HUMANIST REPUDIATION OF EASTERN INFLUENCES IN EARLY MODERN MATHEMATICS

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INTRODUCTION

At the beginning of the seventeenth century mathematics in Europe became radically different from mediaeval and Renaissance traditions. Renaissance humanism had a decisive influence on the transition of mediaeval to early modern mathematics. The scholarly tradition of qualitative arithmetic based on Boethian proportion theory passed into oblivion. A systematic program for reforming mathematics was initiated by the end of the fifteenth century by Johannes **Regiomontanus** and Giorgio **Valla** and further executed by Petrus **Ramus**, Jacques **Peletier**, Francoi **Viète** and Christopher **Clavius**. The aim was to provide new grounds for old traditions such as arithmetic and algebra and to set up sixteenth-century mathematics on Greek foundations. Together with their important contribution to the development of modern mathematics, humanists also created the myth that all mathematics, including algebra, descended from the ancient Greeks. Beginning with Regiomontanus's 1464 lecture at Padua, humanist writers distanced themselves from "barbaric" influences. This paper concerns the humanist repudiation of eastern sources on which the greatest part of mediaeval mathematics depended. The reality was that, with some exceptions, ancient Greek mathematics was more foreign to European mathematics than Eastern influences which were well digested within the vernacular tradition. It was a cultural, not a conceptual distance between European and eastern mathematics that led humanists to repudiate the eastern roots of sixteenth-century mathematics.

In a first part we will describe the state of mediaeval mathematics before 1464. We will make the distinction between scholarly and sub-scientific traditions and show that the greater part of the mathematical practice within the vernacular tradition relies on Arabic and Indian sources. Secondly we will show that from 1464 till end of the sixteenth century the humanist repudiation of eastern influences was part of a systematic program to fabricate a new identity for European mathematics.

The newly shaped identity of European mathematics had a dominant weight on the way historians looked at mathematics. We will illustrate the humanist influence on the historiography of mathematics by two examples. The first one concerns symbolic algebra. The tripartite distinction between rhetorical, syncopated and symbolic algebra by Georg Heinrich Ferdinand **Nesselmann** was a normative distinction to provide **Diophantus** with a privileged status in the development of algebra. We will show that this distinction, which is still found in today's textbooks, cannot be sustained. A second example deals with the reception of classical Indian mathematics during the nineteenth century after the publication of the first translations of Sanskrit mathematical works by Henry Thomas **Colebrooke** in 1817. The historical account of Moritz *Cantor* and

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others was deeply influenced by the humanist prejudice that all higher intellectual culture, in particular all science, had risen from Greek soil.[2]

EUROPEAN SCHOLARLY MATHEMATICS

Before we discuss humanist mathematics let us look at the status of mathematics in Europe before its transformation, and let us take Regiomontanus's Padua lectures from 1464 as a transition point. In order to do so we first have to make the distinction between scholarly mathematics and the vernacular tradition. Scholarly mathematics during the fourteenth and fifteenth century was implicitly defined by the *quadrivium* as consisting of the four disciplines astronomy, arithmetic, geometry and music, with arithmetic as the most dominant one. The basic text on arithmetic was **Boethius**'s *De Institutione Arithmetica*. The Boethian arithmetic strongly relies on **Nichomachus of Gerasa**'s *Arithmetica* from the second century. This basically qualitative arithmetic deals with properties of numbers and ratios. All ratios have a name and operations or propositions on ratios are expressed in a purely rhetorical form. The qualitative aspect is well illustrated by the following proposition from **Jordanus de Nemore**'s *De elementis arithmetice artis* (c. 1250):[3]

> Datis superparticularibus vel multiplicibus superparticularibus multiplices superparticulares et superpartientes et datis superpartientibus aut multiplicibus superpartientibus superpartientes et multiplices superpartientes procreare.

A *superparticular* has the form $(n + 1)/n$ and thus covers proportions such as the common *sesquialter* (3/2) and *sesquiter* (4/3) proportions; a *superpartient* proportion has the form $(m + n)/n$ with $m > 1$ and includes proportions such as 8/3. The proposition describes how to create multiple *superparticular* proportions from a given one. This problem, treated in the most extensive treatise of the period, may serve as an example to illustrate that this arithmetic was of little practical use. It was not applied outside monasteries and universities. It was intended mainly for aesthetic and intellectual pursuit. During the eleventh century a board game named *Rhythmomachia* was designed to meet with these aesthetic aspirations. Originated as the subject of a competition on the knowledge of Boethian arithmetic amongst cathedral schools in Germany, the game was played until the sixteenth century, when the arithmetic tradition passed into oblivion. Despite its limited applicability, Boethian arithmetic evolved into a specific kind of mathematics, typical for the European Middle Ages, and left its mark on early natural philosophy. **Carl Boyer**'s book on the history of calculus demonstrates how fourteenth-century thinkers such as **Bradwardine** and Richard **Suiseth** developed ideas on continuity and acceleration within this framework which influenced the later development of mathematics and natural philosophy.[4]

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[2] Nesselmann, *Versuch einer kritischen*; Cantor, *Vorlesungen*; Colebrooke, *Algebra*.
[3] The quote is from Book IX, proposition LXXI; Busard, *De Elementis*, 199. For a modern edition of Boethius's *De Institutione Arithmetica* see Friedlein, *Boetii De institutione*. Nichomachus of Gerasa's *Arithmetica* has been edited and translated by Robbins and Karpinski, *Introduction to arithmetic*.
[4] See Boyer, *History of the Calculus*, chapter 3. Concerning *Rhythmomachia*, the definitive history, tracing the game back to its roots, has been written by Borst, *Das mittelalterliche Zahlenkampfspiel*. For a more accessible introduction see Moyer, *The Philosopher's Game*.

Geometry
When discussing geometry within the European context we immediately think of **Euclid**. However, the study of Euclidean geometry before the sixteenth century was very limited. Whenever Euclid's *Elements* was studied or taught, it was limited to the first and occasionally the second book. Extant texts before the sixteenth century depended mostly on Arabic sources. While there circulated Latin editions based on Boethius's translation of the sixth century, they hardly found any readers. According to Menso **Folkerts** only **Fibonacci** and **Campanus** used them. All others used the adaptation of **Campanus** which was based on the twelfth-century translation of an Arabic edition by Gerard of Cremona, who also translated the algebra of **al-Khwārizmī**. Campanus's edition was the first printed Euclid edition and appeared in 1482 in Venice. Hence it is no exaggeration to state that almost all knowledge of Euclidean geometry in Medieval Europe was based on translations from the Arabic scholarly tradition. It is only by the end of the fifteenth century, under influence of **Regiomontanus** and **Valla**, that any serious work was undertaken to study the *Elements* and to reconstruct the original text from Greek manuscripts.[5]

Other mathematical disciplines such as spherical trigonometry also strongly depended on Arabic and Hindu sources. Though there existed Greek versions of **Ptolemy**'s *Almagest* (Vat. gr. 1594, 9th century) and direct translations from the Greek (Vat. lat. 2056, c. 1060) the most important medieval Latin version of the *Almagest* on which most manuscripts are based, was translated from Arabic by **Gerard of Cremona** in Spain in 1175. **Nasir al-Din al-Tusi** wrote the *Book of the Transversal Figure,* which was the first treatment of plane and spherical trigonometry as an independent part of the mathematical sciences. **Regiomontanus** based his treatise *De Triangulis omnimodis* on several Arabic sources including Al-Jayyani's *The book of unknown arcs of a sphere*. In the process he copied large parts of the *Correction of the Almagest* by **Jabir ibn Aflah** (better known as **Geber**) from Cremona's translation in the fourth book of the *Triangulis* without any acknowledgment. Such case of plagiarism went not unnoticed to Girolamo **Cardano** who had studied Geber's work in detail and wrote his own little treatise in which he calculated the distance between two places based on latitude and longitude.[6]

SUB-SCIENTIFIC MATHEMATICS

Jens **Høyrup** coined the term sub-scientific mathematics for a long tradition of practice which has been neglected by historians. As a scholar working on a wide period of mathematical practice, from Babylonian algebra to the seventeenth century, Høyrup has always paid much attention to the more informal transmission of mathematical knowledge which he calls sub-scientific structures. By nature, sub-

[5] For the study of Euclid during the Middle ages see Juschkewitsch *Geschichte der Mathematik*, and Folkerts, *Euclid in Medieval Europe*. A modern scholarly edition of Campanus's *Elements* of 1482 has recently been published by Busard, *Campanus of Novara*, who also edited the translation by Gerard of Cremona, *The Latin translation*. Regiomontanus's plan for a new Euclid edition is described in Folkerts, "Regiomontans Euklidhandschriften".
[6] The *De Triangulis omnimodis* was published posthumously in 1533. For an English translation see Hughes, *Regiomontanus on Triangles*. On the sources of Regiomontanus see Lorch, "The Astronomy", and Hairetdinova, "On spherical trigonometry". Cardano mentions his treatise in *De libris propriis*, 1557, 9 that he lends it to a friend who unfortunately died of plague. The manuscript was subsequently lost. See also Morley, *Jerome Cardan*, I, 27.

scientific structures are not communicated in formal writings, making their dissemination very difficult to determine. We will now discuss arithmetic, algebra and geometry within the sub-scientific traditions and show that these disciplines were established as a mixture of cross-cultural influences.[7]

Hindu influences in Renaissance arithmetic

We traced the influences of Hindu algebra and arithmetic to Western problems and problem solving methods.[8] Although no formal influence has been demonstrated there is strong indication that there is some relation between these two traditions. In absence of extant texts that support a direct connection, we present an explanation for the possible influence of Hindu algebra on Renaissance problem-solving methods by means of proto-algebraic rules. A proto-algebraic rule is a procedure or algorithm for solving one specific type of problem. Our main hypothesis is that many recipes or precepts for arithmetical problem solving, in abbaco texts and arithmetic books before the second half of the sixteenth century, are based on proto-algebraic rules. We call these rules proto-algebraic because they are, or could be based originally on algebraic derivations. Yet their explanation, communication and application do not involve algebra at all. Proto-algebraic rules are disseminated together with the problems to which they can be applied. The problem functions as a vehicle for the transmission of this sub-scientific structure. Little attention has yet been given to sub-scientific mathematics or proto-algebraic rules. However, viewed as solidifications of algebraic problems solving, proto algebraic rules function as fossils of algebraic practice in non-algebraic writings. As fossils provide important evolutionary data to paleontologists and archeologists, so do proto-algebraic rules for the historian of mathematics. Analysis and comparison of formulations and variations of these rules allow us to reconstruct possible paths of transmission.

We demonstrate that Hindu arithmetic and algebra influenced Renaissance arithmetic in three types of proto-algebraic rules for solving linear problems:

1) the *regula augmentis* and problems of the type *zuviel – zu wenig* are based on the Hindu rule of *gulikā-antara*: $(ax + b = cx + d)$,

2) the *regula augmentationis* corresponds with the solution of problems in two unknowns appearing in Hindu algebra since the the Bakhshālī Manuscript (c. 700):
$$x + a = c(y - a)$$
$$y + b = d(x - b)$$

3) indeterminate versions of the problem of "men finding a purse" and determine their contents, follow **Mahāvīra**'s solution in the *Gaṇitasārasangraha*. Also the practical context of the purse first appears in India,

[7] Høyrup coined the term in his 1990 article *Sub-scientific mathematics* but it is returns in many of his publications, see also *In Measure, Number and Weight*.
[8] For a comprehensive study see Heeffer, "The Tacit Appropriation".

$$x + p = a(y + z)$$
$$y + p = b(x + z)$$
$$z + p = c(x + y)$$

In addition to these three well-documented examples we provide a framework for further research along these lines and identify other candidates as examples of proto-algebraic rules. One specific class of Renaissance problems deals with a man going on several business trips winning and spending money on each trip. The problem can easily be solved by reversing the order and calculating backwards. In Indian mathematics the method was called *viparītakarma* or *Rule of inversion*. **Āryabhaṭa** prescribes the *viparītakarma* for the four arithmetical operations, but this was extended to include squaring and roots by **Brahmagupta** and Bhāskara. The Bakhshālī Manuscript has the first occurrence of the problem formulated as business trips.[9]

Of course, Hindu arithmetic is just one of the many cultural traditions which influenced mediaeval and Renaissance sub-scientific mathematics. We also discern Arabic, Persian and Byzantine influences as can be easily determined in **Fibonacci**'s *Liber Abbaci* only on basis of the monetary units used. There was also a Latin European tradition concerned with this kind of arithmetical problems. **Alcuin**'s *Propositiones ad Acuendos Juvenes* (Propositions for Sharpening Youths) from the ninth century provides us with an extant witness. This collection contains 53 problems of which many are repeated over and again in mediaeval and Renaissance works. As the title suggests, the problems were to be used for educational purposes and to be read aloud, copied and solved by students.[10]

<u>*Arabic roots of European algebra*</u>

Arithmetical problem solving became much more advanced with the introduction of Arab algebra through the Latin translations of **al-Khwārizmī**'s *Algebra* by Robert of **Chester** (c. 1145), Gerard of **Cremona** (c. 1150) and Guglielmo **de Lunis** (c. 1250). With the possible exception of **Jean de Murs**' *Quadripartitum numerorum* at the Sorbonne, algebra was not practiced or even spoken about at universities for the following three centuries. Algebra continuously developed within the vernacular tradition of abacus schools in fourteenth- and fifteenth-century Italy. Algebra was not only foreign by its Arabic origins; it was completely foreign to the scholarly tradition.[11]

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[9] The name *regula augmentis* is coined by Widmann, *Behende vnd hubsche Rechenung*, f. 110^v. The term *regula augmentationis* originates from the *Algorismus Ratisbonensis* and is mentioned in relation to the problem 138 (c. 1450, see Vogel, *Die Practica*, 70). Both rules have been in use long before they were referred to by those names. For a classification and history of problem types and their labels see Tropfke, *Geschichte der Elementar-Mathematik*, reedited in 1980 and Singmaster, *Sources*. For a modern scholarly edition of *The Bakhshālī Manuscript* see Hayashi. The *Gaṇitasārasangraha* was edited by Padmavathamma and Rangācārya.

[10] The *Liber abbaci* has been edited by Boncompagni and translated into English by Sigler, *Fibonacci*. Translations of the *Propositiones ad Acuendos Juvenes* are quite recent. Folkerts "Die älteste mathematische Aufgabensammlung" translated Alcuin into German. Hadley provided an English translation, annotated by Singmaster, "Problems to Sharpen".

[11] It is only during the past decades that critical editions of the three Latin translations have become available. The translation by Gerard of Cremona was edited by Hughes, "Gerard of Cremona's Translation", based on seven manuscript copies. Hughes, *Robert of Chester's Latin Translation,* provides a critical edition of the second translation from Robert of Chester based on the three extant

At the same time, algebra flourished within the sub-scientific tradition. Although not taught at the abbaco schools, algebraic problem solving was practiced by the abbaco masters who produced hundreds of treatises dealing with the subject. The common view that abbaco schools and algebraic practice was a direct consequence of the *Liber Abbaci* by **Fibonacci** is currently being challenged. Several distinctive characteristics of abbaco algebra direct us to the conclusion that there must have been a European algebraic tradition before Fibonacci. There are basically four arguments that lead us to this belief:

1) The six Arabic types of 'equations' can be found in abbaco algebra but they are usually expanded to (many) more types and they occur in a different order.
2) The rules for solving equations in Arabic algebra apply to normalized cases in which the coefficient of the square term is one. In *all* abbaco treatises, the rules apply to non-normalized cases. The first step in the solution is to divide the terms by the coefficient of the square term, even if this step involves dividing by one.
3) The early abbaco treatises on algebra contain no references to Fibonacci. Only from the fifteenth century do we see occasional references to Fibonacci's *Liber abbaci*.
4) Early Arabic algebra contains anomalies regarding the use of both *māl* (x^2) and *shay'* (x) for the unknown. These anomalies are partially present in Fibonacci *Liber abbaci*. For most of the algebra part, Fibonacci uses the *res* and *census* terminology of Gerard of **Cremona**. However, in the middle of chapter 15 he switches from *census* to *avere* for *māl*. The extant abbaco treatises on algebra do not show these anomalies.[12]

By 1460, there was a 250 years old tradition of algebraic practice, in which Arabic algebra was well digested and several important contributions had been made and the foundations were being laid for what would become the new symbolic algebra of the sixteenth century.

<u>*The long tradition of practical geometry*</u>

The adjective *practical* allows us to distinguish this tradition from the scholarly type of Euclidean geometry. It also refers to its preoccupation with measurement and surveying, the two main practical uses of this kind of geometry. An early example of this discipline is the pseudo-**Heron** *Geometrica*. During the mediaeval period the most important text was the *Liber mensurationem*. The Arabic text is attributed to **Abū Bakr** and the Latin translation is by Gerard of **Cremona** from the twelfth century. Although this work deals primarily with surveying problems it also uses the methods as well as the terminology of the early Arabic *jabr* tradition. **Høyrup**, who named the method "naive geometry" or "the tradition of lay surveyors", has pointed out the relation between this work and Babylonian. Following Hubertus **Busard**, he

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manuscripts. A third translation has been edited by Wolfgang Kaunzner, "Die lateinische Algebra". Although this text (Oxford, Bodleian, Lyell 52) was originally attributed to Gerard, it is now considered to be a translation from Guglielmo de Lunis, see Hughes "The Medieval Latin Translations" and *Robert of Chester's Latin Translation*. It has been argued by several scholars that Gerard of Cremona's translation is the best extant witness of early Arabic algebra in Europe. For a recent transcription of the text by de Murs see L'Huillier, *Le Quadripartitum numerorum*.

[12] These arguments have been put forward by Høyrup in several publications. See his latest *Jacopo da Firenze's* and Heeffer, *A Conceptual Analysis* for a summary.

has convincingly demonstrated that the solution to these problems depend on cut-and-paste operations and thus are concretely geometrical.[13]

While the algebraic aspects of this Babylonian tradition may have been incorporated into the scholarly *jabr* mathematics of **al-Khwārizmī**, this kind of mensuration geometry became a tradition on its own. Most abbaco treatises contain a section on the subject dealing with the measurement of plane figures, solids and distances, sometimes with the use of a quadrant. Several texts also apply algebra to basic geometrical problems involving triangles and circles. Apart from the many abbaco treatises on practical geometry we also have Fibonacci's *De practica geometrie* and Pacioli's *Summa de arithmetica et geometria*. The section on practical geometry by **Pacioli** is in fact a literal reproduction of an anonymous abbaco treatise.[14]

From the second half of the fifteenth century the scope of practical geometry vastly expanded. Typical Renaissance subjects such as perspective, ballistics and the science of weights became part of practical geometry. Also fortification can be placed within this tradition although sixteenth-century engineers who published on the subject started including short introductions on Euclidean geometry in their books. In France the most notable examples were Claude **Flamand** and Jean **Errard**. The latter one subsequently published his own edition of Euclid.[15]

The state of European mathematics in 1460

We can now characterize the state of European mathematics at the middle of the fifteenth century and the processes under which it was to be reformed under humanist influences. Within the scholarly tradition arithmetic had degenerated into a sterile discipline. Since the *Numerus datis* of **Jordanus** little was contributed to the Boethian tradition. The qualitative nature of this arithmetic made it difficult to practice and cumbersome to apply to natural philosophy. Though **Euclid** was known and the first book was studied, all knowledge of Euclidean geometry came from the Arab world. Often, the contribution of Arab and Hindu authors to European mathematics was substantive as in the case of plane and spherical trigonometry.

The sub-scientific tradition was a cross-cultural amalgam of several traditions. Merchant type arithmetic and recreational problems show a strong similarity with

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[13] Heron's text is published by Heiberg in *Heronis Alexandrini Opera*. For a critical edition of *Liber Mensurationem* see Busard, *L'algèbre au Moyen Âge*. Høyrup's arguments for demonstrating its relationship with Babylonian mathematics are expounded in "Al-Khwārizmī, Ibn Turk", "The Formation", "Algebra and Naive Geometry" and his seminal work *Lengths, Widths, Surfaces*.

[14] A Latin edition of Fibonacci's *De practica geometrie* is published by Boncompagni in . An English translation has recently become available from Hughes, *Fibonacci's De practica geometrie*. For acomprehensive overview of abbaco text on practical geometry see Simi and Rigatelli, "Some 14th and 15th Century Texts" . The case of plagiarism by Pacioli is demonstrated by Picutti "Sui plagi matematici", 76: "tutta la «Geometria» della *Summa* dagli inizi a p. 59v (cioè 119 pagine in folio) è trascrizione delle prime 241 carte del codice Palatino 577 della Biblioteca Nazionale di Firenze, di autore ignoto (ma che anni fa abbiamo attribuito e continuiamo ad attribuire tuttora a maestro Benedetto da Firenze)." Simi and Rigatelli "Some 14th and 15th Century Texts", 463, wrongly cite Picutti that 'all the 'geometria' of the Summa, from the beginning on page 59v. (119 folios), is the transcription of the first 241 folios of the Codex Palatino 57". This quote has been repeated by other authors but lead to confusion because the page numbers do not match. I am grateful to Alan Sangster who pointed out the mistake and provided me a with copy of Picutti's article.

[15] Flamand, *La guide des fortifications*, Errard, *La géométrie*. His Euclid edition is *Les six premiers livres*.

Indian sources. Algebra descended from the Arabs. By the time **Regiomontanus** learned algebra in Italy it was practiced by abbaco masters for more than 250 years. The tradition of surveying and mensuration within practical geometry goes back to Babylonian times. We can therefore conclude that classic mathematical Greek texts had hardly any influence on mediaeval European mathematics before 1460. Mediaeval knowledge of mathematics was very much depending on Eastern sources and influences. Humanists like Regiomontanus not only were aware of these Eastern influences it became their main motive to construct a "pure mathematics" founded on Greek classical texts, uncontaminated by the Arabs.

Regiomontanus as a transition point

During the fifteenth century Italian humanists eagerly started collecting editions of Greek mathematics. One of the most industrious was **Cardinal Bessarion** who lived in Venice. By his death in 1472 he had accumulated over five hundred Greek manuscripts. Regiomontanus, who had befriended Bessarion, began to study these Greek texts around 1463, including Diophantus' *Arithmetica*. He reported his find of the six books of the *Arithmetica* in a letter to Giovanni Bianchini. Highly receptive to influences between traditions, he immediately conjectured a relation between Greek and Arabic algebra.[16]

In his *Oratio compendiose declantur scientiae mathematicae et utilitates earum*, part of a series of lectures at the University of Padua in 1464, he defines and presents a sort of history of mathematics in eleven pages. Commencing with an etymological analysis of the Greek origin of the words 'mathematics', 'geometry' and 'arithmetic' he sets up the rhetoric for substantiating that all our knowledge of mathematics descends from Greek antiquity. After discussing the Egyptian origin of astronomy and its development within ancient Greece he moves to arithmetic:

> Although, through his skill and numbers, Pythagoras attained immortality among future generations, both because he submitted himself to wandering Egyptian and Arab teachers, who were greatly skilled in that study, then because he tried to probe all the secrets of nature by the certain connection of numbers.[17]

Here he already assumes a link between Arabic knowledge of arithmetic and algebra and the number theory of Pythagoras, a highly controversial position which is defended up to this day. Twentieth-century historians have used one particular problem, the *Bloom of Thymaridas*, to conjecture a relation between Diophantus and Pythagoras. This single problem, which became known to us through Iamblichus, six centuries after Thymaridas, is very dubious evidence for such relation.[18] Regiomontanus continues:

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[16] Rose, *The Italian Renaissance*, is an extensive study on the acquisition of mathematical manuscripts. For Bessarion see 44-46 and 90-109. Also relevant is Rose, "Humanist Culture". Regiomontanus's find of the *Arithmetica* of Diophantus is reported in his correspondence, see Curtze, "Der Briefwechsel", 256-7.

[17] Regiomontanus, *Oratio*, 46, translation from Byrne, "A Humanist History", p. 54.

[18] This is argued extensively in Heeffer, "The Reception". The main objection is that the *Bloom* and the alleged corresponding problem by Diophantus are in fact two different kinds of problems. The position is defended by Cantor, *Vorlesungen*, I, 148, Heath, *A History*, 94, Cajori, *A History*, 59, van der Waerden, *Science Awakening*, 116, Flegg, *Numbers*, 205 and Kaplan, *The Nothing*, 62. Cantor sees in the Pythagorean school an algebraic practice without the symbols: "Das ist, wie man sieht, volständig

> Nevertheless, Euclid made a much more worthy foundation of numbers in three of his books, the seventh, eighth and ninth, whence Jordanus gathered the ten books of elements of numbers and from this produced his three most beautiful books on given numbers.[19]

As we have previously argued, Euclid had little influence on medieval mathematics, and only the first and the second book were studied. Here, Regiomontanus mentions the seventh book in which a non-axiomatic theory of numbers is introduced. This Euclidean arithmetic is in its principles quite different from the tradition based on Nichomachus's *Arithmetica*, and in contrast to **Jordanus**, rather new to mediaeval scholars. Euclid defines a number as a multitude of units and thus excludes one as a number, not to speak about zero, negative or irrational solutions which frequently show up in the algebraic practice of abbaco masters. Euclidean notions of arithmetic must have been appeared very strange for fifteenth-century practioners of mathematics.[20] He then proceeds to algebra:

> Diophantus, however, produced thirteen most subtle books (which no one has yet translated from Greek into Latin), in which lie the very flower of all arithmetic, namely the art of *rei* and *census*, which today is called algebra after its Arabic name. Indeed, Latin authors treat many fragments of that most beautiful art, but after Giovanni Bianchini, an excellent man, I find a scarcity of greatly learned men in our own time.[21]

This is a crucial passage with lasting effects, not only on European mathematics as such but also on the historiography of mathematics. Here **Regiomontanus** introduces the idea that Arabic algebra descended from Diophantus's *Arithmetica*. His formulation is subtle. He does not claim that the Arabs learned algebra from **Diophantus**, but it can and it was understood as if Arabic algebra was derived from the *Arithmetica*. Regiomontanus was one of the few men who had seen the Greek text of **Diophantus** in 1464 and he was aware of its importance. By then he was also well-acquainted with Arabic algebra. He owned a copy of the Latin translation of the algebra by **al-Khwārizmī**, possibly from his own pen (MS. Plimpton 188). He must have been aware of the very different nature of the two traditions. The "art of *rei* and *census*" is the typical Latin nomenclature only used in the Latin translations of Arabic works. Here however, he uses this terminology to refer to Diophantus and claims this is known today as "algebra, after its Arabic name". The last sentence of the quotation is rather puzzling. He implies that apart from **Bianchini** not many men were versed in

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gesprochene Algebra, welcher nur Symbole fehlen, um mit einer modernen Gleichungsauflösung durchaus übereinzustimmen, und insbesondere ist mit Recht auf die beiden Kunstausdrücke der *gegebene* und *unbekannten* Grösse aufmerksam gemacht worden". Cajori finds in the Thymaridas "investigations of subjects which are really algebraic in their nature". Van der Waerden goes as far as to claim that "we see from this that the Pythagoreans, like the Babylonians, occupied themselves with the solution of systems of equations with more than one unknown".

[19] Regiomontanus, *Oratio*, 46.

[20] An axiomatic interpretation of the seventh book of Euclid's *Elements* was attempted by Malmedier, "Eine Axiomatic". However, for an argumentation of its non-axiomatic aspects see Vandoulakis, "Was Eulid's approach to arithmetic axiomatic?".

[21] Regiomontanus, *Oratio*, p. 46, translation from Byrne, 54-5. *Rei* en *census* are the technical terms for the unknown and the square of the unknown, wrongly translated by Byrne as 'assessing' and 'accounting'.

algebra. We do not know much about how and where Regiomontanus learned algebra. We know that he exchanged letters with **Bianchini** on problems solved by algebra the year preceding his lecture. In these letters he uses a symbolism which was similar to the one developed within abbaco algebra during the first half of the fifteenth century. He also compiled his own list of problems which he solved algebraically.[22] The sources that influenced Regiomontanus's algebraic practice still wait a full assessment, but we cannot imagine that he was unaware of the vernacular tradition of abbaco masters. His reference to "Latin authors" may indeed be a clue that he intentionally discards the vernacular tradition.

The *Oratio* heralded the initiation of a myth cultivated by humanists for centuries. **Diophantus**, first considered to be the source of inspiration for Arabic algebra, became the alleged origin of European algebra. Several humanist writers such as **Ramus**, chose to neglect or reject the Arabic roots of Renaissance algebra altogether. Ramus follows **Regiomontanus** but puts it more strongly:

> Diophantus, by whom we possess six Greek books, promised however by the author to be thirteen, about the admirable art of subtle and complex arithmetic that commonly is called by the Arabic name algebra; whereas from such an ancient author (he is indeed mentioned by **Theon**) the antiquity of the art appears.[23]

As a matter of fact, Diophantus had almost no impact on European mathematical practice before the late sixteenth century. Two decades before the first edition of the *Arithmetica* was published (1575) we see occasional references to Diophantus, such as by Johannes **Scheubel**, **Peletier**, Ramus and Pedro **Nunez**, but it is clear that these are more expressions of wishing being able to consult the text than actual testimonies.[24] A recent assessment of the *Oratio* by **Byrne**, from which we borrowed the English translation, is careful to point out that we should account for the actual texts, but does not question the continuity of Greek mathematics in the fifteenth century:

> It should be noted that given the distinction that Regiomontanus makes between the origins of the mathematical arts and their true founders—men like Euclid and Ptolemy, whose works still survive and are used—

[22] Bianchini is author of the book *De Liber florum Amalgesti*, which is not extant according to Curtze, "Der Briefwechsel" (p. 236, note 3). However at least six copies are now known to exist, Folkerts "Die mathematischen Studien" 206, note 162. Their correspondence is published by Curtze. For an example of the symbolism, see Figure 1 below. The 62 problems appear in Columbia University, Plimpton 188, fols. 82v-96r. Menso Folkerts is still working on a transcription. For a transcription and discussion of the earliest abbaco treatise differentiating between rhetrorical algebra (*per scrittura*) and symbolic algebra (*figuratamente*) see Heeffer, "Text production, reproduction and appropriation".

[23] From the *Scholae mathematica*, 37, cited and translated by Høyrup, "A new art", 33.

[24] For a survey of rejection of Arabic roots by sixteenth-century mathematicians see Høyrup, "A new art". Peletier's reference to Diophantus is based on Scheubel's. Scheubel wrote two treatises on algebra. The first is extant in manuscript form (Columbia University Library MS. X 512, Sch. 2, Q), the second was published as an introduction to Euclid's *Elements*, *Evclidis Megarensis*. The *Algebra* of Ramus was published anonymously and depends very much on Scheubel's. Ramus cites Diophantus on p. 328, (in the Schoner edition of 1592) and on p. 37 of the *Scholae Mathematicae* of 1569. According to Verdonk, *Petrus Ramus en de Wiskunde*, 405, Ramus possessed a Greek manuscript copy of the *Arithmetica*, but I have not been able to find further confirmation of this. Nunez refers to Regiomontanus's *Oratio* as well as to Diophantus in the dedication of his *Algebra*.

> the real continuity in the mathematical disciplines is between the earliest mathematical *texts* and the mathematics of the fifteenth century.[25]

Diophantus inspired authors on algebra such as Guillaume **Gosselin**, Simon **Stevin**, Rafaello **Bombelli** and **Viète** because by then symbolic algebra was well established.[26] By overrating the importance of Diophantus and downgrading the achievements of Arabic algebra, humanist writers created a new mythical identity of European mathematics. Suddenly Greek mathematics became European mathematics.

Viète as a highlight of humanist repudiation

Generally considered to be the father of modern algebra, **Viète** went as far as one can go in repudiating all Arabic influences. In order to build new foundations for the art of algebra he found it necessary to devise a new terminology and to abandon the 'barbaric' name algebra in favor of 'specious logistics'. Viète is often seen as a figure standing apart in the history of modern algebra, but we believe this is mostly so because of the imprint humanists had on the historiography of mathematics and because of his use of a new terminology. Viète was not the isolated figure as he is often depicted. His works and ideas fit very well into evolution which took place in France beginning with Peletier's *Algebre*. His contributions to algebra built on the fundamentals laid out by **Cardano** and **Stifel**. Giovanni **Cifoletti** convincingly demonstrated that the French algebraic tradition with **Peletier** and Gosselin laid the foundations of the *ars analytica* and the study of equations by Viète and later Descartes. Many of the realizations by Viète should therefore be viewed as part of a longer tradition. In more recent work, she argued that the humanist program consisted of establishing new foundations of algebra by applying the Euclidian notions of proof. In particular the common notions (*communes notiones*) from Proclus, later adopted by Valla, provided more certainty to algebra, which was essentially and art rather than a science.[27]

We have demonstrated that the 'specious logistics' of Viète, in which *species* refer to "the forms of things, possibly by means of the elements of the alphabet", is based on a continuous development of the second unknown, also known as the *regula quantitatis*, which emerged within the abbaco tradition. The introduction of letters for multiple unknowns in solving linear problems by Stifel was adopted by Peletier and refined by Johannes **Buteo** in his *Logistica* and later Gosselin. So 'specious logistics' is a culmination of a development which started much earlier.[28]

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[25] Byrne, "A Humanist History", 56.

[26] The title of Gosselin's *De arte magna* refers to Diophantus. He provides algebraic solutions to several Diophantine problems and refers to the Xylander edition. Stevin includes several problems from Diophantus in his *L'arithmetique.* Bombelli writes in the printed edition of his *Algebra* that his first version (manuscript) was based on al-Khwārizmī, Fibonacci and Pacioli (F. 2^{rv}) before he discovered a Greek manuscript of Diophantus' *Arithmetica* (Vaticana MS. Vat Gr. 200). In 1923 Ettoro Borlotti discovered Bombelli's manuscript in Bologna. It shows how he abandoned problems of the abbaco tradition in favour of the "new" problems. See Jayawardene, "The influence" for the abbaco problems left out.

[27] For a general overview of the French algebraic tradition see Cifoletti's PhD, *Mathematics and Rhetoric*; for a discussion of the *communes notiones* see particularly her article "From Valla to Viète". Cardano's *Practica arrithmetice* and the *Ars Magna*, together with Stifel's *Arithmetica Integra* have been very influential on the French tradition.

[28] For a comprehensive history of the second unknown and the evolution of symbolism, see Heeffer, *The Regula quantitatis*.

It is in his *Isagoge* that Viète introduces a whole new terminology for algebra and its operations. But he could build on what others had done before him. The "logistics" in *logistica speciosam* was already used by Buteo and Gosselin. The names *latus* and *quadratus* for the unknown and the square of the unknown were introduced by Ramus. Viète divided the process of algebraic problem solving in three steps: *zetetics*: the translation of a problem into a symbolic equation, *poristics:* the manipulation of the equation by the rules of algebra, and *exegetics:* the interpretation of the solution of the problem. The Arabic operations of *al-jabr* and *al-ikmāl*, translated in Latin as *restauratio* and *reductio* become *antithesis* and *hypobibasmo*. The new terminology had no further impact. Neither did his use of vowels for the unknowns and the use of consonants for constants and coefficients draw any followers. In the beginning of the seventeenth century the cossist symbolism remained dominant until being replaced by the one introduced by Descartes. So why going through all the trouble of devising a new language of algebra? We find the answer in his dedication of the *Isagoge* to Princess Mélusine. The term 'algebra' and all the "pseudo-technical terms" reminded him too much to the Arabic - and thus barbaric - origins of algebra. Algebra was a divine invention by the Greeks which was subsequently "spoiled and defiled by the barbarians". By using a Greek vocabulary it could be cleansed from this "filth" and restored to its original pure form:

> O princess most to be revered, those things which are new are wont in the beginning to be set forth rudely and formlessly and must then be polished and perfected in succeeding centuries. Behold, the art which I present is new, but in truth so old, so spoiled and defiled by the barbarians, that I considered it necessary, in order to introduce an entirely new form into it, to think out and publish a new vocabulary, having gotten rid of all its pseudo-technical terms (pseudo-categorematis) lest it should retain its filth and continue to stink in the old way, but since till now ears have been little accustomed to them, it will be hardly avoidable that many will be offended and frightened away at the very threshold. And yet underneath the Algebra or Almucabala which they lauded and called "the great art", all Mathematicians recognized that incomparable gold lay hidden, though they used to find very little.[29]

This is a sad example of how the humanist reformation of early modern mathematics came accompanied with systematic program to repudiate all eastern influences.

THE MYTH OF SYNCOPATED ALGEBRA

Ever since **Nesselmann**'s study on "Greek algebra" of 1842, historical accounts on algebra draw a distinction in rhetorical, syncopated and symbolic algebra. This tripartite distinction has become such a common-place depiction of the history of algebraic symbolism that modern-day authors even fail to mention their source. The repeated use of Nesselmann's distinction in three *Entwickelungstufen* on the stairs to perfection is odd because it should be considered a highly normative view which cannot be sustained within our current assessment of the history of algebra. Its use in

[29] Viète, *Isagoge*. The translation is from J. W. Smith, published in Klein, *Greek mathematical thought*, 318-9.

present-day text books can only be explained by an embarrassing absence of any alternative models. There are several problems with Nesselmann's approach.[30]

A problem of chronology

Firstly, if seen as steps within a historical development, as is most certainly the view by many who have used the distinction, it suffers from some serious chronological problems. Nesselmann places **Iamblichus**, Arabic algebra, Italian abbacus algebra and **Regiomontanus** under rhetorical algebra ("Die erste und niedrigste Stufe") and thus covers the period from 250 to 1470. A solution to the quadratic problem of **al-Kwārizmī** is provided as an illustration. The second phase, called syncopated algebra, spans from Diophantus's *Arithmetica* to European algebra until the middle of the seventeenth century, and as such includes Viète, Descartes and **van Schooten**. Nesselmann discusses problem III.7 of the *Arithmetica* as an example of syncopated algebra. The third phase is purely symbolic and constitutes modern algebra with the symbolism we still use today. Nesselmann repeats the example of al-Kwārizmī in modern symbolic notation to illustrate the third phase, thereby making the point that it is not the procedure or contextual elements but the use of symbols that distinguishes the three phases.[31]

Though little is known for certain about **Diophantus**, most scholars situate the *Arithmetica* in the third century which is about the same period as Iamblichus (c. 245-325). So, syncopated algebra overlaps with rhetorical algebra for most of its history. This raises serious objections and questions such as "Did these two systems influence each other?" Obviously, historians as **Tropfke** and **Gandz** were struck by this chronological anomaly and formulated an explanation. They claim that Arabic algebra does not rely on Diophantus's syncopated algebra but descends instead from Egyptian and Babylonian problem-solving methods which were purely rhetorical. However, these arguments are now superseded by the discovery of the Arabic translations of the *Arithmetica* (Sesiano 1982). Diophantus was known and discussed in the Arab world ever since **Qustā ibn Lūqā** (c. 860). So if the syncopated algebra of Diophantus was known by the Arabs why did it not affect their rhetorical algebra? If the Greek manuscripts used for the Arab translation of the *Arithmetica* contained symbols, we would expect to find some traces of it in the Arab version.[32]

The role of scribes

The earliest extant Greek manuscript, once in the hands of **Planudes** and used by Tannery, is Codex Matritensis 4678 (ff. 58-135) of the thirteenth century.[33] The extant Arabic translation published independently by **Sesiano** and **Rashed** was completed in 1198. So no copies of the *Arithmetica* before the twelfth century are extant. The ten centuries separating the original text from the earliest extant Greek copy is a huge distance. Two important revolutionary changes took place around the ninth century: the transition of papyrus to paper and the replacement of the Greek uncial or

[30] This paragraph is based on Heeffer, "On the Nature and Origin". Some examples of contemporary histories of algebra which use the distinction without references are Boyer, *A History*, 201; Flegg, Hay and Moss, *Nicolas Chuquet*, and Struik, *A Concise History*..
[31] Nesselmann, *Versuch einer kritischen*, 302 and following pages.
[32] See Tropfke, *Geschichte*, II, 14, and Gandz, "The Sources", 271 for the assumed origins of Arabic algebra. Two independent translations of the Arabic text of the *Arithmetica* have been published by Sesiano, *Books IV and VII*, and Rashed, *Diophante*.
[33] Tannery, *Diophanti Alexandrini opera*.

majuscule script by a new minuscule one. The transition to the new script was very general and drastic to a degree which puzzles today's scholars. From about 850 every scribe copying a manuscript would almost certainly adopt the minuscule script. Transcribing an old text into the new text was a laborious and difficult task, certainly not an undertaking to be repeated when a copy in the new script was already somewhere available. It is therefore very likely that all extant manuscript copies are derived from one Byzantine archetype copy in Greek minuscule. Although contractions where also used in uncial texts, the new minuscule much facilitated the use of ligatures. This practice of combining letters, when performed with some consequence, saved considerable time and therefore money. Imagine the time savings by consistently replacing *άριθμος*, which appears many times for every problem, with ς in the whole of the *Arithmetica*. The role of professional scribes should therefore not be underestimated. Although we find some occurrences of shorthand notations in papyri, the paleographic evidence we now have on a consistent use of ligatures and abbreviations for mathematical words points to a process initiated by mediaeval scribes much more than to an invention by classic Greek authors. Whatever syncopated nature we can attribute to the *Arithmetica* it is mostly an unintended achievement of the scribes.[34] The complete lack of any syncopation in the Arabic translation further supports this thesis. The name for the unknown and the powers of the unknown and even numbers are written by words in Arabic translation. The lack of, at that time, well-established Hindu-Arabic numerals seems to indicate that the Arabic translation was faithful to a Greek majuscule archetype. Sesiano argues that the Arabic version relies on the commentary by **Hypatia** while the Greek versions relate to the original text with some early additions and interpolations. Although the thesis of the reliance on Hypatia's commentaries is strongly opposed by Rashed, and while they disagree on many others issues, both interpretations and translations of the Arabic text concur on the lack of symbolism or syncopation. The *αλογος αριθμος*, or 'untold number' of the Greek text, is translated as *śay'* in Arab, and is thus very similar to the *cosa* of abbaco texts or the *coss* of German cossic texts.[35]

In so far the *Arithmetica* deserves the special status of syncopated algebra it is very likely that the use of ligatures in Greek texts is a practice that developed since the ninth century and not one by **Diophantus** during the third century. This overthrows much of the chronology as proposed by **Nesselmann**.

Symbols or ligatures?

A third problem concerns the interpretation of the qualifications 'rhetorical' and 'syncopated'. Many authors of the twentieth century attribute a highly symbolic nature to the *Arithmetica*. Let us take **Cajori** as the most quoted reference on the history of mathematical notations. Typical for Cajori's approach is the methodological mistake of starting from modern mathematical concepts and operations and looking for corresponding historical ones. He finds in **Diophantus** no symbol for multiplication, and addition is expressed by juxtaposition. For subtraction the symbol is an inverted ψ. As an example he writes the polynomial

$$x^3+13x^2+5x+2 \qquad \text{as} \qquad K^{\Upsilon}\overline{\alpha}\Delta^{\Upsilon}\overline{\iota\chi}\varsigma\overline{\varepsilon}\overset{\circ}{\mathrm{M}}\overline{\beta}$$

[34] For an introduction on the transmission of Greek texts and the conversion of scripts see Reynolds and Wilson, *Scribes and Scholars* (ed. 1996, 66-7). Our view on the role of scribes has recently also been put forward in relation to Archimedes's works in Netz and Noel, *The Archimedes Codex*.

[35] Sesiano, *Books IV and VII*, 75, Rashed, *Diophante*, III: LXII.

where K^Y, Δ^Y, ς are the third, second and first power of the unknown and $\overset{\circ}{M}$ represents the units. Higher order powers of the unknown are used by Diophantus as additive combination of the first to third powers.[36]

Cajori makes no distinction between symbols, notations or abbreviations. In fact, his contribution to the history of mathematics is titled *A History of Mathematical Notations*. In order to investigate the specific nature of mathematical symbolism one has to make the distinction somewhere between symbolic and non-symbolic mathematics. This was, after all, the purpose of Nesselmann's distinction. We take the position together with Thomas **Heath**, Paul **Ver Eecke** and Jacob **Klein**, that the letter abbreviations in the *Arithmetica* should be understood purely as ligatures:[37]

> We must not forget that *all* the signs which Diophantus uses are merely word abbreviations. This is true, in particular for the sign of "lacking", ↑, and for the sign of the unknown number, ς, which (as Heath has convincingly shown) represents nothing but a ligature for *αρ* (*άριθμος*).

Even **Nesselmann** acknowledges that the 'symbols' in the *Arithmetica* are just word abbreviations ("sie bedient sich für gewisse oft wiederkehrende Begriffe und Operationen constanter Abbreviaturen statt der vollen Worte"). In his excellent French literal translation of Diophantus, Ver Eecke consequently omits all abbreviations and provides a fully rhetorical rendering of the text as in "Partager un carré proposé en deux carrés" (II.8, "Divide a given square into two squares"), which makes it probably the most faithful interpretation of the original text.[38]

This objection marks our most important critique on the threefold distinction: symbols are not just abbreviations or practical short-hand notations. Algebraic symbolism is a sort of representation which allows abstractions and new kinds of operations. This symbolic way of thinking can use words, ligatures or symbols. The distinction between words, word abbreviations and symbols is in some way irrelevant with regards to the symbolic nature of algebra.

Counter examples

A final problem for Nesselmann's tripartite distinction is that now, almost two centuries later, we have a much better understanding of the history of symbolic algebra. **Nesselmann** relied mostly on the eighteenth-century Jesuit historian Pietro **Cossali** for a historical account of Italian algebra before the sixteenth century. Except for Rafaello **Canacci**, Cossali does not discuss much the algebra as it was practiced within the abbacus tradition of the fourteenth and fifteenth century. Guillaume **Libri**,

[36] For a typical symbolic interpretation see Kline, *Mathematical Thought*, I, 139-40. Cajori discusses the symbolism of Diophantus in *A History*, I, (ed. 1993, 71-4).

[37] Heath, who only uses modern symbolism in the footnotes of his Euclid edition, presents a complete symbolic rendering of Diophantus's *Arithmetica*. Despite this, he supports the interpretation of symbols as ligatures. Ver Eecke's *Diophante* lacks all symbols and abbreviations in the text. Klein's quotation is from *Greek Mathematical Thought* (ed. 1968, 146).

[38] This problem led Fermat to add the marginal note in his copy of Bachet's translation "Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere". If Fermat had used the 'syncopated' algebra of Diophantus he might have had some marginal space left to add his "marvelous proof" for this theorem.

who had collected many manuscripts from this tradition, describes and published several transcriptions in his *Histoire des sciences mathématiques en Italie* published in 1838. Oddly, the well-informed Nesselmann does not seem to know Libri's work and thus remains ignorant of the continuous practice of algebra in Italy since Fibonacci and the first Latin translations of **al-Kwārizmī**. It is only since the past decades that we have a more complete picture on abbacus algebra thanks to the work of Gino **Arrighi**, Warren **van Egmond**, the *Centro studi della matematica medioevale* of Sienna, and **Høyrup**'s recent book.[39]

In our understanding, symbolic algebra is an invention of the sixteenth century which was prepared by the algebraic practice of the abbacus tradition. At least abbacus algebra has to be called syncopated in the interpretation of **Nesselmann**. Many of abbacus manuscripts use abbreviations and ligatures for *cosa*, the unknown (as c, co. or ρ), censo or cienso, the second power of the unknown (ce. or ç), cubo, the third power (cu.) and beyond. Also plus, minus and the square root are often abbreviated as in p, m and R (with an upper or lower dash). From the fifteenth century we also find manuscripts that explicitly refer to a method of solving problems that is different from the regular rhetorical method. In an anonymous manuscript of c. 1437, the author solves several standard problems in two ways. One he calls symbolical (*figuratamente*) and the other rhetorical (*per scrittura*). Possibly the practice of solving a problem *figuratamente* existed before but in any case it was not found suitable to appear in writing in a treatise. Here however, the anonymous author believes a symbolic notation contributes to a better understanding of the solution as he writes:

> I showed this symbolically as you can understand from the above, not to make things harder but rather for you to understand it better. I intend to give it to you by means of writing as you will see soon.[40]

He then repeats the solution in a rhetorical form as we know from other abbacus texts. This is the first occasion in the history of algebra where an author makes an explicit reference to two different kinds of problem solving, which we would now call symbolical and non-symbolical.

This manuscript or related copies may have influenced the German cossists. **Regiomontanus**, who maintained close contacts with practitioners of algebra in Italy, adopts the same symbolic way of solving problems. In his correspondence with **Bianchini** of 1463 we find problems very similar to the abbacus text: divide 10 into two parts so that one divided by the other together with the other divided by the first equals 25.[41] In modern symbolic notation the problem can be formulated as follows:

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[39] Arrighi edited more than twenty abbaco texts, the most important ones being Piero della Francesca's *Trattato* and Antonio de Mazzinghi's *Trattato di Fioretti*. For an overview of abbaco texts see Franci and Rigatelli, "Towards a history". An almost complete catalogue of extant abbaco text has been published by van Egmond, *Practical Mathematics*. For a current assessment see Høyrup, *Jacopo da Firenze*.

[40] The manuscript is known as Florence, Biblioteca Nazionale, Magl. Cl. XI. 119. See Heeffer, "Text production reproduction and appropriation" for a critical edition of the third part on algebra, f. 59r: "Ora io telo mostrata figuratuiamente come puoi comprendere di sopra bene che e lla ti sia malagievole ma per che tulla intenda meglio. Io intende di dartela a intendere per scrittura come apresso vedrai".

[41] The correspondence is kept in Nürnberg, City Library, Cent. V, 56c, ff. 11r-83v, The transcription is by Curtze 1902, 232-234: "Divisi 10 in duos, quorum maiorem per minorem divis, item minorem per

$$\frac{x}{x-10}+\frac{10-x}{x}=25$$

Regiomontanus solves the problem in the same manner of abacus algebra but adopts only the symbolical version (as shown in Figure 1). He uses symbols for *cosa* and *censo* which we typically find in German cossist algebra from 1460 for a period of about 160 years. Further evidence is found in *De Triangulis Omnimodis* by **Regiomontanus**. The book was published long after his death and omits all the symbolism that Regiomontanus used in his manuscript. From these examples we gather that symbolic calculations were practiced during the fifteenth century but this form of mathematics was not found suitable for publication.[42]

While we see in later abbacus algebra and Regiomontanus the roots of symbolic algebra, **Nesselmann** places both within the stage of rhetorical algebra. According to Nesselmann's own definition these two instances of algebraic practice should at least be called syncopated.

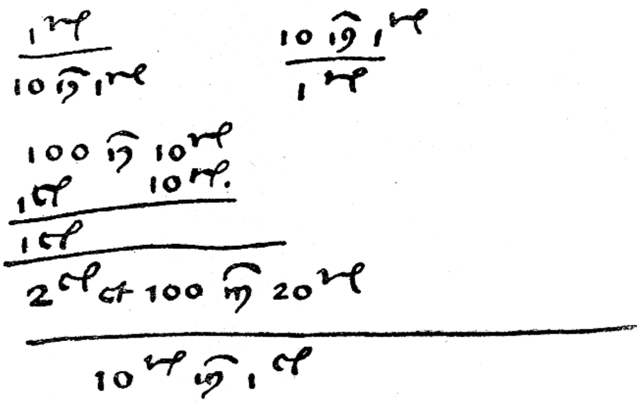

FIGURE 1: THE SOLUTION OF AN ARABIC DIVISION PROBLEM BY REGIOMONTANUS (C 1460, NÜRNBERG CENT. V 56C, F. 23)

$$\frac{x}{10-x},\frac{10-x}{x}$$

$$100-10x$$

$$\frac{x^2-10x}{x^2}$$

$$\frac{2x^2+100-20x}{10x-x^2}$$

Literal transcription in modern symbolism.

THE TROUBLESOME RECEPTION OF INDIAN MATHEMATICS

As a last illustration of the influence of humanist ideas on the historiography of mathematics we will now discuss the reception of classic Indian mathematics during the eighteenth century. The eighteenth century was very important for history of science and the history of mathematics in particular. Many classic works were produced during this century and it was the kind of history in which explanations were expected on the formation of modern mathematics. A most typical example is the monumental work of Moritz **Cantor** which covers the whole of mathematics with much eye for detail and original sources. Like many men of his era Cantor was entrenched with humanist ideas about the cultural superiority of Western knowledge. It is within such context that Europe was struck by Colebrooke's translation of classic Indian works on arithmetic, algebra and trigonometry. Historians of mathematics were

maiorem. Numeros quotiens coniunxi, et fuit summa 25 : quero, que sint partes". The corresponding problem in Magl. Cl. XI. 119 is on f. 61v but uses a sum of 50 instead of 25.

[42] The manuscript was found at the Library of the Russian Academy of Sciences at Moscou, ms. 3 (also nr. 541). The differences between the printed and manuscript version are discussed by Kaunzner, "Über Regiomontanus" who also shows a facsimile of some pages with symbolic calculations.

forced to come with explanations how these newly discovered classic works fit in their grand schemes.[43]

The first descriptions of Indian algebra

In some sense Wallis' *Treatise on Algebra* of 1685 can be considered the first serious historical investigation of the history of algebra. John **Wallis** was well-informed about Arabic writings through **Vossius** and was one of the first to attribute correctly the name algebra to *al-jabr w'al-muchābala*. He also pointed out the mistaken origin of algebra as Geber's name, which was a common misconception before the seventeenth century. Unprecedented, Wallis casted doubts on Diophantes' contribution to modern algebra. He even launched the idea that Arab algebra may have been originated from India:

> However, it is not unlikely that the Arabs, who received from the Indians the numeral figures (which the Greeks knew not), did from them also receive the use of them, and many profound speculations concerning them, which neither Latins nor Greeks know, till that now of late we have learned them from thence. From the Indians also they might learn their algebra, rather than from Diophantus.[44]

So, while in the seventeenth century no Sanskrit mathematics had yet been introduced into Europe, scholars by then were aware of the existence of Indian algebra. Wallis' view persisted in eighteenth-century historical studies, which reiterated the influence from Indian mathematics. Pietro **Cossali**, who wrote an extensive monograph on the history of algebra, concluded his discussion on al-Khwārizmī's *Algebra* with al-Khwārizmī "not having taken algebra from the Greeks, … must have either invented it himself, or taken it from the Indians. Of the two, the second appears to me the most probable". **Hutton**, who included a long entry on algebra in his *Mathematical and Philosophical Dictionary*, wrote:

> But although Diophantus was the first author on algebra that we know of, it was not from him, but from the Moors or Arabians that we received the knowledge of algebra in Europe, as well as that of most other sciences. And it is matter of dispute who were the first inventors of it; some ascribing the invention to the Greeks, while others say that the Arabians had it from the Persians, and these from the Indians.[45]

The first translations

In the early nineteenth century, the English orientalist Henry Thomas **Colebrooke**, who previously published his *Sanskrit Grammar* , undertook the task of translating three classics of Indian mathematics, the *Brāhmasphuṭasiddhānta* of Brāhmagupta (628) and the *Līlāvatī* and the *Bījagaṇita* of **Bhāskara** II (1150). At once European historians had something to reflect upon. In a period that mathematics was hardly practiced in Europe and in the Islam regions, there appeared to have existed this Indian tradition in which algebraic problems were solved with multiple unknowns, in which zero and negative quantities were accepted and in which sophisticated methods

[43] For a more comprehensive discussion on the reception of Indian mathematics, see Heeffer, "The Reception".
[44] Wallis, *Treatise*, 4. His discussion on the etymology of algebra can be found on p. 5.
[45] Cossali, *Origine*, I, 216-9; Hutton, *Dictionary*, I, 66.

were used to solve indeterminate methods. In general, nineteenth-century historians showed an admiration for the Hindu tradition. However, whenever explanations were required, scholars became divided into two opposing camps, which we could call the believers and the non-believers. Non-believers did not grant Indian mathematicians the status of original thought. Indian knowledge must have stemmed from the Greeks, the cradle of Western mathematics, or even mathematics as such. The major non-believer was Moritz **Cantor** who published an influential four-volume work on the history of mathematics (1880-1908). Cantor takes every opportunity to point out the Greek influences on Hindu algebra. Some examples: the Indians learned algebra through traces of algebra within Greek geometry; Brāhmagupta's solution to quadratic equations has Greek origins; or the Indian method for solving linear problems in several unknowns depended on the Greek method of *Epanthema*.[46]

The believers were not convinced by accidental resemblances between Greek and Hindu solution methods and did not see why Indian mathematics could not have been an independent development. Especially Hankel touches the sore spot when he writes:

> That by humanist education deeply inculcated prejudice that all higher intellectual culture in the Orient, in particular all science, is risen from Greek soil and that the only mentally truly productive people have been the Greek, makes it difficult for us to turn around the direction of influence for one instant.[47]

Soon after the Dutch scholar Hendrik **Kern** published the Sanskrit edition of the *Āryabhaṭīya*, the French orientalist Léon **Rodet** was the first to provide a translation in a Western language. Rodet published in the French *Journal Asiatiques* several articles and monographs on Indian mathematics and its relation with earlier and later developments in the Arab and Western world. He is the scholar who displays the most balanced and subtle views on the relations between traditions. Especially his assessment of Hindu and Arab algebra as two independent traditions is still of value today. He certainly was a believer. Concerning Āryabhaṭa's inadequate approximation of the volume of a sphere (prop. 7), he writes somewhat cynically that if **Āryabhaṭa** got his knowledge from the Greeks, then apparently he chose to ignore **Archimedes**.[48] George **Thibaut** who translated several Sanskrit works on astronomy such as **Varāhamihira**'s *Pañcasiddhāntikā*, also wrote an article on Indian mathematics and astronomy in the *Encyclopedia of Indo-Aryan Research*. Concerning influences from

[46] On the dependence of Greek geometry: Cantor, *Vorlesungen*, II, 562: "Spuren griechischer Algebra müssen mit griechischer Geometrie nach Indien gedrungen sein und werden sich dort nachweisen lassen"; on Brāhmagupta: Cantor ,*Vorlesungen*, II, 584: "So glauben wir auch deutlich die griechische Auflösung der quadratischen Gleichung, wie Heron, wie Diophant sie übte, in der mit ihr nicht bloss zufällig übereinstimmenden Regel des Brāhmagupta zu erkennen", on the *Epanthema*, see Heeffer, *The Reception*.

[47] My translation from Hankel, *Zur Geschichte der Mathematik* , 204: "Das uns durch die humanistische Erziehung tief eingeprägte Vorurtheil, dass alle höhere geistige Cultur im Orient, insbesondere alle Wissenschaft aus griechischem Boden entsprungen und das einzige geistig wahrhaft productive Volk das griechische gewezen sei, kann uns zwar einen Augenblick geneigt machen, das Verhältniss umzukehren".

[48] Kern, *The Āryabhaṭīya*; Rodet finished the translation in 1877 but it was only published in 1879 as Rodet, *Leçons de calcul*. For an assessment of Rodet's publications on Indian and Arabic algebra, see Heeffer, *A Conceptual Analysis*. Rodet's comments: *Leçons de calcul* , 409: "Mais elle a, pour l'histoire des mathématiques, d'autant plus de valeur, parce qu'elle nous démonstre que si Āryabhaṭa avait reçu quelque enseignement des Grecs, il ignorait au moins les travaux d'Archimède".

Greek mathematics, he takes a middle position. In discussing Hindu algebra he writes that "In allen diesen Beziehungen erhebt sich die indische algebra erheblich über das von Diophant Geleistete". As on the origins of Indian mathematics, he points out that Indian algebra, especially indeterminate analysis, is closely intertwined with its astronomy. As he argued on the Greek roots of Indian "scientific" astronomy, his evaluation is that Indian mathematics is influenced by the Greeks through astronomy. However, he adds that several arithmetical and algebraic methods are truly Indian.[49]

Despite the existence of several studies and opinions which should provide sufficient counterbalance for Cantor's position as a non-believer, his views remained influential well into the twentieth century. We may say that the "humanist prejudice" is still alive today. The myth that Greek mathematics is our (Western) mathematics has become intertwined with our cultural identity so strongly that it becomes difficult to understand intellectual achievements within mathematics foreign to the Greek tradition.

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[49] Thibaut, *The Panchasiddhantika*. The Encyclopedia is published by Bühler, e.a., *Grundriss*. Rodet's quotation are from p. 73 and 76-77.

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