Utility Representation of an Incomplete Preference Relation*

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Abstract

We consider the problem of representing a (possibly) incomplete preference relation by means of a vector-valued utility function. Continuous and semicontinuous representation results are reported in the case of preference relations that are, in a sense, not "too incomplete." These results generalize some of the classical utility representation theorems of the theory of individual choice, and paves the way towards developing a consumer theory that realistically allows individuals to exhibit some "indecisiveness" on occasion.

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*While I realize that this is highly nonstandard in the profession, I would nevertheless like to dedicate this paper to the glowing memory of my father Ayhan Türker; the bulk of this paper was written by his hospital bed during his last and painful, yet exceptionally courageous days. In addition, I thank James Foster and Mukul Majumdar for bringing to my attention the problem of representing partial orderings by real functions, Jean Pierre Benoit, Juan Dubra, Bezalel Peleg, Ben Polak, and Debraj Ray for their helpful comments, and Joel Spencer for pointing out the connection between this problem and the dimension theory for posets. The comments of the seminar participants at Alicante, Columbia, NYU, Princeton, Rochester, SMU and Yale, and especially those of the two anonymous referees, have improved the exposition substantially. Support from the C. V. Starr Center for Applied Economics at New York University is also gratefully acknowledged.

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1 The Problem and Motivation

Traditionally, a (weak) preference relation on a nonempty choice set X is defined as a complete, reflexive and transitive binary relation (a complete preorder) on X. All of these assumptions are commonly viewed as rationality postulates, the latter two being consistency requirements and the former being a decisiveness prerequisite. As it is of interest to model the behavior of boundedly rational individuals in economics, the implications of relaxing these axioms are naturally motivated. In fact, it appears that the completeness axiom does not really correspond to an unexceptionable trait of rationality, but it is rather useful on the grounds of analytical tractability. As discussed thoroughly by Aumann (1962) and Bewley (1986) among others, there seems to be no *a priori* reason why a rational decision-maker cannot exhibit "indecisiveness" (and not indifference) in certain choice problems; daily practice would provide many instances to this effect.[1] Even for the normative applications of the theory of individual decision making, it is argued elsewhere, the completeness requirement is not all that compelling.[2]

Apart from their realistic appeal, there is another reason why incomplete preferences can be viewed as interesting. Such preferences provide a conservative modeling technique in those situations in which the modeler has only partial information about the preferences of an agent.[3] In such cases it may be interesting to ask what sort of predictions can be derived from what is already known, and this requires one to view the agent "as if" she has incomplete preferences. This position, and a "procedural" way of modeling the same phenomenon, is investigated recently by Dubra and Ok (1999).[4]

[1] The possibility of incompleteness arises when one takes the psychological preferences (and not the revealed preferences) as the primitive of decision theory. As discussed at length by Rabin (1998), there is now mounting evidence that supports this point of view. We also note that the widely observed phenomenon of loss aversion can also be accounted for in a "rational" choice model that allows for incomplete preferences. For an elaborate analysis of these issues, we refer the reader to Mandler (1999).

[2] "Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint. ... For example, certain decisions that [an] individual is asked to make might involve highly hypothetical situations, which he will never face in real life; he might feel that he cannot reach an "honest" decision in such cases. Other decision problems might be extremely complex, too complex for intuitive "insight," and our individual might prefer to make no decision at all in these problems. ... Is it "rational" to force decisions in such cases?" (Aumann, 1962, p. 446). The links between the notion of rationality and incomplete preferences are thoroughly elaborated further in Bewley (1986) and Mandler (1999).

[3] I owe this particular insight to an anonymous referee of this journal.

[4] To illustrate, suppose that an experimenter, who would like to pin down the preferences of an agent, conducts several experiments. The result would be finitely many rankings by the agent. If the experimenter believes (perhaps on the basis of the observed behavior) that the agents' preferences satisfy certain properties (e.g. transitivity), then, a prudent way for her to model the agent's preferences is by using the smallest preorder that extends the observed rankings and that satisfies these properties. This way she could model the preferences of the subject by means of an incomplete preference relation which may well be adequate to derive "safe" behavioral predictions.

To provide an alternative motivation for taking incomplete preference relations seriously, we may think of X as a set of social alternatives and consider a preference relation over this set as a social welfare ordering. Under this interpretation, positing at the outset the completeness of $\succsim$ would again be unduly limiting, for it is only natural to allow social ethics criteria not to be able to rank every social alternative.[5] Moreover, there are many economic instances in which a decision maker is in fact a committee so that the individual choices are to be based on social preferences which are naturally modeled as incomplete. For instance, in coalitional bargaining games, one may specify the preferences of each coalition by using a vector of utility functions (one for each member of the coalition), which forces one to model the preferences of a coalition as an incomplete preorder (cf. Shapley, 1957, and Ray and Vohra, 1997).[6]

Given such considerations, it may perhaps be useful at times to take a (possibly incomplete) preorder as the primitive of analysis. Yet the fact that an incomplete preorder does not admit a utility representation limits the scope of adopting such a viewpoint; this is but the price to pay for relaxing the completeness axiom. It appears to us that the problem of intractability is one of the main reasons why incomplete preference relations are not widely used in choice theory. However, we contend that this problem is not as severe as it first strikes the eye. It is still possible to provide a utility representation for an incomplete preference relation, provided that we suitably generalize the notion of a "utility function." Doing this may be thought of as taking the first step towards developing a satisfactory choice theory that allows for a tractable usage of incomplete preference relations. This is the primary objective of the present paper.

To begin with, let us note that while it is obviously not possible to associate an incomplete preorder $\succsim$ a real function u on X such that $x \succsim y$ iff $u(x) \geq u(y)$ for all $x, y \in X$, we may nevertheless find a function u such that $x \succ (\sim) \, y$ implies $u(x) > (=) \, u(y)$ for all x and y; this approach is explored by Richter (1966, 1971), Peleg (1970), Jaffray (1975) and Sondermann (1980). However, as noted by Majumdar and Sen (1976), such a representation has a serious shortcoming in that it may result in a substantial information loss.[7] Indeed, one cannot recover the original relation

[5] Aumann (1962, p. 447) quotes Shapley on this issue when noting that partial preference orderings are "useful for describing the preferences of groups, since they enable one to distinguish clearly between indecision and indifference." Applied work seems at large to be in concert with this position. After all, the standard social welfare orderings like Pareto dominance and (generalized or ordinary) Lorenz ordering are incomplete preorders (see Fishburn, 1974, and Donaldson and Weymark, 1998, for further discussion.)

[6] Before proceeding any further, we should note that the issue of incomplete preferences is only interesting if we impose transitivity (or some other restriction) on preferences. As noted by a referee, completeness, *in itself*, has no behavioral consequences. For instance, the requirement of transitivity overrules the trivial way of completing a preference relation by declaring incomparable pairs indifferent becomes unacceptable (for then the resulting relation need not be transitive).

[7] To deal with this problem, Majumdar and Sen introduce the notion of "transparent complete representation" of a preorder. This representation concept is, however, rather indirect, and as the authors demonstrate, it has undesirable features relating to the continuity of representation. For instance, the coordinatewise ordering on $\mathbf{R}^n$

$\succsim$ from a Richter-Peleg utility function u; the information contained in u is strictly less than that of $\succsim$. Put differently, the maximization of a Richter-Peleg utility function yields a selection from the choice correspondence of a decision maker, but it does not allow us to determine the choice correspondence entirely. The problem is, of course, due to the fact that the range of a real-valued function is completely ordered while its domain is not.

This suggests that one must either posit the completeness of a preference relation (social welfare ordering etc.) at the outset and thus obtain a convenient numerical representation even though this may be conceptually lacking, or develop a choice theory in terms of possibly incomplete preference orderings even though this may prove analytically difficult due to the absence of a numerical representation. There is, however, a compromise solution. We may allow for the incompleteness of a preference relation and represent it by a *vector-valued* utility function the range of which is contained in some finite dimensional Euclidean space which is naturally incompletely ordered. Indeed, given a preorder $\succsim$, it is sometimes possible to find a function $\mathbf{u} : X \to \mathbf{R}^n$ for some positive integer n such that

$$x \succsim y \text{ if and only if } \mathbf{u}(x) \geq \mathbf{u}(y) \quad \text{for all } x, y \in X. \tag{1}$$

It is natural to say that such a vector-valued function $\mathbf{u}$ "represents" $\succsim$.[8]

This formulation of a utility representation has several advantages. First, it generalizes the usual (Debreu) representation in a very natural manner. Second, it does not cause any information loss; the potential incompleteness of $\succsim$ is fully reflected in the function $\mathbf{u}$ that represents $\succsim$. Third, it makes working with incomplete preference relations analytically less difficult since it is much easier to manipulate vector-valued functions than preorders. For instance, while maximizing a preorder can be an elusive job at times, one may utilize the theory of vector optimization (multi-objective programming) in developing a satisfactory theory of demand which dispenses with the completeness assumption. A representation like (1) may thus link the classical consumer theory to the theory of multicriteria decision making (which is well developed in the operations research literature). This potential application also explains why we require the range of the representing function to be *finite* dimensional. Without this finiteness condition, one loses the substantial economy that such a representation may otherwise provide.

Loosely speaking, our aim in this paper is to demonstrate that under suitable generalizations

lacks a continuous representation in the sense of Majumdar and Sen.

[8] This approach parallels the problem of representing a preorder as the intersection of its completions; see Donaldson and Weymark (1998). More general extension theorems of this sort are obtained by Duggan (1999). Our notion of representation is also alluded to in the seminal von Neumann and Morgenstern (1944, p. 29) in the case of preferences over lotteries: "If the general comparability assumption is not made, a mathematical theory ... is still possible. It leads to what may be described as a many-dimensional vector concept of utility. This is a more complicated and less satisfactory set-up, but we do not propose to treat it systematically here." [This passage is also quoted in Aumann, 1962.]

of the standard assumptions of utility theory one can represent a preference relation by a vector-valued utility function, provided that the preference relation is, in a sense, *not too incomplete.* Put more precisely, we shall show that, under reasonable separability conditions, preorders that are the product of complete preorders, and/or that do not render infinitely many alternatives mutually incomparable, admit a utility representation in the sense of (1). What is more, if one is rather content with an infinite dimensional representation ($x \succsim y$ iff $u(x) \geq u(y)$ for all u that belong to some set $\mathcal{U}$), then this "near-completeness" requirement too can be dropped.

In addition, we shall provide here conditions for the continuity and semicontinuity of these representations thereby generalizing some of the classical representation theorems of the utility theory. Finally, we shall investigate the regularity properties of the choice correspondences induced by continuously representable preference relations, and obtain a generalization of the celebrated maximum theorem. The paper ends with a concluding section in which we outline some open problems that arise from the present exercise, and point to the potential applications of our main results.

2 Preliminary Definitions and Results

2.1 Elements of Poset Theory

Let $\succcurlyeq$ be a (binary) relation on a set X. We say that x and y are **$\succcurlyeq$-comparable** if either $x \succcurlyeq y$ or $y \succcurlyeq x$ holds, otherwise we say that they are **$\succcurlyeq$-incomparable.** In turn, we say that $\succcurlyeq$ is **complete** if all x and y in X are $\succcurlyeq$-comparable, and that it is **incomplete** otherwise. The **strict** (*asymmetric*) **part** of $\succcurlyeq$, denoted $\succ$, is a relation on X defined as $x \succ y$ iff $x \succcurlyeq y$ and $\neg(y \succcurlyeq x)$. The **symmetric part** of $\succcurlyeq$, denoted $\sim$, is defined as $\sim \equiv \succcurlyeq \setminus \succ$. For any $Y \subseteq X$, the relation **induced by $\succcurlyeq$ on** Y, denoted $\succcurlyeq|_Y$, is defined as $\succcurlyeq|_Y \equiv \succcurlyeq \cap (Y \times Y)$. We say that $\succcurlyeq'$ is an **extension** of $\succcurlyeq$, if $\succcurlyeq'$ is a relation on X such that $\succcurlyeq \subseteq \succcurlyeq'$ and $\succ \subseteq \succ'$.

A relation $\succcurlyeq$ is said to be a **preorder** (or a *quasiorder*) if $\succcurlyeq$ is reflexive and transitive, a **partial order** if it is an antisymmetric preorder, a **strict partial order** if it is irreflexive and transitive, and finally, a **linear order** if it is a complete partial order.[9] We say that $(X, \succcurlyeq)$ is a **preordered set** whenever X is any nonempty set and $\succcurlyeq$ is a preorder on X. For any $x, y \in X$, we write $x \bowtie y$ iff x and y are $\succcurlyeq$-incomparable; this defines $\bowtie$ as an irreflexive relation on X. An **open interval** in $(X, \succcurlyeq)$ is any set of the form

$$(a, b)_{\succcurlyeq} \equiv \{x \in X : b \succ x \succ a\}, \qquad a, b \in X.$$

Given another preordered set $(Y, \trianglerighteq)$, a function $f : X \to Y$ is called **isotonic** (*order-preserving*) if

[9] We have $\succcurlyeq \setminus \succ = \{(x, x)\}_{x \in X}$ whenever $\succcurlyeq$ is a partial order and, conversely, $\succ \cup \{(x, x)\}_{x \in X}$ is a partial order whenever $\succ$ is a strict partial order. Consequently, it is immaterial whether one adopts a partial order or a strict partial order as a primitive of analysis.

$x \succcurlyeq y$ implies $f(x) \trianglerighteq f(y)$ for all $x, y \in X$. It is called an **order isomorphism** if it is a bijection such that both f and f^{-1} are isotonic. If such a function exists, we say that $(X, \succcurlyeq)$ and $(Y, \trianglerighteq)$ are **order-isomorphic**.

Let $(X, \succcurlyeq)$ be any preordered set. We say that $Y \subseteq X$ is **$\succcurlyeq$-dense** if, for any $x, y \in X$ with $x \succ y$, there exists a $z \in Y$ such that $x \succ z \succ y$. The preorder $\succcurlyeq$ is then said to be **weakly separable** if there exists a countable $\succcurlyeq$-dense set in X. Clearly, if $\succcurlyeq$ is weakly separable, and $\succ$ is nonempty, then X must be infinite. A well-known result due to Cantor (1895) states that if X is a countable set and $\succcurlyeq$ is a linear order on X such that X is $\succcurlyeq$-dense and does not contain maximum and minimum elements with respect to $\succcurlyeq$, then $(X, \succcurlyeq)$ is order isomorphic to $(\mathbf{Q} \cap (0, 1), \geq)$. A corollary of this fact is that any weakly separable linear order can be represented by a real function.

Many interesting preorders are obtained as a product of a finite number of complete preorders. We say that $\succcurlyeq$ is an **n-dimensional product order** on a set X if $(X_k, \succcurlyeq_k)$ is a complete preorder, $k = 1, ..., n$, $X = \times_{k=1}^{n} X_k$, and

$$(x_1, ..., x_n) \succcurlyeq (y_1, ..., y_n) \text{ iff } x_k \succcurlyeq_k y_k \text{ for all } k = 1, ..., n.$$

In this case we write $(X, \succcurlyeq) = \times_{k=1}^{n} (X_k, \succcurlyeq_k)$. Among the n-dimensional product orders used in economics are the Pareto and stochastic dominance orderings defined on a subset $\mathbf{R}^n$. In decision theory, product orders are particularly relevant in multi-attribute choice problems.

A preordered set $(X, \succcurlyeq)$ is called a **poset** **(***partially ordered set***)** if $\succcurlyeq$ is a partial order on X. A poset $(X, \succcurlyeq)$ is called finite (resp. infinite) when $|X| < \infty$ (resp. $|X| = \infty$). It is called a **$\vee$-semilattice** if every two element subset of X has a lowest upper bound w.r.t. $\succcurlyeq$; as usual, we write $x \vee y$ for $\sup\{x, y\}$. A **$\wedge$-semilattice** is defined similarly. $(X, \succcurlyeq)$ is called a **lattice** if it is both a $\vee$- and a $\wedge$-semilattice. It is readily verified that $x \succcurlyeq y$ iff $x \vee y = x$ in any poset.

A poset $(X, \succcurlyeq)$ is called a **chain** (or a *loset*) if $\succcurlyeq$ is a linear order, and an **antichain** if $\succ = \emptyset$. Clearly, if $(X, \succcurlyeq)$ is a chain, then $\bowtie = \emptyset$, and if it is an antichain, then $\bowtie = \{(x, y) : x \neq y\}$. Abusing the terminology a bit, we mean by a **chain in a poset** $(X, \succcurlyeq)$ a set $Y \subseteq X$ such that the relation $\succcurlyeq|_Y$ is complete. Similarly, a set $Y \subseteq X$ is called an **antichain in a poset** $(X, \succcurlyeq)$ if $\succcurlyeq|_Y = \emptyset$. We say that an antichain Y in $(X, \succcurlyeq)$ is of **maximal cardinality** if the cardinality of Y is at least as large as that of any other antichain in $(X, \succcurlyeq)$. In this case the cardinality of Y is called the **width** of the poset $(X, \succcurlyeq)$, and is denoted by $w(X, \succcurlyeq)$. The width of a poset is a natural measure of the incompleteness of the associated partial order. Clearly, the width of any chain is zero, while the width of any antichain $(X, \succcurlyeq)$ is equal to $|X|$. A fundamental result in the theory of posets is *Dilworth's* (1950) *decomposition theorem* which says that if $(X, \succcurlyeq)$ is a poset with finite width w, then X can be written as the union of w chains in $(X, \succcurlyeq)$.[10]

[10] A simple proof of this theorem when X is finite is given by Perles (1963) who has also shown that the finite width condition is necessary. When X is infinite, one needs to adopt a suitable version of the axiom of choice to prove the result. While Dilworth has used Zorn's Lemma in his original article, Mirsky and Perfect (1966) have proved

For any poset $(X, \succcurlyeq)$, let $\mathcal{L}(X, \succcurlyeq)$ stand for the set of all extensions of $\succcurlyeq$ that are linear orders. By the classic *Szpilrajn's* (1930) *theorem*, any partial order can be extended to a linear order so that we have $\mathcal{L}(X, \succcurlyeq) \neq \emptyset$ for any poset $(X, \succcurlyeq)$. From this observation it follows that

$$\succcurlyeq \ = \bigcap_{R \in \mathcal{L}(X, \succcurlyeq)} R,$$

for any poset $(X, \succcurlyeq)$, that is, every partial order is the intersection of all of its linear extensions. It is not difficult to see that this observation can be generalized to account for preorders (Donaldson and Weymark, 1998).

Following Dushnik and Miller (1941), we define the **order dimension** of a poset $(X, \succcurlyeq)$, denoted $\dim(X, \succcurlyeq)$, as the minimum number of linear extensions of $\succcurlyeq$ the intersection of which is equal to $\succcurlyeq$, provided that this number is finite, and as ∞, otherwise. That is,

$$\dim(X, \succcurlyeq) \equiv \min \left\{ k \in \mathbf{N} : R_i \in \mathcal{L}(X, \succcurlyeq), \ i = 1, ..., k, \text{ and } \succcurlyeq \ = \bigcap_{i=1}^{k} R_i \right\}.$$

It is easy to see that $\dim(X, \succcurlyeq) = 1$ if and only if $(X, \succcurlyeq)$ is a chain. On the other hand, the dimension of any antichain is equal to 2. Indeed, if $(X, \succcurlyeq)$ is any antichain, we have $\succcurlyeq \ = R \cap R'$ where R is any linear order on X (the existence of which is guaranteed by Szpilrajn's theorem) and $R' \equiv \{(x, y) : (y, x) \in R\}$. To give another example, we note that the order dimension of the product of n many chains is exactly n. In particular, for the vector dominance (strong Pareto) ordering $\geq$ on the Euclidean space $\mathbf{R}^n$, $n \in \mathbf{N}$, we have $\dim(\mathbf{R}^n, \geq) = n$.[11]

If X is finite, we trivially have $\dim(X, \succcurlyeq) < \infty$. If, on the other hand, $(X, \succcurlyeq)$ is an infinite poset, then $\dim(X, \succcurlyeq)$ need not be finite; for instance, $\dim(2^A, \supseteq) = \infty$ whenever $|A| = \infty$ (Komm 1948). However, by a theorem of Hiraguchi (1955), we have $w(X, \succcurlyeq) \geq \dim(X, \succcurlyeq)$ for any poset $(X, \succcurlyeq)$. Thus an infinite poset with finite width must have finite dimension. Many other interesting inequalities are obtained for the dimension of a poset in the literature; we refer the interested reader to the excellent surveys by Kelley and Trotter (1982) and Fishburn (1985).

2.2 Order Dimension and Representation of Product Orders

While our main objective is to represent preorders in the sense of (1) here, it will be convenient at times to derive a representation result first for partial orders. Since partial orders (like Pareto dominance) are used frequently in welfare economics, identification of representable partial orders is actually of interest on its own right. Consequently, in what follows, we shall also examine conditions

this theorem by using Rado's selection principle. Several alternative proofs have been since then furnished; the most direct proof that I know deduces the result readily from the finite case by applying Gödel's compactness theorem (Milner, 1990).

[11] In what follows, we adopt the following notation for vector inequalities in $\mathbf{R}^n$: $x \geq y$ iff $x_i \geq y_i$ for all i; $x > y$ iff $x \geq y$ and $x \neq y$; $x \gg y$ iff $x_i > y_i$ for all i.

under which we can associate a partial order $\succcurlyeq$ with a positive integer n and a function $\mathbf{u}: X \to \mathbf{R}^n$ such that

$$x \succcurlyeq y \;\text{ iff }\; \mathbf{u}(x) \geq \mathbf{u}(y) \quad \text{for all } x, y \in X. \tag{2}$$

We will refer to $\succcurlyeq$ as **representable** whenever we can do this. Moreover, $\succcurlyeq$ is called **continuously representable** if (2) holds for some $n \in \mathbf{N}$ and a continuous $\mathbf{u}: X \to \mathbf{R}^n$.

Given this definition, it should be clear that the notion of the order dimension of a poset is closely related to the analysis of representable partial orders (and hence preference relations). To make this connection transparent, we begin by noting the following obvious fact.

Proposition 1. *Let $\succcurlyeq$ be a partial order on a nonempty set X. If $\succcurlyeq$ is representable, then $dim(X, \succcurlyeq) < \infty$. Moreover, if X is countable and $dim(X, \succcurlyeq) < \infty$, then $\succcurlyeq$ is representable.*

Proof. Let (2) hold for some utility function $\mathbf{u}: X \to \mathbf{R}^n$. For each k, write u_k for $\text{proj}_k \mathbf{u}$, and define the strict partial order $\succ_k$ on X as follows: $x \succ_k y$ iff $u_k(x) > u_k(y)$ or $u_i(x) = u_i(y)$, $i = k, ..., s-1 \pmod n$ and $u_s(x) > u_s(y)$ for some $s \pmod n$. We define next $\succcurlyeq_k \equiv \succ_k \cup \{(x,x)\}_{x \in X}$ for each k. It is easy to verify that $\succcurlyeq_k$ is a linear order and that it is an extension of $\succcurlyeq$. Moreover, we have $\succcurlyeq = \cap_{k=1}^n \succcurlyeq_k$ which means that $\dim(X, \succcurlyeq)$ is bounded above by n. The second claim follows from the fact that every linear order on a countable set admits a numerical representation. ■

Thus having finite order dimension is a necessary condition for the representability of a partial order. It is easy to see that the converse of this statement does not hold, however. Indeed, it is well-known that the lexicographic ordering on $\mathbf{R}^2$ is not representable, while, being linear, it is of dimension one.

Proposition 1 suggests that we may be able to represent finite dimensional product orders without much difficulty. This is a useful observation because many partial orders in economics arise as a product of finitely many linear orders. The standard examples include the strong Pareto dominance relation and the stochastic dominance orderings. These partial orders are trivially representable, and this is simply because their projections (on $\mathbf{R}$) are representable linear orders. More generally, it is obvious that $\succcurlyeq$ is continuously representable if $(X, \succcurlyeq) = \times_{k=1}^n (X_k, \succcurlyeq_k)$ and if each $\succcurlyeq_k$ is a continuously representable linear order on X_k. A slightly stronger version of this observation is proved next.

Proposition 2. *Let $n \in \mathbf{N}$ and $\succsim$ be a weakly separable preorder on a nonempty set X such that*

$$(X, \succsim) = \times_{k=1}^n (X_k, \succsim_k)$$

where each X_k is endowed with a topology such that $\{y_k : x_k \succ_k y_k\}$ and $\{y_k : y_k \succ_k x_k\}$ are open for all $x_k \in X_k$. Then, there exists a mapping $\mathbf{u}: X \to \mathbf{R}^n$ that satisfies (1) and which is continuous in the product topology.

Proof. Let Y be a countable $\succsim$-dense set in X, and define the countable set

$$Y_k \equiv \{z_k \in X_k : (z_k, z_{-k}) \in Y \text{ for some } z_{-k} \in X_{-k}\}, \quad k = 1, ..., n.$$

Observe that if $a \succ_k b$ for some $a, b \in X_k$, then $(a, x_{-k}) \succ (b, x_{-k})$ so that there exists a $z \in Y$ such that $(a, x_{-k}) \succ z \succ (b, x_{-k})$. Since it is obvious here that $z_j \sim_j x_j$ for all $j \neq k$, we have $a \succ_k z_k \succ_k b$. But $z_k \in Y_k$ so we may conclude that Y_k is $\succsim_k$-dense set in X_k. Consequently, $\succsim_k$ is a weakly separable complete preorder on X_k and hence, by Lemma II of Debreu (1954), there exists a continuous mapping $v_k : X_k \to \mathbf{R}$ that satisfies $x_k \succsim_k y_k$ iff $v_k(x_k) \geq v_k(y_k)$ for all $x_k, y_k \in X_k$. Defining $u_k(x) \equiv v_k(x_k)$ and letting $\mathbf{u}(x) \equiv (u_1(x), ..., u_n(x))$ for all $x \in X$ yields the claim. ■

Proposition 2 entails the following elementary result in the case of product partial orders.

Corollary 1. *Let $\succcurlyeq$ be a weakly separable partial order on a nonempty set X such that $(X, \succcurlyeq) = \times_{k=1}^{n}(X_k, \succcurlyeq_k)$ where each X_k is endowed with the order topology induced by $\succcurlyeq_k$.*[12] *Then, $\succcurlyeq$ is representable by a mapping $\mathbf{u} : X \to \mathbf{R}^n$ which is continuous in the product topology.*

These results (which are admittedly straightforward implications of what is already known in utility theory) identify the product structure along with weak separability as forces that are capable of yielding satisfactory representation results for incomplete preorders. A natural question at this point is, then, if one can obtain similar results for weakly separable preorders that do not possess a product structure. Our aim in the next section will be to produce some results precisely to this effect.

3 Representation of Near-Complete Preferences

3.1 The Case of Near-Complete Partial Orders

It should be clear that the incomparability part of a product preorder is rather well-behaved; this was in fact the main reason why we were able to obtain the above representation results so easily. Thus, the conditions that should replace the product structure assumed so far should introduce some regularity to the incomparability part of a preorder. One way of doing this is to require that the partial order at hand is not "too incomplete," in the sense that a decision-maker who acts under the guidance of such a preference relation does not appear indecisive very frequently. Of course, there is not an obvious way of defining a "not-too-incomplete" preorder, but it seems quite reasonable to view a preorder as such if any subset A of X, with $x \bowtie y$ for all distinct x, y in A, is finite. In words, this simply says that there do not exist infinitely many alternatives that are *mutually* incomparable for the decision-maker. In the case of finite choice sets, this is of course

[12] The *order topology* on X_k induced by the linear order $\succcurlyeq_k$ is the the topology generated by taking as the basis all the open intervals defined via $\succcurlyeq_k$.

not a restriction at all. In the case of infinite X, on the other hand, it just puts a bound on the amount of indecisiveness an agent may exhibit in his decision making. We shall thus refer to such a preorder in what follows as **near-complete.**

Clearly, a partial order $\succcurlyeq$ on X is **near-complete** if and only if $w(X, \succcurlyeq) < \infty$. This, in turn, motivates our definition of near-completeness further, for the width of a partial order is a natural measure of its incompleteness. Of course, all partial orders defined on a finite set are near-complete, while any linear order is necessarily near complete. On the other hand, an antichain is near-complete if, and only if, it is finite. As for examples, observe that $w(\mathbf{R}^2, \geq) = \infty$ and $w(\mathbf{R} \times \{1, ..., k\}, \geq) = k$ for any $k \in \mathbf{N}$. Thus the vector dominance (coordinatewise) ordering is near-complete on $\mathbf{R} \times \{1, ..., k\}$ but not on $\mathbf{R}^2$.

It is important to note that an agent with a near-complete preference relation may be unable to rank alternatives infinitely many times; near-completeness does *not* mean that one is decisive in all but finitely many cases. Put differently, the set $\{(x, y) : x \bowtie y\}$ may well be infinite even if the preorder in question is near-complete. For example, there are infinitely many $\succcurlyeq$-noncomparable pairs in the poset $(\mathbf{R} \times \{1, 2\}, \geq)$ whose width (and hence its dimension) is 2. Another example will be given after Theorem 1 below.

Since we have $w(X, \succcurlyeq) \geq \dim(X, \succcurlyeq)$ for any poset $(X, \succcurlyeq)$, a near-complete partial order satisfies the finite dimension condition which is necessary for representability (Proposition 1). Consequently, any near-complete partial order is representable on a countable set. In what follows, our aim is to improve upon this preliminary observation. To this end, we need to introduce a final bit of terminology.

Let $(X, \succsim)$ be any preordered set. We say that $Y \subseteq X$ is **upper $\succsim$-dense** if, for any $x, y \in X$ with $x \bowtie y$, there exists a $z \in Y$ such that $x \succ z \bowtie y$, and is called **lower $\succsim$-dense** if, for any $x, y \in X$ with $x \bowtie y$, there exists a $z \in Y$ such that $x \bowtie z \succ y$. In turn, we say that $\succsim$ is **upper (lower) separable** if there exists a countable set in X which is both $\succsim$-dense and upper (lower) $\succsim$-dense. Finally, $\succsim$ is called **separable** if it is both upper and lower separable. For instance, the vector dominance (or any majorization) ordering $\geq$ on $\mathbf{R}^n$ is a separable partial order. The interpretation of separability is analogous to that of weak separability which is a standard axiom in the classical utility theory.[13] Notice that when $\succsim$ is complete, both of the upper and lower separability conditions reduce to the usual requirement that there exists a countable order dense

[13] Intuitively, when $x \bowtie y$ holds, we think of the decision maker as unable to rank x and y. This may be because of the fact that each alternative possesses an attribute superior to the other. In this case it would not be unreasonable to assume that there exists another alternative z which is inferior to x in all relevant attributes, but so slightly inferior that, in those attributes that x dominates y, alternative z also dominates y. For such an alternative z, we would have $x \succ z \bowtie y$, and this is precisely what upper separability requires. (That z must come from a *countable* set is a requirement that limits the richness of the set of indifference and incomparability classes.) The interpretation of lower separability is analogous.

set in X.[14]

Upper separable partial orders are of interest, because any such near-complete order admits a utility representation in the sense of (2). The following is, then, our first main result.

Theorem 1. *Let X be any nonempty set and let $\succcurlyeq$ be a near-complete and upper-separable partial order on X. Then $\succcurlyeq$ is representable.*

The following simple example illustrates the relative strength of this theorem over the preliminary representation results considered in the previous section.

Example 1. Let $X = \{x \in \mathbf{R}^2 : \|x\| = 1 \text{ and } x_1 \neq 0\}$ and define the binary relation $\succcurlyeq$ on X as

$$x \succcurlyeq y \quad \text{iff} \quad \text{sgn}\, x_1 = \text{sgn}\, y_1 \quad \text{and} \quad x_2 \geq y_2.$$

It is readily verified that $\succcurlyeq$ is a partial order on X, but that neither Proposition 1 (for X is not countable) nor Corollary 1 (for $\succcurlyeq$ is not a product order) says anything about the representability of $\succcurlyeq$. Yet $w(X, \succcurlyeq) = 2$ and $X \cap \mathbf{Q}^2$ is both $\succcurlyeq$-dense and upper $\succcurlyeq$-dense in X, and hence we may apply Theorem 1 to conclude that $\succcurlyeq$ is representable. Indeed, the continuous function $\mathbf{u} : X \to \mathbf{R}^2$ defined as $\mathbf{u}(x) = (u_1(x), u_2(x))$, where

$$u_1(x) \equiv \begin{cases} x_2 + 2, & \text{if } x_1 < 0 \\ x_2, & \text{if } x_1 > 0 \end{cases} \quad \text{and} \quad u_2(x) \equiv \begin{cases} x_2 + 2, & \text{if } x_1 > 0 \\ x_2, & \text{if } x_1 < 0 \end{cases}$$

for all $x \in X$, represents $\succcurlyeq$. $\|$

3.2 Semicontinuous Representation

Continuity of the representing utility function in our multidimensional context is not so easy to guarantee. In the standard theory where the representing utility function takes values in $\mathbf{R}$, continuity matters are handled by appealing to the Debreu's *open gap lemma* which states that there exists a strictly increasing real function g defined on any nonempty subset S of the real line such that each component of $\mathbf{R} \backslash g(S)$ is either a singleton or an open interval. Indeed, once a utility representation is found for a preference relation defined on X, say u, we may conclude that $g \circ u$ is a continuous representation of this relation in any topology with respect to which all upper and lower contour sets of $\succ$ are open, where $g : u(X) \to \mathbf{R}$ is given by Debreu's lemma. Unfortunately, this line of logic breaks in our setting since it is not at all obvious how the open gap lemma could be modified, if at all, in the case of vector-valued functions. To make the nature of the difficulty clear, we will show below (in Example 2) that even a preference relation that can be represented

[14] It is clear that separability is a vacuous property for *finite* preorders. But the representation problem that we study here is really interesting only when $|X| = \infty$, because *any* finite preorder $\succsim$ is representable (since $\succsim$ is the intersection of all of its linear extensions and there are only finitely many such extensions when $|X| < \infty$.)

by two utility functions one of which is continuous and the other is upper semicontinuous, need not be continuously representable.

Another classical theorem of utility theory (due to Rader, 1963) gives sufficient conditions for the upper (and lower) semicontinuous representation of a complete preference relation. This is quite useful because upper semicontinuity is often adequate for purposes regarding the maximization of utility functions. Fortunately, Rader's approach can be modified in the present context to ensure the upper semicontinuous representation of incomplete preference relations. In particular, we have the following result.

Theorem 2. *Let X be a topological space and let $\succcurlyeq$ be a near-complete and separable partial order on X such that $\{y : x \succ y\}$ is open for all $x \in X$. Then there exists an upper semicontinuous mapping $\mathbf{u} : X \to [0,1]^n$ such that (2) holds and $n = w(X, \succcurlyeq)$.*

While the detailed proofs of these results are provided in Section 3.5, it may be useful here to indicate the basic line of argument which, in effect, combines the largely known techniques of utility theory with the additional structure brought in by the property of upper/lower separability. The crucial role played here by this property is hardly surprising. It is well known that weak separability is crucially linked to the utility representation of a complete preference relation, so, arguably, the question in the present setting is really to find the "appropriate" notion of separability for partial orders.

The proofs proceed along the following lines. By using upper separability we may find a set of weakly separable partial orders that extend the original relation $\succcurlyeq$ in such a way that their intersection equals $\succcurlyeq$. By near-completeness (and hence Dilworth's theorem) this set of extensions can be chosen to be finite. But it is known that any such extension can further be extended to a preorder that admits a utility representation. Putting these observations together yields Theorem 1. Following the original approach of Rader, Theorem 2 is, on the other hand, obtained by using a suitable limsup of the real functions found in Theorem 1. Interestingly, however, lower separability is used in establishing not only the order-preservation of these new functions, but also their upper semicontinuity.

3.3 The Case of Near-Complete Preorders

Since we would like to identify a preference relation with a preorder in this paper, we need to generalize the previous representation results to the case of preorders. The standard method of doing this is by passing to the quotient sets, and this is precisely what the next corollary does.

Corollary 2. *Let X be a topological space and let $\succsim$ be a near-complete and separable preorder on X such that $\{y : x \succ y\}$ is open for all $x \in X$. Then there exist a positive integer n and an upper semicontinuous mapping $\mathbf{u} : X \to [0,1]^n$ such that (1) holds.*[15]

[15] It is easy to check that the representation is unique up to strictly increasing transformations; that is, $\mathbf{v} : X \to \mathbf{R}^m$

Proof. Let $X/\sim$ denote the quotient set $\{[x]_\sim : x \in X\}$ where $[x]_\sim \equiv \{y \in X : x \sim y\}$ for all $x \in X$. Endow $X/\sim$ with the quotient topology, and define the partial order $R(\succsim)$ on the quotient set $X/\sim$ as $[x]_\sim R(\succsim)\ [y]_\sim$ iff $x \succsim y$. It is easily checked that we may apply Theorem 2 to $R(\succsim)$ to find an upper semicontinuous mapping $\mathbf{v} : X/\sim \to \mathbf{R}^{w(X,R(\succsim))}$ such that $[x]_\sim R(\succsim)\ [y]_\sim$ iff $\mathbf{v}([x]_\sim) \geq \mathbf{v}([y]_\sim)$ for all $x, y \in X$. Letting $p : X \to X/\sim$ be the identification map (i.e. $p(x) = [x]_\sim$ for all x) and setting $\mathbf{u} \equiv \mathbf{v} \circ p$ we are done. ■

As an immediate corollary of this observation, we obtain the following classical result.

Corollary 3. (Rader, 1963) *Let X be a topological space and let $\succsim$ be a weakly separable and complete preorder on X such that the set $\{y : x \succ y\}$ is open for all $x \in X$. Then there exists an upper semicontinuous mapping $u : X \to [0, 1]$ such that $x \succsim y$ iff $u(x) \geq u(y)$ for all $x, y \in X$.*[16]

Proof. Completeness of $\succsim$ allows us to apply Corollary 2 to find an $n \geq 1$ and an upper semicontinuous $\mathbf{u} : X \to [0, 1]^n$ such that $x \succsim y$ iff $\mathbf{u}(x) \geq \mathbf{u}(y)$ for all $x, y \in X$. Define $u \equiv \sum_{k=1}^{n} \text{proj}_k \mathbf{u}$. ■

We would have obtained a lower semicontinuous representation result if we asked for the openness of all the upper contour sets of $\succ$ instead of the openness of its lower contour sets in Corollary 2.[17] However, it appears quite difficult to turn these results to continuous representation results. The following simple example illustrates what may go wrong in this regard. (See Peleg (1971), Jaffray (1975), and Sondermann (1980) for similar examples.)

Example 2. Let $X = [0, 1]^2$ and define the maps $f, g : [0, 1]^2 \to \mathbf{R}$ by $f(x) := x_1 x_2$ and $g(x) := \begin{cases} f(x), & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$. Consider next the preorder $\succsim$ defined on X by $x \succsim y$ iff $(f(x), g(x)) \geq (f(y), g(y))$. By definition, $\succsim$ admits a vector-valued utility representation. Furthermore, one of the two utilities used in this representation is continuous and the other is upper semicontinuous. Yet, $\succsim$ *cannot* be represented continuously. To see this, suppose that there exists a set of continuous functions $\mathcal{U}$ in $\mathbf{R}^X$ such that $x \succsim y$ iff $u(x) \geq u(y)$ for all $\mathcal{U}$. We must then have $u(1, 0) = u(1/m, 0)$ for all $m \in \mathbf{N}$ and all $u \in \mathcal{U}$. Therefore, by continuity, we must have $u(1, 0) = u(0, 0)$ for all $u \in \mathcal{U}$, which contradicts $(0, 0) \succ (1, 0)$. ‖

The problem of obtaining a general continuous utility representation theorem for preorders is left here (and this is not due to lack of effort) as an open problem.

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also satisfies (1) if, and only if, there exists an $\mathbf{f} : \mathbf{u}(X) \to \mathbf{R}^m$ such that (i) $\mathbf{v} = \mathbf{f} \circ \mathbf{u}$, and (ii) for all $a, b \in \mathbf{u}(X)$, $a \geq b$ holds iff $\mathbf{f}(a) \geq \mathbf{f}(b)$. (To prove the "only if" part of this claim, we would define $\mathbf{f} : \mathbf{u}(X) \to \mathbf{R}^m$ by $\mathbf{f}(a) := v(x_a)$, where x_a is an arbitrary member of $\mathbf{u}^{-1}(a)$.)

[16] It is well known that the weak separability requirement can be omitted in this statement provided that X has a countable basis.

[17] This is proved by means of a straightforward modification of the proof of Theorem 2.

3.4 Representation without Near-Completeness

In Theorems 1 and 2 (and hence in Corollary 1) we have used the property of near-completeness in a crucial way. This may cause some discomfort in that the divisibility assumption widely used in economic models often render the near-completeness assumption too demanding. However, it turns out that this property is essential only insofar as one is interested in finding a representation of the form (1) with $\mathbf{u}$ having *finitely* many component maps. If we do not demand a finite dimensional representation, then one can prove the following result which provides a general method of representing a partial order. This result can of course be extended to the case of preorders, by passing to quotient spaces as in the previous section.

Theorem 3. *Let X be any nonempty set and let $\succcurlyeq$ be an upper-separable partial order on X. Then there exists a set $\mathcal{U} \subseteq \mathbf{R}^X$ such that, for each $x, y \in X$, we have*

$$x \succsim y \quad \text{if and only if} \quad u(x) \geq u(y) \quad \text{for all } u \in \mathcal{U}. \tag{3}$$

If $\succcurlyeq$ is separable, and X is endowed with a topology that makes $\{y : x \succ y\}$ open for all $x \in X$, then all members of $\mathcal{U}$ can be chosen upper semicontinuous.

Before providing the proofs of the above representation theorems, we should note that these results are "sufficiency" theorems. Since weak separability is *not* a necessary condition for representability of a linear order, it is clear that none of these results provide us with necessary conditions for the utility representation of partial orders. There is a sense in which this is problematic, for in applications, one would like to use "utilities" instead of partial orders, and hence it would be useful to know exactly what a vector-valued utility representation entails in terms of preferences. The applicability of our representation results instead rests on the weakness of the sufficiency conditions they provide. For instance, if one believes in upper separability alone, then Theorem 3 assures him/her of the existence of a utility representation, and hence justifies adopting vector-valued utilities in applied work. This point is facilitated further by the fact that it is often very easy to check whether or not the preferences implied by the adopted utilities satisfy upper separability. Having said this, however, we note that the absence of the necessity counterparts of our representation results is certainly a shortcoming that should be remedied in future work.

3.5 Proofs of Theorems 1 and 2

We begin with the following reformulation of the Szpilrajn's theorem.

Lemma 1. *Let $(X, \succcurlyeq)$ be any poset. If $\succcurlyeq$ is weakly separable, then there exists a function $u : X \to [0, 1]$ such that $x \succ y$ implies $u(x) > u(y)$ for all $x, y \in X$.*

Proof.[18] If $\succ = \emptyset$, then there is nothing to prove, so let $\succ \neq \emptyset$, and observe that there must exist $a, b \in Y$ such that $b \succ a$, where Y is a countable $\succcurlyeq$-dense set in X. Consequently, $\{(a,b)_{\succcurlyeq} : a, b \in Y$ and $b \succ a\}$ is a countably infinite set which we enumerate as $\{(a_k, b_k)_{\succcurlyeq}\}_{k=1}^{\infty}$. For any $(a_k, b_k)_{\succcurlyeq}$, the set $(a_k, b_k)_{\succcurlyeq} \cap Y$ is partially ordered by $\succcurlyeq$ so that, by Hausdorff Maximal Principle, it contains a maximal linearly ordered subset Z_k. This set must be nonempty since $b_k \succ z_0 \succ a_k$ holds for some $z_0 \in Y$. Proceeding inductively, we observe that Z_k is an infinite set since it must contain a sequence $(z_m)_{m \in \mathbf{Z}}$ that satisfies $b_k \succ z_{m+1} \succ z_m$ for $m \in \mathbf{Z}_+$ and $z_{m+1} \succ z_m \succ a_k$ for $m \in \mathbf{Z}_-$. Thus, Z_k is countably infinite, does not contain maximum or minimum elements, and, by its maximality, it is $\succcurlyeq$-dense in itself. By Cantor's theorem, therefore, there exists a bijection $f_k : Z_k \to (0,1) \cap \mathbf{Q}$ such that $x \succcurlyeq y$ iff $f_k(x) \geq f_k(y)$ for all $x, y \in Z_k$. Define next the mapping $\varphi_k : X \to [0,1]$ by letting $\varphi^k(x) \equiv 0$ if there does not exist a $t \in Z_k$ with $x \succ t$, and by letting

$$\varphi_k(x) \equiv \sup\{f_k(t) : x \succ t \in Z_k\}$$

otherwise. Clearly, we have $\varphi_k(x) = 0$ for all $x \preccurlyeq a_k$ and $\varphi_k(x) = 1$ for all $x \succcurlyeq b_k$. Using this observation and the definition of f_k, one can verify that, for any $x, y \in X$ with $x \succ y$ we have $\varphi_k(x) \geq \varphi_k(y)$. To complete the proof, then, define $u(x) \equiv \sum_{k=1}^{\infty} 2^{-k} \varphi_k(x)$ for all $x \in X$, and notice that $x \succ y$ implies $\varphi_k(x) \geq \varphi_k(y)$ for all k, and that there exist $b_j, a_j \in Y$ such that $x \succ b_j \succ a_j \succ y$ so that $\varphi_j(x) = 1 > 0 = \varphi_j(y)$. Consequently, $x \succ y$ implies $u(x) > u(y)$. ■

Proofs of Theorems 1 and 2. Since $(X, \succcurlyeq)$ is a poset with finite width, say n, we may apply Dilworth's theorem to decompose $(X, \succcurlyeq)$ into n many disjoint chains, that is, there exist a partition of X, say $X_1, ..., X_n$, such that $\succcurlyeq|_{X_k}$ is a linear order on X_k for each k. Fix any k and define the relation R_k on X as follows:

$$x \, R_k y \quad \text{iff} \quad x \notin X_k \text{ and } x \bowtie a \succcurlyeq y \text{ for some } a \in X_k.$$

Define next the relation $\trianglerighteq_k$ on X as

$$\trianglerighteq_k \, \equiv \, \succcurlyeq \cup R_k.$$

The first order of business is to prove that $\trianglerighteq_k$ is a weakly separable partial order. Reflexivity of $\trianglerighteq_k$ is immediate from that of $\succcurlyeq$. To see antisymmetry, let $x \trianglerighteq_k y \trianglerighteq_k x$ and observe that transitivity of $\succcurlyeq$ implies the impossibility of $x \succcurlyeq y \, R_k x$ and $x \, R_k y \succcurlyeq x$. If, on the other hand, $x \, R_k y \, R_k x$, then $x \bowtie a \succcurlyeq y \bowtie b \succcurlyeq x$ for some $a, b \in X_k$. Since $\succcurlyeq|_{X_k}$ is complete, a and b are $\succcurlyeq$-comparable, and this implies that either a is $\succcurlyeq$-comparable to x or b is comparable to y, contradiction. Thus we must have $x \succcurlyeq y \succcurlyeq x$ and $x = y$ obtains from antisymmetry of $\succcurlyeq$. Transitivity of $\trianglerighteq_k$ is also similarly verified by distinguishing between four cases that the statement $x \trianglerighteq_k y \trianglerighteq_k z$ entails. Finally, to

[18] There are several standard methods of proving this lemma, a version of which is first proved by Richter (1966). We adopt here the approach used by Peleg (1970) in a slightly different setting, and provide a short proof. Richter (1971) provides two other methods of proof.

check weak separability, we use upper separability of $\succcurlyeq$ to find a countable subset Y of X which is both $\succcurlyeq$-dense and upper $\succcurlyeq$-dense. We claim that Y is $\trianglerighteq_k$-dense in X. To prove this, let $x \rhd_k y$ where $\rhd_k$ is the strict part of $\trianglerighteq_k$. If $x \succ y$, then $\succcurlyeq$-denseness of Y readily guarantees the existence of a $z \in Y$ with $x \rhd_k z \rhd_k y$. Assume then that $\neg(x \succ y)$ so that $x \, R_k y$. By definition of R_k, then, we must have $y \neq x \notin X_k$ and $x \bowtie a \succcurlyeq y$ for some $a \in X_k$. By upper $\succcurlyeq$-denseness of Y, there exists a $z \in Y \backslash \{y\}$ such that $x \succ z \bowtie a \succcurlyeq y$. Since $\succcurlyeq|_{X_k}$ is complete and $a \in X_k$, we must have $z \notin X_k$, and hence $z \trianglerighteq_k y$. But since $\neg(x \succ y)$, we have $z \neq y$, and this establishes $x \rhd_k z \rhd_k y$ as we sought.

Given that $\trianglerighteq_k$ is a weakly separable partial order on X, we may use Lemma 1 to find a function $\varphi_k : X \to [0,1]$ such that

$$x \rhd_k y \Rightarrow \varphi_k(x) > \varphi_k(y), \qquad \text{for all } x, y \in X, \; k = 1, ..., n. \tag{4}$$

In particular, we have

$$x \succ y \Rightarrow \varphi_k(x) > \varphi_k(y), \qquad \text{for all } x, y \in X, \; k = 1, ..., n. \tag{5}$$

Now if $x \bowtie y$ for any $x, y \in X$, then $x \neq y$, and since $\{X_k : k = 1, ..., n\}$ partitions X and $\succcurlyeq|_{X_k}$ is complete for each k, there must exist distinct i and j such that $x \in X_i \backslash X_j$ and $y \in X_j \backslash X_i$. Thus, $x R_j y R_i x$ so that, by (4), we have $\varphi_j(x) > \varphi_j(y)$ while $\varphi_i(y) > \varphi_i(x)$. Combining this with (5) and letting $\mathbf{u}(x) \equiv (\varphi_1(x), ..., \varphi_n(x))$ for all $x \in X$ completes the proof of Theorem 1.

To complete the proof of Theorem 2, we define next the mapping $u_k : X \to [0,1]$

$$u_k(a) \equiv \begin{cases} \varphi_k(a), & \text{if } U(a) = \emptyset \\ \inf_{b \in U(a)} \sup_{t \in L(b)} \varphi_k(t), & \text{otherwise} \end{cases} \tag{6}$$

where $U(a) \equiv \{b : b \succ a\}$ and $L(a) \equiv \{t : a \succ t\}$ for all $a \in X$. Now take any $x, y \in X$. If $x \succ y$, then, by weak separability, there exists a $z \in U(y)$ with $x \succ z$. Thus, by definition of u_k and (5),

$$u_k(x) \geq \varphi_k(x) > \varphi_k(z) \geq \sup_{t \in L(z)} \varphi_k(t) \geq u_k(y). \tag{7}$$

We may then conclude that $x \succ y$ implies $u_k(x) > u_k(y)$ for each k. Consider next the case $x \bowtie y$. By lower separability, there exists a $z \in X$ such that $x \bowtie z \succ y$. Since $X = X_1 \cup \cdots \cup X_n$ and $\succcurlyeq|_{X_k}$ is complete, it must be the case that $z \in X_k$ and $x \notin X_k$ for some $k = 1, ..., n$. Consequently, $x \, R_k z$ and thus $\varphi_k(x) > \varphi_k(z)$ by (4). But then since we also have $z \in U(y)$, all of the inequalities in (7) hold and we obtain $u_k(x) > u_k(y)$ for some k. Since $\bowtie$ is symmetric, we dually find that $u_\ell(y) > u_\ell(x)$ for some $\ell \neq k$. We may thus conclude that $x \succ (=) \, y$ holds if and only if $u_k(x) > (=) \, u_k(y)$ for each k, which establishes (2). Therefore, if we can prove that each u_k is upper semicontinuous, the proof will be complete by letting $\mathbf{u}(x) = (u_1(x), ..., u_n(x))$ for all $x \in X$.

We claim that $u_k^{-1}[\theta, 1]$ is closed for all real θ. Since this follows trivially when $\theta \leq 0$ or $\theta \geq 1$, we consider only the case $\theta \in (0,1)$. Take any net $(x_\alpha)_{\alpha \in \mathbb{D}}$ in X such that $x_\alpha \to a$ and $u_k(x_\alpha) \geq \theta$

for all $\alpha \in \mathbb{D}$. If $a \succcurlyeq x_\alpha$ for some $\alpha \in \mathbb{D}$, then we have $u_k(a) \geq u_k(x_\alpha) \geq \theta$. Assume then that $x_\alpha(\succ \cup \bowtie)a$ for all $\alpha \in \mathbb{D}$. This ensures that $U(a) \neq \emptyset$ due to lower separability of $\succcurlyeq$. Pick any $b \in U(a)$, and observe that, since $L(b)$ is open, there exists an $\alpha(b) \in \mathbb{D}$ such that $x_{\alpha(b)} \in L(b)$. So, $\sup_{t\in L(b)} \varphi_k(t) \geq u_k(x_{\alpha(b)}) \geq \theta$ for all $b \in U(a)$, and this implies that

$$u_k(a) = \inf_{b\in U(a)} \sup_{t\in L(b)} \varphi_k(t) \geq \theta.$$

Hence u_k is upper semicontinuous for each k, and the proof is complete. ■

3.6 Proof of Theorem 3

Since the proof of the second statement in this theorem is essentially identical to the argument we gave above for Theorem 2, we shall omit it here. To prove the first claim, we let $\mathcal{U}$ be the set of all $u \in \mathbf{R}^X$ such that $x \succ y$ implies $u(x) > u(y)$ for all $x, y \in X$. By Lemma 1, this set is nonempty. We wish to show that (3) holds for this particular choice of $\mathcal{U}$. To this end, pick any $x, y \in X$ such that $x \bowtie y$. Let Z be a countable $\succcurlyeq$- and upper $\succcurlyeq$-dense set in X, and define $S \equiv \{z \in Z : x \succ z \bowtie y\}$ which is a countably infinite set by upper separability. Define next

$$R \equiv \{(t, y) \in X^2 : t = x \text{ or } t \in S\}$$

and let

$$\trianglerighteq \equiv \text{ tran}(\succcurlyeq \cup R)$$

where $\text{tran}(\cdot)$ is the operator of transitive closure. $\trianglerighteq$ is easily checked to be a partial order. In addition, we claim that Z is $\trianglerighteq$-dense in X. To see this, say $a \vartriangleright b$ for some $a, b \in X$ so that there exist $c_1, ..., c_m \in X$, $m \geq 2$, such that

$$a = c_1 \ (\succcurlyeq \cup R) \ c_2 \ (\succcurlyeq \cup R) \ \cdots \ (\succcurlyeq \cup R) \ c_{m-1} \ (\succcurlyeq \cup R) \ c_m = b.$$

If $c_i \succ c_j$ for some i and j here, the claim follows readily. So assume instead that $(c_i, c_{i+1}) \in R\backslash \succ$ for each $i = 1, ..., m-1$. This implies that either $c_1 \in S$ and $c_2 = y$ so that $x \succ c_1 \bowtie c_2$, or that $(c_1, c_2) = (x, y)$ so that $x = c_1 \bowtie c_2$. In the former case by using the upper $\succcurlyeq$-denseness of Z, we can find a $z \in Z$ such that $x \succ c_1 \succ z \bowtie c_2 = y$. But then $z \in S$ so that $c_1 \succ z \ R \ c_2$, which implies that $a = c_1 \vartriangleright z \vartriangleright c_2 \trianglerighteq b$. Since precisely the same reasoning applies to the second case, we may conclude that $\trianglerighteq$ is a weakly separable partial order on X. We now use Lemma 1 again to find a $u \in \mathbf{R}^X$ that represents $\trianglerighteq$ in the sense of Peleg-Richter. Since $\trianglerighteq$ is an extension of $\succcurlyeq$, it must be the case that $u \in \mathcal{U}$. Moreover, since $x \ R \ y$, we have $u(x) > u(y)$. But interchanging the roles of x and y in the above argument we could find another function $v \in \mathcal{U}$ with $v(x) < v(y)$. This proves the theorem. ■

4 Decision Making with Incomplete Preferences

We now briefly turn to the question of modeling the decision making of an individual (or a social planner) who maximizes an incomplete preference relation. A key observation in this regard is that the problem of maximizing an incomplete preorder that actually admits a utility representation can simply be regarded as the problem of maximizing a vector-valued function. Being as such the problem leads itself to a satisfactory formal analysis. In what follows, we sketch the preliminaries of such an analysis, hopefully paving the way towards an extended choice theory that allows for incomplete preferences.

Let $\succsim$ be a preference relation (that is a preorder) on a topological space X. As is conventional, by a *choice problem* we mean here a nonempty compact subset of X, and denote the set of all choice problems by $\mathcal{C}(X)$. We define the **choice correspondence** $C_{\succsim}$ induced by $\succsim$ then as the function that maps any given nonempty compact subset of X to the set of all $\succsim$-maximal elements in this set. That is, we define

$$C_{\succsim}(Z) \equiv \{x \in Z : y \succ x \text{ for no } y \in Z\}, \quad Z \in \mathcal{C}(X).$$

The set $C_{\succsim}(Z)$ is sometimes called the *efficiency frontier* (or the *undominated subset*) of Z. A useful way of thinking about $C_{\succsim}(Z)$ is viewing it as determining the alternatives $Z \backslash C_{\succsim}(Z)$ as the set of all alternatives which would not be chosen by a rational individual whose preferences are summarized by $\succsim$. One may claim that this is an inadequate prediction regarding the actual choices of the decision maker who may have other means of choosing an alternative in $C_{\succsim}(Z)$ (like hiring a consultant, or maintaining the status quo, or postponing the decision etc.). The preliminary analysis we conduct presently ignores such means. Our point of view (which is the prominent one in the theory of multicriteria decision making) is that understanding the structure of the efficiency frontier is essentially a prerequisite for a satisfactory investigation of the second stage problem of choosing an alternative from $C_{\succsim}(Z)$.

Our analysis here therefore concerns the structure of the choice correspondence $C_{\succsim}$. As one would expect, this mapping has rather desirable properties in the case of *representable* preference relations.

Proposition 3. *Let X be a topological space and let $\succsim$ be a preorder on X which is representable by an upper semicontinuous vector-valued utility function. Then:*

(i) *The choice correspondence $C_{\succsim}$ is nonempty and compact-valued.*

(ii) *If $\succsim$ is near-complete, then there can be at most finitely many incomparable alternatives in $C_{\succsim}(Z)$ for any $Z \in \mathcal{C}(X)$.*

Proof. Let an upper semicontinuous mapping $\mathbf{u} : X \to \mathbf{R}^n$ represent $\succsim$ and let $Z \in \mathcal{C}(X)$. By upper semicontinuity of $\mathbf{u}$ and compactness of Z, we have $\emptyset \neq \arg\max_{x \in Z} \sum \text{proj}_k \mathbf{u} \subseteq C_{\succsim}(Z)$. Moreover, by upper semicontinuity of $\mathbf{u}$, $\{x : \mathbf{u}(y) > \mathbf{u}(x)\}$ is an open set in X for each $y \in Z$.

This means that $\{x : y \succ x\}$ is an open set and hence $\{x : \neg(y \succ x)\} \cap Z$ is a closed subset of the compact Z for each $y \in Z$. Thus $A(y) \equiv \{x \in Z : \neg(y \succ x)\}$ is a compact set in X. But $C_{\succsim}(Z) = \cap_{y \in Z} A(y)$ and hence $C_{\succsim}(Z)$ must be compact, which proves (i). The proof of (ii), on the other hand, follows readily from the definitions. ■

To be able to discuss the continuity of choice correspondences, we need to introduce a topology for $\mathcal{C}(X)$. The most standard way of doing this is to endow $\mathcal{C}(X)$ by the **Vietoris topology** which is generated by the basis $\{\langle O_1, ..., O_m \rangle : O_i$ is open in X, $i = 1, ..., m$, $m \in \mathbf{N}\}$, where

$$\langle O_1, ..., O_m \rangle \equiv \{Z \in \mathcal{C}(X) : Z \subset \cup_{i=1}^m O_i \text{ and } Z \cap O_i \neq \emptyset,\ i = 1, ..., m\}$$

for each positive integer m. For an extensive discussion of the suitability of Vietoris topology for studying the continuity properties of correspondences, we refer the reader to Michael (1951) and Klein and Thompson (1984). It should suffice here to note that if X is a compact metric space, then Vietoris topology reduces to the topology induced by the familiar Hausdorff metric.

The following example shows that $C_{\succsim}$ need not be closed or upper hemicontinuous even when the topology of X is well-behaved and $\succsim$ is representable by a continuous vector-valued utility function.

Example 3. Let $X = [0, 1]^2$ and consider the following sequence of compact sets:

$$Z_m \equiv \{(a, b) \in X : b \leq m(1 - a)\}, \quad m \geq 1.$$

It is readily computed that $C_{\geq}(Z_m) = \{(a, b) : 1 - 1/m \leq a \leq 1 \text{ and } b = m(1 - a)\}$ so that $C_{\geq}(Z_m) \to \{1\} \times [0, 1]$ (w.r.t. the Hausdorff metric), whereas $Z_m \to X = [0, 1]^2$ and $C_{\geq}(X) = \{(1, 1)\}$. Thus, $C_{\geq}$ does not have a closed graph. Since $C_{\geq}$ is closed-valued, this also implies that it is not upper hemicontinuous. $\|$

This observation is obviously discouraging. Yet there is an interesting way out of this problem, and that is by strengthening slightly the nature of the representability requirement. Let us say that $\mathbf{u} : X \to \mathbf{R}^n$ **properly represents** the preorder $\succsim$ if, for any $x, y \in X$, we have

$$x \left\{ \begin{array}{c} \succ \\ \sim \\ \bowtie \end{array} \right\} y \quad \text{implies} \quad \mathbf{u}(x) \left\{ \begin{array}{c} \gg \\ = \\ \| \end{array} \right\} \mathbf{u}(y)$$

where we write $a \| b$ for any two coordinatewise noncomparable vectors a and b in $\mathbf{R}^n$. It is readily verified that if $\mathbf{u}$ properly represents $\succsim$, then it also represents it in the sense of (1), but not conversely. Moreover, these two notions of utility representation coincide by reducing to the standard notion of utility representation in the case of complete preference relations.

As for examples, we note that any near-complete and upper separable partial order on any nonempty set X is properly representable. (It is in fact this stronger claim that we demonstrated when proving Theorem 1). Similarly, any preference relation that satisfies the hypotheses of Corollary 2 is properly representable. For another example, we note that the weak Pareto ordering on $\mathbf{R}^n$ is properly representable, while the strong Pareto ordering is *not* properly representable although it admits a utility representation in the sense of (1).

Our interest in proper representability stems from the fact that this notion allows one guarantee the upper hemicontinuity of the choice correspondence $C_{\succsim}$. Indeed, we have the following result which amounts to a generalization of Berge's classical *maximum theorem.*

Theorem 4. *Let X be a topological space and let $\succsim$ be a preorder on X which is properly representable by a continuous vector-valued utility function. Then the choice correspondence $C_{\succsim}$ is nonempty, compact-valued, closed and upper hemicontinuous (in the Vietoris topology).*

Proof. The first two claims are proved in Proposition 3. To prove that $C_{\succsim}$ is closed, observe that we can write, for each $Z \in \mathcal{C}(X)$,

$$C_{\succsim}(Z) \equiv \{x \in Z : \mathbf{u}(y) \gg \mathbf{u}(x) \text{ for no } y \in Z\},$$

where $\mathbf{u} : X \to \mathbf{R}^n$ is continuous and properly represents $\succsim$. Pick an arbitrary $Z \in \mathcal{C}(X)$, and let $x \notin C_{\succsim}(Z)$ so that there exists a $y \in Z$ such that $\mathbf{u}(y) \gg \mathbf{u}(x)$. Since u is upper semicontinuous, we can find an open neighborhood V of x such that $\mathbf{u}(y) \gg \mathbf{u}(V)$. Furthermore, by lower semicontinuity of u, we can find an open neighborhood U of y such that $\mathbf{u}(U) \gg \mathbf{u}(V)$. Fix O to be any open set in X that contains Z, and define $\mathcal{O} \equiv \langle O, U \rangle$. We have $Z \in \mathcal{O}$ because $Z \cap U \neq \emptyset$ (since $y \in Z \cap U$) and $Z \cap O = Z$. Consequently, $\mathcal{O}$ is an open neighborhood of Z. Moreover, by definition, we have $Z' \cap U \neq \emptyset$ for all $Z' \in \mathcal{O}$. It follows from this observation that $C_{\succsim}(Z') \cap V = \emptyset$ for all $Z' \in \mathcal{O}$. We may thus conclude that $C_{\succsim}$ is a closed correspondence on $\mathcal{C}(X)$. Since the identity correspondence (mapping $\mathcal{C}(X)$ to $\mathcal{C}(X)$) is trivially continuous, the closedness and compact-valuedness of $C_{\succsim}$ along with Theorem 7 of Berge (1963, p. 112) entail that $C_{\succsim}$ is upper hemicontinuous. ■

An immediate corollary of Theorem 4 is the well known fact that the choice correspondence that maps any compact utility possibility set to its weakly Pareto efficient subset is upper hemicontinuous, while, by Example 3, the same cannot be said for strong Pareto efficiency. The classical maximum theorem is also an immediate corollary of this result.[19] Another interesting implication concerns the theory of vector optimization. For, Theorem 4 guarantees the closedness of the correspondence

$$Z \mapsto \{x \in Z : \mathbf{f}(y) \gg \mathbf{f}(x) \text{ for no } y \in Z\}$$

[19] For any topological spaces X and A, Berge's maximum theorem states that the correspondence $M : A \rightrightarrows X$ defined as $M(a) \equiv \arg\max_{x \in \Gamma(a)} f(x)$ is compact-valued and upper hemicontinuous, provided that $f : X \to \mathbf{R}$ is a continuous function and $\Gamma : A \rightrightarrows X$ is a compact-valued and upper hemicontinuous correspondence. This statement is proved readily by defining $\Gamma^* : A \to \mathcal{C}(X)$ as $\Gamma^*(a) = \Gamma(a)$, and $C_f(Z) : \mathcal{C}(X) \rightrightarrows X$ as $C_f(Z) \equiv \{y \in Z : f(y) \geq f(x)$ for all $x \in Z\}$, and by using Theorem 4 along with the fact that $M(a) = C_f(\Gamma^*(a))$ for all $a \in A$.

for any $\mathbf{f} : X \to \mathbf{R}^n$, where X is a topological space. More importantly, however, this theorem provides support to the claim that the continuously and properly representable preference relations may well lead to a satisfactory theory of rational choice, and therefore, motivates further investigation of the matter. We note again that the representation results reported in Section 3 fall short of providing a solution for the continuous representation problem (and this sort of a representation is used in Theorem 4).

5 Conclusion and Further Research

In this paper we have argued that utility theory can be beneficially extended to cover incomplete preference relations. This seems to be the first step towards a consumer theory which does not precondition individuals to be decisive at all times. Of course, the predictions that stem from such a theory are likely to be less precise, but this does not mean that one loses all predictive power by allowing for incomplete preferences, especially when incompleteness is bounded by other conditions, such as near-completeness.

Our aim here was, however, not to provide a formal support to this claim, but rather to carry out a utility representation exercise in the case of incomplete preference relations. The analysis leads us to believe that there is good reason to extend this investigation to other parts of choice theory at large, for instance, to the theory of decision making under uncertainty and the classical theory of demand.

As a more concrete problem that relates directly to the present paper, we note that our results fall short of determining necessary conditions for the functional representation of preorders. Perhaps more importantly, none of our results (with the exception of the elementary Proposition 2) provide a continuous representation theorem for incomplete preferences. Identifying a useful set of conditions on preorders that would lead to such a representation result is an open problem worthy of investigation.

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