

$$\phi_{sym} = \frac{1}{|\mathfrak{S}_N|} \sum_{\sigma\in\mathfrak{S}_N} \phi_\sigma. \tag{3.45}$$

As a consequence for any symmetric measure $f^N \in P(E^N)$ we have

$$\left| \langle f^N, R^\ell_\varphi(\mu^N_V)\rangle - \langle f^N, \varphi\rangle \right| \leq \frac{2\,\ell^2\, \|\varphi\|_{L^\infty(E^\ell)}}{N}. \tag{3.46}$$

Proof of Lemma 3.29. For a given $\ell \leq N/2$ we introduce

$$A_{N,\ell} := \left\{ (i_1, ..., i_\ell) \in [|1,N|]^\ell \;:\; \forall\, k \neq k',\; i_k \neq i_{k'} \right\} \qquad \text{and} \qquad B_{N,\ell} := A^c_{N,\ell}.$$

Since there are $N(N-1)\dots(N-\ell+1)$ ways of choosing ℓ distinct indices among $[|1,N|]$ we get

$$\begin{aligned} \frac{|B_{N,\ell}|}{N^\ell} &= 1 - \left(1 - \frac{1}{N}\right) \dots \left(1 - \frac{\ell-1}{N}\right) = 1 - \exp\left(\sum_{i=0}^{\ell-1} \ln\left(1 - \frac{i}{N}\right)\right) \\ &\leq 1 - \exp\left(-2\sum_{i=0}^{\ell-1} \frac{i}{N}\right) \leq \frac{\ell^2}{N}, \end{aligned}$$

where we have used

$$\forall\, x \in [0, 1/2], \quad \ln(1-x) \geq -2\,x \qquad \text{and} \qquad \forall\, x \in \mathbb{R}, \quad e^{-x} \geq 1 - x.$$

Then we compute

$$\begin{aligned} R^\ell_\varphi(\mu^N_V) &= \frac{1}{N^\ell} \sum_{i_1,\dots,i_\ell=1}^N \varphi(v_{i_1}, \dots, v_{i_\ell}) \\ &= \frac{1}{N^\ell} \sum_{(i_1,\dots,i_\ell)\in A_{N,\ell}} \varphi(v_{i_1}, \dots, v_{i_\ell}) + \frac{1}{N^\ell} \sum_{(i_1,\dots,i_\ell)\in B_{N,\ell}} \varphi(v_{i_1}, \dots, v_{i_\ell}) \\ &= \frac{1}{N^\ell}\, \frac{1}{(N-\ell)!} \sum_{\sigma\in\mathfrak{S}_N} \varphi(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) + \mathcal{O}\left(\frac{\ell^2}{N}\, \|\varphi\|_{L^\infty}\right) \\ &= \frac{1}{N!} \sum_{\sigma\in\mathfrak{S}_N} \varphi(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) + \mathcal{O}\left(\frac{2\,\ell^2}{N}\, \|\varphi\|_{L^\infty}\right) \end{aligned}$$

and the proof of (3.44) is complete. Next for any $f^N \in P(E^N)$ we have

$$\langle f^N, \varphi\rangle = \left\langle f^N, \left(\varphi \otimes \mathbf{1}^{\otimes N-\ell}\right)_{sym} \right\rangle,$$

and (3.46) trivially follows from (3.44). $\square$

Step 2: Estimate of the second term $\mathcal{T}_2$. Let us prove that for any $t \in [0,T)$ and any $N \geq 2\ell$ there holds

$$\begin{aligned} \mathcal{T}_2 &:= \left| \left\langle f_0^N, T_t^N \left(R_\varphi^\ell \circ \mu_V^N \right) \right\rangle - \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle \right| \\ &\leq C_{T,m_2}^N \, C_T^\infty \, \|\varphi\|_{\infty, \mathcal{F}_2^2 \otimes (L^\infty)^{\ell-2}} \, \ell^2 \, \varepsilon(N). \end{aligned} \tag{3.47}$$

We start from the following identity

$$T_t^N \pi_N - \pi_N T_t^\infty = -\int_0^t \frac{d}{ds} \left(T_{t-s}^N \, \pi_N \, T_s^\infty \right) \, ds = \int_0^t T_{t-s}^N \left[G^N \pi_N - \pi_N G^\infty \right] \, T_s^\infty \, ds.$$

From assumptions **(A1)** and **(A3)**, we have for any $t \in [0,T)$

$$\begin{aligned} &\left| \left\langle f_0^N, T_t^N \left(R_\varphi^\ell \circ \mu_V^N \right) \right\rangle - \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle \right| \\ &\quad \leq \int_0^T \left| \left\langle M_{m_1}^N \, S_{t-s}^N \left(f_0^N \right), \left(M_{m_1}^N \right)^{-1} \left[G^N \pi_N - \pi_N G^\infty \right] \left(T_s^\infty R_\varphi^\ell \right) \right\rangle \right| \, ds \\ &\quad \leq \left(\sup_{0 \leq t < T} \left\langle f_t^N, M_{m_1}^N \right\rangle \right) \left(\int_0^T \left\| \left(M_{m_1}^N \right)^{-1} \left[G^N \pi_N - \pi_N G^\infty \right] \left(T_s^\infty R_\varphi^\ell \right) \right\|_{L^\infty(\mathbb{E}_N)} ds \right) \\ &\quad \leq \varepsilon(N) \, C_{T,m}^N \sup_{\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}} \int_0^T \left[T_s^\infty R_\varphi^\ell \right]_{C_{\Lambda_1}^{1,\eta}(P_{\mathcal{G}_1,\mathbf{r}})} ds. \end{aligned} \tag{3.48}$$

Now, let us fix $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$. Since $T_t^\infty(R_\varphi^\ell) = R_\varphi^\ell \circ S_t^{NL}$ with $S_t^{NL} \in C_{\Lambda_2}^{1,\eta}(P_{\mathcal{G}_1,\mathbf{r}}; P_{\mathcal{G}_2})$ thanks to assumption **(A4)**, and $R_\varphi^\ell \in C^{1,1}(P_{\mathcal{G}_2}; \mathbb{R})$ because $\varphi \in \mathcal{F}_2^{\otimes \ell}$ (see subsection 2.6), we obtain with the help of Lemma 2.12 that $T_t^\infty(R_\varphi^\ell) \in C_{\Lambda_2^{1+\theta'}}^{1,\eta}(P_{\mathcal{G}_1,\mathbf{r}}; \mathbb{R})$ with

$$\left[T_s^\infty \left(R_\varphi^\ell \right) \right]_{C_{\Lambda_2^{1+\theta'}}^{1,\eta}(P_{\mathcal{G}_1,\mathbf{r}})} \leq \left(\left[S_t^{NL} \right]_{C_{\Lambda_2}^{1,\eta}(P_{\mathcal{G}_1,\mathbf{r}}, P_{\mathcal{G}_2})} + \left[S_t^{NL} \right]_{C_{\Lambda_2}^{0,1}(P_{\mathcal{G}_1,\mathbf{r}}, P_{\mathcal{G}_2})}^{1+\theta'} \right) \left\| R_\varphi^\ell \right\|_{C^{1,\eta}(P_{\mathcal{G}_2})}.$$

With the help of $\Lambda_2 = \Lambda_1^{1/(1+\theta')}$, (2.19) and assumption **(A4)**, we hence deduce

$$\int_0^T [T_s^\infty(R_\varphi^\ell)]_{C_{\Lambda_1}^{1,\eta}(P_{\mathcal{G}_1,\mathbf{r}})} \, ds \leq C_T^\infty \, \ell^2 \left(\|\varphi\|_{\mathcal{F}_2 \otimes (L^\infty)^{\ell-2}} + \|\varphi\|_{\mathcal{F}_2 \otimes (L^\infty)^{\ell-1}} \right) \tag{3.49}$$

Then we go back to the computation (3.48), and plugging (3.49) we deduce (3.47).

Step 3: Estimate of the third term $\mathcal{T}_3$. Let us prove that for any $t \geq 0$, $N \geq \ell$

$$\begin{aligned} \mathcal{T}_3 := \left| \left\langle f_0^N, \left(T_t^\infty R_\varphi^\ell \right) \circ \mu_V^N \right\rangle - \left\langle \left(S_t^\infty(f_0) \right)^{\otimes \ell}, \varphi \right\rangle \right| &\leq \\ \leq [R_\varphi]_{C^{0,1}} \, \Theta_{C_{0,m_3}^N, T} \left(\mathcal{W}_{1,P_{\mathcal{G}_3}} \left(\pi_P^N f_0^N, \delta_{f_0} \right) \right). \end{aligned} \tag{3.50}$$

We shall proceed as in Step 2, using that:

- $\operatorname{supp} \pi_P^N f_0^N \subset \mathcal{K} := \{ f \in P_{\mathcal{G}_3};\ M_{m_3}(f) \leq C_{0,m_3}^N \}$ thanks to assumption **(A1)**,
- S_t^{NL} satisfies some Hölder like estimate uniformly on $\mathcal{K}$ and $[0,T)$ thanks to assumption **(A5)**,
- $R_\varphi^\ell \in C^{0,1}(P_{\mathcal{G}_3}, \mathbb{R})$ because $\varphi \in \mathcal{F}_3^{\otimes \ell}$.

We deduce thanks to the Jensen inequality (for a concave function)

$$\begin{aligned}
\mathcal{T}_3 &= \left|\left\langle f_0^N, R_\varphi^\ell\left(S_t^{NL}(\mu_V^N)\right)\right\rangle - \left\langle f_0^N, R_\varphi^\ell\left(S_t^{NL}(f_0)\right)\right\rangle\right| \\
&= \left|\left\langle f_0^N, R_\varphi^\ell\left(S_t^{NL}(\mu_V^N)\right) - R_\varphi^\ell\left(S_t^{NL}(f_0)\right)\right\rangle\right| \\
&\le \left[R_\varphi\right]_{C^{0,1}(P_{\mathcal{G}_3})} \left\langle f_0^N, \, \mathrm{dist}_{\mathcal{G}_3}\left(S_t^{NL}(f_0), S_t^{NL}(\mu_V^N)\right)\right\rangle \\
&\le \left[R_\varphi\right]_{C^{0,1}(P_{\mathcal{G}_3})} \left\langle f_0^N, \, \Theta_{a,T}\left(\mathrm{dist}_{\mathcal{G}_3}(f_0, \mu_V^N)\right)\right\rangle \\
&\le \left[R_\varphi\right]_{C^{0,1}(P_{\mathcal{G}_3})} \Theta_{a,T}\left(\left\langle f_0^N, \, \mathrm{dist}_{\mathcal{G}_3}(f_0, \mu_V^N)\right\rangle\right).
\end{aligned}$$

Now, by definition of the optimal transport Wasserstein distance we have

$$\forall \, \mu_1, \, \mu_2 \in P\left(P_{\mathcal{G}_3}\right), \quad \mathcal{W}_{1,P_{\mathcal{G}_3}}\left(\mu_1, \mu_2\right) = \inf_{\pi \in \Pi(\mu_1,\mu_2)} \int_{P_{\mathcal{G}_3} \times P_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(\mu, \mu') \, \pi(d\mu, d\mu'),$$

where $\Pi(\mu_1, \mu_2)$ denotes the probabilty measures on the product space $P_{\mathcal{G}_3} \times P_{\mathcal{G}_3}$ with first and second marginals μ_1, μ_2. In the case when $\mu_2 = \delta_{f_0}$ then $\Pi(\mu_1, \delta_{f_0}) = \{\mu_1 \otimes \delta_{f_0}\}$ has only one element, and therefore

$$\begin{aligned}
\mathcal{W}_{1,P_{\mathcal{G}_3}}\left(\pi_P^N f_0^N, \delta_{f_0}\right) &= \inf_{\pi \in \Pi(\pi_P^N f_0^N, \delta_{f_0})} \int_{P_{\mathcal{G}_3} \times P_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f, g) \, \pi(df, dg) \\
&= \iint_{P_{\mathcal{G}_3} \times P_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f, g) \, \pi_P^N f_0^N(dg) \, \delta_{f_0}(df) \\
&= \int_{P_{\mathcal{G}_3}} \mathrm{dist}_{\mathcal{G}_3}(f_0, g) \, \pi_P^N f_0^N(dg) \\
&= \int_{E^N} \mathrm{dist}_{\mathcal{G}_3}(\mu_V^N, f_0) \, f_0^N(dV).
\end{aligned}$$

We then easily conclude. □

4. (True) maxwellian molecules

4.1. **The model.**

Let us consider $E = \mathbb{R}^d$, $d \ge 2$, and a N-particles system undergoing space homogeneous random Boltzmann type collisions according to a collision kernel $B = \Gamma(z)\, b(\cos\theta)$ (see Subsection 1.1). More precisely, given a pre-collisional system of velocity variables $V = (v_1, ..., v_N) \in E^N = (\mathbb{R}^d)^N$, the stochastic process is:

(i) for any $i' \neq j'$, draw a random time $T_{\Gamma(|v_{i'} - v_{j'}|)}$ of collision accordingly to an exponential law of parameter $\Gamma(|v_{i'} - v_{j'}|)$, and then choose the collision time T_1 and the colliding couple (v_i, v_j) (which is a.s. well-defined) in such a way that

$$T_1 = T_{\Gamma(|v_i - v_j|)} := \min_{1 \le i' \neq j' \le N} T_{\Gamma(|v_{i'} - v_{j'}|)};$$

(ii) then draw $\sigma \in S^{d-1}$ according to the law $b(\cos\theta_{ij})$, where $\cos\theta_{ij} = \sigma \cdot (v_j - v_i)/|v_j - v_i|$;

(iii) the new state after collision at time T_1 becomes

$$V_{ij}^* = (v_1, \dots, v_i^*, \dots, v_j^*, \dots, v_N),$$

where only velocities labelled i and j have changed, according to the rotation

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_i - v_j|\, \sigma}{2}, \quad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_i - v_j|\, \sigma}{2}. \tag{4.1}$$

The associated Markov process $(\mathcal{V}_t)$ on the velocity variables on $(\mathbb{R}^d)^N$ is then built by iterating the above construction. After scaling the time (changing $t \to t/N$ in order that the number of interactions is of order $\mathcal{O}(1)$ on finite time interval, see [39]) we denote by f_t^N the law of $\mathcal{V}_t$, S_t^N the associated semigroup, G^N and T_t^N respectively the dual generator and dual semigroup, as in the previous abstract construction. The so-called *Master equation* on the law f_t^N is given in dual form by

$$\partial_t \langle f_t^N, \varphi\rangle = \langle f_t^N, G^N \varphi\rangle \tag{4.2}$$

with

$$(G^N\varphi)(V) = \frac{1}{N} \sum_{i,j=1}^N \Gamma\left(|v_i - v_j|\right) \int_{\mathbb{S}^{d-1}} b(\cos\theta_{ij}) \, \left[\varphi^*_{ij} - \varphi\right] \, d\sigma \tag{4.3}$$

where $\varphi^*_{ij} = \varphi(V^*_{ij})$ and $\varphi = \varphi(V) \in C_b(\mathbb{R}^{Nd})$.

This collision process is invariant under velocities permutations and satisfies the microscopic conservations of momentum and energy at any collision time

$$\sum_{j=1}^N v_j^* = \sum_{j=1}^N v_j \quad \text{and} \quad |V^*|^2 = \sum_{j=1}^N |v_j^*|^2 = \sum_{j=1}^N |v_j|^2 = |V|^2.$$

As a consequence, for any symmetric initial law $f_0^N \in P_{\rm sym}(\mathbb{R}^{Nd})$ the law f_t^N at later times is also a symmetric probability, and it conserves momentum and energy:

$$\forall\, \alpha = 1, \dots, d, \quad \int_{\mathbb{R}^{dN}} \left(\sum_{j=1}^N v_{j,\alpha}\right) \, f_t^N(dV) = \int_{\mathbb{R}^{dN}} \left(\sum_{j=1}^N v_{j,\alpha}\right) \, f_0^N(dV),$$

where $(v_{j,\alpha})_{1\le\alpha\le d}$ denote the components of $v_j \in \mathbb{R}^d$, and

$$\forall\, \phi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \int_{\mathbb{R}^{dN}} \phi(|V|^2) \, f_t^N(dV) = \int_{\mathbb{R}^{dN}} \phi(|V|^2) \, f_0^N(dV) \tag{4.4}$$

(equality between possibly infinite non-negative quantities).

The (expected) limiting nonlinear homogeneous Boltzmann equation is defined by (1.2), (1.3), (1.4). The equation generates a nonlinear semigroup $S_t^{NL}(f_0) := f_t$ for any $f_0 \in P_2(\mathbb{R}^d)$ (probabilities with bounded second moment): for the Maxwell case we refer to [42, 44], for the hard spheres case we refer to [35] in a L^1 setting, and for the generalization of these L^1 solutions to $P_2(\mathbb{R}^d)$ we refer to [14], [17] and [28]. For these solutions, one has the conservation of momentum and energy

$$\forall\, t \ge 0, \quad \int_{\mathbb{R}^d} v \, f_t(dv) = \int_{\mathbb{R}^d} v \, f_0(dv), \qquad \int_{\mathbb{R}^d} |v|^2 \, f_t(dv) = \int_{\mathbb{R}^d} |v|^2 \, f_0(dv).$$

Without restriction, by using the change of variable $\sigma \mapsto -\sigma$, from now on we restrict the angular domain to $\theta \in [-\pi/2, \pi/2]$ for the limiting equation as well as the N-particle system. Therefore we assume $\operatorname{supp} b \subset [0,1]$. We still denote by b the symmetrized version of b by a slight abuse of notation.

4.2. **Statement of the result.** In this section we consider the case of the *Maxwell molecules kernel*. More precisely we shall assume

$$\Gamma \equiv 1, \quad b \in L^\infty_{loc}([0,1)), \quad \forall\, \eta > 1/2, \quad C_\eta(b) := \int_{\mathbb{S}^{d-1}} b(\cos\theta) \, |1 - \cos\theta|^{\eta/2} \, d\sigma < \infty. \tag{4.5}$$

Indeed for any positive real function ψ and any given vector $u \in \mathbb{R}^d$ we have

$$\int_{S^{d-1}} \psi(\hat u \cdot \sigma)\, d\sigma = |\mathbb{S}^{d-2}| \int_0^\pi \psi(\cos\theta)\, \sin^{d-2}\theta\, d\theta$$

and the true Maxwell angular collision kernel **(tMM)** defined in Subsection 1.1 satisfies (in dimension $d = 3$) $b(z) \sim K\,(1-z)^{-5/4}$ as $z \to 1$, which hence fulfills (4.5). These assumptions also trivially include the Grad's cutoff Maxwell molecules **(GMM)** introduced in Subsection 1.1.

Our fluctuations estimate result then states as follows:

Theorem 4.1 (Boltzmann equation for Grad's cut-off/true Maxwell molecules)**.** *Let us consider an initial distribution $f_0 \in P(\mathbb{R}^d)$ with compact support. Let us consider a hierarchy of N-particle distributions $f_t^N = S_t^N(f_0^{\otimes N})$ issued from the tensorized initial data $f_0^N = f_0^{\otimes N}$. Then for any*

$$\varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in \mathcal{F}^{\otimes \ell},$$

with

$$\mathcal{F} := \left\{ \varphi : \mathbb{R}^d \to \mathbb{R};\ \|\varphi\|_{\mathcal{F}} := \int_{\mathbb{R}^d} (1+|\xi|^4)\, |\hat\varphi(\xi)|\, d\xi < \infty \right\},$$

we have

$$\sup_{t\ge 0} \left| \left\langle \left(S_t^N(f_0^N) - \left(S_t^{NL}(f_0)\right)^{\otimes N}\right), \varphi \right\rangle \right| \tag{4.6}$$

$$\le C \left[\ell^2 \frac{\|\varphi\|_\infty}{N} + C_{T,4}^N \frac{C_{\eta,\infty}^\infty}{N^{1-\eta}}\, \ell^2\, \|\varphi\|_{\mathcal{F}^2 \otimes (L^\infty)^{\ell-2}} \right.$$

$$\left. + \ell\, \|\varphi\|_{W^{1,\infty} \otimes (L^\infty)^{\ell-1}}\, \mathcal{W}_{W_2}^N(f_0) \right],$$

$$\le \ell^2 \frac{C_\eta}{N^{\frac{2}{d}-\eta}}\, \|\varphi\|_{\mathcal{F}^{\otimes \ell}}$$

for some constants $C_{\eta,\infty}^\infty$, $C_\eta, \in (0,\infty)$ which may blow up when $\eta \to 0$ (and depend on b and $\mathcal{E} := M_2(f_0)$).

In order to prove Theorem 4.1, we have to establish the assumptions **(A1)-(A2)-(A3)-(A4)-(A5)** of Theorem 3.27 with $T = \infty$.

4.3. **Proof of (A1).** The operators S_t^N, T_t^N and G^N are well defined on $L^2(\mathbb{S}^{dN-1}(\sqrt{\mathcal{E}}))$ for any $\mathcal{E} > 0$ thanks to the facts

$$\left\langle G^N \varphi, \varphi \right\rangle_{L^2(\mathbb{S}^{dN-1}(\sqrt{\mathcal{E}}))} = -\frac{1}{N} \sum_{i,j=1}^N \Gamma\left(|v_i - v_j|\right) \int_{\mathbb{S}^{d-1}} b(\cos\theta_{ij})\ \left[\varphi_{ij}^* - \varphi\right]^2\, d\sigma$$

and $(S_t^N)^* = S_t^N = T_t^N$ on this space, for any $\mathcal{E} > 0$ (see for instance [24, Chapter 9, section 2]).

Then G^N is well defined on the domain $C_b^1(\mathbb{R}^{dN})$, and it is closable on this space by using the fact that for any $\varphi_n \in \mathrm{Dom}(G^N)$, $\varphi_n \to 0$, $G^N\varphi_n \to \psi$, one has for any $\mathcal{E} > 0$ that the restrictions of φ_n, $G^N\varphi_n$ to the sphere of energy $\mathcal{E}$ belongs to L^2 on this compact space, and then, using the preceeding discussion, we deduce that $\psi \equiv 0$ on this subspace. Since we can argue in this way for any $\mathcal{E} > 0$, we deduce that $\psi \equiv 0$, which is the definition of being closable.

Then it remains to prove bounds on the polynomial moments of the N-particles system. We shall prove the following more general lemma:

Lemma 4.2. *Consider the collision kernel $B = |v - v_*|^\gamma \, b(\theta)$ with $\gamma = 0$ or 1 and $b \in L^1([0,1),(1-z)^2)$. This covers the three cases* **(HS)**, **(tMM)** *and* **(GMM)**.

Assume that the initial datum of the N-particle system satisfies:

$$\operatorname{supp} f_0^N \subset \left\{ V \in \mathbb{R}^{Nd};\ M_2^N(V) \le \mathcal{E} \right\}, \quad M_2^N = \frac{1}{N} \sum_{j=1}^N |v_j|^2$$

and

$$\left\langle f_0^N, M_k^N \right\rangle \le C_{0,k} < \infty, \quad M_k^N = \frac{1}{N} \sum_{j=1}^N |v_j|^k, \quad k \ge 2.$$

Then we have

$$\sup_{t \ge 0} \left\langle f_t^N, M_k^N \right\rangle \le \max\left\{ C_{0,k};\ \bar a_k \right\}$$

where $\bar a_k \in (0,\infty)$ depends on k and $\mathcal{E}$.

Proof of Lemma 4.2. From (2.6) we have

$$\operatorname{supp} f_t^N \subset \left\{ V \in \mathbb{R}^{Nd};\ M_2^N(V) \le \mathcal{E} \right\}, \tag{4.7}$$

by taking $\phi(z) := \mathbf{1}_{z > N\,\mathcal{E}}$ in (4.4).

Next, we write the differential equality on k-th moment

$$\frac{d}{dt} \left\langle f_t^N, \frac{1}{N} \sum_{j=1}^N |v_j|^k \right\rangle = \frac{1}{N^2} \sum_{j_1 \ne j_2}^N \left\langle f_t^N, |v_{j_1} - v_{j_2}|^\gamma \, \mathcal{K}\left(v_{j_1}, v_{j_2}\right) \right\rangle,$$

with

$$\mathcal{K}\left(v_{j_1}, v_{j_2}\right) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} b(\theta_{j_1 j_2}) \, \left[|v_{j_1}^*|^k + |v_{j_2}^*|^k - |v_{j_1}|^k - |v_{j_2}|^k \right] \, d\sigma.$$

From the so-called Povner's Lemma proved in [35, Lemma 2.2] (valid for singular collision kernel as in our case), we have

$$\mathcal{K}(v_{j_1}, v_{j_2}) \le C_1 \left(|v_{j_1}|^{k-1} \, |v_{j_2}| + |v_{j_1}| \, |v_{j_2}|^{k-1} \right) - C_1 \left(|v_{j_1}|^k + |v_{j_2}|^k \right)$$

for some constants $C_1, C_2 \in (0,\infty)$ depending only on k and b.

By using the inequalities $|v_{j_1} - v_{j_2}| \ge |v_{j_1}| - |v_{j_2}|$ and $|v_{j_1} - v_{j_2}| \ge |v_{j_2}| - |v_{j_1}|$ in order to estimate the last term when $\gamma = 1$, we then deduce

$$\begin{aligned} &|v_{j_1} - v_{j_2}| \, \mathcal{K}(v_{j_1}, v_{j_2}) \le \\ &\le C_3 \left[(1 + |v_{j_1}|^{k+\gamma-1}) \, (1 + |v_{j_2}|^2) + (1 + |v_{j_1}|^2) \, (1 + |v_{j_2}|^{k+\gamma-1}) \right] - C_1 \left(|v_{j_1}|^{k+\gamma} + |v_{j_2}|^{k+\gamma} \right), \end{aligned}$$

for a constant C_3 depending on C_1 and C_2.

Using (symmetry hypothesis) that

$$\forall\, k \ge 0, \quad \langle f_t^N, |v_1|^k \rangle = \left\langle f_t^N, M_k^N \right\rangle,$$

and (4.7) we get

$$\begin{aligned} \frac{d}{dt} \left\langle f_t^N, |v_1|^k \right\rangle &\le 2\, C_3 \left\langle f_t^N, (1 + M_{k+\gamma-1}^N) \, (1 + M_2^N) \right\rangle - 2\, C_2 \left\langle f_t^N, M_{k+\gamma} \right\rangle \\ &\le C_1 \, (1 + \mathcal{E}) \, \left(1 + \left\langle f_t^N, |v_1|^{k+\gamma-1} \right\rangle \right) - 2\, C_2 \left\langle f_t^N, |v_1|^{k+\gamma} \right\rangle. \end{aligned}$$

Using finally Hölder's inequality

$$\left\langle f_1^N, |v|^{k-\gamma+1} \right\rangle \le \left\langle f_1^N, |v|^{k+\gamma} \right\rangle^{(k-\gamma+1)/(k+\gamma)}$$

we conclude that $y(t) = \langle f_t^N, |v_1|^k \rangle$ satisfies a differential inequality of the following kind

$$y' \le -K_1\, y^{\theta_1} + K_2\, y^{\theta_2} + K_3$$

with $\theta_1 \ge 1$ and $\theta_2 < \theta_1$, which concludes the proof of the lemma. $\square$

Lemma 4.2 proves **(A1)**-(i) with $m_e(v) = |v|^2$, **(A1)**-(ii) with $m_1(v) = |v|^4$ (and we do not need **(A1)**-(iii) in this particular case: we may take $m_3 \equiv 0$).

4.4. **Proof of (A2).** Let us define

$$P_{\mathcal{G}_1} := \left\{ f \in P_4(\mathbb{R}^d)\, ;\ \langle f, m_e \rangle \le \mathcal{E} \right\} \quad \text{endowed with the distance induced by } |\cdot|_2.$$

Let us prove **(A2)**-(i)-(ii) and **(A2)**-(iii) with:

$$\forall\, f_0 \in P_{\mathcal{G}_1}, \quad \forall\, t \in (0,1] \qquad \left| S_t^{NL} f_0 - f_0 - t\, Q(f_0, f_0) \right|_2 \le C\, t^2, \tag{4.8}$$

(one can prove more generally that the application $\mathcal{S}(f_0) : [0,\tau) \to P_{\mathcal{G}_1}$, $t \mapsto \mathcal{S}(f_0)(t) := S_t^{NL}(f_0)$ is $C^{1,1}([0,\tau); P_{\mathcal{G}_1})$, with $\mathcal{S}(f_0)'(0) = Q(f_0, f_0)$ and $\tau > 0$).

Let us recall the following result proved in [44]. We provide its proof for the the sake of completeness and because we will need to modify it in order to obtain similar results in the next sections.

Lemma 4.3. *For any $f_0, g_0 \in P_2(\mathbb{R}^d)$, the associated solutions f_t and g_t to the Boltzmann equation for Maxwellian collision kernel satisfy*

$$\sup_{t \ge 0} |f_t - g_t|_2 \le |f_0 - g_0|_2\,. \tag{4.9}$$

Proof of Lemma 4.3. We recall Bobylev's identity for Maxwellian collision kernel (*cf.* [4])

$$\mathcal{F}\left(Q^+(f,g)\right)(\xi) = \hat{Q}^+(F,G)(\xi) =: \frac{1}{2} \int_{S^{d-1}} b\left(\sigma \cdot \hat{\xi}\right)\, \left[F^+\, G^- + F^-\, G^+\right] d\sigma,$$

with $F = \hat{f}$, $G = \hat{g}$, $F^\pm = F(\xi^\pm)$, $G^\pm = G(\xi^\pm)$, $\hat{\xi} = \xi/|\xi|$ and

$$\xi^+ = \frac{1}{2}(\xi + |\xi|\,\sigma), \quad \xi^- = \frac{1}{2}(\xi - |\xi|\,\sigma).$$

Denoting by $D = \hat{g} - \hat{f}$, $S = \hat{g} + \hat{f}$, the following equation holds

$$\partial_t D = \hat{Q}(S, D) = \int_{S^2} b\left(\sigma \cdot \hat{\xi}\right)\, \left[\frac{D^+\, S^-}{2} + \frac{D^-\, S^+}{2} - D\right]\, d\sigma. \tag{4.10}$$

We perform the following cutoff on the angular collision kernel:

$$\int_{\mathbb{S}^{d-1}} b_K\left(\sigma \cdot \hat{\xi}\right)\, d\sigma = K, \quad b_K = b\, \mathbf{1}_{|\theta| \ge \delta(K)}$$

for some well-chosen $\delta(K)$, and we shall relax this assumption in the end (using uniqueness of measure solutions of [44]). Using that $\|S\|_\infty \le 2$, we deduce in distributional sense

$$\frac{d}{dt}\frac{|D|}{|\xi|^2} + K\, \frac{|D|}{|\xi|^2} \le \left(\sup_{\xi \in \mathbb{R}^d} \frac{|D|}{|\xi|^2}\right)\, \left(\sup_{\xi \in \mathbb{R}^d} \int_{S^{d-1}} b_K\left(\sigma \cdot \hat{\xi}\right)\, \left(\left|\hat{\xi}^+\right|^2 + \left|\hat{\xi}^-\right|^2\right)\, d\sigma\right)$$

with

$$\left|\hat{\xi}^+\right| = \frac{1}{\sqrt{2}}\, \left(1 + \sigma \cdot \hat{\xi}\right)^{1/2}, \qquad \left|\hat{\xi}^-\right| = \frac{1}{\sqrt{2}}\, \left(1 - \sigma \cdot \hat{\xi}\right)^{1/2}.$$

By using $|\hat\xi^+|^2 + |\hat\xi^-|^2 = 1$, we deduce

$$\frac{d}{dt}\frac{|D|}{|\xi|^2} + K\,\frac{|D|}{|\xi|^2} \le K\,\left(\sup_{\xi\in\mathbb{R}^d}\frac{|D|}{|\xi|^2}\right)$$

from which we deduce

$$\left(\sup_{\xi\in\mathbb{R}^d}\frac{|D_t(\xi)|}{|\xi|^2}\right) \le \left(\sup_{\xi\in\mathbb{R}^d}\frac{|D_0(\xi)|}{|\xi|^2}\right)$$

for any value of the cutoff parameter K. Therefore by relaxing the cutoff $K\to\infty$, we deduce (4.9). □

Hence we deduce that S_t^{NL} is $C^{0,1}(P_{\mathcal{G}_1}, P_{\mathcal{G}_1})$ and **(A2)-(i)** is proved.

Lemma 4.4. *For any $f,g\in P_2$ with same momentum and finite second moment, we have*

$$|Q(f,f)|_1 \le C\,\left(\int_{\mathbb{R}^d}(1+|v|)\,df(v)\right)^2 \tag{4.11}$$

$$|Q(f,f)\,v|_1 \le C\,\left(\int_{\mathbb{R}^d}(1+|v|)\,df(v)\right)^2 \tag{4.12}$$

$$|Q(f,f)|_2 \le C\,\left(\int_{\mathbb{R}^d}\left(1+|v|^2\right)\,df(v)\right)^2 \tag{4.13}$$

and

$$|Q(f+g,f-g)|_1 \le C\,\left(\int_{\mathbb{R}^d}(1+|v|)\,\left(df(v)+dg(v)\right)\right)\,\left(|f-g|_1 + \int_{\mathbb{R}^d}(1+|v|)\,|df-dg|(v)\right). \tag{4.14}$$

$$|Q(f+g,f-g)\,v|_1 \le C\,\left(\int_{\mathbb{R}^d}(1+|v|)\,\left(df(v)+dg(v)\right)\right)\,\left(|f-g|_1 + \left|(f-g)\,v\right|_1\right). \tag{4.15}$$

$$|Q(f+g,f-g)|_2 \le C\,\left(\int_{\mathbb{R}^d}(1+|v|)\,\left(df(v)+dg(v)\right)\right)\,\left(|f-g|_2 + \left|(f-g)\,v\right|_1\right). \tag{4.16}$$

Proof of Lemma 4.3. We prove the second inequalities (4.14)-(4.16). The first inequalities (4.11)-(4.13) are then a trivial consequence by using

$$Q(f,f) = Q(f,f) - Q(M,M) = Q(f-M, f+M)$$

where M is the maxwellian distribution with same momentum and energy as f, and then applying (4.14) or (4.16) with $f-M$ and $f+M$.

We write in Fourier:

$$\mathcal{F}\left(Q(f+g,f-g)\right) = \hat{Q}(D,S) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat\xi)\,\left(S(\xi^+)\,D(\xi^-) + S(\xi^-)\,D(\xi^+) - 2\,D(\xi)\right)$$

where $\hat{Q}$ is the Fourier form the symmetrized collision operator Q, which yields

$$\frac{\left|\hat{Q}(D,S)\right|}{|\xi|^2} \le \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$$

with

$$\mathcal{T}_1 \le \int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat\xi)\,\left|S(\xi^+)\right|\,\frac{|D(\xi^-)|}{|\xi^-|^2}\,\frac{|\xi^-|^2}{|\xi|^2}\,d\sigma \le C\,|D|_2$$

and

$$\mathcal{T}_2 \le \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat\xi)\, \frac{|D(\xi^+)|}{|\xi^+|}\, \frac{|S(\xi^-) - 2|}{|\xi^-|}\, \frac{|\xi^-|}{|\xi|}\, d\sigma \le C\, |D|_1 \left(\int_{\mathbb{R}^d} (1+|v|)\, (df(v) + dg(v)) \right)$$

and

$$\mathcal{T}_3 \le 2 \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat\xi)\, \frac{|D(\xi^+) - D(\xi)|}{|\xi|}\, d\sigma$$
$$\le \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \hat\xi)\, \frac{|\xi^-|}{|\xi|} \int_0^1 \frac{|\nabla D(\theta\xi + (1-\theta)\xi^+)|}{|\theta\xi + (1-\theta)\xi^+|}\, d\theta\, d\sigma \le C\, |(f-g)\, v|_1$$

which concludes the proof of (4.14). The proof of (4.15) and (4.16) are similar. □

The proof of **(A2)-(ii)** is a consequence of (4.16), while the proof of **(A2)-(iii)** can be done by repeated use of Lemma 4.4. Indeed, we start from

$$f_t - f_0 = \int_0^t Q(f,f)\, ds$$

from which we deduce thanks to (4.11), (4.12), (4.13)

$$|f_t - f_0|_1 + |(f_t - f_0)\, v|_1 + |f_t - f_0|_2 \le C\, t.$$

Then write

$$f_t - f_0 - t\, Q(f_0,f_0) = \int_0^t \left(Q(f_s,f_s) - Q(f_0,f_0) \right)\, ds = \int_0^t Q(f_s - f_0, f_s + f_0)\, ds,$$

from which we deduce thanks to (4.12), (4.13)

$$|f_t - f_0 - t\, Q(f_0,f_0)|_2 \le C \int_0^t \left(|f_s - f_0|_2 + |v\, (f_s - f_0)|_1 \right)\, ds \le C \int_0^t s\, ds \le C\, t^2.$$

4.5. **Proof of (A3) with $\mathcal{G}_1$.** Define $\Lambda_1(\rho) := \langle \rho, 1 + |v|^4 \rangle^{1-\eta}$ for any $\rho \in P_{\mathcal{G}_1}$ and some $\eta \in (0,1)$. Let us prove that for any $\Phi \in C^{1,\eta}_{\Lambda_1}(P_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$,

$$\left\| M_4^N(V)^{\eta-1} \left(G^N\, \pi^N - \pi^N\, G^\infty \right)\, \Phi \right\|_{L^\infty(\mathbb{E}_N)} \le \frac{C_1\, \mathcal{E}}{N}\, [\Phi]_{C^{1,\eta}_{\Lambda_1}(P_{\mathcal{G}_1,\mathbf{r}};\mathbb{R})}, \tag{4.17}$$

for some constant C_1 and where $\mathbb{E}_N = \{V \in (\mathbb{R}^d)^N\ ,\ |V|^2 \le \mathcal{E}\}$.

First, consider velocities $v, v_*, w, w_* \in \mathbb{R}^d$ such that

$$w = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\, \sigma, \quad w_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\, \sigma, \quad \sigma \in \mathbb{S}^{d-1}.$$

Then $\delta_v + \delta_{v_*} - \delta_w - \delta_{w_*} \in \mathcal{I}P_{\mathcal{G}_1}$. Performing a Taylor expansion up to order two and one, we have

$$\begin{aligned} &e^{iv\cdot\xi} + e^{iv_*\cdot\xi} - e^{iw\cdot\xi} - e^{iw_*\cdot\xi} = \\ &\quad = i\,(w-v)\xi\, e^{iv\cdot\xi} + \mathcal{O}(|w-v|^2\, |\xi|^2) + i\,(w_*-v_*)\xi\, e^{iv_*\cdot\xi} + \mathcal{O}(|w_*-v_*|^2\, |\xi|^2) \\ &= i\,(w-v)\, e^{iv\cdot\xi} + \mathcal{O}(|w-v|^2\, |\xi|^2) + i\,(w_*-v_*)\, (e^{iv.\xi} + \mathcal{O}(|v-v_*|\, |\xi|) + \mathcal{O}(|w_*-v_*|^2\, |\xi|^2) \\ &\qquad = \mathcal{O}(|\xi|^2\, |v-v_*|^2 \cos\theta) \end{aligned}$$

thanks to the impulsion conservation and the fact that

$$|w-v| + |w_*-v_*| \le \sqrt{2}\, |v-v_*|\, (1-\cos\theta).$$

We then deduce

$$|\delta_v + \delta_{v_*} - \delta_w - \delta_{w_*}|_2 \;=\; \sup_{\xi\in\mathbb{R}^d} \frac{\left|e^{iv\cdot\xi} + e^{iv_*\cdot\xi} - e^{iw\cdot\xi} - e^{iw_*\cdot\xi}\right|}{|\xi|^2} \;\le\; C\,|v - v_*|\,(1-\cos\theta).$$

Consider $V \in \mathbb{E}_N$ and define

$$\mathbf{r}_V := \big(\langle \mu^N_V, z_1\rangle, \dots, \langle \mu^N_V, z_d\rangle, \langle \mu^N_V, |z|^2\rangle\big) \in \mathbf{R}_{\mathcal{E}}.$$

Then for a given $\Phi \in C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V};\mathbb{R})$, we set $\phi := D\Phi[\mu^N_V]$, $u_{ij} = (v_i - v_j)$ and we compute:

$$\begin{aligned}
G^N(\Phi\circ\mu^N_V) &= \frac{1}{2N}\sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \left[\Phi\left(\mu^N_{V^*_{ij}}\right) - \Phi\left(\mu^N_V\right)\right] b(\theta_{ij})\,d\sigma \\
&= \frac{1}{2N}\sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \left\langle \mu^N_{V^*_{ij}} - \mu^N_V, \phi\right\rangle b(\theta_{ij})\,d\sigma \\
&+ \frac{[\Phi]_{C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V})}}{2N}\sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} \left[M_4\left(\mu^N_{V^*_{ij}} + \mu^N_V\right)\right]^{1-\eta} \mathcal{O}\left(\left|\mu^N_{V^*_{ij}} - \mu^N_V\right|_2^{1+\eta}\right) d\sigma \\
&=: I_1(V) + I_2(V).
\end{aligned}$$

For the first term $I_1(V)$, thanks to Lemma 2.13, we have

$$\begin{aligned}
I_1(V) &= \frac{1}{2N^2}\sum_{i,j=1}^N \int_{\mathbb{S}^{d-1}} b(\theta_{ij})\,\left[\phi(V_i^*) + \phi(V_j^*) - \phi(V_i) - \phi(V_j)\right]\,d\sigma \\
&= \frac{1}{2N^2}\int_v\int_w\int_{\mathbb{S}^{d-1}} b(\theta)\,\left[\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)\right]\,\mu^N_V(dv)\,\mu^N_V(dw)\,d\sigma \\
&\qquad = \left\langle Q(\mu^N_V,\mu^N_V),\phi\right\rangle = (G^\infty\Phi)\,(\mu^N_V).
\end{aligned}$$

For the second term $I_2(V)$, using that

$$M_4\left(\mu^N_{V^*_{ij}}\right) \le C\,M_4^N(V),$$

we deduce

$$\begin{aligned}
|I_2(V)| \;&\le\; \frac{C}{N^{2+\eta}}\,[M_4^N(V)]^{1-\eta}\,[\Phi]_{C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V})} \sum_{i,j=1}^N \mathcal{O}\left\{\int_{\mathbb{S}^{d-1}} b\,(\cos\theta_{ij})\,\left(1+|v_i|^2+|v_j|^2\right)\,(1-\sigma\cdot\hat{u}_{ij})\,d\sigma\right\} \\
&\le\; C\,[M_4^N(V)]^{1-\eta}\,[\Phi]_{C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V})}\,\frac{1}{N^\eta}\,(1+\mathcal{E}),
\end{aligned}$$

which concludes the proof.

4.6. **Proof of (A4) in Fourier-based norms $|\cdot|_2$ and $|\cdot|_4$.** For $f_0, g_0 \in P_4(\mathbb{R}^d)$, let us define the associated solutions f_t and g_t to the nonlinear Boltzmann equation as well as $h_t := \mathcal{L}^{NL}_t[f_0](g_0 - f_0)$ the solution to the linearized Boltzmann equation around f_t. More precisely, we define

$$\begin{cases}
\partial_t f_t = Q(f_t, f_t), & f_{|t=0} = f_0 \\
\partial_t g_t = Q(g_t, g_t), & g_{|t=0} = g_0 \\
\partial_t h_t = Q(f_t, h_t) + Q(h_t, f_t), & h_{|t=0} = h_0 := g_0 - f_0.
\end{cases}$$

Lemma 4.5. *There exists $\lambda \in (0,\infty)$ and for any $\eta \in (0,1)$ there exists C_η such that for any $f_0, g_0 \in P_{\mathcal{G}_{1,\mathbf{r}}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, we have*

$$|f_t - g_t|_2 \le C_\eta \, e^{-(1-\eta)\,\lambda\,t} \, \max(M_4(f_0), M_4(g_0))^{(1-\eta)/2} \, |f_0 - g_0|_2^\eta \,, \tag{4.18}$$

and

$$|h_t|_2 \le C_\eta \, e^{-(1-\eta)\,\lambda\,t} \, \max(M_4(f_0), M_4(g_0))^{(1-\eta)/2} \, |f_0 - g_0|_2^\eta \,. \tag{4.19}$$

As a consequence, the operator $\mathcal{L}_t^{NL}$ defined by $\mathcal{L}_t^{NL}[f_0](h_0) := h_t$ is such that $\mathcal{L}_t^{NL}[f_0] \in \mathcal{M}^1(\mathcal{G}_{1,\mathbf{r}}, \mathcal{G}_{1,\mathbf{r}})$.

Proof of Lemma 4.5. Step 1. Estimate in $|\cdot|_4$. We proceed in the spirit of [42, 7]. With the notation $\mathcal{M} = \mathcal{M}_4$, $\hat{\mathcal{M}} = \hat{\mathcal{M}}_4$, introduced in Example 2.21, let us define $d := f - g$, $s := f + g$ and $\tilde d := d - \mathcal{M}[d]$, and then $D := \mathcal{F}(d)$, $S := \mathcal{F}(s)$ and

$$\tilde D := \mathcal{F}(\tilde d) = D - \hat{\mathcal{M}}[d].$$

The equation satisfies by $\tilde D$ is

$$\begin{aligned} \partial_t \tilde D &= \hat Q(D,S) - \partial_t \hat{\mathcal{M}}[d] \\ &= \hat Q(\tilde D, S) + \{\hat Q(\hat{\mathcal{M}}[d], S) - \hat{\mathcal{M}}[Q(d,s)]\}. \end{aligned} \tag{4.20}$$

We infer from [42] that for any $j \in \mathbb{N}^d$, there exists some absolute coefficients $(a_{i,j})$ depending on the collision kernel b such that

$$\int_{\mathbb{S}^{d-1}} b(\cos\theta) \left[(v^i)' + (v^i)'_* - (v^i) - (v^i)_*\right] \, d\sigma = \sum_{j,\, |j| \le |i|} a_{i,j} \, \left(v^j\right) \, \left(v^{i-j}\right)_* .$$

We deduce that

$$\forall\, |i| \le 3, \quad \nabla_\xi^i \hat{\mathcal{M}}[Q(d,s)]\Big|_{\xi=0} = M_i[Q(d,s)] = \sum_{j,\, |j| \le |i|} a_{i,j} \, M_j[d] \, M_{i-j}[s]$$

together with

$$\begin{aligned} \forall\, |i| \le 3, \quad \nabla_\xi^i \hat Q(\hat{\mathcal{M}}[d], S)\Big|_{\xi=0} &= M_i[Q(\mathcal{M}[d], s)] \\ &= \sum_{j,\, |j| \le |i|} a_{i,j} \, M_j[\mathcal{M}[d]] \, M_{i-j}[s] = \sum_{j,\, |j| \le |i|} a_{i,j} \, M_j[d] \, M_{i-j}[s] \end{aligned}$$

since $M_i[\mathcal{M}[d]] = M_i[d]$ for any $|i| \le 3$ by construction. As a consequence, we get

$$\forall\, \xi \in \mathbb{R}^d, \qquad \left| \hat{\mathcal{M}}[Q(d,s)] - \hat Q(\hat{\mathcal{M}}[d], S) \right| \le C \, |\xi|^4 \Big(\sum_{|i| \le 3} |M_i[f_t - g_t]| \Big). \tag{4.21}$$

On the other hand, from [42, Theorem 8.1] and its corollary, we know that there exists some constants $C, \lambda \in (0,\infty)$ such that

$$\forall\, t \ge 0 \qquad \Big(\sum_{|i| \le 3} |M_i[f_t - g_t]| \Big) \le C \, e^{-\lambda\,t} \Big(\sum_{|i| \le 3} |M_i[f_0 - g_0]| \Big). \tag{4.22}$$

Gathering (4.20), (4.21) and (4.22) and performing the same cutoff on the angular collision kernel as in the proof of Lemma 4.3, we have

$$\frac{d}{dt}\frac{|\tilde D(\xi)|}{|\xi|^4} + K\,\frac{|\tilde D(\xi)|}{|\xi|^4} \le \left(\sup_{\xi\in\mathbb{R}^d}\frac{|\tilde D(\xi)|}{|\xi|^4}\right)\left(\sup_{\xi\in\mathbb{R}^d}\int_{S^{d-1}} b_K\left(\sigma\cdot\hat\xi\right)\left(\left|\hat\xi^+\right|^4+\left|\hat\xi^-\right|^4\right)d\sigma\right) + C\,e^{-\lambda\,t}\left(\sum_{|i|\le 3}|M_i[f_0-g_0]|\right).$$

Let us compute (the supremum has been droped by the spherical invariance)

$$\lambda_K := \int_{S^{d-1}} b_K\left(\sigma\cdot\hat\xi\right)\left(\left|\hat\xi^+\right|^4+\left|\hat\xi^-\right|^4\right)d\sigma = \int_{S^{d-1}} b_K\left(\sigma\cdot\hat\xi\right)\left(\frac{1+\left(\sigma\cdot\hat\xi\right)^2}{2}\right)d\sigma,$$

so that

$$\lambda_K - K = -\left(\int_{S^{d-1}} b_K\left(\sigma\cdot\hat\xi\right)\left(\frac{1-\left(\sigma\cdot\hat\xi\right)^2}{2}\right)d\sigma\right) \xrightarrow[K\to\infty]{} -\int_{S^{d-1}} b\left(\sigma\cdot\hat\xi\right)\left(\frac{1-\left(\sigma\cdot\hat\xi\right)^2}{2}\right)d\sigma := -\bar\lambda \in (-\infty,0).$$

Thanks to Gronwall lemma, we get

$$\left(\sup_{\xi\in\mathbb{R}^d}\frac{|\tilde D_t(\xi)|}{|\xi|^4}\right) \le e^{(\lambda_K-K)\,t}\left(\sup_{\xi\in\mathbb{R}^d}\frac{|\tilde D_0(\xi)|}{|\xi|^4}\right) + C_3\left(\sum_{|i|\le 3}|M_i[f_0-g_0]|\right)\left(\frac{e^{-\lambda\,t}}{K-\lambda_K-\lambda}-\frac{e^{(\lambda_K-K)\,t}}{K-\lambda_K-\lambda}\right).$$

Therefore, passing to the limit $K\to\infty$ and choosing (without any restriction) $\lambda\in(0,\bar\lambda)$, we obtain

$$\sup_{\xi\in\mathbb{R}^d}\frac{|\tilde D_t(\xi)|}{|\xi|^4} \le C\,e^{-\lambda\,t}\left(\sup_{\xi\in\mathbb{R}^d}\frac{|\tilde D_0(\xi)|}{|\xi|^4} + \sum_{|i|\le 3}|M_i[f_0-g_0]|\right)$$

or equivalently (and with the notations of Example 2.21),

$$|||d_t|||_4 \le C\,e^{-\lambda\,t}\,|||d_0|||_4.$$

Step 2. From the preceding step and a trivial interpolation argument, we have

$$\begin{aligned}|f-g|_2 &\le |f-g-\mathcal{M}[f-g]|_2 + C\left(\sum_{|i|\le 3}|M_i[f-g]|\right)\\ &\le \|f-g-\mathcal{M}[f-g]\|_{M^1}^{1/2}\,|f-g-\mathcal{M}[f-g]|_4^{1/2} + C\left(\sum_{|i|\le 3}|M_i[f-g]|\right)\\ &\le C\,(1+M_4(f_0)+M_4(g_0))\,e^{-(\lambda/2)\,t}.\end{aligned}$$

We conclude by writing

$$|f-g|_2 \le |f-g|_2^\eta \, |f-g|_2^{1-\eta},$$

using Lemma 4.9 for the first term and the previous decay estimates for the second term.

Step 3. The same computations imply at least formally the same estimate on h_t as stated in Lemma 4.5. Now, we proceed to the rigorous statement. We consider a truncated (cut-off) model associated to $b_\varepsilon = b\,\mathbf{1}_{\cos\theta \le 1-\varepsilon} \in L^1(S^{d-1})$ and an initial datum $h_0^\varepsilon \in L^1_2(\mathbb{R}^3)$ such that $\langle h_0^\varepsilon, 1\rangle = 0$. By standard argument there exists a unique solution $h_t^\varepsilon \in C([0,\infty); L^1_2(\mathbb{R}^3))$ to the associated linearized Boltzmann equation arround f_t, and this one fulfills (4.19). Letting ε goes to 0, we get that for some $h_t \in C([0,\infty); \mathcal{S}'(\mathbb{R}^3) - w)$ there holds $h_t^\varepsilon \rightharpoonup h_t$ in $\mathcal{S}'(\mathbb{R}^3)$ and h_t is a weak solution to the linearized Boltzmann equation (corresponding now to b) and which satisfies again (4.19). In other words, we have built a linear flow $\mathcal{L}_t^{NL}[f_0] : \mathcal{G}_1 \to \mathcal{G}_1$ si that for any $h_0 \in \mathcal{G}_1$ the distribution $\mathcal{L}_t^{NL}[f_0](h_0) := h_t$ is the solution to linearized Boltzmann equation. □

Lemma 4.6. *There exists $\lambda \in (0,\infty)$ and for any $\eta \in (0,1)$ there exists C_η such that for any $f_0, g_0 \in P_{\mathcal{G}_1,\mathbf{r}}$, $\mathbf{r} \in \mathbf{R}_{\mathcal{G}_1}$, we have*

$$|\omega_t|_4 \le C\, e^{-(1-\eta)\,\lambda\, t} \left|g_0 - f_0\right|_2^{1+\eta}$$

with

$$\omega_t := g_t - f_t - h_t = S_t^{NL}(g_0) - S_t^{NL}(f_0) - \mathcal{L}_t^{NL}[f_0](g_0 - f_0)$$

(as proved below, ω_t always has vanishing moments up to order 3).

Proof of Lemma 4.6. The following arguments can be fully justified as in [44] by truncating b and passing to the limit. Consider the error term

$$\omega := g - f - h, \qquad \Omega := \hat{\omega}.$$

Again we perform the angular cutoff of Lemma 4.3 with cutoff parameter K, then the evolution equation (in the Fourier side) satisfied by Ω is

$$\partial_t \Omega = \hat{Q}(\Omega, S) + \hat{Q}^+(H, D). \tag{4.23}$$

Let us prove that

$$\forall\,|i| \le 3,\ \forall\, t \ge 0, \quad M_i[\omega_t] := \int_{\mathbb{R}^d} v^i \, d\omega_t(v) = 0.$$

Consider the equation on ω_t:

$$\partial_t \omega_t = Q(\omega, (f+g)) + Q(h, (f-g))$$

and the fact that, for maxwell molecules, the i-th moment of $Q(f_1, f_2)$ is a sum of terms given by product of moments of f_1 and f_2 whose orders sum to $|i|$. Hence using that for some absolute coefficients $(a_{i,j})$ we have

$$\int_{\mathbb{S}^{d-1}} b(\cos\theta)\left[(v^i)' + (v^i)'_* - (v^i) - (v^i)_*\right] d\sigma = \sum_{j,\,|j| \le i} a_{i,j}\,\left(v^j\right)\,\left(v^{i-j}\right)_*,$$

we deduce

$$\forall\,|i| \le 3, \quad \frac{d}{dt} M_i[\omega_t] = \sum_{j \le i} a_{i,j}\, M_j[\omega_t]\, M_{i-j}[f_t + g_t] + \sum_{j \le i} a_{i,j}\, M_j[h_t]\, M_{i-j}[f_t - g_t]$$

and since

$$\forall\,|i| \le 1, \quad M_i[h_t] = M_i[f_t - g_t] = 0$$

we deduce

$$\forall\,|i| \le 3, \quad \frac{d}{dt} M_i[\omega_t] = \sum_{j \le i} a_{i,j}\, M_j[\omega_t]\, M_{i-j}[f_t + g_t]$$

and from the initial data $\omega_0 = 0$ this concludes the proof.

We now consider the equation in Fourier form

$$\partial_t \Omega = \hat{Q}(\Omega, S) + \hat{Q}^+(H, D)$$

and we deduce in distributional sense

$$\left(\frac{d}{dt} \frac{|\Omega(\xi)|}{|\xi|^4} + K\, \frac{|\Omega(\xi)|}{|\xi|^4} \right) \le \mathcal{T}_1 + \mathcal{T}_2,$$

where

$$\begin{aligned}
\mathcal{T}_1 \;&:=\; \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} \frac{b\left(\sigma \cdot \hat{\xi}\right)}{|\xi|^4} \left(\left| \frac{\Omega(\xi^+)\, S(\xi^-)}{2} \right| + \left| \frac{\Omega(\xi^-)\, S(\xi^+)}{2} \right| \right) d\sigma \\
&\le\; \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b\left(\sigma \cdot \hat{\xi}\right) \left(\frac{|\Omega(\xi^+)|}{|\xi^+|^4} \frac{|\xi^+|^4}{|\xi|^4} + \frac{|\Omega(\xi^-)|}{|\xi^-|^4} \frac{|\xi^-|^4}{|\xi|^4} \right) d\sigma \\
&\le\; \left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|^2}{|\xi|^2} \right) \left(\sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b\left(\sigma \cdot \hat{\xi}\right) \left(\left|\hat{\xi}^+\right|^4 + \left|\hat{\xi}^-\right|^4 \right) d\sigma \right) \\
&\le\; \lambda_K \left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|}{|\xi|^4} \right),
\end{aligned}$$

where λ_K was defined in Lemma 4.3, and

$$\begin{aligned}
\mathcal{T}_2 \;&:=\; \frac{1}{2} \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} \frac{b\left(\sigma \cdot \hat{\xi}\right)}{|\xi|^4} \left| H(\xi^+)\, D(\xi^-) + H(\xi^-)\, D(\xi^+) \right| d\sigma \\
&\le\; \frac{1}{2} \sup_{\xi \in \mathbb{R}^3} \int_{\mathbb{S}^{d-1}} b\left(\sigma \cdot \hat{\xi}\right) \left(\frac{|H(\xi^+)|}{|\xi^+|^2} \frac{|D(\xi^-)|}{|\xi^-|^2} \frac{|\xi^-|^2}{|\xi|^2} + \frac{|D(\xi^+)|^2}{|\xi^+|^2} \frac{|H(\xi^-)|^2}{|\xi^-|^2} \frac{|\xi^-|^2}{|\xi|^2} \right) d\sigma \\
&\le\; |h_t|_2\; |d_t|_2 \int_{\mathbb{S}^{d-1}} b\left(\sigma \cdot \hat{\xi}_0\right) \left(1 - \sigma \cdot \hat{\xi}_0\right) d\sigma \\
&=\; C\, e^{-(1-\eta)\,\lambda\, t}\; |h_0|_2\; |d_0|_2^{\eta} \le C\, e^{-(1-\eta)\,\lambda\, t}\; |d_0|_2^{1+\eta}
\end{aligned}$$

by using the estimates of Lemma 4.3.

Hence we obtain

$$\left(\frac{d}{dt} \frac{|\Omega(\xi)|}{|\xi|^4} + K\, \frac{|\Omega(\xi)|}{|\xi|^4} \right) \le \lambda_K \left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega(\xi)|}{|\xi|^4} \right) + C\, e^{-(1-\eta)\,\lambda\, t}\; |d_0|_2^{1+\eta}.$$

We then from the Gronwall inequality and relaxing the cutoff parameter K as in Lemma 4.5 (assuming without restriction $(1-\eta)\lambda \le \bar{\lambda}$

$$\left(\sup_{\xi \in \mathbb{R}^3} \frac{|\Omega_t(\xi)|}{|\xi|^4} \right) \le C\, e^{-(1-\eta)\,\lambda\, t}\; |g_0 - f_0|_2^{1+\eta}.$$

□

4.7. **Proof of (A5) in Wasserstein distance.** We know from [42] that

$$\sup_{t\ge 0} W_2(S_t^{NL} f_0, S_t^{NL} g_0) \le W_2(f_0, g_0).$$

As a consequence, from $W_1 \le W_2 \le C_k\, W_1^{\eta(k)}$ and $W_1 = [\cdot]_1^*$, we deduce that **(A5)** holds with $\Theta(x) = x^{\eta(k)}$, $\mathcal{F}_3 = Lip$ and $P_{\mathcal{G}_3} = P_2$ endowed with $\|\cdot\|_{\mathcal{G}_3} = [\cdot]_1^*$.

4.8. **Proof of (A5) in negative Sobolev norms.** It is also possible to prove easily, in the cutoff case, that the weak stability holds in Sobolev space on finite time:

Lemma 4.7. *For any $T \ge 0$ and f_t, g_t solutions of the Boltzmann equation with maxwellian kernel and initial data f_0 and g_0, that there exists $s \in (d/2, d/2+1)$ and a constant C_k such that*

$$\sup_{[0,T]} \|f_t - g_t\|_{H^{-s}} \le C_{T,s}\, \|f_0 - g_0\|_{H^{-s}}\,.$$

Remark that this proves **(A5)** with $\mathcal{G}_3 = P(\mathbb{R}^d)$, endowed with the $\dot H^{-s}$ norm. This is useful in order to obtain the optimal rate $1/\sqrt{N}$ of the law of large numbers.

Let us only sketch the proof. We integrate (4.10) against $D/(1+|\xi|^2)^s$:

$$\frac{d}{dt}\|D\|^2_{\dot H^{-k}} = \frac{1}{2}\int_\xi \int_{\mathbb{S}^{d-1}} b\left(\sigma\cdot\hat\xi\right)\, \frac{\left[D^- S^+ D + D^+ S^-\, D - 2\,|D|^2\right]}{(1+|\xi|^2)^s}\, d\sigma\, d\xi$$

and we use Young's inequality together with the bounds

$$\left\|S^+\right\|_\infty, \quad \left\|S^-\right\|_\infty \le \|f+g\|_{M^1} \le 2$$

to conclude.

5. Hard spheres

5.1. **The model.** The limit equation was introduced in Subsection 1.1 and the stochastic model has discussed Subsection 4.1. We consider here the case of the Master equation (4.2), (4.3) and the limit nonlinear homogeneous Boltzmann equation (1.2), (1.3), (1.4) with $\Gamma(z) = |z|$.

5.2. **Statement of the result.** We have the following theorem:

Theorem 5.1 (Hard spheres homogeneous Boltzmann equation)**.** *Let us consider $f_0 \in P(\mathbb{R}^d)$ with compact support. We denote by $\mathcal{E}$ and A two positive constants such that*

$$M_2(f_0) \le \mathcal{E} \qquad \text{and} \qquad supp\, f_0 \subset \{v \in \mathbb{R}^d,\ |v| \le A\}.$$

Let us also consider the hierarchy of N-particle distributions $f_t^N = S_t^N(f_0^N)$ issued from the tensorial initial datum $f_0^N = f_0^{\otimes N}$, $N \ge 1$. Let us finally fix some $\delta \in (0,1)$.

Then there are some constants $k_1 \ge 2$, $C > 0$, only depending on δ and $\mathcal{E}$, some constant C_η depending on η, and some constants $a > 0$ depending on the collision kernel such that for any

$$\ell \in \mathbb{N}^*, \quad \varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_\ell \in W^{1,\infty}(\mathbb{R}^d)^{\otimes \ell},$$

we have

$$\text{(5.1)}\quad \forall\, N \ge 2\,\ell, \quad \sup_{[0,T]} \left| \left\langle \left(S_t^N(f_0^N) - \left(S_t^{NL}(f_0)\right)^{\otimes N}\right), \varphi \right\rangle \right| \le C\,\ell\, \|\varphi\|_{W^{1,\infty}(\mathbb{R}^d)^{\otimes \ell}} \left[\frac{\|f_0\|_{M^1_1}}{N} + \frac{\|f_0\|_{M^1_{k_1}}}{N^{1-\delta}} + e^{a\,A}\, \varepsilon_3(N) \right]$$

and the dominant error is the last term given by

$$\varepsilon_3(N) = e^{-\left[\log\left(\frac{C_\eta}{N^{\frac{1}{d}-\eta}}\right)_-\right]^{1/2}}$$

for any $\eta > 0$ small (the constant C_η blows up when $\eta \to 0$). As a consequence we deduce propagation of chaos with explicit rate.

In order to prove Theorem 5.1, we shall prove each assumption **(A1)-(A2)-(A3)-(A4)-(A5)** of Theorem 3.27 step by step. Its application then exactly gives Theorem 5.1. We fix $\mathcal{F}_1 = \mathcal{F}_2 = C_b(\mathbb{R}^d)$ and $\mathcal{F}_3 = \mathrm{Lip}(\mathbb{R}^d)$.

5.3. **Proof of (A1).** In the proof of Lemma 4.2 we have already proved that

$$\forall\, t \ge 0, \quad \mathrm{supp}\, f_t^N \subset \mathbb{E}_N := \left\{ V \in \mathbb{R}^{Nd};\ M_2^N(V) \le \mathcal{E} \right\},$$

which is precisely **(A1)-(i)** with $m_2(v) := |v|^2$, as well as that for any $k \ge 2$,

$$\sup_{t\ge 0} \left\langle f_t^N, M_k^N \right\rangle \le C_k^N$$

where C_k^N depends on k, $\mathcal{E}$, on the collision kernel and on the initial value $\langle f_0^N, M_k^N\rangle$ which is uniformly bounded in N (in function of k and A for instance). That is precisely **(A1)-(ii)** with $m_1(v) := |v|^{k_1}$ where $k_1 \ge 2$ will be chosen (large enough) in section 5.7. As for **(A1)-(iii)**, we remark that for a given N-particle velocity $V = (V_1, ..., V_N) \in \mathbb{R}^{dN}$, we have

$$V \in \mathrm{supp}\, f_0^{\otimes N} \iff \forall\, i = 1, \dots, N,\ V_i \in \mathrm{supp}\, f_0 \ \Rightarrow\ \forall\, i = 1, \dots, N,\ m_3(V_i) \le m_3(A),$$

with $m_3(v) := e^{a\,|v|}$. We conclude that

$$\mathrm{supp}\, f_0^{\otimes N} \subset \left\{ V \in \mathbb{R}^{dN};\ M_{m_3}^N(V) \le m_3(A) \right\},$$

and **(A1)-(iii)** holds.

5.4. **Proof of (A2).** We define

$$P_{\mathcal{G}_1} := \left\{ f \in P_{k_1}(\mathbb{R}^d);\ M_2(f) \le \mathcal{E} \right\}$$

that we endow with the total variation norm $\|\cdot\|_{\mathcal{G}_1} := \|\cdot\|_{M^1}$.

• The proof of assertion (i), that is for any $a \ge a_{k_1} > 0$ and for any $t > 0$, the application $f_0 \mapsto S_t^{NL} f_0$ maps $\mathcal{B}P_{\mathcal{G}_1,a} := \{f \in P_{\mathcal{G}_1},\ M_{k_1}(f) \le a\}$ continuously into itself, is postponed to section 5.6 where it is proved in (5.4) a Hölder continuity of the flow.

• For any $f, g \in P_{\mathcal{G}_1}$ we have

$$\begin{aligned} \|Q(g,g) - Q(f,f)\|_{M^1} &= \|\tilde{Q}(g-f, g+f)\|_{M^1} \\ &\le 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\theta)\,|v - v_*|\,|f-g|\,|f_* + g_*|\,d\sigma\,dv_*\,dv \\ &\le 2\,(1+\mathcal{E})\,\|b\|_{L^1}\,\|(f-g)\,\langle v\rangle\|_{M^1}. \end{aligned} \tag{5.2}$$

We deduce that

$$\|Q(g,g) - Q(f,f)\|_{M^1} \le 2\,(1+\mathcal{E})^{3/2}\,\|b\|_{L^1}\,\|f-g\|_{M^1}^{1/2}$$

which yields $Q \in C^{0,1/2}(P_{\mathcal{G}_1}; \mathcal{G}_1)$.

• Finally, we have for any $f_0 \in P_{\mathcal{G}_1}$

$$\|f_t - f_0\|_{M^1_1} = \left\| \int_0^t Q(f_s,f_s)\,ds \right\|_{M^1_1} \le \int_0^t \|Q(f_s,f_s)\|_{M^1_1}\,ds \le 3\,t\,\|b\|_{L^1}\,(1+\mathcal{E})^2,$$

and then, using (5.2),

$$\begin{aligned}\big\|f_t - f_0 - t\,Q(f_0,f_0)\big\|_{M^1} &= \left\| \int_0^t (Q(f_s,f_s) - Q(f_0,f_0))\,ds \right\|_{M^1} \\ &\le \int_0^t \big\|Q(f_s,f_s) - Q(f_0,f_0)\big\|_{M^1}\,ds \\ &\le \int_0^t 2\,(1+\mathcal{E})\,\|b\|_{L^1}\,\|f_s - f_0\|_{M^1_1}\,ds \le 3\,t^2\,\|b\|^2_{L^1}\,(1+\mathcal{E})^3,\end{aligned}$$

which precisely means that $t \mapsto S^{NL}_t(f_0)$ is $C^{1,1}$ at $t = 0^+$ in the space $P_{\mathcal{G}_1}$. A similar argument would yield the $C^{1,1}([0,\tau], P_{\mathcal{G}_1})$ regularity on a small time interval $[0,\tau]$.

5.5. **Proof of (A3).** For any $k \in \mathbb{N}^*$, any energy bound $\mathcal{E}$ (which has been fixed once for all in the statement of the main result), any energy $r_{d+1} \in [0,\mathcal{E}]$ and any mean velocity $(r_1,...,r_d) \in B_{\mathbb{R}^d}(0,r_{d+1})$ we define

$$P_{\mathcal{G}_1,\mathbf{r}} := \Big\{ f \in P_k(\mathbb{R}^d);\ \langle f, v_j\rangle = r_j,\ j=1,...,d,\ M_2(f) = r_{d+1} \Big\}, \quad \mathbf{r} := (r_1,...,r_{d+1}).$$

We also denote by $\mathbf{R}_{\mathcal{E}}$ the set of all admissible vectors $\mathbf{r} \in \mathbb{R}^{d+1}$ constructed as above. We claim that there exists $C = C_{k_1,\mathcal{E}}$ such that for any $\eta \in (0,1)$ and any function

$$\Phi \in \bigcap_{\mathbf{r}\in\mathbf{R}_{\mathcal{E}}} C^{1,\eta}_{\Lambda}(P_{\mathcal{G}_1,\mathbf{r}};\mathbb{R}),$$

with $\Lambda(f) := M_{k_1}(f)$, we have

$$\forall\, V \in \mathbb{E}_N, \quad \big|G^N(\Phi\circ\mu^N_V) - (G^\infty\Phi)(\mu^N_V)\big| \le C \left(\sup_{\mathbf{r}\in\mathbf{R}_{\mathcal{E}}} [\Phi]_{C^{1,\eta}_{\Lambda}(P_{\mathcal{G}_1,\mathbf{r}})} \right) \frac{M^N_{k_1}(V)}{N^\eta}, \tag{5.3}$$

which is precisely assumption **(A3)** with $\varepsilon(N) = C/N^\eta$.

Consider $V \in \mathbb{E}_N$ and define

$$\mathbf{r}_V := \big(\langle \mu^N_V, z_1\rangle, \dots, \langle \mu^N_V, z_d\rangle, \langle \mu^N_V, |z|^2\rangle\big) \in \mathbf{R}_{\mathcal{E}}.$$

Then for a given $\Phi \in C^{1,\eta}_{\Lambda}(P_{\mathcal{G}_1,\mathbf{r}_V};\mathbb{R})$, we set $\phi := D\Phi[\mu^N_V]$ and we compute:

$$\begin{aligned} G^N(\Phi\circ\mu^N_V) &= \frac{1}{2N}\sum_{i,j=1}^N |V_i - V_j| \int_{\mathbb{S}^{d-1}} \Big[\Phi\left(\mu^N_{V^*_{ij}}\right) - \Phi\left(\mu^N_V\right)\Big]\, b(\theta_{ij})\,d\sigma \\ &= \frac{1}{2N}\sum_{i,j=1}^N |V_i - V_j| \int_{\mathbb{S}^{d-1}} \left\langle \mu^N_{V^*_{ij}} - \mu^N_V, \phi \right\rangle b(\theta_{ij})\,d\sigma \\ &+ \frac{[\Phi]_{C^{1,\eta}_{\Lambda}(P_{\mathcal{G}_1,\mathbf{r}_V})}}{2N} \sum_{i,j=1}^N |V_i - V_j| \int_{\mathbb{S}^{d-1}} \max\left\{ M_{k_1}\left(\mu^N_{V^*_{ij}}\right);\ M_{k_1}\left(\mu^N_V\right)\right\}\, \mathcal{O}\left(\left\| \mu^N_{V^*_{ij}} - \mu^N_V \right\|^{1+\eta}_{M^1} \right) d\sigma \\ &=: I_1(V) + I_2(V). \end{aligned}$$

For the first term $I_1(V)$, thanks to Lemma 2.13, we have

$$I_1(V) = \frac{1}{2N^2} \sum_{i,j=1}^N |V_i - V_j| \int_{\mathbb{S}^{d-1}} b(\theta_{ij}) \, \left[\phi(V_i^*) + \phi(V_j^*) - \phi(V_i) - \phi(V_j)\right] \, d\sigma$$
$$= \frac{1}{2N^2} \int_v \int_w |v-w| \int_{\mathbb{S}^{d-1}} b(\theta) \, [\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)] \, \mu_V^N(dv) \, \mu_V^N(dw) \, d\sigma$$
$$= \left\langle Q(\mu_V^N, \mu_V^N), \phi \right\rangle = (G^\infty \Phi) \, (\mu_V^N).$$

For the second term $I_2(V)$, using that

$$\begin{aligned} M_{k_1}\left(\mu^N_{V^*_{ij}}\right) &:= M^N_{k_1}\left(V^*_{ij}\right) = \frac{1}{N} \left(\left(\sum_{n \neq i,j} |V_n|^{k_1} \right) + |V_i^*|^{k_1} + |V_j^*|^{k_1} \right) \\ &\le \frac{1}{N} \left(\left(\sum_{n \neq i,j} |V_n|^{k_1} \right) + 2 \, \left(|V_i|^2 + |V_j|^2\right)^{k_1/2} \right) \\ &\le \frac{2^{1+k_1/2}}{N} \left(\sum_n |V_n|^{k_1} \right) = C \, M^N_{k_1}(V), \end{aligned}$$

we deduce

$$\begin{aligned} |I_2(V)| &\le C \, M^N_{k_1}(V) \, [\Phi]_{C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V})} \left(\frac{1}{2N} \sum_{i,j=1}^N |V_i - V_j| \, \left(\frac{4}{N}\right)^{1+\eta} \right) \\ &\le C \, M^N_{k_1}(V) \, [\Phi]_{C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V})} \left(\frac{1}{N^\eta} \frac{1}{N^2} \sum_{i,j=1}^N [\langle V_i \rangle + \langle V_j \rangle] \right) \\ &\le C \, M^N_{k_1}(V) \, [\Phi]_{C^{1,\eta}_\Lambda(P_{\mathcal{G}_1,\mathbf{r}_V})} \frac{1}{N^\eta} \, (1 + \mathcal{E}). \end{aligned}$$

We conclude that (5.3) holds by gathering these two estimate.

5.6. **Proof of (A4) on a finite time interval** $[0,T]$.

Lemma 5.2. *For any given energy $\mathcal{E} > 0$ and any $\delta > 0$ there exists some constants $k_1 \ge 2$ (depending on $\mathcal{E}$ and δ) and C (depending on $\mathcal{E}$), such that for any $f_0, g_0 \in P_{\mathcal{G}_1}(\mathbb{R}^d)$, we have*

$$\left\| S^{NL}_t(g_0) - S^{NL}_t(f_0) \right\|_{M^1} \le e^{C\,(1+t)} \sqrt{\max\{M_{k_1}(f_0), M_{k_1}(g_0)\}} \; \|f_0 - g_0\|^{1-\delta}_{M^1}, \tag{5.4}$$

$$\left\| \mathcal{L}S^{NL}_t[f_0] \, (f_0 - g_0) \right\|_{M^1} \le e^{C\,(1+t)} \sqrt{M_{k_1}(f_0)} \; \|f_0 - g_0\|^{1-\delta}_{M^1}, \tag{5.5}$$

$$\begin{aligned} &\left\| S^{NL}_t(g_0) - S^{NL}_t(f_0) - \mathcal{L}S^{NL}_t[f_0](g_0 - f_0) \right\|_{M^1} \\ &\qquad \le e^{C\,(1+t)} \sqrt{\max\{M_{k_1}(f_0), M_{k_1}(g_0)\}} \; \|f_0 - g_0\|^{2-\delta}_{M^1}. \end{aligned} \tag{5.6}$$

Proof of Lemma 5.2. We proceed in several steps. Let us define

$$\forall\, f \in M^1(\mathbb{R}^d), \quad \|f\|_{M^1_k} := \int_{\mathbb{R}^d} \langle v \rangle^k \, |f|(dv), \quad \|f\|_{M^1_{k,\ell}} := \int_{\mathbb{R}^d} \langle v \rangle^k \, (1 + \log\langle v \rangle)^\ell \, |f|(dv).$$

Step 1. The strategy. Let us define

$$\begin{cases} \partial_t f_t = Q(f_t, f_t), & f_{|t=0} = f_0 \\ \partial_t g_t = Q(g_t, g_t), & g_{|t=0} = g_0 \\ \partial_t h_t = Q(f_t, h_t) + Q(h_t, f_t), & h_{|t=0} = h_0 := g_0 - f_0. \end{cases}$$

Existence and uniqueness for f_t, g_t and h_t is a consequence of the following important stability argument that we use several times. This estimate is due to DiBlasio [13] in a L^1 framework, and it has been recently extended to a measure framework in [14, Lemma 3.2]. Let us recall the argument for h. We first write

$$\frac{d}{dt}\int \langle v\rangle^2 \, |h_t|(dv) \; \leq \; \iiint |h_t|(dv)\, f_t(dv_*)\, |u|\, b(\theta) \left[\langle v'\rangle^2 + \langle v'_*\rangle^2 - \langle v\rangle^2 - \langle v_*\rangle^2\right] d\sigma \\ +2 \iiint |h_t|(dv)\, f_t(dv_*)\, |u|\, b(\theta)\, \langle v_*\rangle^2 \, d\sigma\, dv\, dv_* \tag{5.7}$$

(this formal computation can be justified by a regularization proceedure, we refer to [14] for instance). Because the first term vanishes, we conclude with

$$\frac{d}{dt}\|h_t\|_{M^1_2} \leq C\, \|f\|_{M^1_3}\, \|h_t\|_{M^1_2}. \tag{5.8}$$

When $\|f_t\|_{M^1_3} \in L^1(0,T)$, we may integrate this differential inequality and we deduce that h is unique.

More precisely, we have established

$$\sup_{[0,T]} \|h_t\|_{M^1_2} \leq \|g_0 - f_0\|_{M^1_2}\, \exp\left(C \int_0^T \|f_s\|_{M^1_3}\, ds\right), \tag{5.9}$$

and similar arguments imply

$$\sup_{[0,T]} \|f_t - g_t\|_{M^1_2} \leq \|g_0 - f_0\|_{M^1_2}\, \exp\left(C \int_0^T \|f_s + g_s\|_{M^1_3}\, ds\right). \tag{5.10}$$

It is worth mentioning that one cannot prove $\|f_t\|_{M^1_3} \in L^1(0,T)$ under the sole assumption $\|f_0\|_{M^1_2} < \infty$ because since it would contradict the non-uniqueness result of [30]. However, thanks to Povzner's inequality, one may show that $\|f_t\|_{M^1_3} \in L^1(0,T)$ whenever $\|f_0\|_{M^1_{2,1}}$ is finite, with the definition

$$\|f_0\|_{M^1_{k,\ell}} := \int \langle v\rangle^k \, \log\left(\langle v\rangle\right)^\ell \, df_0(v) < +\infty$$

(see (5.13) below or [35, 27]), which will be the key step for establishing (5.4) and (5.5).

Now, our goal is to estimate (in terms of $\|g_0 - f_0\|_{M^1}$) the M^1 norm of

$$\zeta_t := f_t - g_t - h_t.$$

The measure ζ_t satisfies the following evolution equation:

$$\partial_t \zeta_t = Q(f_t, f_t) - Q(g_t, g_t) - Q(h_t, f_t) - Q(f_t, h_t), \quad \zeta_0 = 0.$$

We can rewrite this equation as

$$\partial_t \zeta_t = Q(\zeta_t, f_t + g_t) + Q(h_t, f_t - g_t).$$

The same arguments as in (5.7)-(5.8) yield the following differential inequality

$$\frac{d}{dt}\, \|\zeta_t\|_{M^1_2} \leq C\, \|\zeta_t\|_{M^1_2}\, \|f_t + g_t\|_{M^1_3} + \left\|\tilde Q(h_t, f_t - g_t)\right\|_{M^1_2}, \quad \|\zeta_0\|_{M^1_2} = 0.$$

We deduce

$$\sup_{t\in[0,T]} \|\zeta_t\|_{M^1_2} \le \left(\int_0^T \left\|\tilde Q(h_s, f_s-g_s)\right\|_{M^1_2} ds\right) \exp\left(C\int_0^T \|f_s+g_s\|_{M^1_3}\, ds\right). \tag{5.11}$$

Since

$$\begin{aligned}\int_0^T \left\|\tilde Q(h_s, f_s-g_s)\right\|_{M^1_2} ds \ &\le\ C\left(\sup_{[0,T]}\|h_t\|_{M^1_2}\right)\left(\int_0^T \|f_s-g_s\|_{M^1_3}\, ds\right)\\ &\quad + C\left(\sup_{[0,T]}\|g_t-f_t\|_{M^1_2}\right)\left(\int_0^T \|h_s\|_{M^1_3}\, ds\right),\end{aligned} \tag{5.12}$$

we deduce from (5.9) and (5.10)

$$\sup_{[0,T]}\|\zeta_t\|_{M^1_2} \le C\,\|g_0-f_0\|_{M^1_2}\,\exp\left(\int_0^T \|f_s\|_{M^1_3}+\|g_s\|_{M^1_3}\right)$$

$$\left\{\left(\int_0^T \|g_s-f_s\|_{M^1_3}\right)\exp\left(\int_0^T \|f_s\|_{M^1_3}\right) + \left(\int_0^T \|h_s\|_{M^1_3}\right)\exp\left(\int_0^T \|f_s\|_{M^1_3}+\|g_s\|_{M^1_3}\right)\right\}.$$

Hence the problem is reduced to time integral controls over $\|f_t\|_{M^1_3}$, $\|g_t\|_{M^1_3}$, $\|f_t-g_t\|_{M^1_3}$ and $\|h_t\|_{M^1_3}$.

Step 2. Time integral control of f and g in M^1_3. In this step we prove

$$\int_0^T \|f_t\|_{M^1_{3,\ell-1}}\, dt \le C_{\mathcal E}\,T + C'\,\|f_0\|_{M^1_{2,\ell}} \quad \ell=1,2, \tag{5.13}$$

for f, where $C_{\mathcal E}$ is some energy dependent constant, and C' is a numerical constant. The same estimate obviously holds for g. These estimates are a consequence of the accurate version of the Povzner inequality as one can find in [35, 27]. Indeed it has been proved in [35, Lemma 2.2] that for any convex function $\Psi:\mathbb R^d\to\mathbb R$, $\Psi(v)=\psi(|v|^2)$, the solution f_t to the hard spheres Boltzmann equation satisfies

$$\frac{d}{dt}\int_{\mathbb R^d}\Psi(v)\,f_t(dv) = \int_{\mathbb R^d}\int_{\mathbb R^d} f_t(dv)\,f_t(dv_*)\,|v-v_*|\,K_\Psi(v,v_*)$$

with $K_\Psi = G_\Psi - H_\Psi$, where the term G_Ψ "behaves mildly" and the term H_Ψ is given by (see [35, formula (2.7)])

$$H_\Psi(v,v_*) = 2\pi\int_0^{\pi/2}\Big[\psi(|v|^2\cos^2\theta+|v_*|^2\sin^2\theta) - \cos^2\theta\,\psi(|v|^2) - \sin^2\theta\,\psi(|v_*|^2)\Big]\,d\theta,$$

(note that $H_\Psi\ge 0$ since its integrand is nonnegative from the convexity of ψ). More precisely, in the cases that we are interested with, namely $\Psi(v)=\psi_{2,\ell}(|v|^2)$ with $\psi_{k,\ell}(r) = r^{k/2}\,(\log r)^\ell$ and $\ell=1,2$, it has been established that (with obvious notations)

$$\forall\, v, v_*\in\mathbb R^d,\quad \left|G_{\psi_{2,\ell}}(v,v_*)\right| \le C_\ell\,\langle v\rangle\,(\log(1+\langle v\rangle^2))^\ell\,\langle v_*\rangle\,(\log(1+\langle v_*\rangle^2))^\ell. \tag{5.14}$$

On the other hand, in the case $\ell=1$ we easily compute (with the notation $x:=\cos^2\theta$ and $u=|v_*|/|v|$)

$$\begin{aligned}&\forall\, x,\,u\in\mathbb R,\ x\in[1/4,3/4],\ u\in[0,1/2],\\ &\qquad \psi_{2,1}\Big(|v|^2\cos^2\theta+|v_*|^2\sin^2\theta\Big) - \cos^2\theta\,\psi_{2,1}\left(|v|^2\right) - \sin^2\theta\,\psi_{2,1}\left(|v_*|^2\right) =\\ &\qquad\qquad = |v|^2\Big[(1-x)\,\psi_{2,1}\left(u^2\right) + x\,\psi_{2,1}(1) - \psi_{2,1}\left((1-x)\,u^2+x\right)\Big] \ge \kappa_0\,|v|^2,\end{aligned}$$

for some numerical constant $\kappa_0 > 0$ (note that this lower bound only depends on the strict convexity of the real function $\psi_{2,1}$). We straightforwardly deduce that there exists a numerical constant $\kappa_1 > 0$ such that

$$H_{2,1}(v, v_*) \geq \kappa_1 \, |v|^2 \, \mathbf{1}_{|v| \geq 2\, |v_*|}.$$

Similarly, in the case $\ell = 2$, we have

$$\begin{aligned}
&\forall\, x,\, u \in \mathbb{R},\ x \in [1/4, 3/4],\ u \in [0, 1/2],\\
&\quad \psi_{2,2}\Big(|v|^2 \cos^2\theta + |v_*|^2 \sin^2\theta\Big) - \cos^2\theta\, \psi_{2,2}\left(|v|^2\right) - \sin^2\theta\, \psi_{2,2}\left(|v_*|^2\right) =\\
&\quad = 2\, |v|^2 \, \log |v|^2 \, \left\{ (1-x)\, \psi_{2,1}\left(u^2\right) + x\, \psi_{2,1}(1) - \psi_{2,1}\left((1-x)\, u^2 + x\right) \right\}\\
&\quad + |v|^2 \Big[(1-x)\, \psi_{2,2}\left(u^2\right) + x\, \psi_{2,2}(1) - \psi_{2,2}\left((1-x)\, u^2 + x\right) \Big] \geq 2\, \kappa_0 \, |v|^2 \, \log |v|^2,
\end{aligned}$$

and then

$$H_{2,2}(v, v_*) \geq 2\, \kappa_1 \, |v|^2 \, \log |v|^2 \, \mathbf{1}_{|v| \geq 2\, |v_*|}.$$

Putting together the estimates obtained on $G_{2,\ell}$ and $H_{2,\ell}$ we deduce the Povzner inequality

$$|v - v_*| \, K_{2,\ell} \leq C \left(\langle v\rangle^2 + \langle v_*\rangle^2\right) - \kappa \, |v|^3 \left(\log\langle v\rangle\right)^{\ell-1}, \tag{5.15}$$

and we finally obtain the differential inequality

$$\frac{d}{dt} \|f_t\|_{M^1_{2,\ell}} \leq 2\, C\, (1 + \mathcal{E}) - \kappa\, M_{3,\ell-1},$$

from which (5.13) follows.

Step 3. Exponential time integral control of f and g in M^1_3 (proof of (5.4) *and* (5.5)*).* Let us first prove that

$$e^{C\, \int_0^T \|f_s\|_{M^1_3}\, ds} \leq \sqrt{M_k(f_0)}, \qquad e^{C\, \int_0^T \|g_s\|_{M^1_3}\, ds} \leq \sqrt{M_k(g_0)} \tag{5.16}$$

for any $k \geq k_{\mathcal{E}}$, with $k_{\mathcal{E}}$ big enough.

This is a straightforward consequence of the previous step and the following interpolation argument. For any given probability measure $f \in P_k(\mathbb{R}^d)$ with $M_2(f) \leq \mathcal{E}$, we have for any $a > 2$

$$\begin{aligned}
\|f\|_{M^1_{2,1}} &= \int_{\mathbb{R}^d} \langle v\rangle^2 \, \left(1 + \log(\langle v\rangle^2)\right) \left(\mathbf{1}_{\langle v\rangle^2 \leq a} + \mathbf{1}_{\langle v\rangle^2 \geq a}\right) \, f(dv)\\
&\leq (1 + \mathcal{E})\, (1 + \log a) + \frac{1}{a} \int_{\mathbb{R}^d} \langle v\rangle^4 \, \left(1 + \log(\langle v\rangle^2)\right) \, f(dv)\\
&\leq (1 + \mathcal{E})\, (1 + \log a) + \frac{1}{a} \, \|f\|_{M^1_5}.
\end{aligned}$$

By choosing $a := \|p\|_{M^1_5}$, we get

$$\|p\|_{M^1_{2,1}} \leq 2\, (1 + \mathcal{E}) \, \left(1 + \log \|p\|_{M^1_5}\right). \tag{5.17}$$

On the other hand, the following elementary Hölder inequality holds

$$\forall\, k, k' \in \mathbb{N},\ k' \leq k,\ \forall\, f \in M^1_k, \quad \|f\|_{M^1_{k'}} \leq \|f\|_{M^1}^{1 - k'/k} \, \|p\|_{M^1_k}^{j/k}. \tag{5.18}$$

Then estimate (5.16) follows from (5.13), (5.17) and (5.18) with $k' = 2$ and $k = k_{\mathcal{E}} \geq 5$ large enough in such a way that

$$C'\, 2\, (1 + \mathcal{E}) \, \frac{5}{k} \leq \frac{1}{2},$$

where C' is the constant which appears in (5.13). We then deduce (5.4) from (5.10), and (similarly) (5.5) from (5.9).

Step 4. Time integral control on d and h. Let us prove

$$\int_0^T \left(\|d_t\|_{M^1_3} + \|h_t\|_{M^1_3} \right) dt \tag{5.19}$$
$$\leq e^{C_\mathcal{E}\,(1+T)}\, e^{C' \left(\|f_0\|_{M^1_{2,1}} + \|g_0\|_{M^1_{2,1}} \right)} \left(\|f_0\|_{M^1_{2,2}} + \|g_0\|_{M^1_{2,2}} \right) \|g_0 - f_0\|_{M^1_2} + \|g_0 - f_0\|_{M^1_{2,1}} ,$$

for some energy dependent constant $C_\mathcal{E}$ and some numerical constant C'. Performing similar computations to those leading to (5.7), we obtain

$$\begin{aligned}\frac{d}{dt}\|h_t\|_{M^1_{2,1}} \quad \leq & \quad \iint |h_t|(dv)\, f_t(dv_*)\, |u|\, K_{2,1}(v,v_*) \\ & +2\,C \iiint |h_t|(dv)\, f_t(dv_*)\, |u|\, \langle v_*\rangle^2\, \left(1 + \log\langle v_*\rangle^2\right).\end{aligned}$$

Thanks to the Povzner inequality (5.15), we deduce for some numerical constants $C, \kappa > 0$

$$\frac{d}{dt}\,\|h_t\|_{M^1_{2,1}} \leq C\, \|h_t\|_{M^1_2}\, \|f_t\|_{M^1_{3,1}} - \kappa\, \|h_t\|_{M^1_3}\,.$$

Integrating that differential inequality yields

$$\|h_T\|_{M^1_{2,1}} + \kappa \int_0^T \|h_t\|_{M^1_3}\, dt \leq C \left(\sup_{[0,T]} \|h_t\|_{M^1_2} \right) \left(\int_0^T \|f_t\|_{M^1_{3,1}}\, dt \right) + \|h_0\|_{M^1_{2,1}}\,.$$

Using the previous pointwise control on $\|h_t\|_{M^1_2}$ and (5.13) (with $\ell = 2$) we deduce

$$\int_0^T \|h_t\|_{M^1_3}\, dt \leq e^{C_\mathcal{E}\,(1+T)}\, e^{C_1\,\|f_0\|_{M^1_{2,1}}}\, \|f_0\|_{M^1_{2,2}}\, \|g_0 - f_0\|_{M^1_2} + \|g_0 - f_0\|_{M^1_{2,1}}\,. \tag{5.20}$$

Arguing similarly for d_t, we deduce (5.19).

Step 5. Conclusion. By gathering the estimates (5.11)-(5.12)-(5.16)-(5.19), we obtain

$$\sup_{[0,T]} \|\zeta_t\|_{M^1_2} \leq e^{C_\mathcal{E}\,(1+T)}\, e^{C' \left(\|f_0\|_{M^1_{2,1}} + \|g_0\|_{M^1_{2,1}} \right)}$$
$$\left(\|f_0\|_{M^1_{2,2}} + \|g_0\|_{M^1_{2,2}} \right) \|g_0 - f_0\|_{M^1_2}\, \|g_0 - f_0\|_{M^1_{2,1}}\,,$$

for some energy dependent constant $C_\mathcal{E}$ and some numerical constant C'. Then arguing as in the end of step 3, using (5.17) and (5.18) with k large enough, we get

$$\sup_{[0,T]} \|\zeta_t\|_{M^1_2} \leq e^{C\,(1+t)}\, \sqrt{\max\left\{M_k(f_0), M_k(g_0)\right\}}\, \|f_0 - g_0\|^{2-5/k}_{M^1}\,,$$

from which estimate (5.6) follows. □

5.7. **Proof of (A4) uniformly in time.** Let us start from an auxiliary result. It was proved in [36] that the nonlinear and linearized Boltzmann semigroups for hard spheres satisfy

$$\left\|S^{NL}_t\right\|_{L^1(m_z^{-1})} \leq C_z\, e^{-\lambda\, t}, \quad \left\|e^{\mathcal{L}\, t}\right\|_{L^1(m_z^{-1})} \leq C_z\, e^{-\lambda\, t} \tag{5.21}$$

where $m_z(v) := e^{z\,|v|}$, $z > 0$, $\lambda = \lambda(\mathcal{E})$ is the optimal rate, given by the first non-zero eigenvalue of the linearized operator $\mathcal{L}$ in the smaller space $L^2(M^{-1})$ where M is the maxwellian equilibrium (see [36, Theorem 1.2]), and C_z is some constant depending on z and the energy $\mathcal{E}$.

Lemma 5.3. *For any given energy $\mathcal{E} > 0$, there exists some constants $k_1 \ge 2$ (depending on $\mathcal{E}$ and δ) and C (depending on $\mathcal{E}$) and $\eta \in (0,1)$, such that for any $f_0, g_0 \in P_{\mathcal{G}_1}(\mathbb{R}^d)$ satisfying*

$$\forall\, i = 1, \dots, d, \quad \langle f_0, v_i \rangle = \langle g_0, v_i \rangle \quad \textit{and} \quad \left\langle f_0, |v|^2 \right\rangle = \left\langle g_0, |v|^2 \right\rangle \le \mathcal{E},$$

we have

$$\left\| S^{NL}_t(g_0) - S^{NL}_t(f_0) \right\|_{M^1_2} \le e^{C - \frac{\lambda}{2}\, t} \sqrt{\max\{M_{k_1}(f_0),\, M_{k_1}(g_0)\}}\, \|g_0 - f_0\|_{M^1}^{\frac{1+\eta}{2}}, \tag{5.22}$$

$$\left\| \mathcal{L} S^{NL}_t[f_0](g_0 - f_0) \right\|_{M^1_2} \le e^{C - \frac{\lambda}{2}\, t} \sqrt{M_{k_1}(f_0)}\, \|g_0 - f_0\|_{M^1}^{\frac{1+\eta}{2}}, \tag{5.23}$$

$$\begin{aligned} &\left\| S^{NL}_t(g_0) - S^{NL}_t(f_0) - \mathcal{L} S^{NL}_t[f_0](g_0 - f_0) \right\|_{M^1} \\ &\qquad \le e^{C - \frac{\lambda}{2}\, t} \sqrt{\max\{M_{k_1}(f_0),\, M_{k_1}(g_0)\}}\, \|g_0 - f_0\|_{M^1}^{1+\eta}. \end{aligned} \tag{5.24}$$

Note that (5.24) implies **(A4)** with $T = \infty$, $P_{\mathcal{G}_2} = P_{\mathcal{G}_1}$.

From [1], there exists some constants z, Z (only depending on the collision kernel) such that

$$\sup_{t \ge 1} \|f_t + g_t + h_t\|_{L^1_{m_{2z}}} \le Z, \qquad m_z(v) := e^{2\, z\, |v|}.$$

We also know from (5.21) that (possibly increasing Z)

$$\forall\, t \ge 1 \qquad \|f_t - M\|_{L^1_{m_{2z}}} + \|g_t - M\|_{L^1_{m_{2z}}} \le 2\, Z\, e^{-\lambda\, t},$$

where $M := M_{f_0} = M_{g_0}$ stands for the normalized Maxwellian associated to f_0 and g_0 and

$$\left\| e^{\mathcal{L}\, t} \right\|_{L^1_{m_z}} \le C\, e^{-\lambda\, t}, \quad \mathcal{L} h := 2\, Q(h, M). \tag{5.25}$$

We write

$$\partial_t (f_t - g_t) = Q(f_t - g_t, f_t + g_t) = \mathcal{L}(f_t - g_t) + Q(f_t - g_t, f_t - M) + Q(f_t - g_t, g_t - M)$$

and we deduce for

$$u(t) := \|f_t - g_t\|_{M^1_{m_z}}$$

the following differential inequality (starting at some time T_0)

$$u(t) \le e^{-\lambda\, (t - T_0)}\, u(T_0) + C \int_{T_0}^t e^{-\lambda\, (t-s)}\, \|Q(f_s - g_s, f_s - M) + Q(f_s - g_s, g_s - M)\|_{M^1(m_z)}\, ds$$

(this formal inequality and next ones can easily be justified rigorously by a regularizing proceedure and using a uniqueness result for measure solutions such as [17, 14]). Therefore we obtain

$$u(t) \le e^{-\lambda\, (t - T_0)}\, u(T_0) + C \int_{T_0}^t e^{-\lambda\, (t-s)}\, \|(f_s - M) + (g_s - M)\|_{M^1(\langle v \rangle\, m_z)}\, \|f_s - g_s\|_{M^1(\langle v \rangle\, m_z)}\, ds.$$

By using the control of $M^1(\langle v \rangle\, m_z)$ by $M^1(m_{2z})$, the decay control (5.21) and the trivial estimate

$$e^{-\lambda\, (t - T_0)} \le e^{-\frac{\lambda}{2}\, (t - T_0)}\, e^{-\frac{\lambda}{2}\, (s - T_0)}$$

we get

$$u(t) \le e^{-\lambda\, (t - T_0)}\, u(T_0) + C\, e^{-\frac{\lambda}{2}\, (t - T_0)} \int_{T_0}^t e^{-\frac{\lambda}{2}\, (s - T_0)}\, \|f_s - g_s\|_{M^1(\langle v \rangle\, m_z)}\, ds.$$

We then use the following control for any $a > 0$:

$$\begin{aligned}\|f-g\|_{M^1_{\langle v\rangle\, m_z}} &= \int |f-g|\,\langle v\rangle\, e^{z\,|v|} \\ &\le a \int_{|v|\le a} |f-g|\, e^{z\,|v|} + e^{-z\,a} \int_{|v|\ge a} (f+g)\, e^{2\,z\,|v|} \\ &\le a\, u + e^{-z\,a}\, Z.\end{aligned}$$

Hence we get

$$\|f-g\|_{M^1_{\langle v\rangle\, m_z}} \le \begin{cases} u + e^{-z}\, Z \le (1+Z)\, u \quad \text{when} \quad u \ge 1, \quad (\text{choosing } a := 1) \\ \frac{2}{z}\, |\log\, u|\, u + u\, Z \quad \text{when} \quad u \le 1 \quad (\text{choosing } -\frac{z}{2} a := \log\, u) \end{cases}$$

and we deduce

$$\|f-g\|_{M^1_{\langle v\rangle\, m_z}} \le K\, u\, \left(1 + (\log u)_-\right), \qquad K := 1 + \frac{2}{z} + Z.$$

Then for any $\delta > 0$ small, we conclude, by choosing T_0 large enough, with the following integral inequality

$$u(t) \le e^{-\lambda\,(t-T_0)}\, u(T_0) + \delta\, e^{-\frac{\lambda}{4}\,(t-T_0)} \int_{T_0}^t e^{-\frac{\lambda}{2}\,(s-1)}\, u_s\, \big(1 + (\log u_s)_-\big)\, ds. \tag{5.26}$$

Let us prove that this integral inequality implies

$$\forall\, t \ge 1, \quad u(t) \le C\, e^{-\frac{\lambda}{4}\, t}\, u(T_0)^{1-\delta}. \tag{5.27}$$

Consider the case of equality in (5.26). Then we have $u(t) \ge e^{-\lambda\,(t-1)}\, u(1)$ and therefore

$$\big(1 + (\log u_s)_-\big) \le \big(1 + (\log u(T_0))_- + \lambda\,(s - T_0)\big).$$

We then have

$$\begin{aligned}U(t) := \int_{T_0}^t e^{-\frac{\lambda}{2}\,(s-T_0)}\, u_s\, \big(1 + (\log u_s)_-\big)\, ds &\le \int_{T_0}^t e^{-\frac{\lambda}{2}\,(s-T_0)}\, u_s\, \big(1 + (\log u(T_0))_- + \lambda\,(s-T_0)\big)\, ds \\ &\le \big(1 + (\log u(T_0))_-\big) \int_{T_0}^t e^{-\frac{\lambda}{4}\,(s-T_0)}\, u_s\, ds\end{aligned}$$

and we conlude the proof of the claimed inequality (5.27) by a Gronwall-like argument.

Then estimate (5.22) follows by choosing δ small enough (in relation to η) and then connecting the last estimate (5.27) from time T_0 on together with the previous finite time estimate (5.4) from time 0 until time T_0.

Then estimates (5.23) and (5.24) are proved in the same way by using the equations

$$\partial_t h_t = \mathcal{L}(f_t - g_t) + Q(h_t, f_t - M)$$

(which is even simpler than the equation for $f_t - g_t$) and

$$\partial_t \zeta_t = 2\, \mathcal{L}(h_t) + Q(\zeta_t, f_t - M) + Q(\zeta_t, g_t - M) + Q(h_t, d_t).$$

5.8. **Proof of (A5) uniformly in time.** Let us prove that for any $\bar z, \mathcal{M}_{\bar z} \in (0,\infty)$ there exists some continuous function $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$, $\Theta(0) = 0$, such that for any f_0, $g_0 \in P_{m_{\bar z}}(\mathbb{R}^d)$, with $m_{\bar z}$ defined in the previous section, such that

$$\|f_0\|_{M^1_{m_{\bar z}}} \le \mathcal{M}_{\bar z}, \quad \|g_0\|_{M^1_{m_{\bar z}}} \le \mathcal{M}_{\bar z},$$

there holds

$$\sup_{t\ge 0} W_t \le \Theta(W_0), \tag{5.28}$$

where W_t stands for the Kantorovich-Rubinstein distance

$$W_t = W_1\left(S_t^{NL}(f_0),\, S_t^{NL}(g_0)\right).$$

As we shall see, we may choose

$$\Theta(w) := \bar\Theta \min\left\{1,\, e^{-(\log(w/\theta_0))_-^{1/2}}\right\}, \tag{5.29}$$

for some constants $\bar\Theta$, $\theta_0 > 0$ (only depending on $\bar z$ and $\mathcal{M}_{\bar z}$).

We start with

$$\forall\, t \ge 0 \quad W_t \le \|(f_t - g_t)|v|\|_{M^1} \le \frac{1}{2}\left\|(f_t+g_t)\langle v\rangle^2\right\|_{M^1} = 1 + \mathcal{E} =: \bar\Theta. \tag{5.30}$$

Let us improve this inequality for small value of W_0. On the one hand, it has been proved in [17, Theorem 2.2 and Corollary 2.3] that

$$W_t \le W_0 + K\int_0^t W_s\,\left(1 + (\log W_s)_-\right)\,ds, \tag{5.31}$$

for some constant K. Whenever $W_t \le 1/2$, the integral inequality (5.31) implies (possibly increasing the constant K)

$$W_t \le W_0 + K\int_0^t W_s\,(\log W_s)_-\,ds.$$

From the Gronwall lemma we deduce

$$W_t \le (W_0)^{\exp(-Kt)} \quad \text{whenever} \quad W_t \le 1/2. \tag{5.32}$$

On the other hand, from [5] and [36], there exists $\lambda_2, Z > 0$, $z \in (0,\bar z)$ such that

$$\forall\, t \ge 0 \qquad \|f_t - M_{f_0}\|_{L^1_{m_z}} + \|g_t - M_{g_0}\|_{L^1_{m_z}} \le Z\,e^{-\lambda\,t},$$

where M_{f_0} and M_{g_0} stand again for the normalized Maxwellian associated to f_0 and g_0. Denoting by u_{f_0} and u_{g_0} the mean velocity of f_0 and g_0, by θ_{f_0} and θ_{g_0} the temperature associated to f_0 and g_0, and by $\mathcal{E}_{f_0}$ and $\mathcal{E}_{g_0}$ the energy associated to f_0 and g_0, there also holds

$$\begin{aligned} W_1(M_{f_0}, M_{g_0}) &\le C_s \left(\|M_{f_0} - M_{g_0}\|^2_{H^{-s}} \right)^\eta \\ &\le C_s \left(\int_{\mathbb{R}^d} \frac{\left| e^{-\theta_{f_0} \frac{|\xi|^2}{2} - i\, u_{f_0} \sqrt{\theta_{f_0}}\, \xi} - e^{-\theta_{g_0} \frac{|\xi|^2}{2} - i\, u_{g_0} \sqrt{\theta_{g_0}}\, \xi} \right|^2}{\langle \xi \rangle^{2s}} \, d\xi \right)^\eta \\ &\le C_s \left(\int_{\mathbb{R}^d} \frac{|\theta_{f_0} - \theta_{g_0}|^2\, |\xi|^4 + |u_{f_0} \sqrt{\theta_{f_0}} - u_{g_0} \sqrt{\theta_{g_0}}|^2\, |\xi|^2}{\langle \xi \rangle^{2s}} \, d\xi \right)^\eta \\ &\le C_s \left(|\theta_{f_0} - \theta_{g_0}|^\eta + |u_{f_0} - u_{g_0}|^{2\eta} \right) \\ &\le C_s \left(|\mathcal{E}_{f_0} - \mathcal{E}_{g_0}|^\eta + |u_{f_0} - u_{g_0}|^{2\eta} \right) \\ &\le C_s \left(W_2(f_0, g_0)^{2\eta} + W_1(f_0, g_0)^{2\eta} \right) \\ &\le C_s \left(W_1(f_0, g_0)^{\eta/2} + W_1(f_0, g_0)^{2\eta} \right) \end{aligned}$$

(see also [12] for more general estimates of the Wasserstein distance between two gaussians). Gathering these two estimates, we deduce

$$\forall\, t \ge 0 \qquad W_t \le Z_1\, e^{-\lambda\, t} + Z_2\, W_0^{\eta/2}. \tag{5.33}$$

Let us (implicitly) define $T, \bar{W}_0 \in (0, \frac{1}{4^{2/\eta}})$ by

$$Z_1\, e^{-\lambda\, T} = \frac{1}{4} \quad \text{and next} \quad \left(\bar{W}_0\right)^{\exp(-K\, T)} = \frac{1}{2}.$$

Then, for any $W_0 \in (0, \bar{W}_0)$ we have

$$\forall\, t \ge 0 \qquad W_t \le \min \left\{ (W_0)^{\exp(-Kt)}\, ;\, Z_1\, e^{-\lambda\, t} \right\} + Z_2\, W_0^{\eta/2}.$$

Let us search for t^* such that the two functions involved in the minus function are equal:

$$\phi(t^*) := |\log W_0|\, \exp(-K\, t^*) = \lambda\, t^* - \log Z =: \psi(t^*).$$

The time t^* is unique because ϕ is decreasing while ψ is increasing (and it exists for $\log \bar{W}_0 \le \log Z$). Choosing

$$t^\sharp := \frac{1}{K}\, \log \left(|\log W_0|^{1/2} \right),$$

we find

$$\phi(t^\sharp) = |\log W_0|^{1/2} \ge \frac{\lambda}{K}\, \log \left(|\log W_0|^{1/2} \right) - \log Z = \psi(t^\sharp),$$

at least whenever $W_0 \in (0, W_0^\sharp]$, with $W_0^\sharp \in (0, \bar{W}_0]$ small enough. As a consequence, for any $W_0 \in (0, W_0^\sharp]$ we have

$$\forall\, t \ge 0, \quad W_t \le 2\, \phi(t^\sharp) = 2\, e^{-|\log W_0|^{1/2}}.$$

This concludes the proof.

6. Extensions and complements

In this section first we generalize the chaos propagation results on the Boltzmann (maxwellian or hard spheres molecules) to the case where the limit 1-particle distribution is not compactly supported. This shall rely on a construction due to Kac of the N-particle initial data together with careful study of the dependency of the constants in terms of moments of the data. Second, as a corollary, we use our global in time results to give a new method for studying chaotic convergence of the N-particle equilibria towards the limit 1-particle equilibrium. The old question of connecting the long-time behavior was raised by Kac and it motivated its whole study of chaos propagation for particle systems. In the case of classical gas dynamics, we thus recover a well-known computational results (namely the marginals of the constant probabilities on $\sqrt{N}\mathbb{S}^{Nd-1}$ converges to products of gaussians) without any explicit computations, only using the properties of the Boltzmann semigroups. This new method shall prove highly useful in the context of granular gases where the steady states or homogeneous cooling states are not explicitly known.

6.1. Construction of chaotic initial data f_0^N with prescribed energy.

For the sake of completeness, let us recall, following [22], how to construct a f_0-chaotic sequence of initial data f_0^N (*i.e.*, which has the *"Boltzmann's property"* in the words of Kac).

Lemma 6.4. *Consider $f_0 \in P(\mathbb{R}^d)$ with finite energy $M_{m_e}(f_0) := M_2(f_0) = \mathcal{E} \in (0,\infty)$ and which fulfills the following moments conditions $M_{m_i}(f_0) = M^{NL}_{0,m_i} < \infty$, $i = 0,1,3$, for some radially symmetric and increasing weight functions m_i, $i = 1,3$, and $m_0(x) := \exp(a\,|x|^2)$, $a > 0$. Then for any given increasing sequence $(\alpha_N)_{N\geq 1}$ (which increases as slow as we wish in general and may be chosen constant when f_0 has compact support), there exists a sequence $f_0^N \in P(\mathbb{R}^{dN})$, $N \geq 1$, such that*

(i) *The sequence $(f_0^N)_{N\geq 1}$ is f_0-chaotic.*

(ii) *Its support satisfies*

$$\operatorname{supp} f_0^N \subset S^{Nd-1}\left(\sqrt{N\,\mathcal{E}}\right) := \left\{V \in \mathbb{R}^{dN};\ M_2^N(V) = \mathcal{E}\right\} \subset \mathbb{E}_N.$$

(iii) *It satisfies the following integral moment bound based on m_2:*

$$\forall\, N \in \mathbb{N}^*, \quad \left\langle f_0^N, M^N_{m_3}\right\rangle \leq Cst\,\left(M^{NL}_{0,m_3}\right).$$

(iv) *It satisfies the following moment bound* on the support *involving m_3:*

$$\operatorname{supp} f_0^N \subset \mathcal{K}_N := \left\{V \in \mathbb{R}^{dN};\ M^N_{m_3}(V) \leq \alpha_N\right\}.$$

Sketch of the proof of Lemma 6.4. We essentially recall briefly the key arguments presented in [22, Section 5 "Distributions having Boltzmann's property"] and check that the moment conditions required in the sequel of our paper can be satisfied. For the sake of simplicity, we assume with no loss of generality that the energy $\mathcal{E} = 1$. We restrict to the case when $f_0 \in P(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and we refer to [6] for the relaxation of this condition.

Since $f_0 \in C(\mathbb{R}^d)$, we can define

$$f_0^N(V) := \left.\frac{\prod_{j=1}^N f_0(v_j)}{F_N(\sqrt{N})}\right|_{\mathbb{S}^{Nd-1}(\sqrt{N})} \quad \text{with} \quad F_N(r) := \int_{\mathbb{S}^{Nd-1}(r)} \prod_{j=1}^N f_0(v_j)\,d\omega,$$

so that (ii) holds.

From the gaussian moment bound $M_{m_0}(f_0) < \infty$, we obtain from [22] that there exists some constants $C > 0$ such that

$$F_N(\sqrt{N}) \,\sim\, C\,N,$$

and for $\varphi(v_1,\dots,v_\ell)$, $\ell \le N$:

$$\int_{\mathbb{S}^{Nd-1}(\sqrt{N})} \varphi(v_1,\dots,v_\ell) \prod_{j=1}^N f_0(v_j)\, dS(V) \xrightarrow[N\to\infty]{} C\,N \int_{\mathbb{R}^d} \varphi(v_1,\dots,v_\ell)\, df_0(v_1)\dots df_0(v_\ell)$$

which proves the chaos.

We then deduce (i) thanks to [41, Proposition 2.2]. Point (iii) is just a consequence of the above asymptotic with the choice $\varphi = m_1$, $\psi = 1$. When we assume furthermore that f_0 is compactly supported, say $\operatorname{supp} f_0 \subset [-A,A]^d$, we deduce $\operatorname{supp} f_0^N \subset \{V \in \mathbb{R}^{Nd},\ M^N_{m_3}(V) \le m_3(A)\}$ and (iv) holds.

In the non compactly supported case, for any $k \in \mathbb{N}^*$ and for any constant A_k we define $f_{0,k} := f_0\, \mathbf{1}_{|v|\le A_k}$. It is clear that for any $k\in\mathbb{N}^*$, there exists $f^N_{0,k}$ such that $f^N_{0,k}$ is $f_{0,k}$-chaotic such that (ii) and (iii) hold and $\operatorname{supp} f^N_{0,k} \subset \{V\in\mathbb{R}^{dN};\ M^N_{m_3}(V) \le m_3(A_k)\}$. For any given sequence (α_N) which tends to infinity, we define k_N in such a way that $m_3(A_{k_N}) = \alpha_N$ so that $k_N\to\infty$ when $N\to\infty$. We then easily verify that $f_0^N := f^N_{0,k_N}$ satisfies the properties (i)–(iv). □

6.2. **Chaos propagation without compact support for the Boltzmann equation.**

We may relax the compactly support condition in Theorem 4.1 and Theorem 5.1 thanks to Lemma 6.4. We assume that

$$M_{m_0}(f_0) := \int_{\mathbb{R}^d} e^{a\,|v|^2}\, f_0(dv) < \infty$$

for some $a\in(0,\infty)$ and we define f_0^N as in Lemma 6.4. Instead of (3.41) in Theorem 3.27 we have the following bound. For any increasing sequence $\alpha_N\to\infty$, for any $\varphi \in (\mathcal{F}_1\cap\mathcal{F}_2\cap\mathcal{F}_3)^{\otimes\ell}$, there exists a constant $C_{\ell,\varphi}$ (independent of N) such that for any $N\in\mathbb{N}^*$, with $N\ge 2\ell$,

$$\sup_{[0,T)} \left| \left\langle \left(S^N_t(f^N_0) - \left(S^{NL}_t(f_0)\right)^{\otimes N} \right), \varphi \right\rangle \right| \le C_{\ell,\varphi} \left[\frac{1}{N} + C^N_{T,m_1}\, C^\infty_T\, \varepsilon_2(N) + \Theta_{\alpha_N,T}\left(\mathcal{W}_{dist_{\mathcal{G}_3}}(\pi^N_P f^N_0, \delta_{f_0}) \right) \right]. \tag{6.34}$$

By choosing (α^N) appropriately, we will deduce from (6.34) that $S^N_t(f^N_0)$ is again $S^{NL}_t(f_0)$-chaotic uniformly in time.

In the case of the Boltzmann model for Hard Spheres or true Maxwell Molecules, we can take $\operatorname{dist}_{\mathcal{G}_3} = W_1$ the usual Monge-Kantorovich-Wasserstein distance in $P(\mathbb{R}^d)$. We claim that

$$\mathcal{W}_{W_1}(\pi^N_P f^N_0, \delta_{f_0}) \xrightarrow[N\to\infty]{} 0. \tag{6.35}$$

First, thanks to [41, Proposition 2.2] and the fact that f_0^N is f_0-chaotic from Lemma 6.4, we deduce that $\pi^N_P f^N_0$ converges to δ_{f_0} in the weak sense in $P(P(\mathbb{R}^d))$ (that means taking duality product with functions of $C(P(\mathbb{R}^d))$). Next, thanks to [46, Theorem 7.12], (6.35) boils down to prove that

$$\lim_{R\to\infty} \sup_{N\in\mathbb{N}^*} \int_{W_1(\rho,f_0)\ge R} W_1(\rho,f_0)\, \pi^N_P f^N_0(\rho) = 0, \tag{6.36}$$

which will be a straightforward consequence of the following bound

$$\sup_{N\in\mathbb{N}^*} \int_{E^N} [W_1(\mu^N_V, f_0)]^2\, f^N_0(dV) < \infty. \tag{6.37}$$

Finally, in order to get (6.37), we infer that from [46, Theorem 7.10]

$$\begin{aligned}[W_1(\mu^N_V, f_0)]^2 &\le \|\mu^N_V - f_0\|^2_{M^1_1} \\ &\le 2\|\mu^N_V\|^2_{M^1_1} + 2\,\|f_0\|^2_{M^1_1} \\ &\le 2\,[M^N_1(V)]^2 + 2\,\|f_0\|^2_{M^1_1} \le 2\,M^N_2(V) + 2\,\|f_0\|^2_{M^1_1}.\end{aligned}$$

That implies

$$\int_{E^N} [W_1(\mu^N_V, f_0)]^2\, f^N_0(dV) \le 2\,\|f_0\|^2_{M^1_1} + 2\,\langle f^N_0, M_2\rangle,$$

which, together with (ii) in Lemma 6.4, ends the proof of (6.37) and then of (6.35).

From the fact that the $\Theta_{T,A}$ functions exhibited in Theorem 4.1 and Theorem 5.1 are independent of T and satisfy $\Theta_{T,A}(x) = \Theta_A(x) \to 0$ when $x \to 0$ for any fixed $A \in (0,\infty)$ we may build (thanks to a diagonal process) a sequence (α^N) such that

$$\Theta_{\alpha^N}\left(\mathcal{W}_{W_1}(\pi^N_P f^N_0, \delta_{f_0})\right) \underset{N\to\infty}{\longrightarrow} 0.$$

Coming back to (6.34) we obtain that $S^N_t(f^N_0)$ is $S^{NL}_t(f_0)$-chaotic uniformly in time.

6.3. **Chaoticity of the sequence of steady states.**

Theorem 6.5 (Abstract fluctuation estimate in the infinite time)**.** *Consider a sequence of initial datum f^N_0 which satisfies* **(A1)** *(with $C^N_{0,m_3} = \alpha^N$ may depend on N) and is f_0-chaotic with $f_0 \in P_{\mathcal{G}_1} \cap P_{\mathcal{G}_3}$. Assume moreover that the assumptions of theorem 3.27 hold with $T = \infty$. Assume finally that $f^N_t \rightharpoonup \gamma^N$ when $t \to \infty$ in the weak sense of measures in $P(E^N)$ as well as $f_t \rightharpoonup \gamma$ when $t \to \infty$ in the weak sense of measures in $P(E)$. For any $\ell \in \mathbb{N}^*$ and $\varphi \in (\mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3)^{\otimes \ell}$, there exists a constant $C_{\ell,\varphi}$ such that for any $N \in \mathbb{N}^*$, with $N \ge 2\ell$, we have*

$$\begin{aligned}&\left|\left\langle \Pi_\ell \gamma^N - \gamma^{\otimes \ell}, \varphi \right\rangle\right| \le \\ &\le C_{\ell,\varphi}\left[\frac{1}{N} + C^N_{\infty,m_1}\, C^\infty_\infty\, \varepsilon_2(N) + \Theta_{C^N_{0,m_3},\infty}\left(\mathcal{W}_{\mathrm{dist}_{\mathcal{G}_3}}(\pi^N_P f^N_0, \delta_{f_0})\right)\right].\end{aligned} \tag{6.38}$$

As a consequence, γ^N is γ-chaotic.

The proof of that result is trivial: we just have to apply Theorem 3.27 and to pass to the limit in the left hand side of the inequality (3.41) in order to gat (6.38). Arguing as in section 6.2 we deduce γ^N is γ-chaotic (whenever $C^N_{0,m_3} = \alpha^N$ grows slowly enough).

The application to the Boltzmann equation is the following. Consider a sequence of initial data f^N_0 such that $\operatorname{supp} f^N_0 \subset \mathbb{S}^{Nd-1}(\sqrt{N})$, and such that f^N_0 is f_0-chaotic with $\int |v|^2 f_0 = 1$, $\int v_i\, f_0 = 0$ for any $i = 1, \dots, d$. On the one hand, we know (see [22, 6]) that f^N_t converges in the large time asymptotic to γ^N, the uniform distribution on the sphere $\mathbb{S}^{Nd-1}(\sqrt{N})$ (that holds in $L^2(\gamma^N)$ with rate $\exp(-\lambda_N\, t)$ whenever $f^N_0 \in L^2(\gamma^N)$). We also know (see [36] and the references therein for the hard sphere case and [42, 7] or section 4.1 for the (true) Maxwell Molecules case) that f_t converges in the large time asymptotic to γ, the normalized Gaussian. As a consequence, we get (6.38). That result may seem to be trivial, and it is in some sense, because an explicit computation (which go back at least to Mehler in 1866) yields the same result. However, our proof it is not based on an explicit computation nor a variationnal/entropy optimization principle. The consequence is that it applies to many more situations, in particular in the case of some dissipative Boltzmann equation (linked to the Granular media), see [34].

6.4. **On statistical solutions and the non uniqueness of its steady states.** Let us consider the N-particles system associated to the Boltzmann collision process that we do not write in dual for as we have done before, it writes

$$\partial_t f^N = \frac{1}{N} \sum_{i<j} \int_{S^{d-1}} B(|v_i - v_j|, \cos\theta) \left[f^N(\dots, v_i', \dots, v_j', \dots) - f \right] d\sigma. \tag{6.39}$$

We want to describe how the BBGKY (Bogoliubov, Born, Green, Kirkwood and Yvon) method introduced to derive Boltzmann's equation from Liouville's equation applies in our simpler space homogeneous context. Let us thus also introduce the k-th marginal:

$$f^N_\ell(v_1, ..., v_\ell) := \int_{\mathbb{R}^{d(N-k)}} f^N(v_1, ..., v_\ell, w_{\ell+1}, ..., w_N) \, dw_{\ell+1} \, ... dw_N$$

Integrating the master equation (6.39) leads to

$$\begin{aligned} \partial_t f^N_\ell \quad = \quad & \frac{1}{N} \sum_{i,j \le \ell} \mathcal{Z}^N_{ij} && = \mathcal{O}(\ell^2/N) \\ & + \frac{1}{N} \sum_{i \le \ell < j} \mathcal{Z}^N_{ij} \\ & + \frac{1}{N} \sum_{i,j > \ell} \mathcal{Z}^N_{ij} && = 0, \end{aligned}$$

with

$$\mathcal{Z}^N_{ij} := \int_{\mathbb{R}^{d\,(N-\ell-1)}} \int_{S^{N-1}} B \Big[f^N(\dots, v_i', \dots, v_j', \dots) - f^N \Big] \, d\sigma \, dv_{\ell+1} \, ... \, dv_N.$$

Only the second term does not vanish in the limit $N \to \infty$, so that assuming that $f^N_\ell \to \pi_\ell$, we find that (π_ℓ) is a solution to the infinite dimensional system of linear equation (the Boltzmann equation for a system of an infinite number of particles or the statistical Boltzmann equation)

$$\partial f_\ell = A_{\ell+1}(\pi_{\ell+1}) \tag{6.40}$$

with $\pi_\ell = \pi_\ell(t, v_1, ..., v_\ell) \ge 0$ and

$$V \in \mathbb{R}^{d\ell} \mapsto A_{\ell+1}(\pi_{\ell+1})(V) = \sum_{j=1}^{\ell} \int_{S^{d-1} \times \mathbb{R}^d} \Big\{ \pi_{\ell+1}(V_j') - \pi_{\ell+1}(V) \Big\} \, b(\sigma \cdot (v_j - v_{\ell+1})) \, dv_{\ell+1} d\sigma,$$

with $V_j' = (v_1, ..., v_j', ..., v_\ell, v_{\ell+1}')$, $v_j' = v'(v_j, v_{\ell+1}, \sigma)$, $v_{\ell+1}' = v_*'(v_j, v_{\ell+1}, \sigma)$ where vectors v' and v_*' are defined by (1.4).

Lemma 6.6. *There exists a non chaotic stationary solution to the statistical Boltzmann equation. In other words, there exists $\pi \in P(P(\mathbb{R}^d))$ such that $\pi \ne \delta_p$ for some $p \in P(\mathbb{R}^d)$ and $A_{\ell+1}(\pi_{\ell+1}) = 0$ for any $\ell \in \mathbb{N}$.*

Proof of Lemma 6.6. It is clear that any function on the form

$$V \in \mathbb{R}^{d(\ell+1)} \ \mapsto \ \pi_{\ell+1}(V) = \phi(|V|^2)$$

is a stationary solution got equation (6.40), that is $\mathcal{C}_{\ell+1}(\pi_{\ell+1}) = 0$. Now we define, with $d = 1$ for the sake of simplicity, the sequence

$$V \in \mathbb{R}^\ell \mapsto \pi_\ell(V) = \frac{c_\ell}{(1 + |V|^2)^{m+\ell/2}}$$

with c_1 such that π_1 is a probability measure and $c_2 = c_1\,\alpha_2$ with α_2 chosen in the following way:

$$\begin{aligned}\int_{\mathbb{R}} \frac{\alpha_2}{(1+v^2+v_*^2)^{m+1}}\,dv_* &= \frac{\alpha_2}{(1+v^2)^{m+1}}\int_{\mathbb{R}} \frac{1}{(1+\frac{v_*^2}{1+v^2})^{m+1}}\,dv_* \\ &= \frac{\alpha_2}{(1+v^2)^{m+1/2}}\int_{\mathbb{R}} \frac{1}{(1+w_*^2)^2}\,dw_* = \frac{1}{(1+v^2)^{m+1/2}}.\end{aligned}$$

By an iterative process we may chose the constants c_ℓ in such a way that π_ℓ is a solution to $A_\ell(\pi_\ell) = 0$ (because it is a function of the energy) and satisfies the compatibility condition:

$$\pi_\ell(V) = \int_{\mathbb{R}} \pi_{\ell+1}(V, v_*)\,dv_*.$$

We have exhibit a solution which is not chaotic. □

We come back to the abstract setting. We start with the N-particles system equation,

$$\partial_t f^N = A^N\,f^N,$$

that we write in dual form

$$\begin{aligned}\partial_t\langle f^N_\ell, \varphi\rangle &= \partial_t\langle f^N, \varphi\otimes\mathbf{1}^{N-\ell}\rangle \\ &= \langle f^N, G^N(\varphi\otimes\mathbf{1}^{N-\ell})\rangle = \langle f^N_{\ell+1}, G^N_{\ell+1}(\varphi)\rangle.\end{aligned}$$

Lemma 6.7. *Assume that*

(A6) *(f^N_ℓ) is tight in $P(E^\ell)$ for any $\ell \geq \mathbb{N}$ (or equivalently, f^N is tight in $P(P(E))$);*

(A7) *$G^N_{\ell+1}\varphi \to G^\infty_{\ell+1}\varphi$ when $N\to\infty$, for any fixed $\varphi\in C_b(E^\ell)$.*

Then, up to extraction a subsequence, (f^N) converges (in the sense of any ℓ-th marginals) to a solution $\pi = (\pi_\ell) \in P(P(E))$ to the infinite Hierachy

$$\partial_t\pi = A^\infty\pi \quad in \quad P(P(E)),$$

which simply means

$$\partial_t\langle\pi_\ell, \varphi\rangle = \langle\pi_{\ell+1}, G^\infty_{\ell+1}\varphi\rangle \quad for\ any \quad \ell\in\mathbb{N}^*.$$

6.5. **Uniqueness of statistical solutions and chaos.** Assuming **(A2i)** and

(A2iii') $[0,\infty) \to P_{\mathcal{G}_1}$, $t\mapsto S^{NL}_t f$ uniform continuously for any $f\in P_{\mathcal{G}_1}$,

which is a weak version of **(A2iii)**, we have that (S^{NL}_t) is a c_0-semigroup. As it has proved in step 1 of Lemma 2.13, for any $\Phi\in C_b(P_{\mathcal{G}_1},\mathbb{R})$ we may define $T^\infty\Phi$ by

$$(T^\infty\Phi)(f) = \Phi(S^{NL}_t f),$$

and we build in that way a c_0-semigroup (T^∞_t) on $C_b(P_{\mathcal{G}_1},\mathbb{R})$. The Hille-Yosida theory imply that there exists an closed operator G^∞ with dense domain $\mathrm{dom}(G^\infty)$ in $C_b(P_{\mathcal{G}_1},\mathbb{R})$ so that (T^∞_t) is the semigroup associated to the generator G^∞.

Now, on the one hand, for any $\pi_0\in P(P_{\mathcal{G}_1})$ we may define the semigroup (S_t) on $P(P_{\mathcal{G}_1})$ and the flow $(\bar\pi_t)$ by setting $\bar\pi_t = S^\infty_t\pi_0$ and (duality formula)

$$\forall\,\Phi\in C_b(P_{\mathcal{G}_1};\mathbb{R}) \qquad \langle S^\infty_t\pi_0, \Phi\rangle = \langle\pi_0, T^\infty_t\Phi\rangle.$$

Remark (see [3]) that $S^\infty_t\pi_0 \in (C_b(P(V))'_+ \neq P(P(E))$. Under the additional assumption that $P_{\mathcal{G}_1,a}$ is compact for any a (that is true in our application cases when $\|\cdot\|_{\mathcal{G}_1}$ metrizes the weak measures topology) that relation defined a unique probability $S^\infty_t\pi_0\in P(P_{\mathcal{G}_1})$, and again (S^∞_t) is a c_0-semigroup on $P(P_{\mathcal{G}_1})$.

On the other hand, we say that $\pi_t\in C(\mathbb{R}_+;P(P_{\mathcal{G}_1}))$ is a solution to the equation

$$\partial_t\pi_t = A^\infty\,\pi_t, \tag{6.41}$$

if for any $\Phi \in \mathrm{dom}(G^\infty)$ there holds

$$\frac{d}{dt}\langle \pi_t, \Phi\rangle = \langle \pi_t, G^\infty \Phi\rangle \quad \text{in } \mathcal{D}'([0,\infty)).$$

Theorem 6.8. *Assume that* **(A2)** *and* **(A4)** *hold, as well as that $C^1(P_{\mathcal{G}_1};\mathbb{R})$ is dense in $C_b(P_{\mathcal{G}_1};\mathbb{R})$. For any initial datum $\pi_0 \in P(P_{\mathcal{G}_1})$ the flow $\bar\pi_t$ is the unique solution in $C([0,\infty); P(P_{\mathcal{G}_1}))$ to (6.41) starting from π_0. Moreover, if π_0 is f_0-chaotic, then π_t is $S^{NL}_t f_0$-chaotic for any $t \ge 0$.*

Proof of Theorem 6.8. Step 1: Chaos propagation. From Hewitt-Savage's theorem [21], for any $\pi \in P(P(E))$ there exists a unique sequence $(\pi^\ell) \in P(E^\ell)$ such that the identities

$$\begin{aligned}\langle \pi^\ell, \varphi\rangle &= \int_{P(E)} \langle f^{\otimes \ell}, \varphi\rangle \, \pi(df) \\ &= \int_{P(E)} R^\ell_\varphi(f)\, \pi(df) = \langle \pi, R^\ell_\varphi\rangle,\end{aligned}$$

hold for any $\varphi \in C_b(E)^{\otimes \ell}$. As a consequence, if π_0 is f_0-chaotic,

$$\begin{aligned}\langle \bar\pi^\ell_t, \varphi\rangle &= \langle \bar\pi_t, R^\ell_\varphi\rangle = \langle \pi_0, T^\infty_t R^\ell_\varphi\rangle = (T^\infty_t R^\ell_\varphi)(f_0) \\ &= R^\ell_\varphi(S^{NL}_t f_0) = \langle S^{NL}_t\, f_0, \varphi_1\rangle \dots \langle S^{NL}_t\, f_0, \varphi_\ell\rangle,\end{aligned}$$

which means that $\bar\pi^\ell_t = f_t^{\otimes \ell}$, or equivalently $\bar\pi_t = \delta_{f_t}$, and the statistical solution $\bar\pi_t$ is f_t-chaotic.

Step 2: Uniqueness. For any $t > 0$ and $n \in \mathbb{N}^*$ owe define $\varepsilon := t/n$ and $t_k = \varepsilon\, k$, $s_k = t - t_k$. Then for any $\Phi \in C^1_b(P_{\mathcal{G}_1};\mathbb{R})$ we define $\Phi_t := T^\infty_t \Phi$. The very fundamental point is that thanks to Lemma 2.13 we have $\Phi_t \in C^1_b(P_{\mathcal{G}_1};\mathbb{R}) \subset \mathrm{dom}(G^\infty)$ for any $t \ge 0$. We write

$$\begin{aligned}&\langle \pi_t, \Phi\rangle - \langle \bar\pi_t, \Phi\rangle = \langle \pi_t, \Phi\rangle - \langle \pi_0, \Phi_t\rangle \\ &\quad= \sum_{k=0}^{n-1} \big\{ \big[\langle \pi_{t_{k+1}}, \Phi_{s_{k+1}}\rangle - \langle \pi_{t_{k+1}}, \Phi_{s_k}\rangle\big] + \big[\langle \pi_{t_{k+1}}, \Phi_{s_k}\rangle - \langle \pi_{t_k}, \Phi_{s_k}\rangle\big]\big\} \\ &\quad= \mathcal{T}_1 + \mathcal{T}_2 = \sum_{k=0}^{n-1} \{\mathcal{T}_{1,k} + \mathcal{T}_{2,k}\}.\end{aligned}$$

On the one hand, we have

$$\begin{aligned}\mathcal{T}_{1,k} &= \langle \pi_{t_{k+1}}, \Phi_{s_{k+1}} - T^\infty_\varepsilon \Phi_{s_{k+1}}\rangle = -\langle \pi_{t_{k+1}}, \int_0^\varepsilon \frac{d}{ds}[T^\infty_s \Phi_{s_{k+1}}]\, ds\rangle \\ &= -\langle \pi_{t_{k+1}}, \int_0^\varepsilon [G^\infty \Phi_{s_{k+1}+s}]\, ds\rangle = -\int_{s_{k+1}}^{s_k} \langle \pi_{t-[s+1,\varepsilon]}, G^\infty \Phi_s\rangle\, ds,\end{aligned}$$

where $[s,\varepsilon] = [s/\varepsilon]\,\varepsilon$. Passing to the limit $n \to \infty$, we get

$$\mathcal{T}_1 = -\int_0^t \langle \pi_{t-[s+1,\varepsilon]}, G^\infty \Phi_s\rangle\, ds \underset{n\to\infty}{\longrightarrow} -\int_0^t \langle \pi_{t-s}, G^\infty \Phi_s\rangle\, ds.$$

On the other hand, we have

$$\begin{aligned} \mathcal{T}_{2,k} &= \int_0^\varepsilon \frac{d}{d\tau}\langle \pi_{t_k+\tau}, \Phi_{s_k}\rangle\, d\tau \\ &= \int_0^\varepsilon \langle \pi_{t_k+\tau}, G^\infty \Phi_{s_k}\rangle\, d\tau \\ &= \int_{t_k}^{t_{k+1}} \langle \pi_\tau, G^\infty \Phi_{t-[\tau,\varepsilon]}\rangle\, d\tau. \end{aligned}$$

Passing to the limit $n \to \infty$, we get

$$\mathcal{T}_2 = \int_0^t \langle \pi_\tau, G^\infty \Phi_{t-[\tau,\varepsilon]}\rangle\, d\tau \underset{n\to\infty}{\longrightarrow} \int_0^t \langle \pi_\tau, G^\infty \Phi_{t-\tau}\rangle\, d\tau.$$

As a conclusion, for any $\Phi \in C^1(P_{\mathcal{G}_1};\mathbb{R})$, we have proved

$$\langle \pi_t, \Phi\rangle = \langle \bar\pi_t, \Phi\rangle.$$

From a density argument we conclude that $\pi_t = \bar\pi_t$. □

Gathering Lemma 6.7 and Theorem 6.8 we obtain a propagation to the chaos result.

Corollary 6.9 (Abstract chaos propagation)**.** *Assume* **(A2)***,* **(A4)***,* **(A6)***,* **(A7)** *and the following compatibility between G^∞ and the sequence G^∞_ℓ:*

$$\forall\, \ell \in \mathbb{N}^*,\ \forall\, \varphi \in \mathcal{F}_1^{\otimes \ell} \qquad \langle G^N, R_\varphi\rangle \underset{N\to\infty}{\longrightarrow} \langle G^\infty, R_\varphi\rangle.$$

Assume furthermore that f_0^N is f_0-chaotic. Then f_t^N is $S_t^{NL} f_0$-chaotic.

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