ACYCLIC COLOURINGS OF PLANAR GRAPHS WITH LARGE GIRTH

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Abstract

A proper vertex-colouring of a graph is *acyclic* if there are no 2-coloured cycles. It is known that every planar graph is acyclically 5-colourable, and that there are planar graphs with acyclic chromatic number $\chi_a = 5$ and girth $g = 4$. It is proved here that a planar graph satisfies $\chi_a \leqslant 4$ if $g \geqslant 5$ and $\chi_a \leqslant 3$ if $g \geqslant 7$.

1. *Introduction*

An *acyclic colouring* of a graph G is a proper vertex-colouring of G such that every union of two colour classes induces an acyclic subgraph of G, and $\chi_a = \chi_a(G)$ denotes the smallest number of colours in an acyclic colouring of G. Clearly $\chi_a(C) = 3$ if C is a cycle and $\chi_a(F) \leqslant 2$ if F is a forest, with equality unless F is edgeless.

For a planar graph G, Grünbaum [**5**] conjectured that $\chi_a(G) \leqslant 5$ and proved that $\chi_a(G) \leqslant 9$. This bound was sharpened by Mitchem [**9**] to 8, by Albertson and Berman [**1**] to 7, by Kostochka [**7**] to 6, and by Borodin [**3**, **4**] to 5, which is best possible since the double 5-wheel $C_5 + \bar{K}_2$ is planar and (it is easy to see) has $\chi_a = 5$.

The *girth* $g = g(G)$ of a graph G is the length of its shortest cycle. The purpose of the present paper is to prove the following two results, which were partly inspired by J. Nešetřil telling us of Fact 4 (below).

Theorem 1. *If G is planar with girth $g \geqslant 5$ then $\chi_a \leqslant 4$.*

Theorem 2. *If G is planar with girth $g \geqslant 7$ then $\chi_a \leqslant 3$.*

Kostochka and Melnikov [**8**] have constructed planar 2-degenerate bipartite graphs, necessarily with girth $g = 4$, having $\chi_a = 5$. (For example, in $C_5 + \bar{K}_2$, replace each edge uv of C_5 by a copy of $K_{2,4}$ with u, v as the vertices of degree 4.) Thus our condition $g \geqslant 5$ is best possible to imply $\chi_a \leqslant 4$. However, we do not know whether $\chi_a \leqslant 3$ whenever $g \geqslant 6$ (or even $g \geqslant 5$).

Theorems 1 and 2 have several corollaries, in view of the following facts.

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FACT 1 (obvious). If $\chi_a(G) \leqslant k$ then G contains an induced forest on at least $2/k$ of its vertices.

FACT 2 (S. L. Hakimi, J. Mitchem and E. S. Schmeichel (see [**6**])). If $\chi_a(G) \leqslant k$ then $E(G)$ can be partitioned into k 'star forests' (forests in which each component is a star).

FACT 3 (Grünbaum [**5**]). If $\chi_a(G) \leqslant k$ then the *star chromatic number* $\chi_s(G) \leqslant k \cdot 2^{k-1}$.

FACT 4 (Raspaud and Sopena [**10**]). If $\chi_a(G) \leqslant k$ then the *oriented chromatic number* $\chi_o(G) \leqslant k \cdot 2^{k-1}$.

By Fact 2, Borodin's 5-colour theorem implies the truth of the conjecture of Algor and Alon [**2**] that the edges of every planar graph can be partitioned into five star forests. By Facts 3 and 4, it also implies that $\chi_s(G) \leqslant 80$ and $\chi_o(G) \leqslant 80$ for every planar graph G; these bounds remain the best known. For girth $g \geqslant 5$, Theorem 1 gives $\chi_s(G) \leqslant 32$ and $\chi_o(G) \leqslant 32$; for $g \geqslant 7$, Theorem 2 gives $\chi_s(G) \leqslant 12$ and $\chi_o(G) \leqslant 12$.

2. *Preliminaries*

The proofs of the two theorems have a similar structure. In each case we let G be a smallest counterexample to the theorem, which we assume is already embedded in the plane, and we note that clearly G is 2-connected. Our proof uses an application of Euler's formula (Lemma 1) and some structural information derived from the minimality of G (Lemmas 2–5); we then use the method of redistribution of charge in order to obtain a contradiction.

Throughout, G has n vertices, m edges and r faces, the sets of which are denoted by V, E and F respectively. The degree of vertex v is denoted by $d(v)$, a *d-vertex* is a vertex v with $d(v) = d$, and a *$d(b)$-vertex* is a d-vertex that is adjacent to exactly b vertices of degree 2. The number of edges incident to face f is denoted by $r(f)$, and an *r-face* or *$>r$-face* is a face f with $r(f) = r$ or $r(f) > r$, respectively. An (*alternating*) *i,j-path* is a path whose vertices are coloured alternately i and j. A cycle C *separates* two vertices if one of the vertices is inside C and the other is outside C, and a *separating cycle* is a cycle that separates some two vertices. The following lemma holds for every connected planar graph.

LEMMA 1.
(i) $\sum_{v\in V}(3d(v)-10)+\sum_{f\in F}(2r(f)-10) = -20$.
(ii) $\sum_{v\in V}(5d(v)-14)+\sum_{f\in F}(2r(f)-14) = -28$.

Proof. Euler's formula $n-m+r = 2$ can be rewritten in the form $(6m-10n)+(4m-10r) = -20$, which implies (i), and in the form $(10m-14n)+(4m-14r) = -28$, which implies (ii). □

3. *Proof of Theorem* 1 ($g \geqslant 5$)

Let G be a smallest counterexample to Theorem 1. As noted above, G is 2-connected and so has minimum degree at least 2.

LEMMA 2. (i) *No* 2*-vertex is adjacent to a* 2*-vertex or* 3*-vertex.*

(ii) *G contains no* $d(d)$*-vertices* $(2 \leqslant d \leqslant 15)$, *no* $d(d-1)$*-vertices* $(2 \leqslant d \leqslant 9)$ *and no* $d(d-2)$*-vertices* $(3 \leqslant d \leqslant 4)$.

(iii) *If w is a* 5(3)*-vertex, then the three* 2*-vertices occur consecutively in cyclic order round w, and both of the two faces between consecutive* 2*-vertices are* >5*-faces.*

(iv) *If a* 5(2)*-vertex is adjacent to three* 3*-vertices, then it is incident to at least one* >5*-face.*

(v) *A* 5(3) *or* 6(4)*-vertex is not adjacent to any* 3*-vertices.*

Proof. (i): (i) follows immediately from (ii). In proving (ii)–(v), we assume throughout that w is a $d(b)$-vertex with neighbours $v_1, \ldots, v_b, z_1, \ldots, z_{d-b}$ where $v_1, \ldots, v_b$ have degree 2 and are adjacent to $u_1, \ldots, u_b$ respectively. The neighbours of z_i other than w will be referred to as the *outer neighbours* of z_i $(1 \leqslant i \leqslant d-b)$. By the minimality of G, we may suppose that $G-v_1$ has an acyclic 4-colouring $c\colon V\backslash\{v_1\} \longrightarrow \{1,2,3,4\}$ in which without loss of generality $c(w)=1$. If we can convert this into an acyclic 4-colouring of G by colouring v_1 (perhaps after first recolouring some other vertices), then this contradiction will complete the proof. Note that if $c(u_i) \neq c(w)$ then we can give v_i either of the other colours since no 2-coloured cycle can possibly use v_i. Thus we may suppose that $c(u_1)=1$, and that for $j=2,3,4$ there is an alternating $1,j$-path connecting u_1 to w (since otherwise we could set $c(v_1)=j$).

(ii) and (iii): If $b=d<4^2$, then choose a colour j that appears on at most three of $u_1, \ldots, u_b$. Set $c(w)=j$, give the intervening v_i distinct colours not equal to j, and give the remaining v_i any proper colours; this colouring is clearly acyclic.

If $b=d-1<3^2$, then choose a colour $j \neq c(z_1)$ that appears on at most two of $u_1, \ldots, u_b$. Set $c(w)=j$, and proceed as before. If $b=d-2<2^2$, then the same trick works provided that $c(z_1) \neq c(z_2)$, but if $c(z_1)=c(z_2)$ then we dare not recolour w for fear of creating a 2-coloured cycle. However, if at most two of $u_1, \ldots, u_b$ have colour 1, which must be the case if $d-2 \leqslant 2$, then we can colour the corresponding v_i with distinct colours not in $\{1, c(z_1)\}$. This proves (ii), and it also shows that in proving (iii) we may assume that $c(z_1)=c(z_2)=2$, say, and that $c(u_i)=1$ for all i. Hence if v_i, v_j occur consecutively in cyclic order round w, then there is a >5-face between them (otherwise $u_i u_j \in E$).

If the v_i are not consecutive in cyclic order round w, assume that v_1 is between z_1 and z_2. Because of the 1,4-path connecting u_1 to w, there can be no 2,3-path from z_1 to z_2. Thus we may give w colour 3 and the v_i any proper colours. This proves (iii).

(iv): Suppose that $(d,b)=(5,2)$, $d(z_i)=3$ $(i=1,2,3)$ and w is incident to five 5-faces. If $c(u_2)=1$ then, because of the 5-faces, v_1 and v_2 are not consecutive in cyclic order round w, and at most one of z_1, z_2, z_3 has an outer neighbour coloured 1, but this contradicts the existence of the three $1,j$-paths connecting u_1 to w, so we may suppose that $c(u_2) \neq 1$. Then without loss of generality $c(z_i)=i+1$ and z_i has an outer neighbour coloured 1 (because of the $1,(i+1)$-path, $i=1,2,3$). Choose a colour $j \notin \{1, c(u_2)\}$ that occurs on at most one of the outer neighbours of z_1, z_2 and z_3, set $c(w)=j$ and give z_{j-1}, v_1 and v_2 any proper colours.

(v): Suppose that $(d,b)=(5,3)$ or $(6,4)$ and $d(z_1)=3$. First suppose that $c(z_1)=c(z_2)$. If the two outer neighbours of z_1 have the same colour j, we may choose $c(w) \notin \{j, c(z_1)\}$ such that $c(w)$ occurs on at most two of $u_1, \ldots, u_b$; the v_i are now easily coloured. If the two outer neighbours of z_1 have distinct colours, we may recolour first z_1 and then w, and so we may assume from now on that $c(z_1) \neq c(z_2)$, without loss of generality $c(z_i)=i+2$ $(i=1,2)$. If $c(u_i)=1$ for at most one i, put $c(w)=1$ and

$c(v_i) = 2$. The same works with 1 and 2 interchanged, and so we may suppose that $(d, b) = (6, 4)$, $c(u_1) = c(u_2) = 1$ and $c(u_3) = c(u_4) = 2$. If z_1 has no outer neighbour coloured 1, we may put $c(w) = 1$, $c(v_1) = 2$, $c(v_2) = 3$. The same again works with 1 and 2 interchanged, and so we may suppose that z_1 has outer neighbours coloured 1 and 2. Now put $c(z_1) = 4$, $c(w) = 3$ and give $v_1, \dots, v_4$ any proper colours. □

By a *weak* vertex we mean a vertex of degree 2 or 3 or a 4-vertex that is adjacent to both a 2-vertex and a 3-vertex.

LEMMA 3. *Each 3-vertex is adjacent to at most one weak vertex*

Proof. Let w be a 3-vertex adjacent to x, y, z where x, y are weak, with degree 3 or 4 (by Lemma 2(i)). Let the outer neighbours of x (that is, its neighbours other than w) be x_1, x_2 and, if $d(x) = 4$, x_3, where $d(x_3) = 2$ and the other neighbour of x_3 is x_3'. To avoid referring to non-existent vertices, if $d(x) = 3$ add isolated vertices x_3, x_3' to G. Deal with y analogously. Let c be an acyclic 4-colouring of $G - \{w, x_3, y_3\}$. In what follows, whenever we describe how to colour x_3, we assume implicitly that $c(x_3') = c(x)$, since if $c(x_3') \neq c(x)$ then we can use either of the other colours for $c(x_3)$ with impunity; similarly with y_3. Assume that $c(z) = 1$. By interchanging x, y and permuting the other colours if necessary, we have only four cases to consider.

Case 1: $c(x) = 2$, $c(y) = 3$. Set $c(w) = 4$, choose $c(x_3) \notin \{c(x), c(x_1), c(x_2)\}$, and colour y_3 similarly.

Case 2: $c(x) = c(y) = 2$. If $c(x_1) \neq c(x_2)$ and $\{c(x_1), c(x_2)\} \neq \{3, 4\}$, then change $c(x)$ to get case 1. Hence we may assume that $c(x_1) = c(x_2)$ or $\{c(x_1), c(x_2)\} = \{3, 4\}$, and similarly for y_1, y_2. If there is no 2, 3-path connecting x to y, set $c(w) = 3$, if $c(x_1) = c(x_2)$ choose $c(x_3) \notin \{c(x), c(x_1), c(w)\}$, if $\{c(x_1), c(x_2)\} = \{3, 4\}$ set $c(x_3) = 1$, and colour y_3 similarly. We can do the same if there is no 2,4-path connecting x to y; hence we may suppose that both paths exist and $c(x_1) = c(y_1) = 3$, $c(x_2) = c(y_2) = 4$. Now, either the 2,3-path (completed to a cycle through w) separates x_2 from z or the 2,4-path (similarly completed) separates x_1 from z. Suppose the former, so that there is no 1,4-path connecting x_2 to z; set $c(w) = 4$, $c(x) = 1$, $c(x_3) = 2$ and $c(y_3) = 1$.

Case 3: $c(x) = 1$, $c(y) = 2$. If $c(x_1) \neq c(x_2)$ we can change $c(x)$ to get case 1 or case 2. Hence assume that $c(x_1) = c(x_2) \neq 3$ and choose $c(w) = 3$, $c(x_3) \notin \{c(w), c(x), c(x_1)\}$, $c(y_3) \notin \{c(y), c(y_1), c(y_2)\}$.

Case 4: $c(x) = c(y) = 1$. As in case 3, we may suppose that $c(x_1) = c(x_2)$, and similarly $c(y_1) = c(y_2)$. Choose $c(w) \notin (1, c(x_1), c(y_1)\}$, $c(x_3) \notin \{c(w), c(x), c(x_1)\}$ and $c(y_3) \notin \{c(w), c(y), c(y_1)\}$. □

We now show that Lemmas 2 and 3 contradict the supposition that $g \geqslant 5$. Assign a 'charge' of $3d(v) - 10$ units to each vertex v of G and of $2r(f) - 10$ units to each face f of G. By Lemma 1(i), the total charge assigned is negative. We now redistribute the charge, without changing its sum, in such a way that the sum is provably non-negative, and this contradiction will prove the theorem. Note that the charge on each face is non-negative, by the supposition that $r(f) \geqslant g \geqslant 5$, and vertices of degree 2, 3, 4, 5, ... start with charge $-4, -1, 2, 5, \dots$.

The rules for redistribution are as follows:

(R1) Each 2-vertex receives 2 from each adjacent vertex.

(R2) Each 3-vertex receives $\frac{1}{2}$ from each adjacent non-weak vertex.

(R3) Each face f with $r(f) > 5$ and bounding cycle $v_1 v_2 \ldots v_{r(f)} v_1$ gives $\frac{1}{2}$ to each vertex v_i for which $d(v_{i-1}) \leqslant 3$ and $d(v_{i+1}) \leqslant 3$ (subscripts modulo $r(f)$).

It is easy to see that the charge on each face f is still non-negative: by Lemmas 2(i) and 3, the boundary of f cannot contain three consecutive vertices with degree $\leqslant 3$, and so f cannot contribute $\frac{1}{2}$ to two adjacent vertices in its boundary; thus f gives up at most $\frac{1}{4}r(f)$, whereas its initial charge was $2r(f)-10 > \frac{1}{4}r(f)$ if $r(f) > 5$.

It remains to prove that the charge on each vertex v is also non-negative. If $d(v) = 2$ then v started with charge -4 and has gained 4, and so now has charge 0. If $d(v) = 3$ then v started with -1 and has gained at least 1 by Lemma 3, and so it now has non-negative charge. Suppose that $d(v) = 4$, so that v started with charge 2. By Lemma 2(ii) and the definition of a weak vertex, if v is adjacent to a 2-vertex then it gave 2 to only one 2-vertex and nothing to 3-vertices; otherwise it gave $\frac{1}{2}$ to at most four 3-vertices. In either case its charge is still non-negative.

Suppose that $d(v) = 5$, so that v is a $5(b)$-vertex where $b \leqslant 3$ by Lemma 2(ii). If $b = 3$ then, by Lemma 2(iii) and (v), v received $\frac{1}{2}$ from two >5-faces, between pairs of 2-vertices, and gave nothing to 3-vertices; thus v started with charge 5, gave 6 to three 2-vertices, received 1 from faces, and now has 0. If $b = 2$ then v gave 4 to 2-vertices and, by Lemma 2(iv), it either gave at most 1 to 3-vertices or gave $1\frac{1}{2}$ to 3-vertices and received $\frac{1}{2}$ from a >5-face. If $b \leqslant 1$ then v gave at most 2 to a 2-vertex plus 2 to four 3-vertices.

If $d(v) = 6$ then v started with 8 and, by Lemma 2(ii) and (v), gave at most 8, either to four 2-vertices, or to at most three 2-vertices and three 3-vertices. If $7 \leqslant d(v) \leqslant 9$ then, by Lemma 2(ii), v gave to at most $d(v)-2$ 2-vertices and two 3-vertices, making a total of at most $2d(v)-3 \leqslant 3d(v)-10$. Finally, if $d(v) \geqslant 10$ then v gave at most $2d(v) \leqslant 3d(v)-10$. Thus every vertex now has non-negative charge, and this contradiction completes the proof of Theorem 1. □

4. *Proof of Theorem* 2 ($g \geqslant 7$)

Let G be a smallest counterexample to Theorem 2; G is 2-connected, with minimum degree at least 2.

LEMMA 4. (i) *G does not contain two adjacent 2-vertices.*

(ii) *G contains no $d(d)$-vertices* ($2 \leqslant d \leqslant 8$) *or $d(d-1)$-vertices* ($2 \leqslant d \leqslant 4$).

(iii) *No 3-vertex is adjacent to three 3(1)-vertices.*

(iv) *No 3(1)-vertex is adjacent to two 3(1)-vertices.*

Proof. (i) and (ii): With the terminology of Lemma 2, if $b = d < 3^2$ then choose a colour j that occurs on at most two of $u_1, \ldots, u_b$. If $b = d-1 < 2^2$ then choose a colour $j \neq c(z_1)$ that occurs on at most one of $u_1, \ldots, u_b$. In each case, set $c(w) = j$ and proceed as in Lemma 2(ii). This proves (ii), and (i) immediately follows.

(iii): For $i = 1, 2, 3$, let G contain paths $wx_i v_i u_i$ where $d(w) = 3$, $d(v_i) = 2$, x_i has another neighbour y_i, and distinct labels denote distinct vertices. Let c be an acyclic 3-colouring of $G-\{w, v_1, v_2, v_3, x_1, x_2, x_3\}$.

Suppose that we colour w. If $c(w) \neq c(y_i)$, say $c(w) = 1$ and $c(y_i) = 2$, we can colour the path $wx_i v_i u_i$ either 1321 or 1312 or 1313 depending on the colour of u_i, and only in the last case is there an alternating path through x_i; this is a $c(w), c(u_i)$-path and requires w, y_i and u_i to have three different colours. If $c(w) = c(y_i)$ then, by choosing $c(v_i) \neq c(w)$ if $c(x_i) = c(u_i)$, we can ensure that there is only the inevitable $c(w), c(x_i)$-path through $c(x_i)$; this works for either of the two possible choices for $c(x_i)$.

We now colour w as follows; in each case, by the above remarks, we can colour the x_i and v_i so as to create no 2-coloured cycles. If $c(y_i) = 1$, say, for each i, let $c(w)$ be whichever of 2, 3 occurs on more of the u_i (so that the other occurs on at most one u_i). If $c(y_1) = c(y_2) = 1$ and $c(y_3) = 2$, set $c(w) = 3$ unless $c(u_1) = c(u_2) = 2$, in which case set $c(w) = 2$. If $c(y_i) = i$ for each i, set $c(w) = j$ where j is chosen so that $\{j, c(y_i), c(u_i)\} = \{1, 2, 3\}$ for at most one i, and choose $c(x_j) \neq c(u_i)$ if there is such an i.

(iv): This is essentially the same as (iii) with u_3, v_3 removed and $c(u_3)$ interpreted as 1, say, whenever it occurs in the above argument. □

Recall that G has girth $g(G) = g \geqslant 7$. An *r-cycle*, *⩽r-cycle* or *<r-cycle* is a cycle with length $l = r$, $l \leqslant r$ or $l < r$, respectively. A **-cycle* is a separating r-cycle, where $r = 7$ or 8. If G contains a *-cycle, then let S be a *-cycle with as few vertices as possible inside it, and describe every vertex inside S as *distinguished*; otherwise, every vertex of G is *distinguished*.

LEMMA 5. (i) *If a *-cycle C passes through a distinguished vertex, then C is an* 8-*cycle*.

(ii) *If two distinguished* 3(1)-*vertices b_1, b_2 are adjacent then edge $b_1 b_2$ is incident with a* >7-*face*.

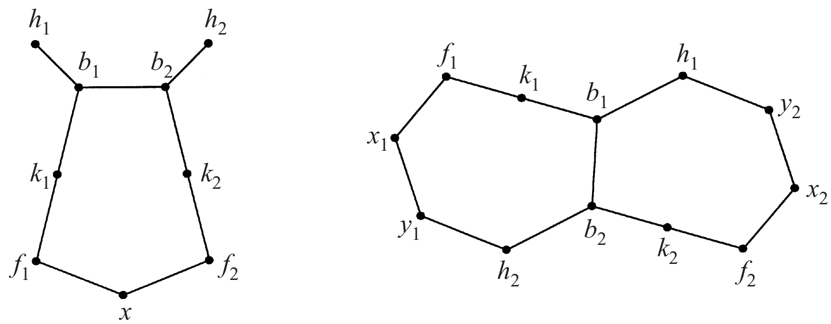
Proof. (i): If such a C exists then clearly S exists and $C \cap S \neq \emptyset$. Suppose that C is a 7-cycle. If only one vertex of C is inside or outside S, then combined with a segment of S it gives a $\leqslant$6-cycle, contradicting $g \geqslant 7$. Thus either two or three vertices of C are inside S, $|V(S)| = 8$, and C splits S into two equal segments, creating two 7-cycles or 8-cycles with fewer vertices inside them than S. Clearly these cycles can have no chords, and since no two 2-vertices of G are adjacent by Lemma 4(i), at least one of the cycles must be separating, contradicting the definition of S.

(ii): For $i = 1, 2$, let b_i be adjacent to h_i and k_i where $d(k_i) = 2$. There are two cases.

Case 1: k_1, k_2 are incident with the same face. Assume that this is labelled as in Figure 1(a). Form G_i from $G' = G - \{k_1, b_1, b_2, k_2\}$ by adding a new 2-vertex z_i adjacent to f_i and h_{3-i} $(i = 1, 2)$.

CLAIM 1. *Either $g(G_1) \geqslant 7$ or $g(G_2) \geqslant 7$.*

Proof. Suppose that $g(G_1) \leqslant 6$ and $g(G_2) \leqslant 6$. Then G' contains paths $f_1 u_1 \ldots h_2$ and $f_2 u_2 \ldots h_1$ of length at most 4. These paths must cross, at a vertex v, say. The distances from v along these paths satisfy $d(v, f_1) + d(v, h_2) \leqslant 4$ and $d(v, f_2) + d(v, h_1) \leqslant 4$ by assumption, and also $d(v, f_1) + d(v, h_1) \geqslant 4$, $d(v, f_2) + d(v, h_2) \geqslant 4$ and $d(v, h_1) + d(v, h_2) \geqslant 4$ because $g(G) \geqslant 7$. It follows that either $d(v, f_1) = d(v, f_2) = 1$ and $d(v, h_1) = d(v, h_2) = 3$, or else all four distances equal 2. In the first case, $v = u_1 = u_2$ and we have a 4-cycle



(a) (b)

FIGURE 1.

unless $v = x$. In the second, $xf_1u_1vu_2f_2x$ is a closed walk of length 6, which contains a $\leqslant 6$-cycle unless $u_1 = u_2 = x$. In either case we may suppose that $u_1 = x$. Then there is a 7-cycle $h_2b_2k_2f_2x\ldots h_2$, which is separating because $d(f_2) \neq d(k_2) = 2$ by Lemma 4(i). This contradicts Lemma 5(i), and so completes the proof of the claim. □

By Claim 1, we may suppose without loss of generality that $g(G_1) \geqslant 7$, which means that G_1 has an acyclic 3-colouring c by the minimality of G. We now show that this can be modified into an acyclic 3-colouring of G. As in Lemma 3, whenever we describe how to colour k_i, we assume implicitly that $c(b_i) = c(f_i)$, since otherwise $c(k_i)$ is uniquely determined and no 2-coloured cycle can possibly use k_i.

Without loss of generality $c(f_1) = 1$. If $c(h_2) \neq 1$, say $c(h_2) = 2$, we can colour $b_1b_2k_2$ so that $h_1b_1b_2k_2$ is coloured 1231, 2313 or 3213, depending on $c(h_1)$. Thus we may suppose that $c(h_2) = c(f_1) = 1$ and, by symmetry, that $c(h_1) = c(f_2) = j$, say. If $j = 1$, set $c(b_1) = 2$, $c(b_2) = 3$. Suppose $j \neq 1$, say $j = 2$. If in G_1, $c(z_1) = 3$, colour $k_1b_1b_2$ with 313. Otherwise, $c(z_1) = 2$, and we colour $k_1b_1b_2$ with 213 or 312 according to whether there is or is not a 1, 2-path connecting h_1 to h_2; note that if there is, then there is no 1, 2-path connecting h_1 to f_1, since there is none in $G_1 - z_1$ connecting f_1 to h_2. Thus in every case we have constructed an acyclic 3-colouring of G, and this contradiction completes the discussion of case (1).

Case 2: k_1, k_2 are not incident with the same face. Assume that the two faces incident to b_1b_2 are labelled as in Figure 1(b).

Let c be an acyclic 3-colouring of $G' = G - \{k_1, b_1, b_2, k_2\}$. If $c(f_1) \neq c(h_2)$, say $c(f_1) = 1$, $c(h_2) = 2$, then we can colour $b_1b_2k_2$ so that $h_1b_1b_2k_2$ is coloured 1231, 2313 or 3213, depending on $c(h_1)$ (with the usual convention about colouring 2-vertices). Thus we may suppose that in every colouring of G', $c(f_1) = c(h_2)$. This means that identifying f_1 with y_1 in G' must create a $\leqslant 6$-cycle, and likewise identifying x_1 with h_2.

Therefore G' contains paths P_1, P_2 of length at most 6 connecting f_1 to y_1 and x_1 to h_2, and P_1 and P_2 must cross, at a vertex v, say. The distances from v along these paths satisfy $d(v,f_1) + d(v,y_1) \leqslant 6$ and $d(v,x_1) + d(v,h_2) \leqslant 6$, and also $d(v,f_1) + d(v,x_1) \geqslant 6$, $d(v,x_1) + d(v,y_1) \geqslant 6$, $d(v,y_1) + d(v,h_2) \geqslant 6$ and $d(v,f_1) + d(v,h_2) \geqslant 4$ because $g(G) \geqslant 7$. It follows that either all four distances equal 3, or else $d(v,f_1) = d(v,h_2) = 2$ and

$d(v, x_1) = d(v, y_1) = 4$. Let C_1, C_2 and C_3 be the three cycles generated by adding $f_1 x_1$, $x_1 y_1$ and $y_1 h_2$, respectively, to $P_1 \cup P_2$, and let C_4 be their mod-2-sum, which is a cycle including v and the path $f_1 x_1 y_1 h_2$. The lengths of $C_1, \ldots, C_4$ are either $7, 7, 7, 9$ or $7, 9, 7, 7$; hence these cycles have no chords. C_4 is certainly separating. Since no two 2-vertices of G are adjacent by Lemma 4(i), either C_1 and C_3 are both separating or C_2 is separating. Either way, each of x_1 and y_1 lies on a separating 7-cycle, and so S exists and, by Lemma 5(i), neither of these vertices is inside S. However, b_1 and b_2 are inside S, and so all vertices in Figure 1(b) are inside S or on S. Hence x_1 and y_1 are on S.

Similarly, x_2 and y_2 are on S. Thus S contains at least two internally disjoint paths between $\{x_1, y_1\}$ and $\{x_2, y_2\}$, at least one of which, say P, has at most two internal vertices. Without loss of generality P connects x_1 to y_2. Then we have a $\leqslant 8$-cycle $x_1 f_1 k_1 b_1 h_1 y_2 P x_1$ which is strictly enclosed in S, and is separating because $d(f_1) \neq d(k_1) = 2$ by Lemma 4(i). This contradicts the definition of S and so completes the proof of Lemma 5. □

We now show that Lemmas 4 and 5 give a contradiction. If G contains a *-cycle, form H from G by deleting all vertices outside S; otherwise let $H = G$. Assign a charge of $5d(v) - 14$ units to each vertex v of H and of $2r(f) - 14$ units to each face f of H. By Lemma 1(ii), the total charge assigned is -28. We now redistribute the charge so that its sum is provably greater than -28, and this contradiction will prove the theorem. Note that the charge on each face is non-negative, by the supposition that $r(g) \geqslant g \geqslant 7$; and vertices of degree $2, 3, 4, 5, \ldots$ start with charge $-4, 1, 6, 11, \ldots$.

Our rules for redistributing the charge are as follows:

(R1) Each distinguished 2-vertex receives 2 from each adjacent vertex.

(R2) Each distinguished 3(1)-vertex receives $\frac{1}{2}$ from each adjacent vertex that is not a distinguished 2-vertex or a distinguished 3(1)-vertex.

(R3) For each pair b_1, b_2 of adjacent 3(1)-vertices, b_1 and b_2 each receive $\frac{1}{2}$ from each >7-face incident to edge $b_1 b_2$.

It is easy to see that the charge on each face f is still non-negative: by Lemma 4(iv) the boundary of f contains at most $\frac{1}{3}r(f)$ pairs of adjacent 3(1)-vertices, and so f gives up at most $\lfloor \frac{1}{3}r(f) \rfloor \leqslant 2r(f) - 14$ if $r(f) > 7$.

We now prove that each distinguished vertex v has non-negative charge. If $d(v) = 2$, then v started with -4 and gained 4, so now has 0. If $d(v) = 3$ then v is a $3(b)$-vertex ($b \in \{0, 1\}$) by Lemma 4(ii). If $b = 0$ then v started with 1 and gave $\frac{1}{2}$ to at most two 3(1)-vertices by Lemma 4(iii). If $b = 1$ let v have neighbours v_1, v_2, v_3 where $d(v_1) = 2$. If v_2, say, is a distinguished 3(1)-vertex then, by Lemma 4(iv), v received $\frac{1}{2}$ from v_3 and $\frac{1}{2}$ from the >7-face incident with edge vv_2 whose existence was proved in Lemma 5(ii); otherwise, v received $\frac{1}{2}$ from each of v_2, v_3. In each case v started with 1, received at least 1 and gave at most 2 to v_1.

If $d(v) = 4$ then v started with 6 and, by Lemma 4(ii), gave up at most 4 to two 2-vertices plus 1 to two 3-vertices. If $d(v) \geqslant 5$, then v gave up at most $2d(v) \leqslant 5d(v) - 15$.

Now we already have a contradiction if $H = G$, when all vertices are distinguished, since in this case the sum of all charges is non-negative. If $H \neq G$ then we must also consider the vertices on S. Each such vertex v has given at most $2(d(v) - 2)$ to distinguished vertices and so now has at least $5d(v) - 14 - 2(d(v) - 2) = 3d(v) - 10$.

This is -4 if $d(v) = 2$, -1 if $d(v) = 3$ and otherwise is positive. Since G is 2-connected, $d(v) > 2$ for at least two $v \in S$, and since $|S| \leqslant 8$ the sum of all the charges, which should be -28, is at least $6 \times (-4) + 2 \times (-1) = -26$. This contradiction completes the proof of Theorem 2. □

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