The Explicit Chaotic Representation of the powers of increments of Lévy Processes

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Abstract

An explicit formula for the chaotic representation of the powers of increments, $(X_{t+t_0} - X_{t_0})^n$, of a Lévy process is presented. There are two different chaos expansions of a square integrable functional of a Lévy process: one with respect to the compensated Poisson random measure and the other with respect to the orthogonal compensated powers of the jumps of the Lévy process. Computationally explicit formulae for both of these chaos expansions of $(X_{t+t_0} - X_{t_0})^n$ are given in this paper. Simulation results verify that the representation is satisfactory. The CRP of a number of financial derivatives can be found by expressing them in terms of $(X_{t+t_0} - X_{t_0})^n$ using Taylor's expansion.

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1 Introduction

The chaotic representation of a square integrable functional of a Lévy process is an expansion via its expectation plus a sum of iterated stochastic integrals, see (1) for a recent review of such representations. There are two different types of chaos expansions: Ito (2) proved a Chaotic Representation Property (CRP) for any square integrable functional for a general Lévy process. This representation is written using multiple integrals with respect to a two-parameter random measure associated with the Lévy process. Nualart and Schoutens (3) proved the existence of a new version of the CRP, which states that every square integrable Lévy functional can be represented as its expectation plus an infinite sum of stochastic integrals with respect to the orthogonalized compensated power jump processes of the underlying Lévy process. Benth et al. (4) and Solé et al. (1) derived the relationships between these two representations. However, these representations are computationally intractable. For the powers of increments, $(X_{t+t_0} - X_{t_0})^n$, of a Lévy process, we instead derive computationally explicit formulae for the integrands of these two chaotic expansions. Hence we have all the results necessary to construct arbitrarily accurate computational formulae for the Lévy functionals themselves.

Power jump processes are important in mathematical finance. Barndorff-Nielsen and Shephard (5) performed hypothesis tests on exchange data under the null of no jumps and found that the tests were rejected frequently. In fact, at intraday scales, prices move essentially by jumps and even at the scale of months, the discontinuous behavior cannot be ignored in general. Only after coarse-graining their behavior over longer time scales do we obtain something similar to Brownian motion. Jumps can be understood both in terms of a Poisson random measure, or equivalently, by using the Power jump processes. Note that Nualart and Schoutens (3, Proposition 2) proved that all square integrable random variables, adapted to the filtration generated by the Lévy process denoted by X, can be represented as a linear combination of powers of increments of X, see Section 2.1 below. In fact, for any square integrable random variable, F, with derivatives of all order, we can apply Taylor's Theorem to express F in terms of a polynomial of powers of increments of X. Thus, the CRP of a number of financial derivatives can be found using this method, as is discussed further in Section 5.

The derivation of an explicit formula for the CRP has been the focus of considerable study, see for example (6), (7), (8) and (9). All the explicit formulae for general Lévy functionals derived in these papers use the Malliavin type derivatives to derive explicit representations of stochastic processes for applications in finance. The derivative operator D is, in all of these cases, defined by its action on the chaos expansions. In other words, the explicit chaos expansion must in fact be known before D can be applied to find the explicit form of the predictable or chaotic representation, thus yielding a circular specification. For example, Léon et al. (7, Definition 1.7) defined the derivative of F in the l-direction by

$$D_t^{(l)}F = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} \sum_{k=1}^{n} 1_{\{i_k=l\}} J_{n-1}^{(i_1,\ldots,\widehat{i_k},\ldots,i_n)} \left(f_{i_1,\ldots,i_n}(\cdots,t,\cdots)\, 1_{\Sigma_n^{(k)}(t)}(\cdot) \right),$$

and Løkka (8, Section 3) defined the derivative operator by

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}\left(f_n(\cdot,t,z)\right),$$

where

$$I_n(f_n) = \int_{[0,T]^n \times \mathbb{R}_0^n} f_n(t_1,...,t_n,z_1,...,z_n)\, \mathrm{d}(\mu - \pi)^{\otimes n}.$$

(Please refer to the corresponding papers for notation). Note that both of these definitions require the knowledge of the functions $\{f_{i_1,\ldots,i_n}\}$'s or $f_n(t_1,...,t_n,z_1,...,z_n)$'s, which are the integrands of the chaos expansion of F.

Jamshidian (10) extended the CRP in (3) to a large class of semimartingales and derived the explicit representation of the power of a Lévy process with respect to the corresponding non-compensated power jump processes, which is discussed further after Theorem 4 in this paper. Note that Lévy processes are included in the class of semimartingales, see (11, Corollary 2.3.21, p.92). Our formula gives the explicit representation with respect to the orthogonalized compensated power jump processes. Our result is therefore complementary to Jamshidian's formula, since our explicit formula gives the CRP with respect to the orthogonalized processes, as defined by Nualart and Schoutens (3).

In practical applications, it is often convenient to truncate the representation given by the PRP. The truncated representation of a stochastic process would yield a practically implementable approximation to the stochastic process. This approximation would be used for simulating the process, or with a finite number of traded higher order options, providing pricing formulae. The truncation would be chosen with minimal variance constraint. The advantage of expressing the sum in terms of stochastic integrals with respect to the orthogonalized processes is that the error terms omitted will be uncorrelated with the terms remained in the approximation. Jamshidian's result holds for general semimartingales (a larger class than ours) but our formula is designed for those with compensators equal to a constant times t only (which is satisfied by all Lévy processes). Our results can be easily extended to semimartingales when the form of the compensators is known.

The rest of the paper is arranged as follow: Section 2 gives the background information about the CRP for Lévy processes. We give the explicit formulae for the CRP for $(X_{t+t_0} - X_{t_0})^n$ of a Lévy process in terms of power jump processes in Section 3 and in terms of Poisson random measure in Section 4. We show that in the Lévy case, our formula complements Jamshidian's formula. Section 5 gives the representation of a common kind of Lévy functionals with the use of Taylor's Theorem. Simulation results for the explicit formulae are given in Section 6. In Section 7, some concluding remarks are provided. Proofs and plots are included as appendices at the end.

2 Background

2.1 Lévy processes and their properties

We give a brief account of Lévy processes and refer the reader to the work by Sato (12) for a more detailed account. A real-valued cádlág stochastic process $X = \{X_t, t \geq 0\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ on $\mathbb{R}^d$ is called *Lévy process* if X has stationary and independent increments with $X_0 = 0$, where $\mathcal{F}$ is the filtration generated by $X : \mathcal{F}_t = \sigma\{X_t, 0 \leq s \leq t\}$. Denote the *left limit process* by $X_{t-} = \lim_{s\to t, s<t} X_s$, $t > 0$, and the *jump size* at time t by $\Delta X_t = X_t - X_{t-}$.

A Lévy process is fully specified by its characteristic function. Let $\phi_{X_1}(u)$ be the *characteristic function* of the Lévy process at $t = 1$, X_1, that is, $\phi_{X_1}(u) = E\left[e^{\mathrm{i}uX_1}\right]$. The characteristic function of X_t is then given by $\left(\phi_{X_1}(u)\right)^t$ since the distribution of a Lévy process is infinitely divisible, see (12, chapter 2). The cumulant characteristic function $\psi(u) = \log \phi_{X_1}(u)$ is often called the *characteristic exponent*, which satisfies the *Lévy-Khintchine formula*:

$$\psi(u) = \mathrm{i}\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} \left(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux 1_{\{|x|<1\}}\right) \nu(\mathrm{d}x), \tag{1}$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\setminus\{0\}$ with $\nu(\{0\}) = 0$ and

$$\int_{-\infty}^{+\infty} \left(1 \wedge x^2\right) \nu(\mathrm{d}x) < \infty.$$

In general, a Lévy process consists of three independent components: a linear deterministic component, a Brownian component and a pure jump component. The Lévy measure $\nu(\mathrm{d}x)$ dictates the jump process: jumps of sizes in set A occur according to a Poisson process with intensity parameter $\int_A \nu(\mathrm{d}x)$. To model a generic Lévy process, only γ, σ and a form for $\nu(\mathrm{d}x)$ need to be specified.

In the rest of the paper, we assume that all Lévy measures concerned satisfy, for some $\varepsilon > 0$ and $\lambda > 0$,

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|) \nu(\mathrm{d}x) < \infty. \tag{2}$$

This condition implies that for $i \geq 2$, $\int_{-\infty}^{+\infty} |x|^i \nu(\mathrm{d}x) < \infty$, and that the characteristic function $E[\exp(\mathrm{i}uX_t)]$ is analytic in a neighborhood of 0.

Denote the i-th *power jump process* by $X_t^{(i)} = \sum_{0<s\leq t} (\Delta X_s)^i$, $i \geq 2$, and for completeness let $X_t^{(1)} = X_t$. In general, it is not true that $X_t = \sum_{0<s\leq t} \Delta X_s$; this holds only in the bounded vari-

ation case, with $\sigma^2 = 0$. By definition, the quadratic variation of X_t, $[X,X]_t = \sum_{0<s\le t}(\Delta X_s)^2 = X_t^{(2)}$ when $\sigma^2 = 0$. The power jump processes are also Lévy processes and jump at the same time as X_t, but with jump sizes equal to the i-th powers of those of X_t, see (3).

Clearly $E[X_t] = E[X_t^{(1)}] = m_1 t$, where $m_1 < \infty$ is a constant and by (13, p.32), we have

$$E[X_t^{(i)}] = E[\sum_{0<s\le t}(\Delta X_s)^i] = t\int_{-\infty}^{\infty} x^i \nu(\mathrm{d}x) = m_i t < \infty, \quad \text{for } i \ge 2, \tag{3}$$

thus defining m_i. Nualart and Schoutens (3) introduced the *compensated power jump process* (or *Teugels martingale*) of order i, $\left\{Y_t^{(i)}\right\}$, defined by

$$Y_t^{(i)} = X_t^{(i)} - E[X_t^{(i)}] = X_t^{(i)} - m_i t \quad \text{for } i = 1,2,3,.... \tag{4}$$

$Y_t^{(i)}$ is constructed to have a zero mean. It was shown by Nualart and Schoutens (3, Section 2) that there exist constants $a_{i,1}, a_{i,2}, ..., a_{i,i-1}$ such that the processes defined by

$$H^{(i)} = Y^{(i)} + a_{i,i-1}Y^{(i-1)} + \cdots + a_{i,1}Y^{(1)}, \tag{5}$$

for $i \ge 1$ are a set of pairwise strongly orthogonal martingales, and this implies that for $i \ne j$, the process $H^{(i)}H^{(j)}$ is a martingale, see (7). For convenience, we define $a_{i,i} = 1$. Nualart and Schoutens proved that this strong orthogonality is equivalent to the existence of an orthogonal family of polynomials with respect to the measure

$$\mathrm{d}\eta\,(x) = \sigma^2 \mathrm{d}\delta_0\,(x) + x^2\nu(\mathrm{d}x),$$

where $\delta_0\,(x) = 1$ when $x = 0$ and zero otherwise, that is, the polynomials p_n defined by

$$p_n\,(x) = \sum_{j=1}^{n} a_{n,j} x^{j-1}$$

are orthogonal with respect to the measure η:

$$\int_{\mathbb{R}} p_n\,(x)\, p_m\,(x)\,\mathrm{d}\eta\,(x) = 0, \quad n \ne m.$$

We now state some key related results in the representation of stochastic processes given in (3).

- *Denseness of polynomials* ((3, Proposition 2)): Let $\mathcal{P} = \{X_{t_1}^{k_1}(X_{t_2} - X_{t_1})^{k_2}\cdots(X_{t_n} - X_{t_{n-1}})^{k_n} : n \ge 0,\ 0 \le t_1 < t_2 < \cdots < t_n, k_1, ..., k_n \ge 1\}$ be a family of stochastic processes. Then we have that $\mathcal{P}$ is a total family in $L^2\,(\Omega, \mathcal{F}_T, P)$, that is, the linear subspace spanned by $\mathcal{P}$ is dense in $L^2\,(\Omega, \mathcal{F}_T, P)$; each element in $L^2(\Omega, \mathcal{F})$ can be represented as a linear combination of elements in $\mathcal{P}$.

- *Chaotic Representation Property* (CRP): Every random variable F in $L^2(\Omega, \mathcal{F})$ has a representation of the form

$$F = E(F) + \sum_{j=1}^{\infty} \sum_{i_1,...,i_j \geq 1} \int_0^{\infty} \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,...,i_j)}(t_1,...,t_j) \mathrm{d}H_{t_j}^{(i_j)} \cdots \mathrm{d}H_{t_2}^{(i_2)} \mathrm{d}H_{t_1}^{(i_1)}, \quad (6)$$

where the $f_{(i_1,...,i_j)}$'s are functions in $L^2(\mathbb{R}_+^j)$. This result means that every random variable in $L^2(\Omega, \mathcal{F}_T, P)$ can be expressed as its expectation plus an infinite sum of zero mean stochastic integrals with respect to the orthogonalized compensated power jump processes of the underlying Lévy process. Note that this representation does not explicitly allow for calculation of the integrands.

- *Predictable Representation Property* (PRP) : From the CRP stated above, we note that every random variable F in $L^2(\Omega, \mathcal{F}_T, P)$ has a representation of the form

$$F = E[F] + \sum_{i=1}^{\infty} \int_0^{\infty} \phi_s^{(i)} \, \mathrm{d}H_s^{(i)}, \quad (7)$$

where $\phi_s^{(i)}$'s are predictable, that is, they are $\mathcal{F}_{s-}$-measurable.

2.2 Jamshidian's notation

In (10), which extends the CRP to semimartingales, the power jump processes and compensators were denoted and defined differently from (3). The power jump processes were defined in (10) by

$$[X]_t^{(2)} = [X^c]_t + \sum_{s \leq t} (\Delta X_s)^2 \text{ and } [X]_t^{(n)} = \sum_{s \leq t} (\Delta X_s)^n \text{ for } n = 3, 4, 5, ..., \quad (8)$$

where $[X^c]_t = [X]_t^c$ is the continuous finite-variation (not martingale) part of $[X]_t^{(2)}$. Note that Jamshidian suppressed the time index t, but we add it here for clarification. Jamshidian denoted the compensator of $[X]_t^{(n)}$ by $\langle X \rangle_t^{(n)}$. The compensator, $\langle X \rangle_t^{(n)}$, is the predictable right-continuous finite variation process such that $[X]_t^{(n)} - \langle X \rangle_t^{(n)}$ is a uniformly integrable martingale. The compensated power jump process, denoted by $X_t^{(n)}$, is thus defined by

$$X_t^{(n)} = [X]_t^{(n)} - \langle X \rangle_t^{(n)} \text{ for } n = 2, 3, 4, ... \quad (9)$$

For Lévy processes, the compensators have the form $m_i t$, where $m_1 t = E[X_t]$ and $m_i t = t \int_{-\infty}^{\infty} x^i \nu(\mathrm{d}x)$ for $i = 2, 3, 4, ...$ However, for semimartingales, the general form of the compensators is not known.

3 The chaotic representation with respect to power jump processes

In this section we firstly derive the explicit formulae for the CRP when F in equation (6) is the power increment of a pure jump Lévy process and extend it to a general Lévy process.

3.1 Pure jump case

Let us first outline the form of the representation to introduce the reader to the flavour of results in this paper. Suppose $t_0 \geq 0$ and let G_t be a pure jump Lévy process with no Brownian part (that is, $\sigma^2 = 0$), $G_t^{(i)}$ be its i-th power jump process and $\hat{G}_t^{(i)}$ be its i-th compensated power jump process. Based on the structure of the expressions for $(G_{t+t_0} - G_{t_0})^3$ and $(G_{t+t_0} - G_{t_0})^4$ calculated using equations (A.4)-(A.6), we desire to derive a general formula for $(G_{t+t_0} - G_{t_0})^k$, $k = 1, 2, 3, ...$, as this forms a starting point for the representation of X_t. We notice that the number of stochastic integrals in each of the above representation is less than the possible full representation specified in the CRP by Nualart and Schoutens (3):

$$\begin{aligned}(X_{t+t_0} - X_{t_0})^k &= f^{(k)}(t, t_0) + \sum_{j=1}^{k} \sum_{\substack{(i_1,...,i_j) \\ \in \{1,...,k\}^j}} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \cdots \int_{t_0}^{t_{j-1}-} f^{(k)}_{(i_1,...,i_j)}(t, t_0, t_1, ..., t_j) \\ &\quad \times \mathrm{d}Y_{t_j}^{(i_j)} \cdots \mathrm{d}Y_{t_2}^{(i_2)} \mathrm{d}Y_{t_1}^{(i_1)},\end{aligned}$$

where the $f^{(k)}_{(i_1,...,i_j)}$'s are deterministic functions in $L^2\left(R_+^j\right)$ and Y's are defined in equation (4). In $(G_{t+t_0} - G_{t_0})^2$, we have $\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \mathrm{d}\hat{G}_{t_2}^{(1)} \mathrm{d}\hat{G}_{t_1}^{(1)}$, $\int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(1)}$ and $\int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(2)}$, that we shall represent via the list $\{(1,1), (1), (2)\}$. We can do an equivalent representation of $(G_{t+t_0} - G_{t_0})^3$ and $(G_{t+t_0} - G_{t_0})^4$ to get the following two lists:

$$\begin{aligned}&\{(1,1,1), (1,1), (1,2), (2,1), (1), (2), (3)\}. \\ &\{(1,1,1,1), (1,1,1), (1,1,2), (1,2,1), (2,1,1), \\ &(1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (1), (2), (3), (4)\}.\end{aligned}$$

In general, the list of the orders of the compensated power jump processes of the stochastic integrals in $(G_{t+t_0} - G_{t_0})^k$ depends on the collection of numbers

$$\mathcal{I}_k = \left\{ (i_1, i_2, ..., i_j) \mid j \in \{1, 2, ..., k\},\ i_p \in \{1, 2, ..., k\} \text{ and } \sum_{p=1}^{j} i_p \leq k \right\}. \tag{10}$$

This construction is explained in the beginning of the proof of Theorem 1 (Appendix C) using induction. A typical element $(i_1, i_2, ..., i_j)$ in $\mathcal{I}_k$ therefore indexes a multiple stochastic integral j-times repeated with respect to the power jump processes with powers $i_1, i_2, ..., i_j$ and indexed

$t_1, t_2, ..., t_j$. That is, $(i_1, i_2, ..., i_j)$ indexes the integral

$$\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \cdots \int_{t_0}^{t_{j-1}} \mathrm{d}\hat{G}_{t_j}^{(i_1)} \cdots \mathrm{d}\hat{G}_{t_2}^{(i_{j-1})} \mathrm{d}\hat{G}_{t_1}^{(i_j)}.$$

Next we consider the terms in the representation not involving any stochastic integrals. That is, in $(G_{t+t_0} - G_{t_0})^2$, $m_1^2 t^2 + m_2 t$ is considered; in $(G_{t+t_0} - G_{t_0})^3$, $m_1^3 t^3 + 3m_1 m_2 t^2 + m_3 t$ is considered, and in $(G_{t+t_0} - G_{t_0})^4$, $m_1^4 t^4 + 6m_1^2 m_2 t^3 + \left(4m_1 m_3 + 3m_2^2\right) t^2 + m_4 t$ is considered. We use equations (A.4)-(A.6), given in Appendix A, to derive the representation. This time the representation can be simplified a great deal since we are not considering any stochastic integrals. Denote the terms which do not contain any stochastic integrals in $(G_{t+t_0} - G_{t_0})^k$ by $C_t^{(k)}$.

Proposition 1 $C_0^{(r)} = 0$ for all r, $C_t^{(0)} = 1$, $C_t^{(1)} = m_1 t$, and for $k = 2, 3, 4, ...$,

$$C_t^{(k)} = \sum_{j=1}^{k-1} \binom{k}{j} m_j t C_t^{(k-j)} - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_0^t t_1 \, \mathrm{d}C_{t_1}^{(k-j)} + m_k t. \tag{11}$$

Proof. The results for $C_0^{(r)}$ and $C_t^{(0)}$ are trivial. For $k = 1$, $(G_{t+t_0} - G_{t_0}) = \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(1)} + m_1 t$ and hence $C_t^{(1)} = m_1 t$. For $k \geq 2$, the terms in equation (A.4) are equal to zero since G_t has no Brownian part ($\sigma^2 = 0$). The first term in equation (A.5) contains a stochastic integral and hence from the second term of equation (A.5) and equation (A.6), we have

$$C_t^{(k)} = \sum_{j=1}^{k-1} \binom{k}{j} m_j (t + t_0) C_t^{(k-j)} - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_{t_0}^{t+t_0} t_1 \, \mathrm{d}C_{t_1-t_0}^{(k-j)} + m_k t.$$

Putting $u = t_1 - t_0$ in the second term, we have

$$\begin{aligned} C_t^{(k)} &= \sum_{j=1}^{k-1} \binom{k}{j} m_j (t + t_0) C_t^{(k-j)} - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_0^t (u + t_0) \, \mathrm{d}C_u^{(k-j)} + m_k t \\ &= \sum_{j=1}^{k-1} \binom{k}{j} m_j t C_t^{(k-j)} - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_0^t t_1 \, \mathrm{d}C_{t_1}^{(k-j)} + m_k t. \end{aligned}$$

Note that $C_t^{(k)}$ is independent of t_0. □

Thus, given Proposition 1, $C_t^{(k)}$ can be expressed in terms of m_i's for any given k and easily coded. We will show in the followings that in the calculation of $(G_{t+t_0} - G_{t_0})^k$, all the $C_t^{(j)}$'s, $j = 0, 1, ..., k$ are required. In fact the coefficients of the stochastic integrals in the representation depend only on $C_t^{(j)}$'s, $j = 0, 1, ..., k$, as stated in Theorem 1 below.

The next proposition gives the representation for $C_t^{(k)}$ in a non-recursive form. Let

$$\mathcal{L}_k = \left\{ (i_1, i_2, ..., i_l) \,|\, l \in \{1, 2, ..., k\}, i_q \in \{1, 2, ..., k\}, i_1 \geq i_2 \geq \cdots \geq i_l \text{ and } \sum_{q=1}^{l} i_q = k \right\}. \quad (12)$$

The number of distinct values in a tuple $\phi_k = \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)$ in $\mathcal{L}_k$ is less than or equal to l. When it is less than l, it means some of the value(s) in the tuple are repeated. Let the number of times $r \in \{1, 2, 3, .., k\}$ appears in the tuple $\phi_k = \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)$ be $p_r^{\phi_k}$.

Proposition 2

$$C_t^{(k)} = \sum_{\phi_k = \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right) \in \mathcal{L}_k} \frac{1}{l!} \left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)! \left(p_1^{\phi_k}, p_2^{\phi_k}, ..., p_k^{\phi_k}\right)! \left[\prod_{q \in \phi_k} m_q\right] t^l \quad (13)$$

where $i_1^{(k)}, ..., i_l^{(k)}$ are the elements of ϕ_k, $p_j^{\phi_k}$'s are defined above and $\left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)!$ is the multinomial coefficient: $\left(i_1^{(k)}, i_2^{(k)}, ..., i_l^{(k)}\right)! = \frac{\left(\sum_{j=1}^{l} i_j^{(k)}\right)!}{i_1^{(k)}! i_2^{(k)}! \cdots i_l^{(k)}!}$

Proof. Proof is included in Appendix B. □

Let $\Pi_{(i_1, i_2, ..., i_j), t}^{(k)}$ be the coefficient of $\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \cdots \int_{t_0}^{t_{j-1}-} \mathrm{d}\hat{G}_{t_j}^{(i_1)} \cdots \mathrm{d}\hat{G}_{t_2}^{(i_{j-1})} \mathrm{d}\hat{G}_{t_1}^{(i_j)}$ in $\left(G_{t+t_0} - G_{t_0}\right)^k$. We then have the following result.

Proposition 3

$$\Pi_{(i_1, i_2, ..., i_j), t}^{(k)} = (i_1, i_2, ..., i_j, n)! C_t^{(n)} \text{ where } n = k - \sum_{p=1}^{j} i_p. \quad (14)$$

Proof. The proof of Proposition 3 is contained in the proof of Theorem 1. □

For example, say we want to find the coefficient of $\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \mathrm{d}\hat{G}_{t_2}^{(1)} \mathrm{d}\hat{G}_{t_1}^{(1)}$ in $\left(G_{t+t_0} - G_{t_0}\right)^4$, that is, we want to find $\Pi_{(1,1),t}^{(4)}$. To derive this coefficient, we first note that $n = 2$ and so $\Pi_{(1,1),t}^{(4)} = \frac{4!}{1!1!2!} C_t^{(2)} = 12\left(m_2 t + m_1^2 t^2\right)$, which can be easily verified by calculating $\left(G_{t+t_0} - G_{t_0}\right)^4$ using equations (A.4)-(A.6). Now we put the above results together to get a general formula for $\left(G_{t+t_0} - G_{t_0}\right)^k$.

Theorem 1 Let G_t be a Lévy process with no Brownian part satisfying condition given in equation (2). Then the power of its increment can be expressed by:

$$\left(G_{t+t_0} - G_{t_0}\right)^k = \sum_{\theta_k \in \mathcal{I}_k} \Pi_{\theta_k, t}^{(k)} \mathcal{S}_{\theta_k, t, t_0} + C_t^{(k)}, \quad (15)$$

where $\mathcal{I}_k$ is defined in equation (10), $\Pi^{(k)}_{\theta_k,t}$ is defined in Proposition 3, the $C^{(k)}_t$ are constants defined in Proposition 2 and $\mathcal{S}_{(i_1,i_2,...,i_j),t,t_0}$ is defined as the integral:

$$\mathcal{S}_{(i_1,i_2,...,i_j),t,t_0} = \int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\int_{t_0}^{t_{j-1}-} \mathrm{d}\hat{G}^{(i_1)}_{t_j}\cdots \mathrm{d}\hat{G}^{(i_{j-1})}_{t_2}\mathrm{d}\hat{G}^{(i_j)}_{t_1}.$$

Proof. Proof is included in Appendix C. □

To derive the explicit formula for the power of increment of a Lévy process, $(X_{t+t_0} - X_{t_0})^n$, with respect to orthogonalized compensated power jump processes, we need the following proposition.

Proposition 4 The n-th compensated power jump processes, $Y^{(n)}$, of a general Lévy processes satisfying condition given in equation (2), can be expressed in terms of the orthogonalized compensated power jump processes, $H^{(i)}$ for $i = 1, 2, ..., n$, by

$$Y^{(n)} = H^{(n)} + \sum_{k=1}^{n-1} b_{n,k}H^{(k)},$$

where $b_{n,k}$ denotes the sum of the set $\mathcal{M}^{n,k}$, which is defined by

$$\mathcal{M}^{n,k} = \left\{(-1)^{j-1}\, a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{j-1},i_j} : i_1 = n, i_j = k, i_p > i_q \text{ if } p < q, i_p \in \mathbb{N} \text{ for all } p\right\},$$

and $\mathcal{M}^{n,n} = \{1\}$.

Proof. Proof is included in Appendix D. □

Theorem 2 Let G_t be a Lévy process with no Brownian part satisfying condition given in equation (2). Then the power of its increment in terms of stochastic integrals with respect to the orthogonal martingales, H, is given by the following equation:

$$(G_{t+t_0} - G_{t_0})^k = \sum_{\theta_k\in\mathcal{I}_k} \Pi^{(k)}_{\theta_k,t}\mathcal{S}^{(H)}_{\theta_k,t,t_0} + C^{(k)}_t, \tag{16}$$

where $\mathcal{I}_k$ is defined in equation (10), $\Pi^{(k)}_{\theta_k,t}$ is defined in Proposition 3, $C^{(k)}_t$ is defined in Proposition 2 and $\mathcal{S}^{(H)}_{(i_1,i_2,...,i_j),t,t_0}$ is defined as the integral:

$$\mathcal{S}^{(H)}_{(i_1,i_2,...,i_j),t,t_0} = \sum_{k_1=1}^{i_1}\cdots\sum_{k_{j-1}=1}^{i_{j-1}}\sum_{k_j=1}^{i_j} b_{i_1,k_1}\cdots b_{i_{j-1},k_{j-1}}b_{i_j,k_j}\int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\int_{t_0}^{t_{j-1}-}\mathrm{d}H^{(k_1)}_{t_j}\cdots\mathrm{d}H^{(k_{j-1})}_{t_2}\mathrm{d}H^{(k_j)}_{t_1},$$

$b_{n,k}$ is defined in Proposition 4.

Proof. From Proposition 4, we have

$$\begin{aligned}
&\mathcal{S}_{(i_1,i_2,...,i_j),t,t_0}\\
&= \int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\int_{t_0}^{t_{j-1}-} \mathrm{d}\hat{G}_{t_j}^{(i_1)}\cdots \mathrm{d}\hat{G}_{t_2}^{(i_{j-1})}\mathrm{d}\hat{G}_{t_1}^{(i_j)}\\
&= \int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\int_{t_0}^{t_{j-1}-} \mathrm{d}\left[\sum_{k_1=1}^{i_1} b_{i_1,k_1}H_{t_j}^{(k_1)}\right]\cdots \mathrm{d}\left[\sum_{k_{j-1}=1}^{i_{j-1}} b_{i_{j-1},k_{j-1}}H_{t_2}^{(k_{j-1})}\right]\mathrm{d}\left[\sum_{k_j=1}^{i_j} b_{i_j,k_j}H_{t_1}^{(k_j)}\right]\\
&= \sum_{k_1=1}^{i_1}\cdots\sum_{k_{j-1}=1}^{i_{j-1}}\sum_{k_j=1}^{i_j} b_{i_1,k_1}\cdots b_{i_{j-1},k_{j-1}}b_{i_j,k_j}\int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\int_{t_0}^{t_{j-1}-}\mathrm{d}H_{t_j}^{(k_1)}\cdots \mathrm{d}H_{t_2}^{(k_{j-1})}\mathrm{d}H_{t_1}^{(k_j)}.
\end{aligned}$$

Hence, by using Theorem 1, we finish the proof. □

Corollary 1 *By Theorem 1,*

$$\begin{aligned}
\left(G_{t+t_0}-G_{t_0}\right)^m\left(G_{t+t_0}-G_{t_0}\right)^n &= \left(\sum_{\theta_m\in\mathcal{I}_m}\Pi_{\theta_m,t}^{(m)}\mathcal{S}_{\theta_m,t,t_0}^{(H)}+C_t^{(m)}\right)\left(\sum_{\theta_n\in\mathcal{I}_n}\Pi_{\theta_n,t}^{(n)}\mathcal{S}_{\theta_n,t,t_0}^{(H)}+C_t^{(n)}\right)\\
&= \sum_{\theta_{m+n}\in\mathcal{I}_{m+n}}\Pi_{\theta_{m+n},t}^{(m+n)}\mathcal{S}_{\theta_{m+n},t,t_0}^{(H)}+C_t^{(m+n)}.
\end{aligned}$$

Hence, we can find out how to express the product of two iterative stochastic integrals of orders m and n as a weighted sum of iterative stochastic integrals of order $m+n,\ m+n-1,...,2,1$.

Note in Theorems 1 and 2, the integrands of the stochastic integrals do **not** involve t_0 nor any of the integrating variables $t_1,t_2,...,t_j$. They are completely characterized by $C_t^{(p)}$'s, where $p=0,1,...,k$. Hence to find the chaotic representation of $\left(G_{t+t_0}-G_{t_0}\right)^k$, we only need to know the moments of G_t, $m_1t=E\left[X_t\right]$ and $m_p=\int_{-\infty}^{\infty}x^p\nu\left(\mathrm{d}x\right)$ for $p=2,...,k$. This result is intuitive as $\left(G_{t+t_0}-G_{t_0}\right)$ is a stationary process.

3.2 General case

Next we want to derive the formula for the power of the increments of Lévy processes with $\sigma\neq 0$. Recall $X=\{X_t,t\geq 0\}$ denotes a general Lévy process, $X_t^{(i)}$ denotes its i-th power jump process and $Y_t^{(i)}$ denotes its i-th compensated power jump process as defined in equation (4). We define $A_1\left(X_{t+t_0},X_{t_0};k\right)$ and $A_2\left(X_{t+t_0},X_{t_0};k\right)$ such that $\left(X_{t+t_0}-X_{t_0}\right)^k=A_1\left(X_{t+t_0},X_{t_0};k\right)+A_2\left(X_{t+t_0},X_{t_0};k\right)$, where $A_1\left(X_{t+t_0},X_{t_0};k\right)$ comprises all the terms not containing σ in $\left(X_{t+t_0}-X_{t_0}\right)^k$.

From equations (A.4)-(A.6), it may be noted:

$$\begin{aligned}
\left(X_{t+t_0}-X_{t_0}\right)^k &= \frac{\sigma^2}{2}k\left(k-1\right)\left(\left(X_{t+t_0}-X_{t_0}\right)^{k-2}t-\int_{t_0}^{t+t_0}\left(s-t_0\right)\mathrm{d}\left(X_s-X_{t_0}\right)^{k-2}\right)\\
&\quad+\sum_{j=1}^{k}\binom{k}{j}\int_{t_0}^{t+t_0}A_2\left(X_{s-},X_{t_0};k-j\right)\mathrm{d}Y_s^{(j)}\\
&\quad+\sum_{j=1}^{k-1}\binom{k}{j}m_j\left(t+t_0\right)A_2\left(X_{t+t_0},X_{t_0};k-j\right)\\
&\quad-\sum_{j=1}^{k-1}\binom{k}{j}m_j\int_{t_0}^{t+t_0}s\ \mathrm{d}\left[A_2\left(X_s,X_{t_0};k-j\right)\right]+A_1\left(X_{t+t_0},X_{t_0};k\right). \qquad (17)
\end{aligned}$$

Proposition 5 For any Lévy process X_t satisfying condition given in equation (2),

$$\left(X_{t+t_0}-X_{t_0}\right)^k=A_1\left(X_{t+t_0},X_{t_0};k\right)+\sum_{n=1}^{\lfloor k/2\rfloor}\frac{k!}{(k-2n)!}\frac{1}{n!}\frac{1}{2^n}\sigma^{2n}A_1\left(X_{t+t_0},X_{t_0};k-2n\right)t^n.$$

Proof. The proof uses the same techniques as in the proof of Theorem 1. Note that $A_1\left(X_{t+t_0},X_{t_0};p\right)$, where $p=1,2,...,k$, are given by Theorem 1. □

Proposition 5 gives the formula of $\left(X_{t+t_0}-X_{t_0}\right)^k$ in terms of a summation of A_1, where $\lfloor k/2\rfloor+1$ calculations of A_1 are needed. The next theorem gives the formula in an alternative form which requires A_1 to be computed once only.

Definition 1 Let $C_{t,\sigma}^{(k)}$ be the terms obtained by replacing m_2 with $m_2+\sigma^2$ in $C_t^{(k)}$ (Proposition 2) and $\Pi_{(i_1,i_2,\ldots,i_j),t,\sigma}^{(k)}$ be the terms obtained by replacing $C_t^{(k)}$ with $C_{t,\sigma}^{(k)}$ in $\Pi_{(i_1,i_2,\ldots,i_j),t}^{(k)}$ (Proposition 3). We then note the following theorem.

Theorem 3 For any Lévy process X_t with $\sigma^2\neq 0$ and satisfying condition given in equation (2), the representation of $\left(X_{t+t_0}-X_{t_0}\right)^n$ is given by Theorem 1 with m_2 replaced by $\left(m_2+\sigma^2\right)$, i.e.

$$\left(X_{t+t_0}-X_{t_0}\right)^n=\sum_{\theta_n\in\mathcal{I}_n}\Pi_{\theta_n,t,\sigma}^{(n)}\mathcal{S}'_{\theta_n,t,t_0}+C_{t,\sigma}^{(n)},$$

where $\mathcal{I}_n$ is defined in equation (10), $\Pi_{\theta_n,t,\sigma}^{(n)}$ and $C_{t,\sigma}^{(n)}$ are defined above and $\mathcal{S}'_{(i_1,i_2,\ldots,i_j),t,t_0}$ is defined to be the integral:

$$\mathcal{S}'_{(i_1,i_2,\ldots,i_j),t,t_0}=\int_{t_0}^{t+t_0}\int_{t_0}^{t_1-}\cdots\int_{t_0}^{t_{j-1}-}\mathrm{d}Y_{t_j}^{(i_1)}\cdots\mathrm{d}Y_{t_2}^{(i_{j-1})}\mathrm{d}Y_{t_1}^{(i_j)}.$$

Proof. We define a new class of power jump processes by

$$\begin{aligned}\widetilde{X}_t^{(2)} &= X_t^{(2)} + \sigma^2 t, \\ \widetilde{X}_t^{(j)} &= X_t^{(j)} \quad \text{for } j = 1 \text{ and } j = 3, 4, 5, ...\end{aligned} \tag{18}$$

We also define a new class of compensators

$$\begin{aligned}\tilde{m}_2 t &= \left(m_2 + \sigma^2\right) t, \\ \tilde{m}_j t &= m_j t \quad \text{for } j = 1 \text{ and } j = 3, 4, 5, ...\end{aligned}$$

Hence, by definition, the compensated power jump processes, $\tilde{Y}_t^{(i)} = \tilde{X}_t^{(i)} - \tilde{m}_i t = X_t^{(i)} - m_i t = Y_t^{(i)}$ for all $i \geq 1$. Therefore, the representation of $(X_{t+t_0} - X_{t_0})^k$ in terms of the stochastic integrals with respect to $Y_t^{(i)}$ is the same no matter we start from using $X_t^{(i)}$ or $\widetilde{X}_t^{(i)}$. To calculate the expression using $\widetilde{X}_t^{(i)}$, we use equation (2) in (3):

$$\begin{aligned}
&(X_{t+t_0} - X_{t_0})^k \\
&= \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}X_s^{(j)} \\
&\quad + \frac{\sigma^2}{2} k(k-1) \left((X_{t+t_0} - X_{t_0})^{k-2} t - \int_0^t s \, \mathrm{d}\left(X_{s+t_0} - X_{t_0}\right)^{k-2} \right) \\
&= \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}X_s^{(j)} + \frac{\sigma^2}{2} k(k-1) \int_0^t \left(X_{(s+t_0)-} - X_{t_0}\right)^{k-2} \mathrm{d}s \\
&= \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}X_s^{(j)} + \frac{\sigma^2}{2} k(k-1) \int_{t_0}^{t+t_0} (X_{u-} - X_{t_0})^{k-2} \, \mathrm{d}u \\
&= \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}X_s^{(j)} + \binom{k}{2} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-2} \, \mathrm{d}\left(\sigma^2 s\right).
\end{aligned}$$

By equation (18), we have

$$(X_{t+t_0} - X_{t_0})^k = \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}\tilde{X}_s^{(j)}.$$

Using exactly the same calculation as the one leading to equations (A.4)-(A.6), we have

$$\begin{aligned}
(X_{t+t_0} - X_{t_0})^k &= \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}Y_s^{(j)} + \sum_{j=1}^{k-1} \binom{k}{j} \tilde{m}_j (t+t_0) (X_{t+t_0} - X_{t_0})^{k-j} \\
&\quad - \sum_{j=1}^{k-1} \binom{k}{j} \tilde{m}_j \int_{t_0}^{t+t_0} s \, \mathrm{d}(X_s - X_{t_0})^{k-j} + \tilde{m}_k t.
\end{aligned}$$

This is exactly the equations (A.5)-(A.6) we based on in the derivation of Theorem 1, except that

m_j is replaced by $\tilde{m}_j$. Hence we now have a simple formula for the representation of $(X_{t+t_0} - X_{t_0})^k$ in terms of the stochastic integrals with respect to $Y_t^{(i)}$ by replacing m_j with $\tilde{m}_j$ in the formula given by Theorem 1. In other words, we have

$$(X_{t+t_0} - X_{t_0})^n = \sum_{\theta_n \in \mathcal{I}_n} \Pi^{(n)}_{\theta_n,t,\sigma} \mathcal{S}'_{\theta_n,t,t_0} + C^{(n)}_{t,\sigma},$$

where $\Pi^{(n)}_{\theta_n,t,\sigma}$ and $C^{(n)}_{t,\sigma}$ are defined above. Note that this representation does not depend on the power jump processes directly since it is in terms of the compensated power jump processes, $Y_t^{(j)}$. So it does not matter if we change the definition of the power jump processes, as long as we change the compensators accordingly, we will get the same compensated power jump processes. □

Theorem 4 For any Lévy process X_t with $\sigma^2 \neq 0$ and satisfying condition given in equation (2), the representation of $(X_{t+t_0} - X_{t_0})^n$ is given by Theorem 2 with m_2 replaced with $\left(m_2 + \sigma^2\right)$, i.e.

$$(X_{t+t_0} - X_{t_0})^n = \sum_{\theta_n \in \mathcal{I}_n} \Pi^{(n)}_{\theta_n,t,\sigma} \mathcal{S}'^{(H)}_{\theta_n,t,t_0} + C^{(n)}_{t,\sigma},$$

where $\mathcal{I}_n$ is defined in equation (10), $\Pi^{(n)}_{\theta_n,t,\sigma}$ and $C^{(n)}_{t,\sigma}$ are defined above and $\mathcal{S}'_{(i_1,i_2,\ldots,i_j),t,t_0}$ is defined to be the integral:

$$\mathcal{S}'^{(H)}_{(i_1,i_2,\ldots,i_j),t,t_0} = \sum_{k_1=1}^{i_1} \cdots \sum_{k_{j-1}=1}^{i_{j-1}} \sum_{k_j=1}^{i_j} b_{i_1,k_1} \cdots b_{i_{j-1},k_{j-1}} b_{i_j,k_j} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \cdots \int_{t_0}^{t_{j-1}-} \mathrm{d}H^{(k_1)}_{t_j} \cdots \mathrm{d}H^{(k_{j-1})}_{t_2} \mathrm{d}H^{(k_j)}_{t_1},$$

$b_{n,k}$ is defined in Proposition 4.

Proof. It follows directly from Theorems 2 and 3. □

Remark As noted in Section 2.2, Jamshidian (10) derived an explicit formula for the chaotic representation of X_t^k in terms of the non-compensated power jump processes, $X_t^{(j)}$, when X_t is a semimartingale. Our explicit formula gives the representation in terms of orthogonalized compensated power jump processes, $H_t^{(j)}$. In the following, we show that in the Lévy case, our formula complements Jamshidian's one. We note the notation used by Jamshidian in Section 2.2. If X is a Lévy process, we can see that $[X^c]_t = [X]_t^c = \sigma^2 t$ (where the superscript c stands for continuous part of the process) and hence $[X]_t^{(2)} = \sigma^2 t + \sum_{s \leq t} (\Delta X_s)^2$. With Jamshidian's notation, the σ^2 is implicitly included in the $[X]_t^{(2)}$.

Jamshidian defined $\mathcal{C} = \mathcal{C}^* \cap \mathcal{C}_*$, where $\mathcal{C}^*$ is the set of semimartingales of finite moments with continuous compensators adapted to a Brownian filtration, and $\mathcal{C}_*$ is the set of processes with exponentially decreasing law. Jamshidian generalized the CRP from Lévy processes to the set $\mathcal{C}$.

In proposition 8.2 of (10), an explicit formula for the chaotic representation with respect to the non-compensated power jump processes for the semimartingales in $\mathcal{C}$ when $t_0 = 0$ was derived. Jamshidian defined the power jump processes using the power brackets, see equation (8) and equation (9). The multi-indices were denoted by $I = (i_1, ..., i_p) \in \mathbb{N}^p$, where $\mathbb{N}$ is the set of natural numbers, and for integers $1 \leq p \leq n$,

$$\mathbb{N}_n^p = \{I = (i_1, ..., i_p) \in \mathbb{N}^p : i_1 + \cdots + i_p = n\}, \quad p, n \in \mathbb{N}. \tag{19}$$

Note that from equation (10), $\mathcal{I}_k = \bigcup_{n=1}^{k} \bigcup_{p=1}^{n} \mathbb{N}_n^p$. Proposition 8.2 of (10) states that, for a semi-martingale X_t with $X_0 = 0$, we have, for all $n \in N$

$$X_t^n = \sum_{p=1}^{n} \sum_{I \in \mathbb{N}_n^p} \frac{n!}{i_1! \cdots i_p!} \int_0^t \int_0^{t_1 -} \cdots \int_0^{t_{p-1}-} \mathrm{d}\,[X]_{t_p}^{(i_1)} \cdots \mathrm{d}\,[X]_{t_2}^{(i_p - 1)}\, \mathrm{d}\,[X]_{t_1}^{(i_p)}. \tag{20}$$

Since Jamshidian only considered non-compensated processes, we substitute all the m_j in equation (11) by zeros (since the compensators in the Lévy case are $m_j t$), which makes $C_t^{(k)} = 0$ for all $k \neq 0$. So $\Pi^{(k)}_{(i_1, i_2, ..., i_j), t}$ is non-zero only when $\sum_{p=1}^{j} i_p = k$, as defined in equation (19). Hence in the Lévy case, Theorem 3 reduces to equation (20).

Corollary 2 *The expectation of* $(X_{t+t_0} - X_{t_0})^k$ *is given by* $C_{t,\sigma}^{(n)}$*, which can be obtained by replacing* m_2 *with* $m_2 + \sigma^2$ *in* $C_t^{(k)}$*, given by equation (13).*

Proof. As the expectations of all the stochastic integrals are zero, this follows directly from Theorem 3. □

Corollary 3 *The expectation of* $\left[H_t^{(1)}\right]^k = \left[\int_0^t \mathrm{d}H_{t_1}^{(1)}\right]^k$ *can be obtained by replacing* m_2 *with* $m_2 + \sigma^2$ *and* m_1 *with* 0 *in* $C_t^{(k)}$*, given by Proposition 2.*

From Corollary 2, $E\left[X_t^k\right]$ can be obtained by replacing m_2 with $m_2 + \sigma^2$ in $C_t^{(k)}$. Since $H_t^{(1)} = X_t - m_1 t$ and

$$[X_t]^k = \left[\int_0^t \mathrm{d}H_{t_1}^{(1)} + m_1 t\right]^k, \tag{21}$$

by putting $m_1 = 0$ in equation (21), we can conclude that the expectation of $\left[\int_0^t \mathrm{d}H_{t_1}^{(1)}\right]^k$ can be obtained by replacing m_2 with $m_2 + \sigma^2$ and m_1 with 0 in $C_t^{(k)}$.

In the next section, we extend our results to chaos expansions in terms of the Poisson random measure, with the use of the relationship between the two chaos expansions derived by Benth et al. (4).

4 The Chaos Expansion with respect to the Poisson random measure

Ito (2) proved a chaos expansion for general Lévy processes in terms of multiple integrals with respect to the compensated Poisson random measure. Note that it is trivial to covert the representation to iterated integrals as done by Løkka (8). The *compensated Poisson measure* is defined to be $\tilde{N}(\mathrm{d}t, \mathrm{d}x) = N(\mathrm{d}t, \mathrm{d}x) - \nu(\mathrm{d}x)\,\mathrm{d}t$, where $\nu(\mathrm{d}x)$ is the Lévy measure of the underlying Lévy process, X, and

$$N(B) = \#\{t : (t, \Delta X_t) \in B\}, \quad B \in \mathcal{B}([0,T] \times \mathbb{R}_0),$$

where $\mathcal{B}([0,T] \times \mathbb{R}_0)$ is the *Borel σ-algebra* of $[0,T] \times \mathbb{R}_0$ and $\mathbb{R}_0 = \mathbb{R} - \{0\}$, is the jump measure of the process and its compensator is known to be

$$E[N(\mathrm{d}t, \mathrm{d}x)] = \nu(\mathrm{d}x)\,\mathrm{d}t.$$

Let f be a real function on $([0,T] \times \mathbb{R})^n$. Its *symmetrization* $\tilde{f}$ with respect to the variables $(t_1, x_1), ..., (t_n, x_n)$ is defined by

$$\tilde{f}(t_1, x_1, ..., t_n, x_n) = \frac{1}{n!} \sum_{\boldsymbol{\pi}} f(t_{\pi_1}, x_{\pi_1}, ..., t_{\pi_n}, x_{\pi_n}), \tag{22}$$

where the sum is taken over all permutations $\boldsymbol{\pi}$ of $\{1, ..., n\}$. f is said to be *symmetric* if $f = \tilde{f}$. The definition of symmetrization is used to represent the CRP in terms of multiple integrals instead of iterative integrals.

4.1 Pure jump case

We first consider the representation for pure jump Lévy processes as in (8). Let $\tilde{L}_2((\lambda \times \nu)^n)$ be the space of all square integrable symmetric functions on $([0,T] \times \mathbb{R})^n$. In an iterative integral such as equation (6), the time variables $t_1, ..., t_n$ are monotonic. For ease of notation, we let

$$G_n = \{(t_1, x_1, ..., t_n, x_n) : 0 \leq t \leq \cdots \leq t_n \leq T; x_i \in \mathbb{R}, i = 1, ..., n\}, \tag{23}$$

and let $L_2(G_n)$ be the space of functions g such that

$$\|g\|^2_{L_2(G_n)} = \int_{G_n} g^2(t_1, x_1, ..., t_n, x_n)\,\mathrm{d}t_1 \nu(\mathrm{d}x_1) \cdots \mathrm{d}t_n \nu(\mathrm{d}x_n) < \infty.$$

For $f \in L_2(G_n)$, let

$$J_n(f) = \int_0^T \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, x_1, ..., t_n, x_n)\,\tilde{N}(\mathrm{d}t_1, \mathrm{d}x_1) \cdots \tilde{N}(\mathrm{d}t_n, \mathrm{d}x_n).$$

For $f \in \tilde{L}_2\left((\lambda \times \nu)^n\right)$, let

$$I_n(f) = \int_{([0,T]\times\mathbb{R})^n} f(t_1, x_1, ..., t_n, x_n)\, \tilde{N}^{\otimes n}(\mathrm{d}\mathbf{t}, \mathrm{d}\mathbf{x}) = n! J_n(f).$$

- *Chaos Expansion for pure jump Lévy processes:* Let F be a square integrable random variable adapted to the underlying pure jump Lévy process, X_t. Ito (2) proved that there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ where $f_n \in \tilde{L}_2([0,T]\times\mathbb{R})^n$ such that

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n). \tag{24}$$

Benth et al. (4) derived relations between the two chaos expansions, that is, between the expansion in terms of compensated power jump processes and the expansion in terms of the Poisson random measure. Benth et al. showed that the compensated power jump process defined in equation (4) satisfies the equation

$$Y^{(i)}(t) = \int_0^t \int_{\mathbb{R}} x^i \tilde{N}(\mathrm{d}s, \mathrm{d}x), \quad 0 \le t \le T,\ i = 1, 2, ... \tag{25}$$

Hence, the CRP can be written as

$$\begin{aligned}
F &= E(F) + \sum_{j=1}^{\infty} \sum_{i_1,...,i_j \ge 1} \int_0^{\infty} \int_0^{t_1-} \cdots \int_0^{t_{j-1}-} f_{(i_1,...,i_j)}(t_1,...,t_j)\mathrm{d}Y_{t_j}^{(i_j)} ...\mathrm{d}Y_{t_2}^{(i_2)}\mathrm{d}Y_{t_1}^{(i_1)} && (26)\\
&= E(F) + \sum_{j=1}^{\infty} \sum_{i_1,...,i_j \ge 1} \int_0^{\infty} \int_{\mathbb{R}} \int_0^{t_1-} \int_{\mathbb{R}} \cdots \int_0^{t_{j-1}-} \int_{\mathbb{R}} x_j^{i_j} \cdots x_2^{i_2} x_1^{i_1} \\
&\quad \times f_{(i_1,...,i_j)}(t_1,...,t_j)\tilde{N}(\mathrm{d}t_j, \mathrm{d}x_j) \cdots \tilde{N}(\mathrm{d}t_2, \mathrm{d}x_2)\tilde{N}(\mathrm{d}t_1, \mathrm{d}x_1) && (27)\\
&= E(F) + \sum_{j=1}^{\infty} \int_0^{\infty} \int_{\mathbb{R}} \int_0^{t_1-} \int_{\mathbb{R}} \cdots \int_0^{t_{j-1}-} \int_{\mathbb{R}} g_j(t_1, x_1, ..., t_j, x_j) \\
&\quad \times \tilde{N}(\mathrm{d}t_j, \mathrm{d}x_j) \cdots \tilde{N}(\mathrm{d}t_2, \mathrm{d}x_2)\tilde{N}(\mathrm{d}t_1, \mathrm{d}x_1) \\
&= E(F) + \sum_{j=1}^{\infty} J_j(g_j) = E(F) + \sum_{j=1}^{\infty} n! J_j(\tilde{g}_j) = E(F) + \sum_{j=1}^{\infty} I_j(\tilde{g}_j),
\end{aligned}$$

where $\tilde{g}_j$ is the symmetrization (defined in (22)) of the function g_j given by

$$\begin{aligned}
&g_j(t_1, x_1, ..., t_j, x_j) \\
&= \begin{cases} \sum_{i_1,...,i_j \ge 1} x_1^{i_1} \cdots x_j^{i_j} f_{(i_1,...,i_j)}(t_1,...,t_j), & \text{on } G_n \\ 0 & \text{on } ([0,T]\times\mathbb{R})^n - G_n. \end{cases}
\end{aligned} \tag{28}$$

Therefore, by uniqueness, $\{f_n\}_{n=0}^{\infty}$ in equation (24) is given by

$$f_n = \tilde{g}_n, \qquad n = 1, 2, ...$$

This equation provides a simple relationship between the two expansions. From Theorem 3, of course,

$$(X_{t+t_0} - X_{t_0})^n = \sum_{\theta_n \in \mathcal{I}_n} \Pi^{(n)}_{\theta_n,t,\sigma} \mathcal{S}'_{\theta_n,t,t_0} + C^{(n)}_{t,\sigma}. \tag{29}$$

We can now use this relationship to derive a form for $\tilde{g}_n$ in terms of $\mathcal{I}_n$, $\Pi^{(n)}_{\theta_n,t,\sigma}$ and $C^{(n)}_{t,\sigma}$. Let $\mathcal{K}_{l,s} = \left\{(i_1, ..., i_l) \,|i_j \in \{1, 2, ..., s\} \text{ and} \sum_{j=1}^{l} i_j = s\right\}$. Since the length of a tuple must not be greater than the sum of all the elements in the tuple (because an element must be at least 1), $l \leq s$. By definition, we have $\mathcal{I}_n = \bigcup_{s=1}^{n} \bigcup_{l=1}^{s} \mathcal{K}_{l,s}$. So we can write

$$(X_{t+t_0} - X_{t_0})^n = \sum_{l=1}^{n} \sum_{s=l}^{n} \sum_{\theta_n \in \mathcal{K}_{l,s}} \Pi^{(n)}_{\theta_n,t,\sigma} \mathcal{S}'_{\theta_n,t,t_0} + C^{(n)}_{t,\sigma},$$

where θ_n is the tuple $\left(i_1^{\theta_n}, ..., i_l^{\theta_n}\right)$ with l elements which sum up to s. Therefore, we deduce that for $F = (X_{t+t_0} - X_{t_0})^n$ in equation (26), $f_{(i_1,...,i_j)}(t_1, ..., t_j)$ is given by

$$f_{(i_1,...,i_j)}(t_1, ..., t_j) = \Pi^{(n)}_{\theta_n,t,\sigma}. \tag{30}$$

By equation (28), we have then proved the following proposition.

Proposition 6 For any pure jump Lévy process X_t satisfying condition given by equation (2),

$$(X_{t+t_0} - X_{t_0})^n = \sum_{l=1}^{n} I_l\left(\tilde{g}_l^{(n)}\right) + C^{(n)}_{t,\sigma},$$

where $\tilde{g}_l^{(n)}$ is the symmetrization of the function $g_l^{(n)}$ defined by

$$\begin{aligned} & g_l^{(n)}(t_1, x_1, ..., t_l, x_l) \\ & = \begin{cases} \sum_{s=l}^{n} \sum_{\theta_n \in \mathcal{K}_{l,s}} x_1^{i_1^{\theta_n}} \cdots x_j^{i_j^{\theta_n}} \Pi^{(n)}_{\theta_n,t,\sigma}, & \text{on } G_n \\ 0 & \text{on } ([0,T] \times \mathbb{R})^n - G_n, \end{cases} \end{aligned}$$

where $C^{(n)}_{t,\sigma}$ and $\Pi^{(n)}_{\theta_n,t,\sigma}$ are defined in Definition 1.

The following proposition gives a more straightforward representation.

Proposition 7 For any pure jump Lévy process X_t satisfying condition given by equation (2),

$$\begin{aligned} (X_{t+t_0} - X_{t_0})^n \;\; = \;\; & \sum_{\theta_n \in \mathcal{I}_n} \int_0^\infty \int_{\mathbb{R}} \int_0^{t_1-} \int_{\mathbb{R}} \cdots \int_0^{t_{j-1}-} \int_{\mathbb{R}} x_j^{i_j^{\theta_n}} \cdots x_2^{i_2^{\theta_n}} x_1^{i_1^{\theta_n}} \\ & \times \Pi^{(n)}_{\theta_n,t,\sigma} \tilde{N}(\mathrm{d}t_j, \mathrm{d}x_j) \cdots \tilde{N}(\mathrm{d}t_2, \mathrm{d}x_2)\, \tilde{N}(\mathrm{d}t_1, \mathrm{d}x_1) + C^{(n)}_{t,\sigma}, \end{aligned} \tag{31}$$

where $C_{t,\sigma}^{(n)}$ and $\Pi_{\theta_n,t,\sigma}^{(n)}$ are defined in Definition 1.

Proof. This follows directly by replacing $f_{(i_1,...,i_j)}(t_1,...,t_j)$ in equation (27) by equation (30). $\square$

Note that both chaos expansions, that is, the expansion in terms of compensated power jump processes and the expansion in terms of random measure, depend on $\mathcal{I}_n$, $\Pi_{\theta_n,t,\sigma}^{(n)}$ and $C_{t,\sigma}^{(n)}$. From equation (25), we note the relationship between $Y^{(i)}(t)$ and $\tilde{N}(\mathrm{d}s, \mathrm{d}x)$. Because of the simple form of this relationship, we can use Theorem 1 to derive the explicit representation of equation (31).

4.2 General case

We shall now discuss the general relationship between the two representations. Ito (2) proved the chaos expansion for general Lévy functionals. Benth et al. (4) gave the relationship between the chaos expansions in the case with both a continuous (Wiener process) component and a pure jump (Poisson random measure) component. In this general case, the stochastic integrals are in terms of both Brownian motion, W, and the compensated Poisson measure, $\tilde{N}(\cdot,\cdot)$. Hence, to unify notation, Benth et al. defined:

$$\begin{aligned}
U_1 &= [0,T] && \text{and} \quad U_2 = [0,T]\times\mathbb{R} \\
\mathrm{d}Q_1(\cdot) &= \mathrm{d}W(\cdot) && \text{and} \quad Q_2(\cdot) = \tilde{N}(\cdot,\cdot) \\
\mathrm{d}\langle Q_1\rangle &= \mathrm{d}\lambda && \text{and} \quad \mathrm{d}\langle Q_2\rangle = \mathrm{d}\lambda\times\mathrm{d}\nu \\
\int_{U_1} g\left(u^{(1)}\right) Q_1\left(\mathrm{d}u^{(1)}\right) &= \int_0^t g(s)\, W(\mathrm{d}s) && \text{and} \\
\int_{U_2} g\left(u^{(2)}\right) Q_2\left(\mathrm{d}u^{(2)}\right) &= \int_0^t \int_{\mathbb{R}} g(s,x)\, \tilde{N}(\mathrm{d}s,\mathrm{d}x),
\end{aligned}$$

where $\lambda(\mathrm{d}t) = \mathrm{d}t$ is the Lebesgue measure on $[0,T]$. Let F be a square integrable random variable adapted to the filtration generated by the underlying Lévy process, X. Benth et al. proved that

$$F = E[F] + \sum_{n=1}^{\infty} \sum_{j_1,...,j_n=1,2} \tilde{J}_n\left(g_n^{(j_1,...,j_n)}\right) \tag{32}$$

for a unique sequence $g_n^{(j_1,...,j_n)}$ $(j_1,...,j_n = 1,2;\ n = 1,2,...)$ of deterministic functions in the corresponding L_2-space, $L_2(G_n) = L_2\left(G_n, \otimes_{i=1}^n d\langle Q_{j_i}\rangle\right)$, where

$$G_n = \left\{\left(u_1^{(j_1)},...,u_n^{(j_n)}\right) \in \Pi_{i=1}^n U_{j_i} : 0 \le t_1 \le \cdots \le t_n \le T\right\}$$

with $u^{(j_i)} = t$ if $j_i = 1$, and $u^{(j_i)} = (t, x)$ if $j_i = 2$, and

$$\begin{aligned}&\tilde{J}_n\left(g_n^{(j_1,...,j_n)}\right)\\ &= \int_{\Pi_{i=1}^n U_{j_i}} g_n^{(j_1,...,j_n)}\left(u_1^{(j_1)},...,u_n^{(j_n)}\right) 1_{G_n}\left(u_1^{(j_1)},...,u_n^{(j_n)}\right) Q_{j_1}\left(\mathrm{d}u_1^{(j_1)}\right)\cdots Q_{j_n}\left(\mathrm{d}u_n^{(j_n)}\right).\end{aligned}$$

Similar to the pure jump case, we can derive the explicit formula for the chaos expansion with respect to the Poisson random measure of a general Lévy process, i.e. $\sigma \neq 0$. In this case, we have

$$\begin{aligned}Y^{(1)}(t) &= \sigma\int_0^t \mathrm{d}W(ds) + \int_0^t\int_{\mathbb{R}} x\tilde{N}(\mathrm{d}s,\mathrm{d}x)\\ Y^{(i)}(t) &= \int_0^t\int_{\mathbb{R}} x^i\tilde{N}(\mathrm{d}s,\mathrm{d}x), \quad 0\le t\le T,\ i=2,3,....\end{aligned}$$

To derive the relation between the two chaos expansions, we introduce the following notation:

$$\begin{aligned}R^{(1)}(\mathrm{d}s,\mathrm{d}x) &= \sigma\mathrm{d}W(\mathrm{d}s) + \int_{\mathbb{R}} x\tilde{N}(\mathrm{d}s,\mathrm{d}x)\\ R^{(i)}(\mathrm{d}s,\mathrm{d}x) &= \int_{\mathbb{R}} x^i\tilde{N}(\mathrm{d}s,\mathrm{d}x), \quad i=2,3,....\end{aligned}$$

Hence the CRP with respect to the power jump processes can be written as

$$\begin{aligned}F &= E(F) + \sum_{j=1}^{\infty}\sum_{i_1,...,i_j\ge 1}\int_0^{\infty}\int_0^{t_1-}\cdots\int_0^{t_{j-1}-} f_{(i_1,...,i_j)}(t_1,...,t_j)\mathrm{d}Y_{t_j}^{(i_j)}...\mathrm{d}Y_{t_2}^{(i_2)}\mathrm{d}Y_{t_1}^{(i_1)}\\ &= E(F) + \sum_{j=1}^{\infty}\sum_{i_1,...,i_j\ge 1}\int_0^{\infty}\int_0^{t_1-}\cdots\int_0^{t_{j-1}-} f_{(i_1,...,i_j)}(t_1,...,t_j)\\ &\quad \times R^{(i_j)}(\mathrm{d}t_j,\mathrm{d}x_j)...R^{(i_2)}(\mathrm{d}t_2,\mathrm{d}x)R^{(i_1)}(\mathrm{d}t_1,\mathrm{d}x).\end{aligned}$$

From Theorem 3,

$$\begin{aligned}(X_{t+t_0} - X_{t_0})^n &= \sum_{\theta_n\in\mathcal{I}_n}\Pi_{\theta_n,t,\sigma}^{(n)}\mathcal{S}'_{\theta_n,t,t_0} + C_{t,\sigma}^{(n)}\\ &= \sum_{\theta_n\in\mathcal{I}_n}\int_0^{\infty}\int_0^{t_1-}\cdots\int_0^{t_{j-1}-}\Pi_{\theta_n,t,\sigma}^{(n)}\\ &\quad \times R^{(i_j^{\theta_n})}(\mathrm{d}t_j,\mathrm{d}x_j)...R^{(i_2^{\theta_n})}(\mathrm{d}t_2,\mathrm{d}x)R^{(i_1^{\theta_n})}(\mathrm{d}t_1,\mathrm{d}x) + C_{t,\sigma}^{(n)}.\end{aligned}$$

We have then proved the following proposition.

Proposition 8 For any Lévy process X_t satisfying condition given by equation (2),

$$(X_{t+t_0} - X_{t_0})^n = \sum_{\theta_n \in \mathcal{I}_n} \int_0^{\infty} \int_0^{t_1 -} \cdots \int_0^{t_{j-1}-} \Pi^{(n)}_{\theta_n, t, \sigma} \times R^{(i_j^{\theta_n})}(\mathrm{d}t_j, \mathrm{d}x_j) \ldots R^{(i_2^{\theta_n})}(\mathrm{d}t_2, \mathrm{d}x) \, R^{(i_1^{\theta_n})}(\mathrm{d}t_1, \mathrm{d}x) + C^{(n)}_{t,\sigma},$$

where $C^{(n)}_{t,\sigma}$ and $\Pi^{(n)}_{\theta_n, t, \sigma}$ are defined in Definition 1.

5 The explicit chaos expansions for a common kind of Lévy functionals

Note that we have only found the explicit representations for powers of increments of Lévy processes. In this section, we explain how the explicit formulae for a common kind of Lévy functionals might be obtained using multivariate Taylor expansion.

Assume that a real function g, possessing derivatives of all orders, is such that

$$F = g\left(X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}\right), \tag{33}$$

where the indices $0 \le t_1 < t_2 < \cdots < t_n$ are known and n is finite. By expressing F in terms of power of increments of X, we can use our explicit formula to obtain the CRP of F. For example, in financial applications, g corresponds to all pricing functions of contingent claims which depend on the underlying asset at a finite number of time points. Suppose $\{X_t, 0 \le t \le T\}$ is the background driving Lévy process and time is now $t = t_n$. Suppose the underlying asset, $\{S_t, 0 \le t \le T\}$, is given by the exponential-Lévy model, see (14, Chapter 8.4):

$$S_t = S_0 \exp(X_t),$$

where S_0 is the initial value of the underlying asset at time $t = 0$. Then, for example, we can represent F as the pricing functions of a number of contingent claims listed in Table 1.

In equation (33), let $x_1 = X_{t_1}, x_2 = X_{t_2} - X_{t_1}, ..., x_n = X_{t_n} - X_{t_{n-1}}$. If g is not a linear combination of powers of x_i, we need to use the multivariate Taylor series, see (15), about the points $x_i = 0, i = 1, ..., n$ to obtain such a representation:

$$g(x_1, ..., x_n) = \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[\sum_{k=1}^{n} x_k \frac{\partial}{\partial x'_k} \right]^j g(x'_1, ..., x'_n) \right\}_{x'_1 = 0, ..., x'_n = 0}. \tag{34}$$

Note that this representation exists when g possesses derivatives of all orders at zero. To show typical elements in this representation, we note the special case of $n = 2$:

$$\begin{aligned} g(x_1, x_2) &= \sum_{j=0}^{\infty} \left\{ \frac{1}{j!} \left[x_1 \frac{\partial}{\partial x_1'} + x_2 \frac{\partial}{\partial x_2'} \right]^j g(x_1', x_2') \right\}_{x_1'=0, x_2'=0} \\ &= g(0,0) + \left[x_1 \left. \frac{\partial g}{\partial x_1'} \right|_{x_1'=0, x_2'=0} + x_2 \left. \frac{\partial g}{\partial x_2'} \right|_{x_1'=0, x_2'=0} \right] \\ &\quad + \frac{1}{2!} \left[x_1^2 \left. \frac{\partial^2 g}{\partial x_1'^2} \right|_{x_1'=0, x_2'=0} + 2 x_1 x_2 \left. \frac{\partial^2 g}{\partial x_1' \partial x_2'} \right|_{x_1'=0, x_2'=0} + x_2^2 \left. \frac{\partial^2 g}{\partial x_2'^2} \right|_{x_1'=0, x_2'=0} \right] + \cdots . \end{aligned}$$

Name	**Formula**
Forward and future contracts on a security providing no income	$F_t = S_t \exp(r(T-t)) = S_0 \exp(X_t + r(T-t))$, where r is the risk free interest rate and T is the maturity of the contract.
Forward and future contracts on a security providing a known cash income	$F_t = (S_t - I)\exp(r(T-t))$ $= (S_0 \exp(X_t) - I)\exp(r(T-t))$, where I is the present value of the perfectly predictable income on S.
Forward and future contracts on a foreign currency	$F_t = S_t \exp((r - r_f)(T-t))$ $= S_0 \exp(X_t + (r - r_f)(T-t))$, where r_f is the risk free interest rate of the foreign currency.
Forward and future contracts on commodity	$F_t = (S_t + U)\exp(r(T-t))$ $= (S_0 \exp(X_t) + U)\exp(r(T-t))$, where U is the present value of all storage costs.
European call options	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[(S_T - K)^+ \mid \mathcal{F}_t \right]$, where K is the strike, T is the maturity, Q is the risk neutral measure and $\mathcal{F}_t$ is the filtration of S.
Up-and-out barrier call options	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[(S_T - K)^+ 1_{\{M_T^S < H\}} \right]$, where H is the barrier and $M_t^S = \sup\{S_u; 0 \le u \le t\},\ 0 \le t \le T$.
Up-and-in barrier call options	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[(S_T - K)^+ 1_{\{M_T^S \ge H\}} \right]$.
Down-and-out barrier call options	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[(S_T - K)^+ 1_{\{m_T^S > H\}} \right]$, where $m_t^S = \inf\{S_u; 0 \le u \le t\},\ 0 \le t \le T$.
Down-and-in barrier call options	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[(S_T - K)^+ 1_{\{m_T^S \le H\}} \right]$.
Lookback options with a floating strike	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[M_T^S - S_T \right]$.
Lookback options with a fixed strike	$F(t, S_t) = \exp(-r(T-t)) E_Q \left[(M_T^S - K)^+ \right]$.
Asian call options	$F(t, S_t) = \frac{\exp(-r(T-t))}{n} E_Q \left[\left(\sum_{k=1}^n S_{t_k} - nK \right)^+ \middle\vert \mathcal{F}_t \right]$.

Table 1: The contingent claims and their pricing formulae to which Taylor's expansion can be

applied at some values of S_t.

Let $g^{(l)}_{j_1,j_2,\dots,j_l}(\mathbf{0}) = \frac{1}{l!}\left.\frac{\partial^l g}{\partial x'_{j_1}\partial x'_{j_2}\cdots\partial x'_{j_l}}\right|_{x'_1=0,\dots,x'_n=0}$. As in (16, Lemma 2), we assume that

$$\sum_{l=2}^{\infty}\sum_{j_1,\dots,j_l\in\{1,\dots,n\}}\left|g^{(l)}_{j_1,j_2,\dots,j_l}(\mathbf{0})\right|R^l<\infty, \tag{35}$$

for all $R>0$. The multivariate Taylor series equation (34) expresses F in terms of sum of products of powers of increments of X. From Theorem 4, we can substitute $x_i,\ i=1,2,\dots$with the iterated integrals with respect to the orthogonal martingales.

For all $F\in L^2(\Omega,\mathcal{F})$ having the form equation (33), then

$$\begin{aligned} F &= \sum_{j=0}^{\infty}\left\{\frac{1}{j!}\left[\sum_{k=1}^{n}\left(X_{t_k}-X_{t_{k-1}}\right)\frac{\partial}{\partial x'_k}\right]^j g\left(x'_1,\dots,x'_n\right)\right\}_{x'_1=0,\dots,x'_n=0} \\ &= g(0,0,\dots,0)+\sum_{j=1}^{n}\left(X_{t_j}-X_{t_{j-1}}\right)g^{(1)}_j(\mathbf{0})+\frac{1}{2!}\sum_{j=1}^{n}\left(X_{t_j}-X_{t_{j-1}}\right)^2 g^{(2)}_{j,j}(\mathbf{0}) \\ &\quad+\frac{1}{2!}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}1_{\{j_1\neq j_2\}}\left(X_{t_{j_1}}-X_{t_{j_1-1}}\right)\left(X_{t_{j_2}}-X_{t_{j_2-1}}\right)g^{(2)}_{j_1,j_2}(\mathbf{0}) \\ &\quad+\frac{1}{3!}\sum_{j=1}^{n}\left(X_{t_j}-X_{t_{j-1}}\right)^3 g^{(3)}_{j,j,j}(\mathbf{0}) \\ &\quad+\frac{3}{3!}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}1_{\{j_1\neq j_2\}}\left(X_{t_{j_1}}-X_{t_{j_1-1}}\right)^2\left(X_{t_{j_2}}-X_{t_{j_2-1}}\right)g^{(3)}_{j_1,j_1,j_2}(\mathbf{0}) \\ &\quad+\frac{1}{3!}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n}1_{\{j_1\neq j_2\neq j_3\}}\left(X_{t_{j_1}}-X_{t_{j_1-1}}\right)\left(X_{t_{j_2}}-X_{t_{j_2-1}}\right) \\ &\quad\times\left(X_{t_{j_3}}-X_{t_{j_3-1}}\right)g^{(3)}_{j_1,j_2,j_3}(\mathbf{0})+\cdots, \end{aligned} \tag{36}$$

where $\left(X_{t_i}-X_{t_{i-1}}\right)^n$'s are given by Theorem 4 and we assume $X_{t_0}=0$. The sums converge for every $\omega\in\Omega$ because of equation (35).

Since $0\le t_1<t_2<\cdots<t_n$, the product of two iterated integrals with non-overlapping limits results in an iterated integral: if $i\le j-1,\ u,v\in\{1,2,3,\dots\}$ and ϕ_i,ϕ_j are the predictable integrands,

$$\begin{aligned} \int_{t_{i-1}}^{t_i}\phi_i\,\mathrm{d}H^{(u)}_{s_1}\times\int_{t_{j-1}}^{t_j}\phi_j\,\mathrm{d}H^{(v)}_{r_1} &= \int_{t_{j-1}}^{t_j}\int_{t_{i-1}}^{t_i}\phi_i\phi_j\,\mathrm{d}H^{(u)}_{s_1}\mathrm{d}H^{(v)}_{r_1} \\ &= \int_0^{t_j}\int_0^{t_i}1_{\{s_1>t_{i-1}\}}1_{\{r_1>t_{j-1}\}}\phi_i\phi_j\,\mathrm{d}H^{(u)}_{s_1}\mathrm{d}H^{(v)}_{r_1} \\ &= \int_0^{t_j}\int_0^{r_1}1_{\{t_i>s_1>t_{i-1}\}}1_{\{r_1>t_{j-1}\}}\phi_i\phi_j\,\mathrm{d}H^{(u)}_{s_1}\mathrm{d}H^{(v)}_{r_1}, \end{aligned}$$

since $r_1 > t_{j-1} \geq t_i$, giving an iterated integral. Hence, we get a chaos expansion of F in terms of iterated integrals with respect to orthogonalized compensated power jump processes.

Note that in some applications, it is only necessary to apply Taylor's Theorem directly to F to obtain a PRP representation. While the approach given in this section gives the CRP of F, each $\left(X_{t_{jn}} - X_{t_{jn-1}}\right)^m$ consists of an infinite sum and therefore equation (36) is composed of two levels of infinite sums. Yip et al. (17) applied Taylor's Theorem directly to obtain the PRP of European and exotic option prices for hedging.

6 Simulations using the explicit formula

To verify the theoretical results given in Section 3, we simulate the underlying Lévy processes and compare the values of $(X_{t+t_0} - X_{t_0})^n$ with the value given by its chaos expansion. In simulations we apply the stochastic Euler scheme for the stochastic differential equations (SDEs) of general Lévy processes. The rate of convergence of this scheme for Lévy processes was discussed by Protter and Talay (18). For an up-to-date introduction to numerical solutions of SDEs, see for example (19), (20), (21) and (22).

The processes considered are Gamma process and a combination of Wiener and Gamma processes. For illustration, we run simulations for $k = 4$ and $k = 9$ in the pure jump case and $k = 5$ and $k = 8$ for the combined case. The plots produced are shown in Figures 1, 3, 5, 7 in Appendix E respectively. In the second and fourth simulations, we set $t_0 = 0.0099$ and $t_0 = 0.0019$ respectively. These simulations substantiate our explicit formula for the CRP for $t_0 \geq 0$. We see that processes generated using the CRP and those generated directly from the Gamma process jump at the same time points. The differences between the two are plotted in Figures 2, 4, 6, 8 accordingly. Note that the axis of Figures 2, 4, 6, 8 are in much smaller scales than those in Figures 1, 3, 5, 7. We deduce that the difference is due to approximation errors of the stochastic Euler scheme. The errors decrease with the step size Δ. In each of the Figures 1, 3, 5, 7, independent realizations of the Gamma and Wiener processes are used.

7 Conclusion

Lévy processes were introduced in mathematical finance to improve the performance of some of the financial models which are based on using Brownian motion as the underlying process and to model stylized features observed in financial processes. The derivation of an explicit formula for the CRP has been of the focus of considerable study, for previous work, see (7), (4), (8) and (9). In this paper, we have derived a computational explicit formula for the construction of CRP of the powers of increments of Lévy processes in terms of orthogonal compensated power jump processes and its CRP in terms of Poisson random measures. Jamshidian (10) extended the CRP in terms of power jump processes to a large class of semimartingales and we have shown that in the Lévy case, our formula complements the one given by Jamshidian. Our explicit formula shows that the integrands of the stochastic integrals in the CRP of the powers of increments of Lévy processes do not depend on the integrating variables nor the starting time, which makes the construction and simulation of the CRP much easier. The coefficients of the CRP depend on m_i's which represent the moments of the process with respect to its Lévy measure. In this paper, we consider only Lévy processes and their compensators are always of the form $m_i t$. Using the same calculation, it is trivial to extend the representation to semimartingales which stochastic compensators have known representations. The CRP of the pricing functions for some common financial derivatives can be found by expressing the pricing functions in terms of powers of increments of the underlying Lévy process using Taylor's expansion.

APPENDICES

A A note on the Nualart and Schoutens representation

Nualart and Schoutens (3) derived the basic result for representing $(X_{t+t_0} - X_{t_0})^k$ when $t_0 = 0$ and $k = 2$. In the proof of the CRP, Nualart and Schoutens (3) made use of Proposition 2 in their paper, given in Section 2.1 and the following equation derived from the Ito formula (equation (5) in (3)):

$$\begin{aligned}
&(X_{t+t_0} - X_{t_0})^k \\
&= \frac{\sigma^2}{2} k(k-1) \left((X_{t+t_0} - X_{t_0})^{k-2} t - \int_0^t s \;\, \mathrm{d}(X_{s+t_0} - X_{t_0})^{k-2} \right) && \text{(A.1)} \\
&\quad + \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}Y_s^{(j)} + \sum_{j=1}^{k-1} \binom{k}{j} m_j t (X_{t+t_0} - X_{t_0})^{k-j} && \text{(A.2)} \\
&\quad - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_{t_0}^{t+t_0} s \;\, \mathrm{d}(X_s - X_{t_0})^{k-j} + m_k t. && \text{(A.3)}
\end{aligned}$$

There is a small inaccuracy in this equation and we provide the corrected one necessary for the derivation of the explicit formula. The second term in equation (A.2) should be

$$\sum_{j=1}^{k-1} \binom{k}{j} m_j (t + t_0) (X_{t+t_0} - X_{t_0})^{k-j}$$

rather than $\sum_{j=1}^{k-1} \binom{k}{j} m_j t (X_{t+t_0} - X_{t_0})^{k-j}$. The error propagates from equation (4) in (3). By integration by parts, $\sum_{j=1}^{k} \binom{k}{j} m_j \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}s$ should give

$$\sum_{j=1}^{k-1} \binom{k}{j} m_j (t + t_0) (X_{t+t_0} - X_{t_0})^{k-j} - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_{t_0}^{t+t_0} s \;\, \mathrm{d}(X_s - X_{t_0})^{k-j} + m_k t$$

rather than the term

$$\sum_{j=1}^{k-1} \binom{k}{j} m_j t (X_{t+t_0} - X_{t_0})^{k-j} - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_{t_0}^{t+t_0} s \, \mathrm{d}(X_s - X_{t_0})^{k-j} + m_k t$$

stated in (3, p.114). Omitting t_0 makes the constant term of the representation not equal to the expectation of $(X_{t+t_0} - X_{t_0})^k$ since it depends on t_0. Equation (5) in (3) should in fact be:

$$\begin{aligned}
&(X_{t+t_0} - X_{t_0})^k \\
&= \frac{\sigma^2}{2} k(k-1)\left((X_{t+t_0} - X_{t_0})^{k-2} t - \int_0^t s \ \mathrm{d}(X_{s+t_0} - X_{t_0})^{k-2} \right) && \text{(A.4)} \\
&\quad + \sum_{j=1}^{k} \binom{k}{j} \int_{t_0}^{t+t_0} (X_{s-} - X_{t_0})^{k-j} \, \mathrm{d}Y_s^{(j)} + \sum_{j=1}^{k-1} \binom{k}{j} m_j (t+t_0)(X_{t+t_0} - X_{t_0})^{k-j} && \text{(A.5)} \\
&\quad - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_{t_0}^{t+t_0} s \ \mathrm{d}(X_s - X_{t_0})^{k-j} + m_k t. && \text{(A.6)}
\end{aligned}$$

As an illustration of this representation, we derive $(G_{t+t_0} - G_{t_0})^2$ using equations (A.1)-(A.3) to inspect the constant terms. Note that G_t is a Lévy process with $\sigma^2 = 0$, so the terms in equation (A.1) are equal to zero. We have

$$\begin{aligned}
(G_{t+t_0} - G_{t_0})^2 &= 2\int_{t_0}^{t+t_0} (G_{t_1-} - G_{t_0}) \, \mathrm{d}\hat{G}_{t_1}^{(1)} + \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(2)} + 2m_1 t (G_{t+t_0} - G_{t_0}) \\
&\quad -2m_1 \int_{t_0}^{t+t_0} t_1 \ \mathrm{d}(G_{t_1} - G_{t_0}) + m_2 t \\
&= 2\int_{t_0}^{t+t_0} \left[\left(\hat{G}_{t_1-}^{(1)} - \hat{G}_{t_0}^{(1)} \right) + m_1 (t_1 - t_0) \right] \mathrm{d}\hat{G}_{t_1}^{(1)} + \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(2)} \\
&\quad +2m_1 t \left[\left(\hat{G}_{t+t_0}^{(1)} - \hat{G}_{t_0}^{(1)} \right) + m_1 t \right] - 2m_1 \int_{t_0}^{t+t_0} t_1 \ \mathrm{d}\left[\hat{G}_{t_1}^{(1)} + m_1 t_1 \right] + m_2 t \\
&= 2\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \mathrm{d}\hat{G}_{t_2}^{(1)} \mathrm{d}\hat{G}_{t_1}^{(1)} + \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(2)} + 2m_1 (t - t_0) \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(1)} \\
&\quad +m_1^2 t^2 + m_2 t - 2m_1^2 t t_0.
\end{aligned}$$

The expectation of $2\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \mathrm{d}\hat{G}_{t_2}^{(1)} \mathrm{d}\hat{G}_{t_1}^{(1)} + \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(2)} + 2m_1 (t - t_0) \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(1)}$ is zero since the compensated processes $\hat{G}_t^{(1)}$ and $\hat{G}_t^{(2)}$ have zero means. We see that $m_1^2 t^2 + m_2 t - 2m_1^2 t t_0$ depends on t_0 which in fact cannot be the expectation of $(G_{t+t_0} - G_{t_0})^2$ since the increments of G_t are stationary. Starting from equations (A.4)-(A.6), we can find that

$$(G_{t+t_0} - G_{t_0})^2 = 2\int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \mathrm{d}\hat{G}_{t_2}^{(1)} \mathrm{d}\hat{G}_{t_1}^{(1)} + 2m_1 t \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(1)} + \int_{t_0}^{t+t_0} \mathrm{d}\hat{G}_{t_1}^{(2)} + m_1^2 t^2 + m_2 t. \quad \text{(A.7)}$$

B Proof of Proposition 2

We prove this result using strong induction. Clearly, the proposition is true for $k = 1$ and 2. Assume the proposition is true for $k = n$, where n is an integer ≥ 1. Then for $k = n + 1$, firstly we prove that the sum of the indices of all the m_q's appear in each of the terms of $C_t^{(n+1)}$ (given by Proposition 1) are equal to $n + 1$. We have

$$C_t^{(n+1)} = \sum_{j=1}^{n} \binom{n+1}{j} m_j t C_t^{(n+1-j)} - \sum_{j=1}^{n} \binom{n+1}{j} m_j \int_0^t t_1 \ \mathrm{d}C_{t_1}^{(n+1-j)} + m_{n+1} t. \quad \text{(B.1)}$$

By the induction step, the tuples of the indices of all the m_q's appearing in each of the terms of $C_t^{(n+1-j)}$ are elements of $\mathcal{L}_{n+1-j}$ defined in equation (12). Since we have $m_j C_t^{(n+1-j)}$ appearing in the first term of equation (B.1), $m_j C_{t_1}^{(n+1-j)}$ in the second term and m_{n+1} in the last term, it is clear that the tuples of the indices of all the m_q's appearing in each of the terms of $C_t^{(n+1)}$ are elements of $\mathcal{L}_{n+1}$. Now from equation (B.1), the first term can be proven to be

$$\begin{aligned}
&\sum_{j=1}^{n} \binom{n+1}{j} m_j t C_t^{(n+1-j)} \\
&= \sum_{j=1}^{n} \sum_{\phi_{n+1-j}=\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}\right) \in \mathcal{L}_{n+1-j}} \frac{1}{l!} \left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}, j\right)! \\
&\quad \times \left(p_1^{\phi_{n+1-j}}, p_2^{\phi_{n+1-j}}, \dots, p_{n+1-j}^{\phi_{n+1-j}}\right)! \left[\prod_{q \in \phi_{n+1-j} \cup \{j\}} m_q\right] t^{l+1}
\end{aligned}$$

and the second term can be shown to be

$$\begin{aligned}
&-\sum_{j=1}^{n} \binom{n+1}{j} m_j \int_0^t t_1 \, \mathrm{d}C_{t_1}^{(n+1-j)} \\
&= -\sum_{j=1}^{n} \sum_{\phi_{n+1-j}=\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}\right) \in \mathcal{L}_{n+1-j}} \frac{1}{l!} \left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}, j\right)! \\
&\quad \times \left(p_1^{\phi_{n+1-j}}, p_2^{\phi_{n+1-j}}, \dots, p_{n+1-j}^{\phi_{n+1-j}}\right)! \left[\prod_{q \in \phi_{n+1-j} \cup \{j\}} m_q\right] \frac{l}{l+1} t^{l+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
C_t^{(n+1)} &= \sum_{j=1}^{n} \sum_{\phi_{n+1-j}=\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}\right) \in \mathcal{L}_{n+1-j}} \frac{1}{l!} \left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}, j\right)! \\
&\quad \times \left(p_1^{\phi_{n+1-j}}, p_2^{\phi_{n+1-j}}, \dots, p_{n+1-j}^{\phi_{n+1-j}}\right)! \left[\prod_{q \in \phi_{n+1-j} \cup \{j\}} m_q\right] t^{l+1} \frac{1}{l+1} + m_{n+1} t.
\end{aligned}$$

Next we are going to prove that

$$\begin{aligned}
&\sum_{\phi_{n+1}=\left(i_1^{(n+1)}, i_2^{(n+1)}, \dots, i_{l+1}^{(n+1)}\right) \in \mathcal{L}_{n+1}} \frac{1}{(l+1)!} \left(i_1^{(n+1)}, i_2^{(n+1)}, \dots, i_{l+1}^{(n+1)}\right)! \\
&\times \left(p_1^{\phi_{n+1}}, p_2^{\phi_{n+1}}, \dots, p_{n+1}^{\phi_{n+1}}\right)! \left[\prod_{q \in \phi_{n+1}} m_q\right] t^{l+1} \\
&= \sum_{j=1}^{n} \sum_{\phi_{n+1-j}=\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}\right) \in \mathcal{L}_{n+1-j}} \frac{1}{(l+1)!} \left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, \dots, i_l^{(n+1-j)}, j\right)! \\
&\quad \times \left(p_1^{\phi^{n+1-j}}, p_2^{\phi^{n+1-j}}, \dots, p_{n+1-j}^{\phi^{n+1-j}}\right)! \left[\prod_{q \in \phi_{n+1-j} \cup \{j\}} m_q\right] t^{l+1} + m_{n+1} t. \qquad \text{(B.2)}
\end{aligned}$$

On the R.H.S., we are adding a j to each tuple $\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, ..., i_l^{(n+1-j)}\right)$ such that $\sum_{q=1}^{l} i_q^{(n+1-j)} + j = n+1$. Suppose $\phi_{n+1} = \left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$ has one extra element compared to the tuple $\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, ..., i_l^{(n+1-j)}\right)$ and otherwise they are the same. Since $\sum_{q=1}^{l+1} i_q^{(n+1)} = n+1$, to obtain $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$ from $\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, ..., i_l^{(n+1-j)}\right)$, we are adding an element j to the latter such that the sum of the tuple is equal to $n+1$. Suppose there are r distinct value(s) in $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$. Let $x_1, x_2, ..., x_r$ be the distinct values in $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$ and let $f_i, i = 1, ..., r$ be the number of times x_i appears in $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$. Note that $\sum_{q=1}^{r} f_q$ is equal to the length of the tuple $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$, that is, $\sum_{q=1}^{r} f_q = l+1$. Since $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right)$ can be obtained by adding an element j to a tuple $\left(i_1^{(n+1-j)}, i_2^{(n+1-j)}, ..., i_l^{(n+1-j)}\right)$ whose elements add up to $n+1-j$, j can take one of the r distinct value(s): $x_1, x_2, ..., x_r$. For example, suppose $j = x_i$, then the corresponding term on the R.H.S. of equation (B.2) is

$$\frac{(n+1)!}{(x_1!)^{f_1}(x_2!)^{f_2}\cdots(x_i!)^{f_i-1}\cdots(x_r!)^{f_r}\, x_i! f_1! \cdots (f_i-1)! \cdots f_r!}\left[\prod_{q\in\phi_{n+1}} m_q\right] t^{l+1}\frac{1}{\sum_{q=1}^{r} f_q}.$$

Summing up all the possible $j \in \{x_1, x_2, ..., x_r\}$,

$$\begin{aligned}
&\sum_{i=1}^{r}\frac{(n+1)!}{(x_1!)^{f_1}(x_2!)^{f_2}\cdots(x_i!)^{f_i-1}\cdots(x_r!)^{f_r}\, x_i! f_1! \cdots (f_i-1)! \cdots f_r!}\left[\prod_{q\in\phi_{n+1}} m_q\right] t^{l+1}\frac{1}{\sum_{q=1}^{r} f_q}\\
&= \frac{(n+1)!}{(x_1!)^{f_1}\cdots(x_r!)^{f_r}\, f_1!\cdots f_r!}\frac{1}{\sum_{q=1}^{r} f_q}\left[\prod_{q\in\phi_{n+1}} m_q\right] t^{l+1}\sum_{i=1}^{r} f_i\\
&= \frac{(n+1)!}{(x_1!)^{f_1}\cdots(x_r!)^{f_r}\, f_1!\cdots f_r!}\left[\prod_{q\in\phi_{n+1}} m_q\right] t^{l+1}.
\end{aligned}$$

For the case $\phi_{n+1} = \left(i_1^{(n+1)}\right)$, it is clear that the L.H.S. of equation (B.2) is equal to $m_{n+1}t$. Hence, by applying the same argument to each possible tuple $\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_{l+1}^{(n+1)}\right) \in \mathcal{L}_{n+1}$, we have proven equation (B.2) and therefore

$$\begin{aligned}
C_t^{(n+1)} \quad = \quad & \sum_{\phi_{n+1}=\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_l^{(n+1)}\right)\in\mathcal{L}_{n+1}} \frac{1}{l!}\left(i_1^{(n+1)}, i_2^{(n+1)}, ..., i_l^{(n+1)}\right)!\\
& \times \left(p_1^{\phi_{n+1}}, p_2^{\phi_{n+1}}, ..., p_{n+1}^{\phi_{n+1}}\right)!\left[\prod_{q\in\phi_{n+1}} m_q\right] t^l.
\end{aligned}$$

C Proof of Theorem 1

We prove the result using strong induction. Firstly we need to consider $\mathcal{I}_k$ defined by equation (10). We need to know what tuples are in $\mathcal{I}_{k+1}$ but not in $\mathcal{I}_k$, and these correspond to those elements adding up exactly to $k+1$. Let $\mathcal{J}_{k+1}$ be the collection of these tuples, that is, $\mathcal{I}_{k+1} \equiv \mathcal{I}_k \cup \mathcal{J}_{k+1}$. We have

$$\mathcal{J}_{k+1} = \left\{(i_1, i_2, ..., i_j) \mid j \in \{1, 2, ..., k+1\},\ i_p \in \{1, 2, ..., k+1\} \text{ and } \sum_{p=1}^{j} i_p = k+1\right\}.$$

To construct $\mathcal{J}_{k+1}$ from $\mathcal{I}_k$, we can simply add an element to the end of each tuple in $\mathcal{I}_k$ so that the elements of each new tuple add up exactly to $k+1$, and finally including the tuple $(k+1)$ in $\mathcal{J}_{k+1}$.

We are going to prove by strong induction that $(G_{t+t_0} - G_{t_0})^k = \sum_{\theta_k \in \mathcal{I}_k} \Pi^{(k)}_{\theta_k,t} \mathcal{S}_{\theta_k,t,t_0} + C^{(k)}_t$ for any non-negative integer k. For $k = 0$, clearly both sides equal 1. For $k = 1$ and 2, it can be checked easily that the proposition is true. Assume the proposition is true for $k = 0, 1, 2, ..., n$, where n is a positive integer. Note that it is sufficient to prove the representation for G^{n+1}_t only since we can always let $G_{t+t_0} - G_{t_0} = F_t$, which is also a Lévy process and we have, the i-th power jump process of $\{F_t, t \geq 0\}$, $F^{(i)}_t = G^{(i)}_{t+t_0} - G^{(i)}_{t_0}$ for $i = 1, 2, 3, ...$ Since both $\{F_t, t \geq 0\}$ and $\{G_t, t \geq 0\}$ are created by the same infinitely divisible distribution, the compensators for their i-th power jump processes are both equal to $m_i t$. Hence, we have the i-th compensated power jump process of $\{F_t, t \geq 0\}$,

$$\hat{F}^{(i)}_t = \hat{G}^{(i)}_{t+t_0} - \hat{G}^{(i)}_{t_0}. \tag{C.1}$$

For $k = n + 1$, by equations (A.4)-(A.6),

$$\begin{aligned} G^{n+1}_t &= \sum_{j=1}^{n+1} \binom{n+1}{j} \int_0^t G^{n+1-j}_{t_1-} \, \mathrm{d}\hat{G}^{(j)}_{t_1} + \sum_{j=1}^{n} \binom{n+1}{j} m_j t G^{n+1-j}_t \\ & \quad - \sum_{j=1}^{n} \binom{n+1}{j} m_j \int_0^t t_1 \, \mathrm{d}G^{n+1-j}_{t_1} + m_{n+1} t. \end{aligned} \tag{C.2}$$

Firstly, we want to prove that all the stochastic integrals in G^{n+1}_t is of the form $\mathcal{S}_{\theta_{n+1},t,0}$, where $\theta_{n+1} \in \mathcal{I}_{n+1}$. From equation (C.2), it is clear that the first term is the only term introducing new stochastic integrals which are not in $\mathcal{I}_n$. The general term of the stochastic integrals in the first term is

$$\int_0^t G^{n+1-j}_{t_1-} \, \mathrm{d}\hat{G}^{(j)}_{t_1}, \quad j = 1, 2, ..., n+1. \tag{C.3}$$

By assumption,

$$G^{n+1-j}_{t_1-} = \sum_{\theta_{n+1-j} \in \mathcal{I}_{n+1-j}} \Pi^{(n+1-j)}_{\theta_{n+1-j},t_1} \mathcal{S}_{\theta_{n+1-j},t_1,0} + C^{(n+1-j)}_{t_1}, \ j = 1, 2, ..., n+1.$$

When $j = 1$ in equation (C.3), we have $\int_0^t G^n_{t_1-} \mathrm{d}\hat{G}^{(1)}_{t_1}$, meaning that we are adding a 1 to the end of all tuples in $\mathcal{I}_n$. Since by definition

$$\mathcal{I}_n = \left\{ (i_1, i_2, ..., i_j) \mid j \in \{1, 2, ..., n\}, \ i_p \in \{1, 2, ..., n\} \text{ and } \sum_{p=1}^{j} i_p \leq n \right\},$$

we know that the sum of the elements of the new tuples we get from adding a 1 to the end of each tuple of $\mathcal{I}_n$ is less than or equal to $n + 1$. Similarly, when $j = 2$, we have $\int_0^t G^{n-1}_{t_1-} \mathrm{d}\hat{G}^{(2)}_{t_1}$, meaning that we are adding a 2 to the end of all tuples in $\mathcal{I}_{n-1}$ and since by definition

$$\mathcal{I}_{n-1} = \left\{ (i_1, i_2, ..., i_j) \mid j \in \{1, 2, ..., n-1\}, \ i_p \in \{1, 2, ..., n-1\} \text{ and } \sum_{p=1}^{j} i_p \leq n-1 \right\},$$

we know that the sum of the elements of the new tuples we get from adding a 2 to the end of each tuple of

$\mathcal{I}_{n-1}$ is less than or equal to $n+1$. We can continue the same argument until $j=n$. When $j=n+1$, we have $\int_0^t \mathrm{d}\hat{G}_{t_1}^{(n+1)}$. Since $\mathcal{I}_n \supset \mathcal{I}_{n-1} \supset ... \supset \mathcal{I}_2 \supset \mathcal{I}_1$, the above way of introducing new stochastic integrals is the same as adding an element to the end of each tuple in $\mathcal{I}_n$ so that the elements of each new tuple add up exactly to $n+1$. Hence all the elements in $\mathcal{J}_{n+1}$ have been created and since $\mathcal{I}_{n+1} \equiv \mathcal{I}_n \cup \mathcal{J}_{n+1}$, we have proved that all the stochastic integrals in G_t^{n+1} have the form $\mathcal{S}_{\theta_{n+1},t,0}$, where $\theta_{n+1} \in \mathcal{I}_{n+1}$.

By definition, $C_t^{(n+1)}$ is the term in G_t^{n+1} not containing any stochastic integral. Hence it is correct to write $C_t^{(n+1)}$ as the final term.

Finally, we want to consider the coefficients of the stochastic integrals, that is, we are going to prove Proposition 3. By assumption of the induction step, we have

$$\begin{aligned}
G_t^{n+1} &= \sum_{j=1}^{n} \binom{n+1}{j} \sum_{\theta_{n+1-j} \in \mathcal{I}_{n+1-j}} \int_0^t \Pi_{\theta_{n+1-j},t_1}^{(n+1-j)} \mathcal{S}_{\theta_{n+1-j},t_1,0} \, \mathrm{d}\hat{G}_{t_1}^{(j)} \\
&\quad + \sum_{j=1}^{n} \binom{n+1}{j} m_j t \sum_{\theta_{n+1-j} \in \mathcal{I}_{n+1-j}} \Pi_{\theta_{n+1-j},t}^{(n+1-j)} \mathcal{S}_{\theta_{n+1-j},t,0} \\
&\quad - \sum_{j=1}^{n} \binom{n+1}{j} m_j \sum_{\theta_{n+1-j} \in \mathcal{I}_{n+1-j}} \int_0^t t_1 \, \mathrm{d}\left[\Pi_{\theta_{n+1-j},t_1}^{(n+1-j)} \mathcal{S}_{\theta_{n+1-j},t_1,0}\right] \\
&\quad + \sum_{j=1}^{n} \binom{n+1}{j} \int_0^t C_{t_1}^{(n+1-j)} \, \mathrm{d}\hat{G}_{t_1}^{(j)} + \sum_{j=1}^{n} \binom{n+1}{j} m_j t C_t^{(n+1-j)} \\
&\quad - \sum_{j=1}^{n} \binom{n+1}{j} m_j \int_0^t t_1 \, \mathrm{d}\left[C_{t_1}^{(n+1-j)}\right] + \int_0^t \mathrm{d}\hat{G}_{t_1}^{(n+1)} + m_{n+1} t \\
&= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + \int_0^t \mathrm{d}\hat{G}_{t_1}^{(n+1)} + m_{n+1} t. \qquad \text{(C.4)}
\end{aligned}$$

Let $\mathcal{K}_{l,s} = \left\{ (i_1, ..., i_l) \,|\, i_j \in \{1, 2, ..., s\} \text{ and } \sum_{j=1}^{l} i_j = s \right\}$. Since the length of a tuple must not be greater than the sum of all the elements in the tuple (because an element must be at least 1), $l \leq s$. By definition, we have $\mathcal{I}_n = \bigcup_{s=1}^{n} \bigcup_{l=1}^{s} \mathcal{K}_{l,s}$. For any $\theta_{l,s} \in \mathcal{K}_{l,s}$, let $\theta_{l,s} = \left(i_1^{\theta_{l,s}}, i_2^{\theta_{l,s}}, ..., i_l^{\theta_{l,s}} \right)$. It is obvious from Proposition 1 that $C_t^{(k)}$ has the form $C_t^{(k)} = q_0^{(k)} + q_1^{(k)} t + q_2^{(k)} t^2 + ... + q_k^{(k)} t^k$. Note that $q_0^{(k)}$ is non-zero only when $k=0$. When $k=0$, by definition $C_t^{(k)} = 1$, so we have $q_0^{(0)} = 1$. We need to find out the recursive relationships between the $q_r^{(k)}$'s. From equation (11), for $k > 1$,

$$\begin{aligned}
q_1^{(k)} t + q_2^{(k)} t^2 + \cdots + q_k^{(k)} t^k &= \sum_{j=1}^{k-1} \binom{k}{j} m_j t \left[q_1^{(k-j)} t + q_2^{(k-j)} t^2 + \cdots + q_{k-j}^{(k-j)} t^{k-j} \right] \\
&\quad - \sum_{j=1}^{k-1} \binom{k}{j} m_j \int_0^t t_1 \, \mathrm{d}\left[q_1^{(k-j)} t_1 + q_2^{(k-j)} t_1^2 + \cdots + q_{k-j}^{(k-j)} t_1^{k-j} \right] + m_k t \\
&= m_k t + \sum_{j=1}^{k-1} \binom{k}{j} m_j \left[q_1^{(k-j)} t^2 + q_2^{(k-j)} t^3 + \cdots + q_{k-j}^{(k-j)} t^{k-j+1} \right] \\
&\quad - \sum_{j=1}^{k-1} \binom{k}{j} m_j \left[\frac{1}{2} q_1^{(k-j)} t^2 + \frac{2}{3} q_2^{(k-j)} t^3 + \cdots + \frac{k-j}{k-j+1} q_{k-j}^{(k-j)} t^{k-j+1} \right].
\end{aligned}$$

By comparing the coefficients of t, $q_1^{(k)} = m_k$. By comparing the coefficients of t^r, $r = 2, ..., k$,

$$q_r^{(k)} = \sum_{j=1}^{k+1-r} \binom{k}{j} m_j q_{r-1}^{(k-j)} - \sum_{j=1}^{k+1-r} \binom{k}{j} m_j \frac{r-1}{r} q_{r-1}^{(k-j)} = \frac{1}{r} \sum_{j=1}^{k+1-r} \binom{k}{j} m_j q_{r-1}^{(k-j)}. \tag{C.5}$$

To ease notation, we let

$$\begin{aligned} \mathbf{F}_1 &= \frac{(n+1)!}{i_1^{\theta_{l,s}}! i_2^{\theta_{l,s}}! \cdots i_l^{\theta_{l,s}}! j!}, \ \mathbf{F}_2 = \frac{\mathbf{F}_1}{(n+1-j-s)!}, \ \mathbf{G}_i^j = \hat{G}_{t_i}^{\left(i_j^{\theta_{l,s}}\right)}, \ \mathbf{I}_1 = \int_0^{t_1-} \cdots \int_0^{t_l-} \mathrm{d}\mathbf{G}_{l+1}^1 \cdots \mathrm{d}\mathbf{G}_2^l, \\ \mathbf{I}_2 &= \int_0^{t_1-} \cdots \int_0^{t_{l-1}-} \mathrm{d}\mathbf{G}_l^1 \cdots \mathrm{d}\mathbf{G}_2^{l-1}, \ \mathbf{I}_3 = \int_0^t \mathbf{I}_2 \ \mathrm{d}\mathbf{G}_1^l, \ \tilde{\mathbf{q}}_i = q_i^{(n+1-j-s)}. \end{aligned}$$

Note that it is only for simplicity in writing out the equations. When doing calculation, we should always use the long but clear notation. So, we have

$$\begin{aligned} L_1 &= \sum_{j=1}^{n} \sum_{s=1}^{n+1-j} \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \int_0^t \mathbf{F}_2 \sum_{w=0}^{n+1-j-s} \tilde{\mathbf{q}}_w t_1^w \mathbf{I}_1 \ \mathrm{d}\hat{G}_{t_1}^{(j)}. \\ L_2 &= \sum_{j=1}^{n} m_j \sum_{s=1}^{n+1-j} \left\{ 1_{\{s=n+1-j\}} t \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \mathbf{F}_1 \mathbf{I}_3 + 1_{\{s \le n-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w t^{w+1} \mathbf{I}_3 \right\} \\ L_3 &= -\sum_{j=1}^{n} m_j \sum_{s=1}^{n+1-j} \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \mathbf{F}_2 \left\{ \tilde{\mathbf{q}}_0 \int_0^t t_1 \mathbf{I}_2 \ \mathrm{d}\mathbf{G}_1^l + \frac{1}{2} \tilde{\mathbf{q}}_1 t^2 \mathbf{I}_3 + \frac{1}{2} \tilde{\mathbf{q}}_1 \int_0^t t_1^2 \mathbf{I}_2 \ \mathrm{d}\mathbf{G}_1^l \right. \\ &\quad \left. + \sum_{w=2}^{n+1-j-s} \tilde{\mathbf{q}}_w \left\{ \frac{w}{w+1} t^{w+1} \mathbf{I}_3 + \frac{1}{w+1} \int_0^t t_1^{w+1} \mathbf{I}_2 \ \mathrm{d}\mathbf{G}_1^l \right\} \right\}. \\ L_4 &= \sum_{j=1}^{n} \binom{n+1}{j} \int_0^t \sum_{w=1}^{n+1-j} q_w^{(n+1-j)} t_1^w \ \mathrm{d}\hat{G}_{t_1}^{(j)}. \\ L_5 &= \sum_{j=1}^{n} \binom{n+1}{j} m_j \sum_{w=1}^{n+1-j} q_w^{(n+1-j)} t^{w+1}. \\ L_6 &= -\sum_{j=1}^{n} \binom{n+1}{j} m_j \sum_{w=1}^{n+1-j} q_w^{(n+1-j)} \left\{ \frac{w}{w+1} t^{w+1} \right\}. \end{aligned}$$

Next, consider L_1 and L_3. Let $u, v \in \{1, 2, ..., n-1\}$ and $u + v \le n$. In L_1, when $j = u$, $s = v$ (hence $s \le n - j$),

$$\begin{aligned} L_1 &= \sum_{l=1}^{v} \sum_{\theta_{l,v} \in \mathcal{K}_{l,v}} \int_0^t \frac{(n+1)!}{i_1^{\theta_{l,v}}! i_2^{\theta_{l,v}}! \cdots i_l^{\theta_{l,v}}! u!} \frac{1}{(n+1-u-v)!} \left\{ m_{n+1-u-v} t_1 + \sum_{w=2}^{n+1-u-v} q_w^{(n+1-u-v)} t_1^w \right\} \\ &\quad \times \int_0^{t_1-} \int_0^{t_2-} \cdots \int_0^{t_l-} \mathrm{d}\hat{G}_{t_{l+1}}^{\left(i_1^{\theta_{l,v}}\right)} \cdots \mathrm{d}\hat{G}_{t_2}^{\left(i_l^{\theta_{l,v}}\right)} \mathrm{d}\hat{G}_1^{(u)}. \end{aligned} \tag{C.6}$$

Since $s = v$, $l \in \{1, 2, ..., v\}$, we have by definition $\left(i_1^{\theta_{l,v}}, i_2^{\theta_{l,v}}, ..., i_l^{\theta_{l,v}}\right) \in \mathcal{J}_v$. In L_3, when $j = n + 1 - u - v$

(hence $j \in \{1,...,n-1\}$), $s = u+v$ (hence $s = n+1-j$) and $i_l^{\theta_{l,s}} = u$ (hence $i_l^{\theta_{l,s}} < s$),

$$L_3 = -m_{n+1-u-v} \sum_{l=1}^{u+v} \sum_{\theta_{l,u+v} \in \mathcal{K}_{l,u+v}} \frac{(n+1)!}{i_1^{\theta_{l,u+v}}! i_2^{\theta_{l,u+v}}! \cdots i_{l-1}^{\theta_{l,u+v}}! u! (n+1-u-v)!} \times \int_0^t t_1 \int_0^{t_1-} \int_0^{t_2-} \cdots \int_0^{t_{l-1}-} d\hat{G}_{t_l}^{\left(i_1^{\theta_{l,u+v}}\right)} \cdots d\hat{G}_{t_2}^{\left(i_{l-1}^{\theta_{l,u+v}}\right)} d\hat{G}_{t_1}^{(u)}.$$

Since $s = u+v$ and $i_l^{\theta_{l,s}} = u$, $\sum_{p=1}^{l-1} i_p^{\theta_{l,u+v}} = v$, we have by definition $\left(i_1^{\theta_{l,u+v}}, i_2^{\theta_{l,u+v}}, ..., i_{l-1}^{\theta_{l,u+v}}\right) \in \mathcal{J}_v$. Hence the terms

$$\sum_{l=1}^{v} \sum_{\theta_{l,v} \in \mathcal{K}_{l,v}} \int_0^t \frac{(n+1)!}{i_1^{\theta_{l,v}}! i_2^{\theta_{l,v}}! \cdots i_l^{\theta_{l,v}}! u!} \frac{1}{(n+1-u-v)!} m_{n+1-u-v} t_1 \times \int_0^{t_1-} \int_0^{t_2-} \cdots \int_0^{t_l-} d\hat{G}_{t_{l+1}}^{\left(i_1^{\theta_{l,v}}\right)} \cdots d\hat{G}_{t_2}^{\left(i_l^{\theta_{l,v}}\right)} d\hat{G}_1^{(u)}$$

in L_1 and

$$-m_{n+1-u-v} \sum_{l=1}^{u+v} \sum_{\theta_{l,u+v} \in \mathcal{K}_{l,u+v}} \frac{(n+1)!}{i_1^{\theta_{l,u+v}}! i_2^{\theta_{l,u+v}}! \cdots i_{l-1}^{\theta_{l,u+v}}! u! (n+1-u-v)!} \times \int_0^t t_1 \int_0^{t_1-} \int_0^{t_2-} \cdots \int_0^{t_{l-1}-} d\hat{G}_{t_l}^{\left(i_1^{\theta_{l,u+v}}\right)} \cdots d\hat{G}_{t_2}^{\left(i_{l-1}^{\theta_{l,u+v}}\right)} d\hat{G}_{t_1}^{(u)}$$

cancel each other. So we now have

$$L_1 = \sum_{j=1}^{n} \left\{ 1_{\{j \le n-1\}} \sum_{s=1}^{n+1-j} \left\{ 1_{\{s \le n-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \int_0^t \mathbf{F}_2 \sum_{w=2}^{n+1-j-s} \tilde{\mathbf{q}}_w t_1^w \mathbf{I}_1 \, d\hat{G}_{t_1}^{(j)} + 1_{\{s=n+1-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \int_0^t \mathbf{F}_1 \mathbf{I}_1 \, d\hat{G}_{t_1}^{(j)} \right\} + 1_{\{j=n\}} (n+1) \int_0^t \int_0^{t_1-} d\hat{G}_{t_2}^{(1)} d\hat{G}_{t_1}^{(n)} \right\}.$$

Since $q_0^{(k)} = 0$ for $k > 0$,

$$L_3 = -\sum_{j=1}^{n} m_j \left\{ 1_{\{j \le n-1\}} \sum_{s=1}^{n+1-j} \left\{ 1_{\{s \le n-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s} \in \mathcal{K}_{l,s}} \mathbf{F}_2 \left\{ \frac{1}{2} \tilde{\mathbf{q}}_1 t^2 \mathbf{I}_3 + \frac{1}{2} \tilde{\mathbf{q}}_1 \int_0^t t_1^2 \mathbf{I}_2 \, d\mathbf{G}_1^l + \sum_{w=2}^{n+1-j-s} \tilde{\mathbf{q}}_w \left\{ \frac{w}{w+1} t^{w+1} \mathbf{I}_3 + \frac{1}{w+1} \int_0^t t_1^{w+1} \mathbf{I}_2 \, d\mathbf{G}_1^l \right\} \right\} + 1_{\{s=n+1-j\}} \frac{(n+1)!}{(n+1-j)! j!} \int_0^t t_1 \, d\hat{G}_{t_1}^{(n+1-j)} \right\} + 1_{\{j=n\}} m_n (n+1) \int_0^t t_1 \, d\hat{G}_{t_1}^{(1)} \right\}.$$

Next, consider L_3 and L_4. Let $u \in \{1, 2, ..., n\}$. In L_3, when $j = n+1-u$ (hence $j \in \{1,...,n\}$), $s = u$ (hence $s = n+1-j$) and $i_l^{\theta_{l,s}} = u$ (hence $i_l^{\theta_{l,s}} = s$), we have

$$L_3 = m_{n+1-u} \frac{(n+1)!}{u! (n+1-u)!} \int_0^t t_1 \, d\hat{G}_{t_1}^{(u)}.$$

In L_4, when $j=u$, we have

$$L_4 = \binom{n+1}{u}\int_0^t \left\{ m_{n+1-u}t_1 + \sum_{w=2}^{n+1-u} q_w^{(n+1-u)} t_1^w \right\} \mathrm{d}\hat{G}_{t_1}^{(u)}.$$

Hence the terms

$$m_{n+1-u}\frac{(n+1)!}{u!\,(n+1-u)!}\int_0^t t_1 \ \mathrm{d}\hat{G}_{t_1}^{(u)}$$

in L_3 and

$$\binom{n+1}{u}\int_0^t m_{n+1-u}t_1 \ \mathrm{d}\hat{G}_{t_1}^{(u)}$$

cancel each other. In L_3, since the terms where $(j=n)$ and $(j \le n-1,\ s=n+1-j)$ get cancelled, we can sum j from 1 to $n-1$ and sum s from 1 to $n-j$.

$$\begin{aligned} L_3 &= -\sum_{j=1}^{n-1} m_j \sum_{s=1}^{n-j}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \left\{ \frac{w}{w+1}t^{w+1}\mathbf{I}_3 + \frac{1}{w+1}\int_0^t t_1^{w+1}\mathbf{I}_2 \ \mathrm{d}\mathbf{G}_1^l \right\}. \\ L_4 &= \sum_{j=1}^{n-1} \left\{ \binom{n+1}{j} \int_0^t \sum_{w=2}^{n+1-j} \frac{1}{w} \left[\sum_{z=1}^{n+2-j-w} \binom{n+1-j}{z} m_z q_{w-1}^{(n+1-j-z)} \right] t_1^w \ \mathrm{d}\hat{G}_{t_1}^{(j)} \right\}. \end{aligned}$$

by equation (C.5).

Consider L_4 and L_3. Let $u \in \{1,2,...,n-1\}$, $v \in \{1,2,...,n-1\}$, $u+v \le n$, $x \in \{1,2,...,v\}$ and hence $x+u \le n$. In L_4, when $j=u$, $w=n+2-u-v$ (hence $w \in \{2,...,n+1-j\}$), $z=x$ (hence $z \in \{1,...,n+2-j-w\}$),

$$L_4 = \binom{n+1}{u}\int_0^t \frac{1}{n+2-u-v}\binom{n+1-u}{x} m_x q_{n+1-u-v}^{(n+1-u-x)} t_1^{n+2-u-v} \ \mathrm{d}\hat{G}_{t_1}^{(u)}. \tag{C.7}$$

In L_3, when $j=x$ (hence $j \in \{1,...,n-1\}$), $s=u$ (hence $s \le n-j$), $i_l^{\theta_{l,s}}=u$ (hence $i_l^{\theta_{l,s}}=s$), $w=n+1-u-v$ (hence $w \le n+1-s-j$),

$$\begin{aligned} L_3 &= m_x \frac{(n+1)!}{u!x!}\frac{1}{(n+1-x-u)!} q_{n+1-u-v}^{(n+1-u-x)} \\ &\quad \times \left\{ \frac{n+1-u-v}{n+2-u-v} t^{n+2-u-v} \int_0^t \mathrm{d}\hat{G}_{t_1}^{(u)} + \frac{1}{n+2-u-v}\int_0^t t_1^{n+2-u-v} \ \mathrm{d}\hat{G}_{t_1}^{(u)} \right\} \end{aligned}$$

where the 2^{nd} term cancels equation (C.7). So now we have

$$\begin{aligned} L_4 &= 0. \\ L_3 &= -\sum_{j=1}^{n-1} m_j \sum_{s=1}^{n-j} \left\{ 1_{\{s=1\}} \frac{(n+1)!}{j!}\frac{1}{(n-j)!}\sum_{w=1}^{n-j} q_w^{(n-j)} \frac{w}{w+1} t^{w+1}\int_0^t \mathrm{d}\hat{G}_{t_1}^{(1)} \right. \\ &\quad + 1_{\{2\le s\le n-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \left\{ 1_{\left\{i_l^{\theta_{l,s}}<s\right\}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \left\{ \frac{w}{w+1}t^{w+1}\mathbf{I}_3 + \frac{1}{w+1}\int_0^t t_1^{w+1}\mathbf{I}_2 \ \mathrm{d}\mathbf{G}_1^l \right\} \right. \\ &\quad \left.\left. + 1_{\left\{i_l^{\theta_{l,s}}=s\right\}} \frac{(n+1)!}{s!j!}\frac{1}{(n+1-j-s)!} \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \frac{w}{w+1} t^{w+1} \int_0^t \mathrm{d}\hat{G}_{t_1}^{(s)} \right\}\right\}. \end{aligned}$$

Next, consider L_1 and L_3. By the equation for $q_w^{(n+1-j-s)}$ given in equation (C.5), we have

$$\begin{aligned} L_1 &= \sum_{j=1}^{n}\left\{1_{\{j\le n-1\}}\sum_{s=1}^{n+1-j}\left\{1_{\{s\le n-1-j\}}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}}\int_0^t \mathbf{F}_2\sum_{w=2}^{n+1-j-s}\frac{1}{w}\right.\right. \\ &\quad\times\sum_{z=1}^{n+2-j-s-w}\binom{n+1-j-s}{z}m_z q_{w-1}^{(n+1-j-s-z)}t_1^w\mathbf{I}_1\,\mathrm{d}\hat{G}_{t_1}^{(j)} \\ &\quad\left.\left.+1_{\{s=n+1-j\}}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}}\int_0^t \mathbf{F}_1\mathbf{I}_1\,\mathrm{d}\hat{G}_{t_1}^{(j)}\right\}+1_{\{j=n\}}\,(n+1)\int_0^t\int_0^{t_1-}\mathrm{d}\hat{G}_{t_2}^{(1)}\mathrm{d}\hat{G}_{t_1}^{(n)}\right\}. \end{aligned}$$

Let $u\in\{1,2,...,n-2\}$, $v\in\{1,2,...,n-2\}$, $u+v\le n-1$, $x\in\{1,2,...,v\}$, $\beta\in\{1,2,...,v+1-x\}$. In L_1, when $j=u$, $s=n-u-v$ (hence $s\in\{1,...,n-1-j\}$), $w=x+1$ (hence $w\in\{2,...,n+1-j-s\}$), $z=\beta$ (hence $z\in\{1,...,n+2-j-s-w\}$),

$$\begin{aligned} L_1 &= \sum_{l=1}^{n-u-v}\sum_{\theta_{l,n-u-v}\in\mathcal{K}_{l,n-u-v}}\int_0^t\frac{(n+1)!}{i_1^{\theta_{l,n-u-v}}!i_2^{\theta_{l,n-u-v}}!\cdots i_l^{\theta_{l,n-u-v}}!u!}\frac{1}{(v+1)!}\frac{1}{x+1}\binom{v+1}{\beta} \\ &\quad\times m_\beta q_x^{(v+1-\beta)}t_1^{x+1}\int_0^{t_1-}\cdots\int_0^{t_l-}\mathrm{d}\hat{G}_{t_{l+1}}^{\left(i_1^{\theta_{l,n-u-v}}\right)}\cdots\mathrm{d}\hat{G}_{t_2}^{\left(i_l^{\theta_{l,n-u-v}}\right)}\mathrm{d}\hat{G}_{t_1}^{(u)}. \end{aligned} \tag{C.8}$$

By definition, since $s=n-u-v$ and $l\in\{1,2,...,n-u-v\}$, $\left(i_1^{\theta_{l,n-u-v}},i_2^{\theta_{l,n-u-v}},...,i_l^{\theta_{l,n-u-v}}\right)\in\mathcal{J}_{n-u-v}$. In L_3, when $j=\beta$ (hence $j\in\{1,...,n-2\}$), $s=n-v$ (hence $s\in\{2,...,n-j\}$), $i_l^{\theta_{l,s}}=u$ (hence $i_l^{\theta_{l,s}}<s$), $w=x$ (hence $w\in\{1,...,n+1-j-s\}$),

$$\begin{aligned} L_3 &= -m_\beta\sum_{l=1}^{n-v}\sum_{\theta_{l,n-v}\in\mathcal{K}_{l,n-v}}\frac{(n+1)!}{i_1^{\theta_{l,n-v}}!i_2^{\theta_{l,n-v}}!\cdots i_{l-1}^{\theta_{l,n-v}}!u!\beta!}\frac{1}{(v+1-\beta)!} \\ &\quad\times q_x^{(v+1-\beta)}\left\{\frac{x}{x+1}t^{x+1}\int_0^t\int_0^{t_1-}\cdots\int_0^{t_{l-1}-}\mathrm{d}\hat{G}_{t_l}^{\left(i_1^{\theta_{l,n-v}}\right)}\cdots\mathrm{d}\hat{G}_{t2}^{\left(i_{l-1}^{\theta_{l,n-v}}\right)}\mathrm{d}\hat{G}_{t_1}^{(u)}\right. \\ &\quad\left.+\frac{1}{x+1}\int_0^t t_1^{x+1}\int_0^{t_1-}\cdots\int_0^{t_{l-1}-}\mathrm{d}\hat{G}_{t_l}^{\left(i_1^{\theta_{l,n-v}}\right)}\cdots\mathrm{d}\hat{G}_{t2}^{\left(i_{l-1}^{\theta_{l,n-v}}\right)}\mathrm{d}\hat{G}_{t_1}^{(u)}\right\}. \end{aligned}$$

The final term in L_3

$$\begin{aligned} &-m_\beta\sum_{l=1}^{n-v}\sum_{\theta_{l,n-v}\in\mathcal{K}_{l,n-v}}\frac{(n+1)!}{i_1^{\theta_{l,n-v}}!i_2^{\theta_{l,n-v}}!\cdots i_{l-1}^{\theta_{l,n-v}}!u!\beta!}\frac{1}{(v+1-\beta)!} \\ &\times q_x^{(v+1-\beta)}\frac{1}{x+1}\int_0^t t_1^{x+1}\int_0^{t_1-}\cdots\int_0^{t_{l-1}-}\mathrm{d}\hat{G}_{t_l}^{\left(i_1^{\theta_{l,n-v}}\right)}\cdots\mathrm{d}\hat{G}_{t2}^{\left(i_{l-1}^{\theta_{l,n-v}}\right)}\mathrm{d}\hat{G}_{t_1}^{(u)} \end{aligned}$$

clearly cancels equation (C.8) in L_1. So now we can write

$$
\begin{aligned}
L_1 &= \sum_{j=1}^{n} 1_{\{j\le n\}} \sum_{s=1}^{n+1-j} 1_{\{s=n+1-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \int_0^t \mathbf{F}_1\mathbf{I}_1 \, \mathrm{d}\hat{G}_{t_1}^{(j)}. \\
L_3 &= -\sum_{j=1}^{n-1} m_j \sum_{s=1}^{n-j} \Bigg\{ 1_{\{s=1\}} \frac{(n+1)!}{j!} \frac{1}{(n-j)!} \sum_{w=1}^{n-j} q_w^{(n-j)} \frac{w}{w+1} t^{w+1} \int_0^t \mathrm{d}\hat{G}_{t_1}^{(1)} \\
&\quad +1_{\{2\le s\le n-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \Bigg\{ 1_{\left\{i_l^{\theta_{l,s}}<s\right\}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \frac{w}{w+1} t^{w+1} \mathbf{I}_3 \\
&\quad +1_{\left\{i_l^{\theta_{l,s}}=s\right\}} \frac{(n+1)!}{s!j!} \frac{1}{(n+1-j-s)!} \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \frac{w}{w+1} t^{w+1} \int_0^t \mathrm{d}\hat{G}_{t_1}^{(s)} \Bigg\} \Bigg\}.
\end{aligned}
$$

We can now simplify it as

$$
L_3 = -\sum_{j=1}^{n-1} m_j \sum_{s=1}^{n-j} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \frac{w}{w+1} t^{w+1} \, \mathbf{I}_3.
$$

All together, we have

$$
\begin{aligned}
L_1 &= \sum_{j=1}^{n} 1_{\{j\le n\}} \left\{ \sum_{s=1}^{n+1-j} 1_{\{s=n+1-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \int_0^t \mathbf{F}_1\mathbf{I}_1 \, \mathrm{d}\hat{G}_{t_1}^{(j)} \right\}. \\
L_2 &= \sum_{j=1}^{n} m_j \sum_{s=1}^{n+1-j} \left\{ 1_{\{s=n+1-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} t\mathbf{F}_1\mathbf{I}_3 + 1_{\{s\le n-j\}} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w t^{w+1} \mathbf{I}_3 \right\}. \\
L_3 &= -\sum_{j=1}^{n-1} m_j \sum_{s=1}^{n-j} \sum_{l=1}^{s} \sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \mathbf{F}_2 \sum_{w=1}^{n+1-j-s} \tilde{\mathbf{q}}_w \frac{w}{w+1} t^{w+1} \mathbf{I}_3. \\
L_4 &= 0. \\
L_5 &= \sum_{j=1}^{n} \binom{n+1}{j} m_j \sum_{w=1}^{n+1-j} q_w^{(n+1-j)} t^{w+1}. \\
L_6 &= -\sum_{j=1}^{n} \binom{n+1}{j} m_j \sum_{w=1}^{n+1-j} q_w^{(n+1-j)} \frac{w}{w+1} t^{w+1}.
\end{aligned}
$$

Since at the beginning of the proof, we have already showed that the stochastic integrals of G_t^{n+1} are of the form $\mathcal{S}_{\theta_{n+1,t},0}$ where $\theta_{n+1,t} \in \mathcal{I}_{n+1}$. We are now going to show that the coefficient of each $\mathcal{S}_{\theta_{n+1,t},0}$ is $\Pi_{\theta_{n+1,t}}^{(n+1)}$. Consider $\int_0^{t_1-}\int_0^{t_2-}\cdots\int_0^{t_l-} \mathrm{d}\hat{G}_{t_{l+1}}^{\left(i_1^{\theta_{l,s}}\right)} \cdots \mathrm{d}\hat{G}_{t_2}^{\left(i_l^{\theta_{l,s}}\right)} \mathrm{d}\hat{G}_{t_1}^{(j)}$ where $\theta_{l,s} \in \mathcal{K}_{l,s}$, $j \in \{1,2,...,n\}$, $s = n+1-j$. This stochastic integral only appears in L_1. And its coefficient is $\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!j!}$. And from equation (14), since $n+1-s-j = n+1-(n+1-j)-j = 0$,

$$
\Pi^{(n+1)}_{\left(i_1^{\theta_{l,s}},i_2^{\theta_{l,s}},\ldots,i_l^{\theta_{l,s}},j\right)} = \frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!j!0!} C_t^{(0)} = \frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!j!}
$$

since $C_t^{(0)} = 1$ by definition given in equation (11). Hence we have proved that the coefficient is given by

$\Pi^{(n+1)}_{\left(i_1^{\theta_{l,s}},i_2^{\theta_{l,s}},\ldots,i_l^{\theta_{l,s}},j\right)}$. Next, we change the summation sign of j and s in L_2.

$$\begin{aligned} L_2 &= \sum_{s=1}^{n}\left\{\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} m_{n+1-s}t\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!\,(n+1-s)!}\,\mathbf{I}_3 \right.\\ &\left. +\sum_{j=1}^{n-s}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} m_j\mathbf{F}_2\sum_{w=1}^{n+1-j-s}\tilde{\mathbf{q}}_w t^{w+1}\mathbf{I}_3\right\}. \end{aligned}$$

Similarly, by changing the summation sign of j and w, we have

$$\begin{aligned} L_2 &= \sum_{s=1}^{n}\left\{\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} m_{n+1-s}t\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!\,(n+1-s)!}\mathbf{I}_3 \right.\\ &\left. +\sum_{w=1}^{n-s}\sum_{j=1}^{n+1-w-s}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} m_j\mathbf{F}_2\tilde{\mathbf{q}}_w t^{w+1}\mathbf{I}_3\right\}. \end{aligned}$$

By equation (C.5), $\frac{1}{w+1}\sum_{j=1}^{n+1-w-s}\frac{(n+1-s)!}{j!(n+1-j-s)!}m_j q_w^{(n+1-s-j)} = q_{w+1}^{(n+1-s)}$, so we have

$$\begin{aligned} L_2 &= \sum_{s=1}^{n}\left\{\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} m_{n+1-s}t\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!\,(n+1-s)!}\mathbf{I}_3 \right.\\ &\left. +\sum_{w=1}^{n-s}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\,(w+1)\,\frac{1}{(n+1-s)!}q_{w+1}^{(n+1-s)}t^{w+1}\mathbf{I}_3\right\}. \end{aligned}$$

Changing $\sum_{w=1}^{n-s}$ to $\sum_{w=2}^{n+1-s}$, we have

$$\begin{aligned} L_2 &= \sum_{s=1}^{n}\left\{\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} m_{n+1-s}t\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!\,(n+1-s)!}\mathbf{I}_3 \right.\\ &\left. +\sum_{w=2}^{n+1-s}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\frac{w}{(n+1-s)!}q_w^{(n+1-s)}t^{w}\mathbf{I}_3\right\}. \end{aligned}$$

Similarly,

$$L_3 = -\sum_{s=1}^{n-1}\sum_{w=1}^{n-s}\sum_{j=1}^{n+1-w-s} m_j\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}}\mathbf{F}_2\tilde{\mathbf{q}}_w\frac{w}{w+1}t^{w+1}\mathbf{I}_3.$$

By equation (C.5), $\frac{1}{w+1}\sum_{j=1}^{n+1-w-s}\frac{(n+1-s)!}{j!(n+1-j-s)!}m_j q_w^{(n+1-s-j)} = q_{w+1}^{(n+1-s)}$, so we have

$$L_3 = -\sum_{s=1}^{n-1}\sum_{w=2}^{n+1-s}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}}\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\frac{w-1}{(n+1-s)!}q_w^{(n+1-s)}t^{w}\mathbf{I}_3$$

For $s=1$, the stochastic integral $\int_0^t \mathrm{d}\hat{G}_{t_1}^{(1)}$ appears in both L_2 and L_3. Its coefficient is given by

$$\begin{aligned}&\sum_{w=2}^{n}(n+1)\,wq_w^{(n)}t^w+m_n\,(n+1)\,t-(n+1)\sum_{w=2}^{n}(w-1)\,q_w^{(n)}t^w\\&=(n+1)\left[m_nt+\sum_{w=2}^{n}q_w^{(n)}t^w\right]=(n+1)\,C_t^{(n)}.\end{aligned}$$

By equation (14),

$$\Pi_{(1)}^{(n+1)}=\frac{(n+1)!}{(n+1-1)!}C_t^{(n+1-1)}=(n+1)\,C_t^{(n)}.$$

For $s\in\{2,3,...,n-1\}$, the coefficients of the stochastic integral

$$\int_0^t\int_0^{t_1-}\cdots\int_0^{t_{l-1}-}\mathrm{d}\hat{G}_{t_l}^{\left(i_1^{\theta_{l,s}}\right)}\cdots\mathrm{d}\hat{G}_{t_2}^{\left(i_{l-1}^{\theta_{l,s}}\right)}\mathrm{d}\hat{G}_{t_1}^{\left(i_l^{\theta_{l,s}}\right)}$$

is given by

$$\begin{aligned}&m_{n+1-s}t\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!\,(n+1-s)!}+\sum_{w=2}^{n+1-s}\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\frac{w}{(n+1-s)!}q_w^{(n+1-s)}t^w\\&-\sum_{w=2}^{n+1-s}\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\frac{(w-1)}{(n+1-s)!}q_w^{(n+1-s)}t^w\\&=\sum_{w=1}^{n+1-s}\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\frac{1}{(n+1-s)!}q_w^{(n+1-s)}t^w\\&=\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}\frac{1}{(n+1-s)!}C_t^{(n+1-s)}\quad=\Pi^{(n+1)}_{\left(i_1^{\theta_{l,s}},i_2^{\theta_{l,s}},...,i_l^{\theta_{l,s}}\right)}\end{aligned}$$

by equation (14). For $s=n$, the stochastic integral appears in L_2 only and its coefficient is given by

$$m_1t\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}=\frac{(n+1)!}{i_1^{\theta_{l,s}}!i_2^{\theta_{l,s}}!\cdots i_l^{\theta_{l,s}}!}C_t^{(1)}=\Pi^{(n+1)}_{\left(i_1^{\theta_{l,s}},i_2^{\theta_{l,s}},...,i_l^{\theta_{l,s}}\right)}.$$

The stochastic integral $\int_0^t \mathrm{d}\hat{G}_{t_1}^{(n+1)}$ appears only once in G_t^{n+1} and its coefficient is equal to one. By equation (14),

$$\Pi_{(n+1)}^{(n+1)}=\frac{(n+1)!}{(n+1)!}C_t^{(0)}=1.$$

Finally, we have to show that $L_5+L_6+m_{n+1}t=C_t^{(n+1)}$. By equation (C.5), $\frac{1}{w+1}\sum_{j=1}^{n+1-w}\binom{n+1}{j}m_jq_w^{(n+1-j)}=q_{w+1}^{(n+1)}$,

$$\begin{aligned}L_5&=\sum_{j=1}^{n}\binom{n+1}{j}m_j\sum_{w=1}^{n+1-j}q_w^{(n+1-j)}t^{w+1}=\sum_{w=1}^{n}\sum_{j=1}^{n+1-w}\binom{n+1}{j}m_jq_w^{(n+1-j)}t^{w+1}=\sum_{w=1}^{n}(w+1)\,q_{w+1}^{(n+1)}t^{w+1}.\\L_6&=-\sum_{w=1}^{n}\sum_{j=1}^{n+1-w}\binom{n+1}{j}m_jq_w^{(n+1-j)}\frac{w}{w+1}t^{w+1}=-\sum_{w=1}^{n}wq_{w+1}^{(n+1)}t^{w+1}.\end{aligned}$$

Hence

$$L_5 + L_6 + m_{n+1}t = \sum_{w=1}^{n} q_{w+1}^{(n+1)} t^{w+1} + m_{n+1}t = \sum_{w=2}^{n+1} q_{w}^{(n+1)} t^{w} + m_{n+1}t = \sum_{w=1}^{n+1} q_{w}^{(n+1)} t^{w} = C_{n+1}^{(k)}.$$

Thus, we have proved that

$$G_t^{n+1} = \sum_{\theta_{n+1}\in\mathcal{I}_{n+1}} \Pi_{\theta_{n+1},t}^{(n+1)} \mathcal{S}_{\theta_{n+1},t,0} + C_t^{(n+1)}.$$

As explained in equation (C.1), since $F_t = G_{t+t_0} - G_{t_0}$ is also a Lévy process, we can write

$$F_t^{n+1} = \sum_{s=1}^{n+1}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \Pi_{\theta_{l,s},t}^{(n+1)} \int_0^t \int_0^{t_1-} \cdots \int_0^{t_{l-1}-} \mathrm{d}\hat{F}_{t_l}^{\left(i_1^{\theta_{l,s}}\right)} \cdots \mathrm{d}\hat{F}_{t_2}^{\left(i_{l-1}^{\theta_{l,s}}\right)} \mathrm{d}\hat{F}_{t_1}^{\left(i_l^{\theta_{l,s}}\right)}$$

and since $\mathrm{d}\hat{F}_t^{(i)} = \mathrm{d}\left(\hat{G}_{t+t_0}^{(i)} - \hat{G}_{t_0}^{(i)}\right) = \mathrm{d}\hat{G}_{t+t_0}^{(i)}$, by changing of variables, we have

$$\begin{aligned}
(G_{t+t_0} - G_{t_0})^{n+1} &= \sum_{s=1}^{n+1}\sum_{l=1}^{s}\sum_{\theta_{l,s}\in\mathcal{K}_{l,s}} \Pi_{\theta_{l,s},t}^{(n+1)} \int_{t_0}^{t+t_0} \int_{t_0}^{t_1-} \cdots \int_{t_0}^{t_{l-1}-} \mathrm{d}\hat{G}_{t_l}^{\left(i_1^{\theta_{l,s}}\right)} \cdots \mathrm{d}\hat{G}_{t_2}^{\left(i_{l-1}^{\theta_{l,s}}\right)} \mathrm{d}\hat{G}_{t_1}^{\left(i_l^{\theta_{l,s}}\right)} \\
&= \sum_{\theta_{n+1}\in\mathcal{I}_{n+1}} \Pi_{\theta_{n+1},t}^{(n+1)} \mathcal{S}_{\theta_{n+1},t,t_0} + C_t^{(n+1)}.
\end{aligned}$$

Therefore, by the principle of strong induction,

$$(G_{t+t_0} - G_{t_0})^{k} = \sum_{\theta_k\in\mathcal{I}_k} \Pi_{\theta_k,t}^{(k)} \mathcal{S}_{\theta_k,t,t_0} + C_t^{(k)}$$

for all non-negative integers k.

D Proof of Proposition 4

We prove by induction. Assume the proposition is true for all $k \geq n$. Now, consider $n+1$,

$$\begin{aligned}
Y^{(n+1)} &= H^{(n+1)} - \sum_{l=1}^{n} a_{n+1,l} Y^{(l)} = H^{(n+1)} - \sum_{l=1}^{n} a_{n+1,l}\left\{H^{(l)} + \sum_{k=1}^{l-1} b_{l,k} H^{(k)}\right\} \\
&= H^{(n+1)} - \sum_{l=1}^{n} a_{n+1,l} \sum_{k=1}^{l} b_{l,k} H^{(k)} = H^{(n+1)} + \sum_{k=1}^{n} b_{n+1,k} H^{(k)},
\end{aligned}$$

which completes the proof.

E Plots

Figure 1: G_t^4 generated using CRP and directly from the Gamma process.Figure 2: The difference of the two series in Figu

Figure 3: $(G_{t+t_0} - G_{t_0})^9$ generated using CRP and directly from the Gamma process.Figure 4: The difference of the two se

Figure 5: X_t^5 generated using CRP and directly from the Wiener and Gamma processes.Figure 6: The difference of the two

Figure 7: $(X_{t+t_0} - X_{t_0})^8$ generated using CRP and directly from the Wiener and Gamma processes.Figure 8: The differenc

Figures 1-8: Solid line is generated using the CRP and the dotted line is generated by the Wiener and Gamma processes. Time step $= \frac{1}{10000}$, $a = 10$, $b = 20$. In Figure 3, $t_0 = 0.0099$; in Figure 5, $\sigma = 0.01$; in Figure 7, $t_0 = 0.0019$ and $\sigma = 0.02$.

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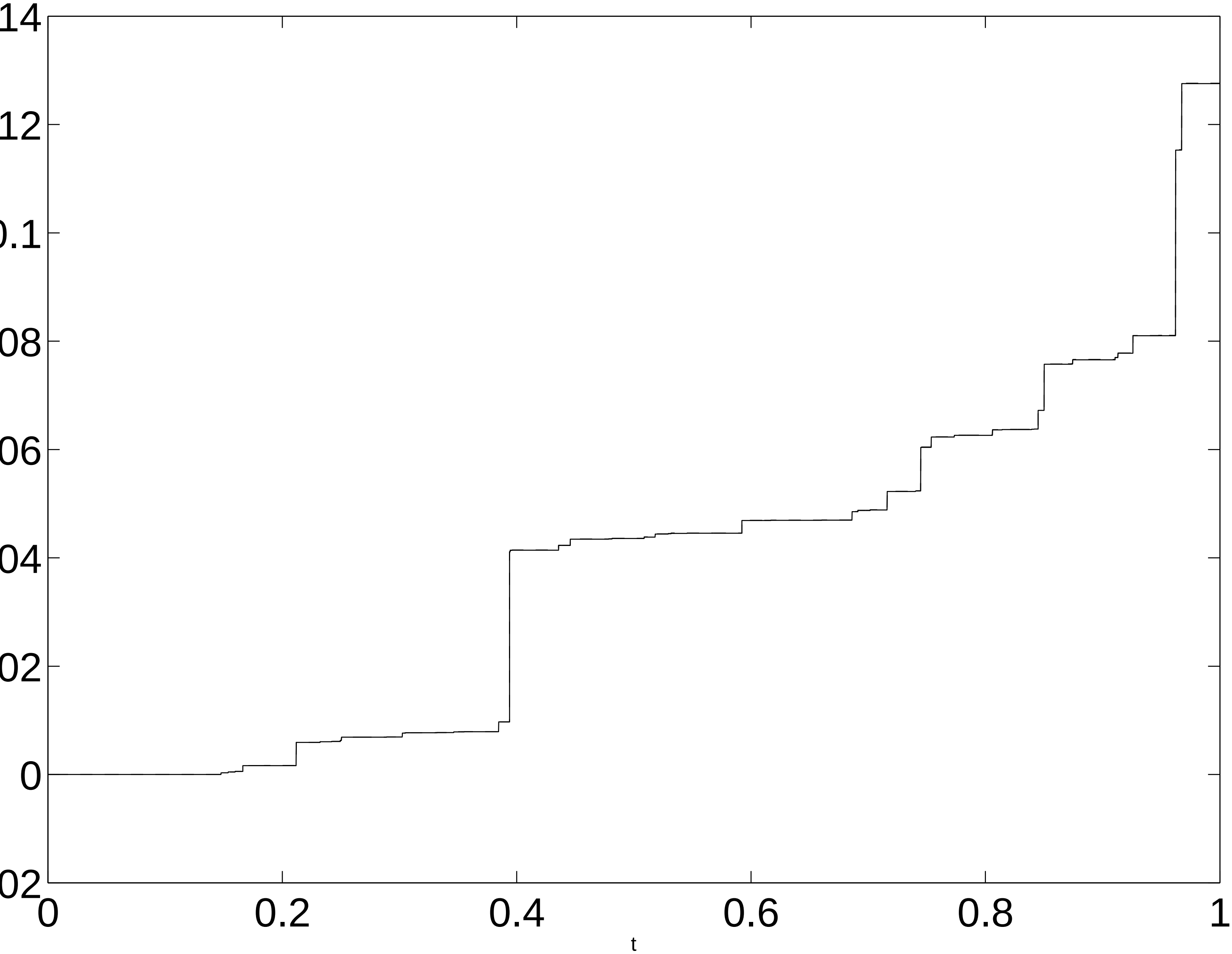

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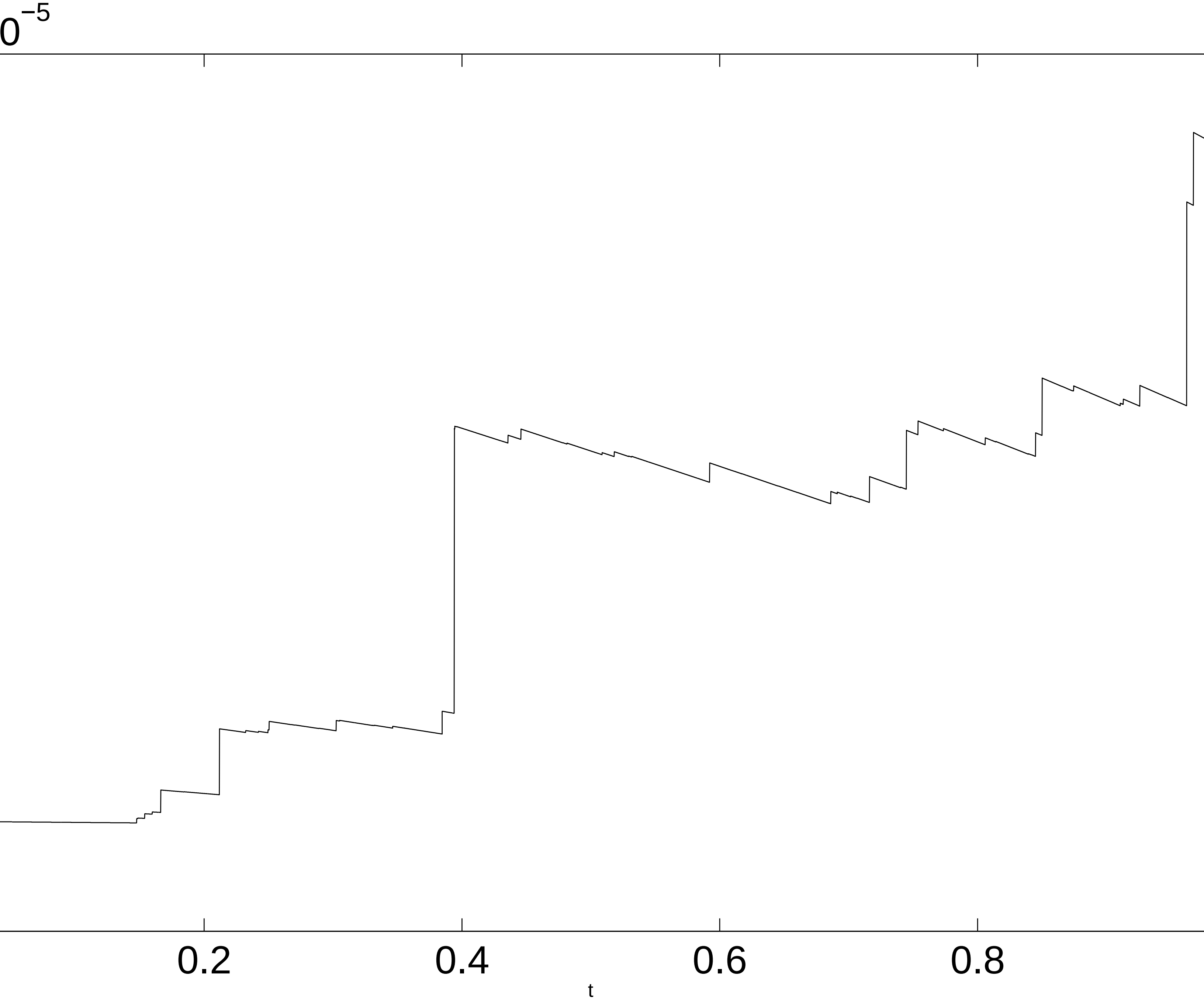

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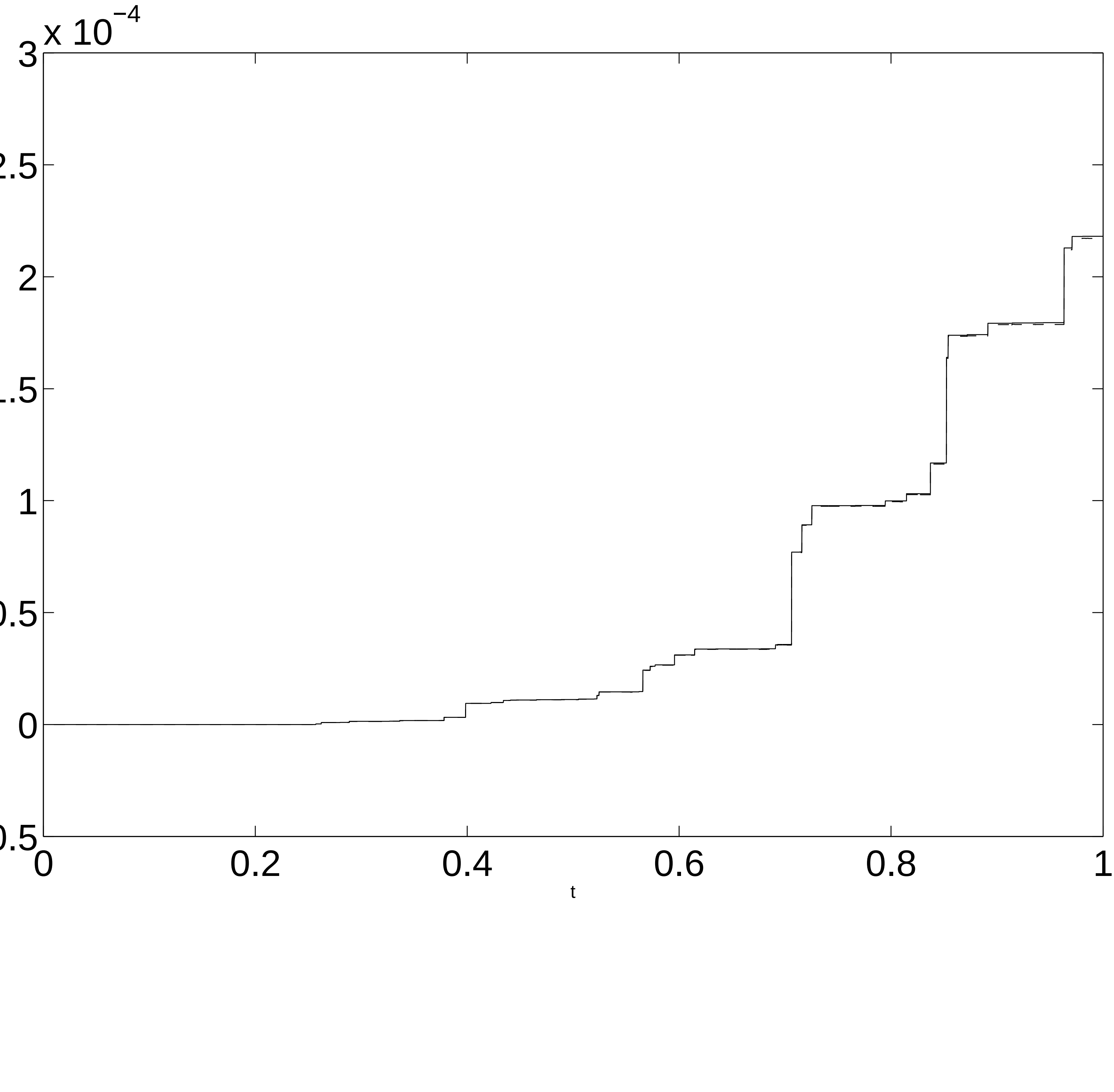

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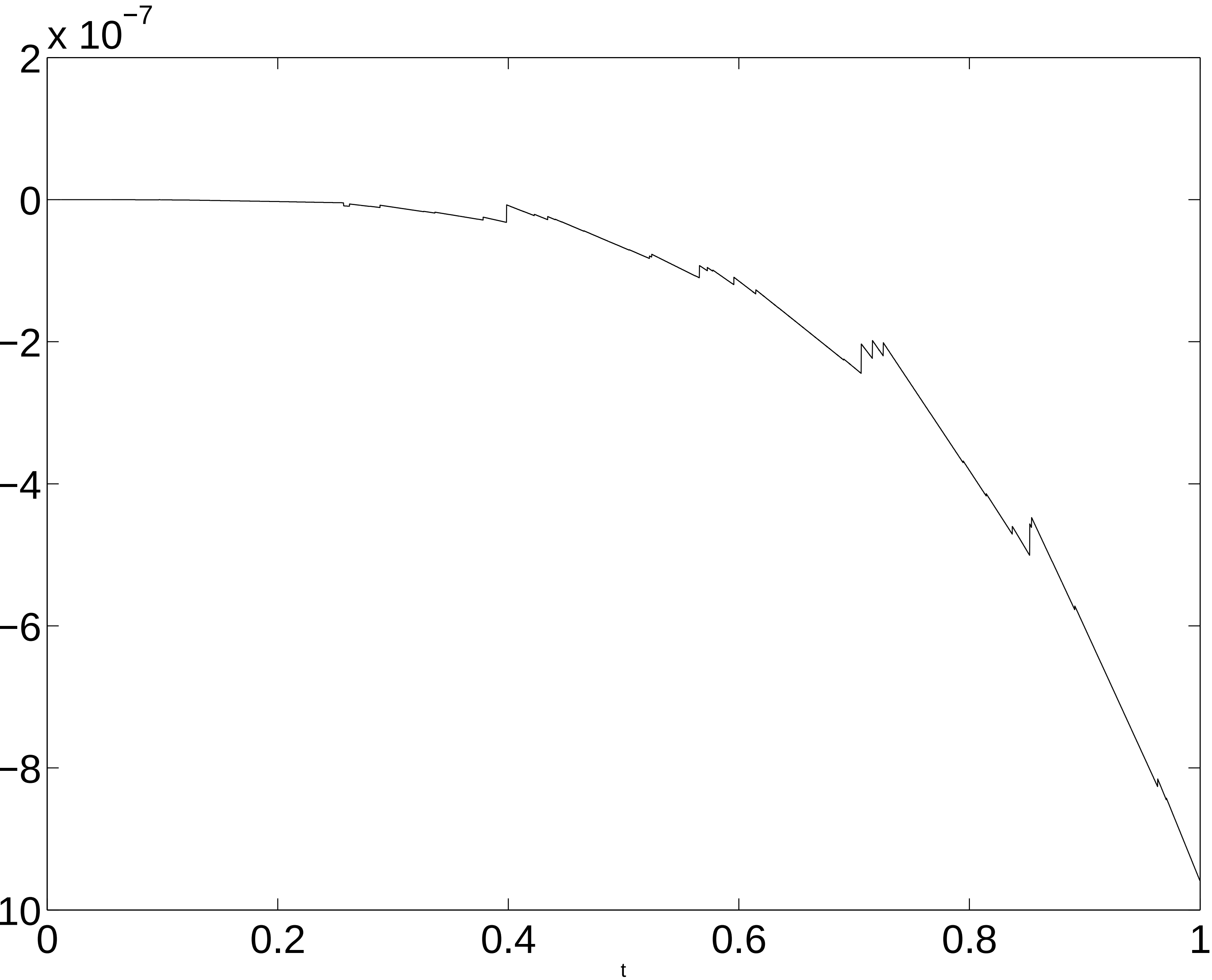
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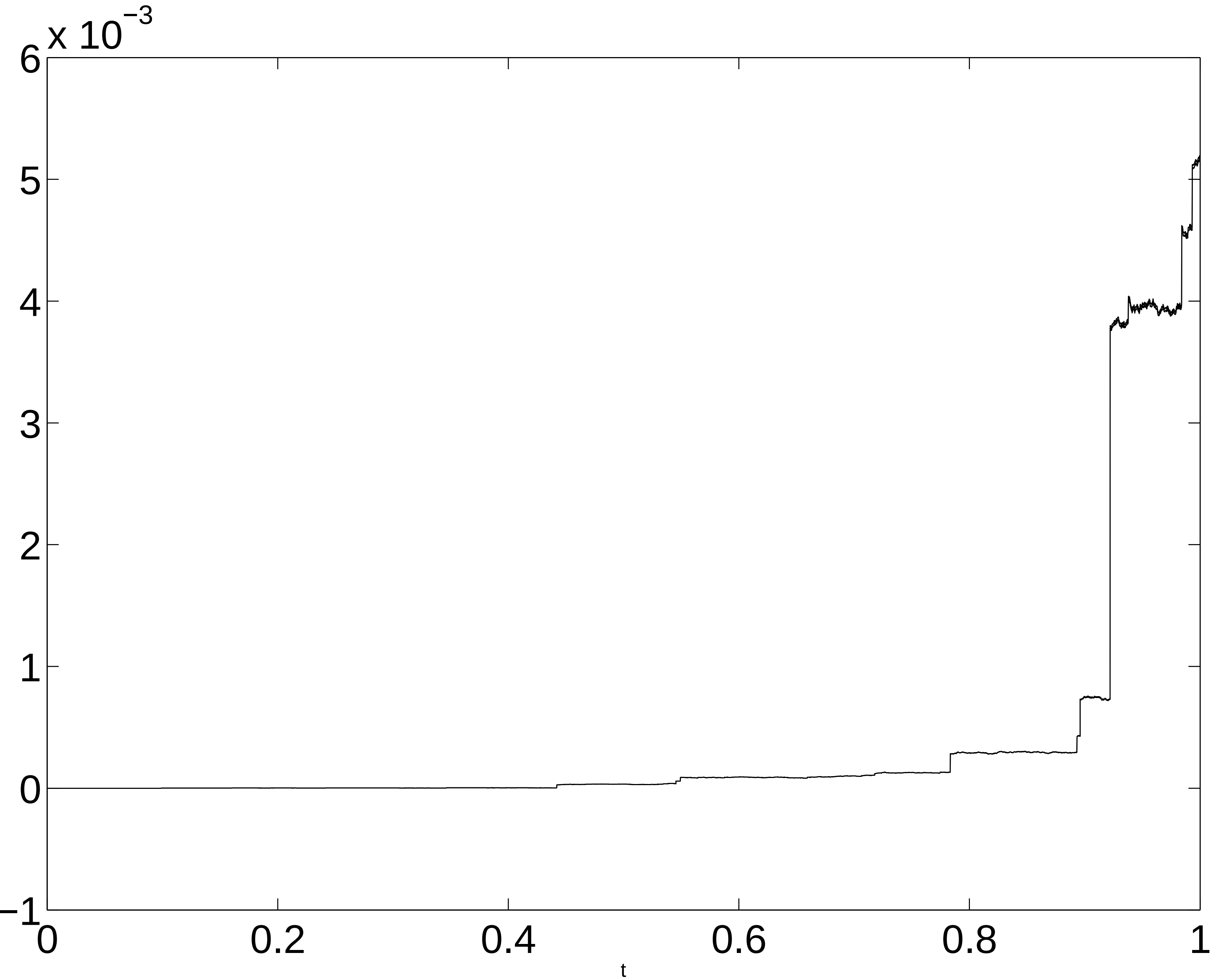
t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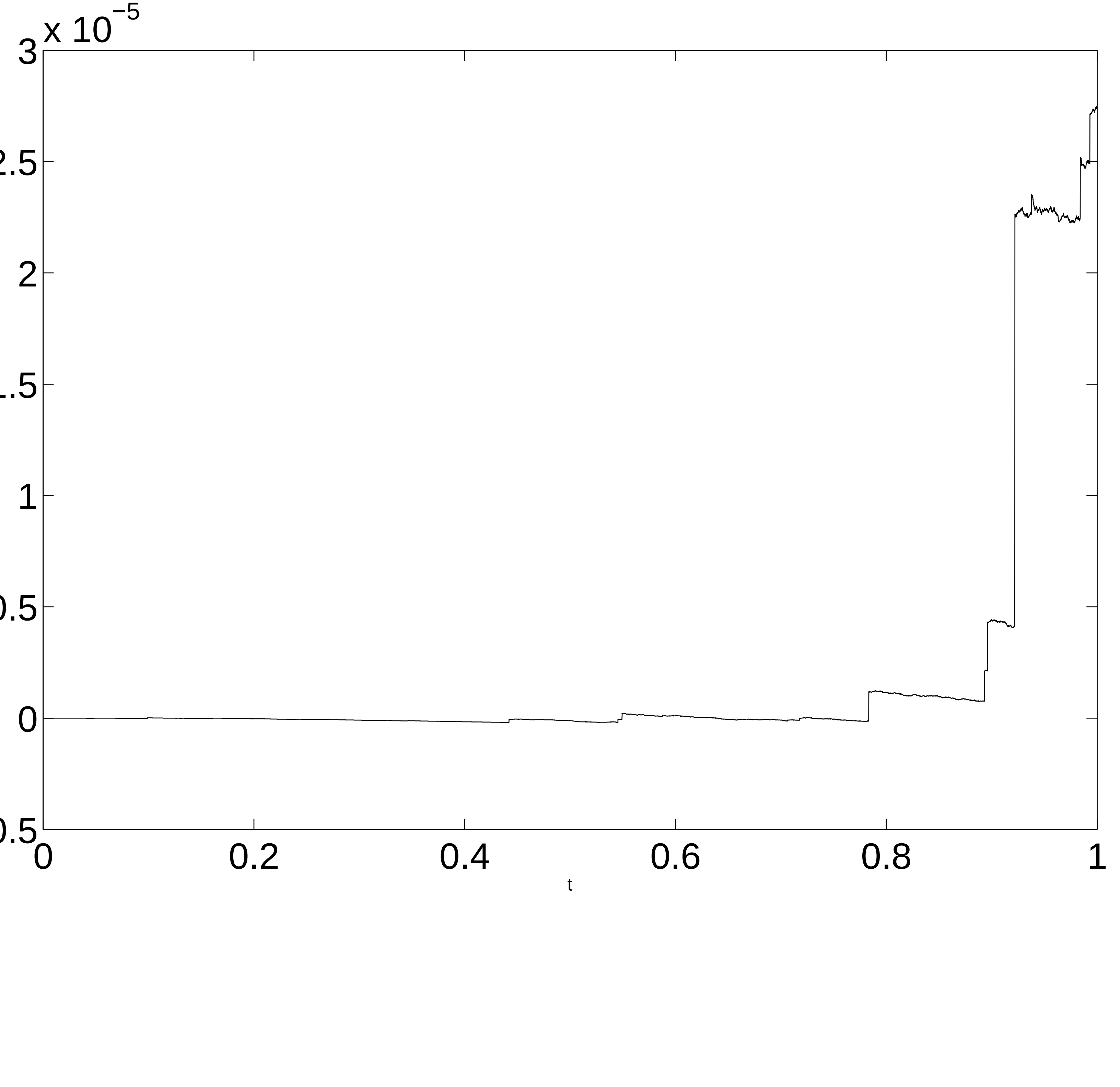
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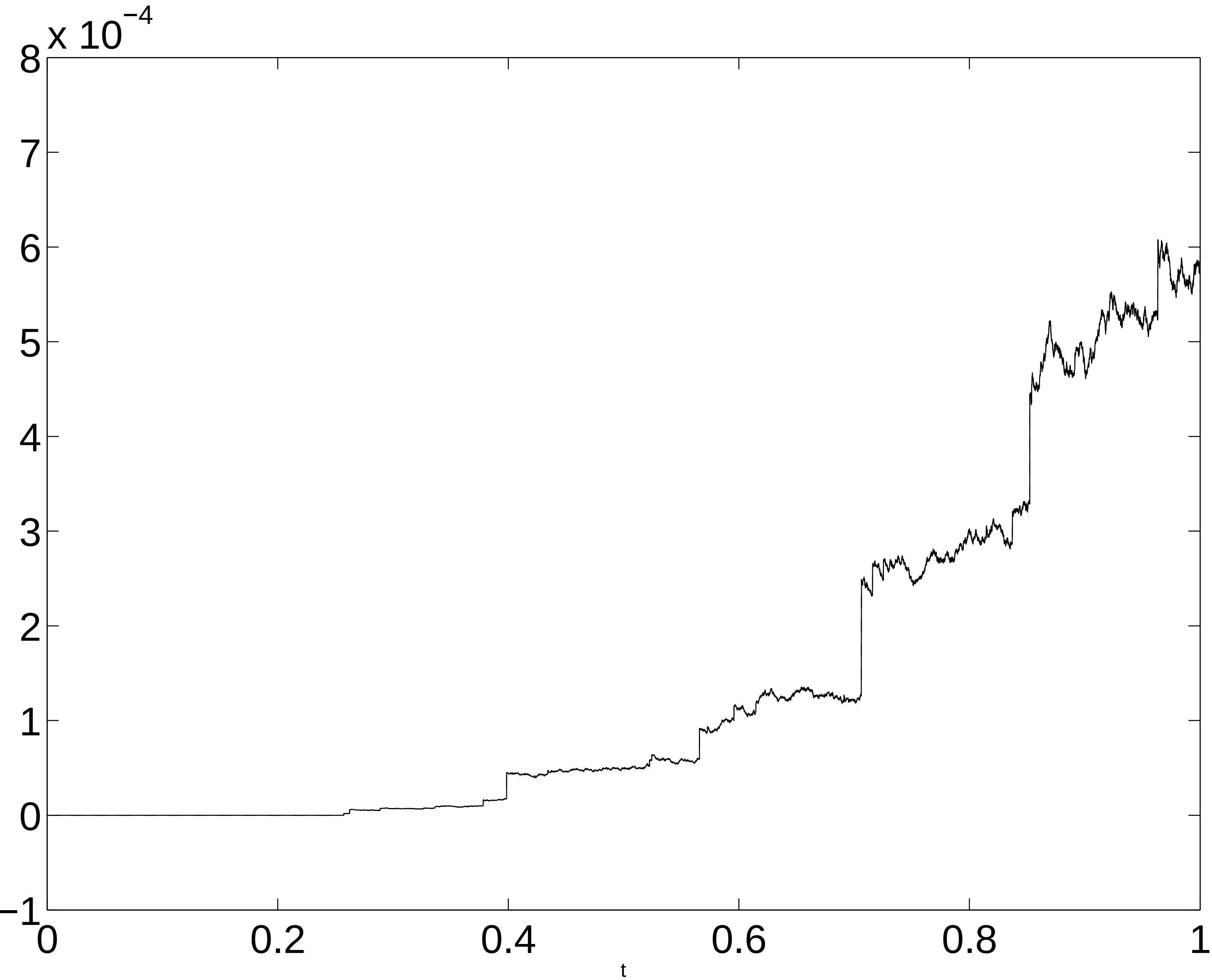

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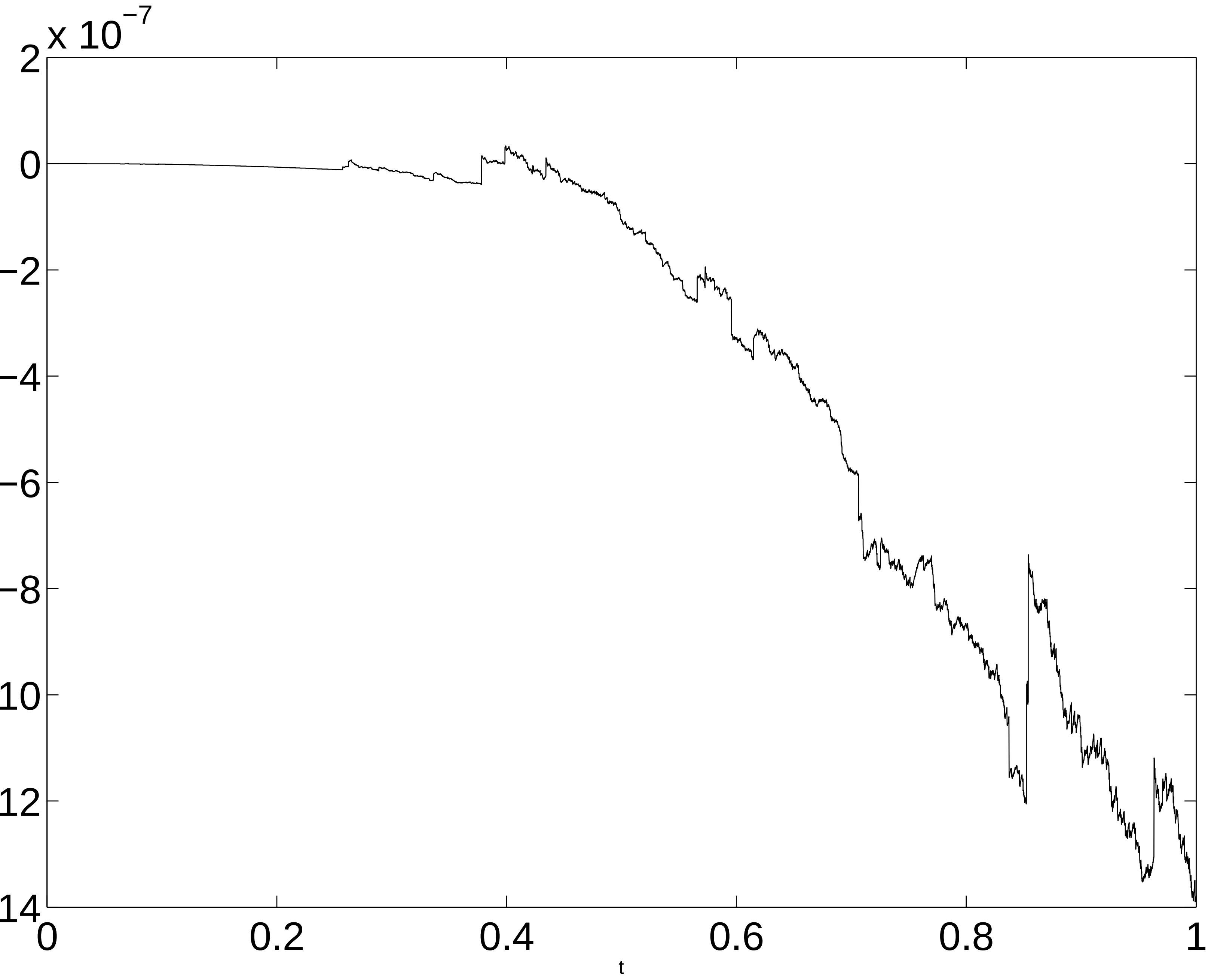

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−6
−8
10
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1
t