A Weak Calculus with Explicit Operators for Pattern Matching and Substitution

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Abstract. In this paper we propose a **Weak Lambda Calculus** called λP_w having explicit operators for **Pattern Matching** and **Substitution**. This formalism is able to specify functions defined by cases via pattern matching constructors as done by most modern functional programming languages such as OCAML. We show the main property enjoyed by λP_w, namely subject reduction, confluence and strong normalization.

1 Introduction

In this paper we propose a **Weak Lambda Calculus** with **Pattern Matching** and **Explicit Substitution** called λP_w. The calculus λP_w is inspired by calculi of explicit substitutions [ACCL90, BBLRD96, BR95, Kes96] and by calculi of patterns [KPT96, CK99, CK]. The weak nature of λP_w allows us to denote variables by names without requiring α-conversion to implement correctly the notion of reduction.

Theoretical study of functional programming has been enriched by the introduction of typed λ-calculi, explicit substitutions [ACCL90, BBLRD96, BR95, Kes96] and pattern matching [KPT96, CK99, CK]. These three notions are the main ingredients of the formalism we propose in this paper to model typed functional languages with function definition by cases.

In the early thirties, Church proposed the λ-*calculus* as a general theory of functions and logic. Typed versions of λ-calculus were then defined by Curry and Church, they became the standard theoretical tool for defining and implementing *typed functional programming languages*. There is however an important gap between λ-calculus and modern functional languages:

– On one hand, the operation of substitution is not incorporated in the language level but is left as a meta-operation.
– On the other hand, most popular functional languages (resp. proof assistants) allow the definition of functions (resp. proofs) by cases via pattern-matching mechanisms, while λ-calculus does not incorporate at all these constructs.

The first problem is solved by incorporating the so-called explicit substitutions into the language used to implement functional programming. To do this, one simply adds a new construction to denote substitution and new reduction rules to describe the interaction between substitution and other constructors of the language. Many calculi with explicit substitutions [ACCL90, BBLRD96, BR95, Kes96] have been proposed in the literature, operational and logic properties of these calculi were extensively studied [Les94, CHL96, Kes96].

The second problem is solved by allowing abstractions of function not only with respect to variables but also with respect to patterns. Thus, the form of the arguments of a given function can be specified in a very precise form; for instance, a term having the form $\lambda\langle x, y\rangle.M$ specify that the expected argument is a pair.

In the early nineties, Kesner, Tannen and Puel [KPT96] proposed a **Calculus of Pattern Matching** as a tool of theoretical study of pattern matching à la ML. In 1999, Kesner and Cerrito [CK99, CK] refined the ideas in [KPT96] and defined the calculus TPC_{ES} as a formalism with **explicit pattern matching** and **explicit substitution**. Other languages with explicit pattern matching, such as for example the ρ-calculus [CK98, CKL01], were recently proposed in the literature to model other programming paradigms.

The calculus presented in this paper, called λP_w, is a calculus with explicit pattern matching and explicit substitutions. This calculus is not designed as a *user level language* but as the *output calculus* of a pattern matching compilation algorithm. Such an algorithm is supposed to take a pattern matching function definition and to return an equivalent one where all the ambiguities between overlapping patterns have disappeared and where incomplete patterns definitions have been detected and completed. Such an hypothesis is not really restrictive since all the functional languages with pattern matching features ([Obj]) apply such an algorithm before evaluating programs.

The calculus λP_w is based on [CK99, CK], but have the following new features:

- λP_w is a **weak calculus** of explicit substitutions, that is, functions are lazyly evaluated. To implement this correctly, substitutions are not allowed to cross lambda constructors - so that α-conversion is no more needed to achieve correct reduction of terms - and composition of substitutions is incorporated into the substitution language in order to guarantee confluence. The syntax of λP_w is based on the weak σ-calculus with names [CHL96] in contrast to TPC_{ES}, which is a **strong** calculus based on the substitution formalism with names called `x` [BR95, Blo95].
- In contrast to TPC_{ES} which treats "ordinary" substitutions explicitly but the so-called "sum" substitutions implicitly [1], λP_w treats *all* the substitutions as explicit. This choice makes the formalism (typing rules and typing

[1] In fact, the first version presented in [CK99] treats sum substitutions explicitly, but the revised and corrected version in [CK] only keeps ordinary substitution as an explicit operation by moving sum substitution to the meta-level.

reduction rules) and the proofs more involved than those in [CK99], but results in a complete and self-contained formalism which is able to describe different implementations of functional languages with pattern matching.

The formalism that we present in this paper enjoys all the classical properties of typed λ-calculi.

- It is **confluent** on all terms.
- It has the **subject reduction** property.
- It is **strongly normalizing** on all well-typed terms.

The paper is organized as follows. We will first give in Section 2 a formal definition of λP_w and give some basic properties such as *preservation of free variables* by reduction and confluence on all ground terms. We then introduce in Section 4 a typing system for λP_w and show that λP_w enjoys the *subject reduction* property. We then study in Section 5 strong normalization of $\lambda_{\sigma w}$ in order to introduce the technique of *reducibility modulo*. We finally show *strong normalization* of well-typed λP_w-terms in Section 6 before the conclusion given in Section 7.

2 Definition of λP_w

We first define the *raw* expressions of the calculus λP_w by giving three different sets to denote respectively raw terms, raw substitutions and raw sum terms. The notion of raw expression is refined by first defining the set of free variables of any raw expression which allows us to define (well-formed) expressions such as terms, substitutions and sum terms. Reduction rules of λP_w are given in Figure 1. These rules are showed to preserve free variables of expressions.

2.1 The Grammar

We fix two distinct infinite sets of variables: the set of *usual variables*, noted $x, y, z, \ldots$, which are used to denote ordinary terms, and the set of *sum variables*, noted $\xi, \psi, \ldots$, which are used to denote disjunction. We also fix two constants `L` and `R` and we use the notation `K` to denote indistinctly one or the other one. We will also use the notation `T` to denote indistinctly `L`, `R` or a sum variable.

Types of λP_w are given by the following grammar:

$$
\begin{array}{lll}
(\textbf{Types})\ A ::= & \iota & \text{Base type} \\
 & \mid A \times A & \text{Product Type} \\
 & \mid A + A & \text{Sum Type} \\
 & \mid A \rightarrow A & \text{Functional Type}
\end{array}
$$

Patterns of λP_w are given by the following grammar:

$$
\begin{array}{lll}
(\textbf{Pattern})\ P := & \_ & \text{Wildcard} \\
 & \mid x & \text{Variable} \\
 & \mid \langle P, P \rangle & \text{Pair} \\
 & \mid @(P, P) & \text{Contraction} \\
 & \mid (P \mid_\xi P) & \text{Sum}
\end{array}
$$

The notations $\_$, x and $\langle P, Q\rangle$ are standard while the notation $@(P, Q)$ is similar (indeed more general) to the `as` constructor of Ocaml [Obj]. The pattern $(P \mid_\xi Q)$ is used to specify two different structures P and Q (of types A and B) corresponding to a pattern of sum type $A + B$. The sum variable ξ appearing in a pattern $(P \mid_\xi Q)$ is used to propagate the result of any matching w.r.t the pattern $(P \mid_\xi Q)$ all along the term where this variable occurs.

Raw Substitutions of λP_w are given by the following grammar:

$$\begin{array}{rll}
(\textbf{Raw Substitution})\ s ::= & id & \text{Identity} \\
\mid & (x/M).s & \text{Cons_usual_var} \\
\mid & (\xi^{P,A}/\mathtt{K}).s & \text{Cons_sum_var} \\
\mid & s \circ s & \text{Concatenation}
\end{array}$$

In order to mark which branch of a sum pattern has been chosen we use special syntax that we called *sum terms*. A sum term is either a constant (there is one for each possible choice), a sum variable (no choice has been made), or a substituted sum variable (the full evaluation has not been made).

$$\begin{array}{rll}
(\textbf{Raw Sum Terms})\ \Xi ::= & \xi & \text{Sum Variable} \\
\mid & \mathtt{L} & \text{Left Constant} \\
\mid & \mathtt{R} & \text{Right Constant} \\
\mid & \xi[s] & \text{Sum Substitution}
\end{array}$$

We are now able to introduce λP_w-terms. The main difference between λ-calculus and pattern calculi is that the notation $\lambda x.M$ is generalized to $\lambda P.M$ where P is a pattern as given by the previous grammar. Thus, λP_w-terms are given by the following grammar:

$$\begin{array}{rll}
(\textbf{Raw Terms})\ M ::= & x & \text{Usual Variable} \\
\mid & (M\ N) & \text{Application} \\
\mid & \langle M, M\rangle & \text{Pair} \\
\mid & \mathtt{inl}_B(M) & \text{Left injection} \\
\mid & \mathtt{inr}_A(M) & \text{Right injection} \\
\mid & [M \mid_\xi M] & \text{Case} \\
\mid & [M \mid^s_\Xi M] & \text{Frozen Case} \\
\mid & \lambda P{:}A.M & \text{Abstraction} \\
\mid & M[s] & \text{Closure}
\end{array}$$

All along the paper we may sometimes omit types from expressions in order to simplify the notation, but expressions are supposed to be as defined by this grammar. A *Case* constructor of the form $[M \mid_\xi N]$ is used to specify two different terms M and N corresponding respectively to two different patterns P and Q of a sum pattern $(P \mid_\xi Q)$ appearing somewhere in the program. The communication between the case constructor $[M \mid_\xi N]$ and its corresponding sum pattern $(P \mid_\xi Q)$ is achieved via the sum variable ξ. The introduction of the Frozen Case constructor is purely technical, the idea is to prevent reduction of the sub-term M (resp. N) inside a case constructor of the form $[M \mid_\xi N]$ where a left (resp. right) choice has been already made.

Example 1. A simple λP_w-term is $\lambda(x \mid_\xi y) : A + B.[\lambda y' : B.\langle x, y'\rangle \mid_\xi \lambda x' : A.\langle x', y\rangle]$. For a more interesting example let us suppose that we have encoded the recursive type *nat* as a sum type, and that $(0 \mid_\xi S\ m)$ is a pattern of type *nat* representing either 0 or a positive natural number of the form $S\ m$. We refer the reader to [CK99] for more examples and more details about encoding of recursive types in the formalism of pattern calculi. Indeed, the following Ocaml [Obj] term:

```
match n with
| 0     -> 0
| (S m) -> m
```

is given by the λP_w-term $\lambda(0 \mid_\xi (S\ m)) : nat.[0 \mid_\xi m]$.

Definition 21 (Raw expression) A *raw expression* is either a raw term, a raw substitution or a raw sum term.

As usually done in λ-calculus we will work modulo α-conversion. This notion must be defined with care since bound variables are not only *all* the variables appearing in *complex* patterns, but also, *all* the variables bound by substitutions.

Definition 22 (Binding Variables) The set of *Binding Variables* of a pattern (resp. a substitution) is defined as:

$$\begin{array}{ll} BVar(\_) & = \emptyset \\ BVar(x) & = \{x\} \\ BVar(\langle P, Q\rangle) & = BVar(P) \cup BVar(Q) \\ BVar((P \mid_\xi Q)) & = BVar(P) \cup BVar(Q) \cup \{\xi\} \\ BVar(@(P, Q)) & = BVar(P) \cup BVar(Q) \end{array} \qquad \begin{array}{ll} BVar(id) & = \emptyset \\ BVar((x/M).s) & = BVar(s) \cup \{x\} \\ BVar((\xi^P/\mathtt{K}).s) & = BVar(s) \cup BVar(P) \cup \{\xi\} \\ BVar(s \circ t) & = BVar(s) \cup BVar(t) \end{array}$$

We have for example, $BVar((x \mid_\xi y)) = \{\xi, x, y\}$ and $BVar((x/M).(\xi^y/\mathtt{L}).id) = \{x, y, \xi\}$.

Definition 23 (Free Variables) The set of *Free Variables* of an expression e is given by:

$$\begin{array}{ll} FV(x) & = \{x\} \\ FV(\mathtt{inl}(M)) = FV(\mathtt{inr}(M)) & = FV(M) \\ FV(M\ N) = FV(\langle M, N\rangle) & = FV(M) \cup FV(N) \\ FV(\lambda P.M) & = FV(M) \setminus BVar(P) \\ FV(M[s]) & = (FV(M) \setminus BVar(s)) \cup FV(s) \\ FV([M \mid_\xi N]) & = FV(M) \cup FV(N) \cup \{\xi\} \\ FV([M \mid^s_\Xi N]) & = ((FV(M) \cup FV(N)) \setminus BVar(s)) \cup FV(s) \cup FV(\Xi) \\ FV(id) = FV(\mathtt{K}) & = \emptyset \\ FV((x/M).s) & = FV(M) \cup FV(s) \\ FV((\xi^P/\mathtt{K}).s) & = FV(s) \\ FV(s \circ t) & = (FV(s) \setminus BVar(t)) \cup FV(t) \\ FV(\xi) & = \{\xi\} \\ FV(\xi[s]) & = (\{\xi\} \setminus BVar(s)) \cup FV(s) \end{array}$$

Thus for example, $FV(\lambda(x \mid_\xi y).[x \mid_\psi t]) = \{\psi, t\}$ and $FV([x \mid^{(\xi^y/\mathtt{L}).id}_{\xi[(x/t).id]} y]) = \{x, t, \xi\}$.

We define the set of *free sum variables* (FSV) of a raw expression e as the set of *sum variables* of e which are in $FV(e)$.

Definition 24 (Bound variables) The *Bound Variables* of a raw expression e are those variables appearing in e but not free in e.

We are now ready to define α-conversion on λP_w-expressions as simply renaming of bound variables. Thus, for example $\lambda(x \mid_\xi x).x$ and $\lambda(y \mid_\psi y).y$ are α-equivalent, but neither $\lambda(x \mid_\xi x).y$ and $\lambda(y \mid_\xi y).y$ nor $\lambda(x \mid_\xi x).x$ and $\lambda(x \mid_\xi y).x$ are α-equivalent.

2.2 The Reduction system

We are now ready to introduce the reduction rules which are given in Figure 1.

The **Pattern matching** rules are the rules implementing the pattern matching.

The **Propagation of Substitutions**, **Substitutions and Variables and Constants**, **Substitutions and Composition** rules are a natural extension of those of the σ-calculus.

The **Case** rules explain the mechanism to distribute a substitution s with respect to a case term $[M \mid_\xi N]$ which consists in:

- We first transform the case term into a frozen case term using the rule $(Freeze)$.
- We then treat the part $\xi[s]$ of the obtained frozen case term until a result (*i.e.* the variable ξ, the constants $\mathtt{L}$ or $\mathtt{R}$) is obtained.
- We can then distribute the substitution in the frozen case term using one of the rules $(Left)$, $(Right)$ or (Xi).

The reduction system generated by the rules *Abs_id*, *Abs_pair*, *Abs_contr*, *Abs_left*, *Abs_right*, *Abs_var* and *Abs_wild* is used to implement the pattern matching operation and is noted by $\longrightarrow_P$. All the other rules generate the reduction system used to implement the behavior of substitution and is noted by $\longrightarrow_{es}$. The reduction relation $\longrightarrow_{\lambda P_w}$ is generated by $\longrightarrow_{es} \cup \longrightarrow_P$. To simplify the notation we may simply note $\longrightarrow$ for $\longrightarrow_{\lambda P_w}$ in the rest of the paper.

Example 2. We show one way to propagate the substitution $s = (x/M_3).(\xi^{P\text{-}A}/\mathtt{L}).id$ inside the term $M = [M_1 \mid_\xi M_2]$.

- First of all we reduce the case term into a frozen case term by $M[s] \longrightarrow_{Freeze} [M_1 \mid^s_{\xi[s]} M_2]$
- We then "evaluate" the part $\xi[s]$: $\xi[s] \longrightarrow_{Sub\_sum\_var_4} \xi[(\xi^{P\text{-}A}/\mathtt{L}).id] \longrightarrow_{Sub\_sum\_var_2} \mathtt{L}$

Start Rule

$(\lambda P.M)\ N$	$\longrightarrow$	$(\lambda P.M)[id]\ N$	(Abs_id)

Pattern Matching

$(\lambda\langle P_1, P_2\rangle.M)[s]\ \langle N_1, N_2\rangle$	$\longrightarrow$	$((\lambda P_1.\lambda P_2.M)[s]\ N_1)\ N_2$	(Abs_pair)
$(\lambda @(P_1, P_2).M)[s]\ N$	$\longrightarrow$	$((\lambda P_1.\lambda P_2.M)[s]N)\ N$	(Abs_contr)
$(\lambda(P_1 \mid_\xi P_2).M)[s]\ \texttt{inl}(N)$	$\longrightarrow$	$(\lambda P_1.M)[(\xi^{P_2}/\texttt{L}).s]\ N$	(Abs_left)
$(\lambda(P_1 \mid_\xi P_2).M)[s]\ \texttt{inr}(N)$	$\longrightarrow$	$(\lambda P_2.M)[(\xi^{P_1}/\texttt{R}).s]\ N$	(Abs_right)
$(\lambda x.M)[s]\ N$	$\longrightarrow$	$M[(x/N).s]$	(Abs_var)
$(\lambda\_.M)[s]\ N$	$\longrightarrow$	$M[s]$	(Abs_wild)

Case

$[M \mid_\xi N][s]$	$\longrightarrow$	$[M \mid^s_{\xi[s]} N]$	$(Freeze)$
$[M \mid^s_{\texttt{L}} N]$	$\longrightarrow$	$M[s]$	$(Left)$
$[M \mid^s_{\texttt{R}} N]$	$\longrightarrow$	$N[s]$	$(Right)$
$[M \mid^s_\xi N]$	$\longrightarrow$	$[M[s] \mid_\xi N[s]]$	(Xi)

Propagation of Substitutions

$(MN)[s]$	$\longrightarrow$	$M[s]N[s]$	(Sub_app)
$\texttt{inl}(M)[s]$	$\longrightarrow$	$\texttt{inl}(M[s])$	(Sub_left)
$\texttt{inr}(M)[s]$	$\longrightarrow$	$\texttt{inr}(M[s])$	(Sub_right)
$\langle M_1, M_2\rangle[s]$	$\longrightarrow$	$\langle M_1[s], M_2[s]\rangle$	(Sub_pair)

Substitutions and Variables and Constants

$x[id]$	$\longrightarrow$	x	(Sub_var_1)
$x[(x/N).s]$	$\longrightarrow$	N	(Sub_var_2)
$y[(x/N).s]$	$\longrightarrow$	$y[s]$ if $y \neq x$	(Sub_var_3)
$x[(\xi^P/\texttt{K}).s]$	$\longrightarrow$	$x[s]$	(Sub_var_4)
$\xi[id]$	$\longrightarrow$	ξ	$(Sub\_sum\_var_1)$
$\xi[(\xi^P/\texttt{K}).s]$	$\longrightarrow$	$\texttt{K}$	$(Sub\_sum\_var_2)$
$\xi[(\psi^P/\texttt{K}).s]$	$\longrightarrow$	$\xi[s]$ if $\xi \neq \psi$	$(Sub\_sum\_var_3)$
$\xi[(x/M).s]$	$\longrightarrow$	$\xi[s]$	$(Sub\_sum\_var_4)$

Substitutions and Composition

$M[s][t]$	$\longrightarrow$	$M[s \circ t]$	(Sub_clos)
$(s \circ t) \circ u$	$\longrightarrow$	$s \circ (t \circ u)$	(Sub_ass_env)
$((x/M).s) \circ t$	$\longrightarrow$	$(x/M[t]).(s \circ t)$	(Sub_concat_1)
$((\xi^P/K).s) \circ t$	$\longrightarrow$	$(\xi^P/K).(s \circ t)$	(Sub_concat_2)
$id \circ s$	$\longrightarrow$	s	(Sub_id)

Fig. 1. Reduction Rules for λP_w

– Thus we have $[M_1 \mid^s_{\xi[s]} M_2] \longrightarrow^+ [M_1 \mid^s_{\mathtt{L}} M_2]$, and thus applying the rule $(Left)$, we obtain $M[s] \longrightarrow^+ M_1[s]$.

Remark 1. Let s and s' be raw substitutions such that $s \longrightarrow s'$, then $BVar(s) = BVar(s')$.

However, the reduction system in Figure 1 is not really correct in the sense that $\longrightarrow_{\lambda P_w}$ does *not* preserve free variables. This is shown by the following example:

Example 3. Let M be a term such that $FV(M) = \emptyset$ and let $U = (\lambda(x \mid_\xi y).x\ \mathtt{inr}(M))$. Then $U \longrightarrow^*_{\lambda P_w} x[(\xi^x/\mathtt{R}).id] \longrightarrow_{\lambda P_w} x$ and $FV(U) = \emptyset$ but $FV(x) = \{x\}$.

In order to avoid this problem we restrict the set of raw expressions in order to guarantee that no new free variable does appear along reduction sequences. The notion of *acceptable* expression, or simply *expression*, is obtained via the introduction of the following concepts:

Definition 25 (Localized Free Variables) Given a sum variable ξ, a sum constant $\mathtt{K}$ and a raw expression e, we define the set of *localized free variables* of e w.r.t. ξ and $\mathtt{K}$, written as $FV^{\mathtt{K}}_\xi(e)$, as the subset of $FV(e)$ define exactly as for $FV(e)$ except for the following cases:

$$\begin{array}{lll}
FV^{\mathtt{L}}_\xi([M \mid_\xi N]) & = FV^{\mathtt{L}}_\xi(M) \cup \{\xi\} & \\
FV^{\mathtt{R}}_\xi([M \mid_\xi N]) & = FV^{\mathtt{R}}_\xi(N) \cup \{\xi\} & \\
FV^{\mathtt{L}}_\xi([M \mid^s_\xi N]) & = FV^{\mathtt{L}}_\xi(M[s]) \cup \{\xi\} & \\
FV^{\mathtt{R}}_\xi([M \mid^s_\xi N]) & = FV^{\mathtt{R}}_\xi(N[s]) \cup \{\xi\} & \\
FV^{\mathtt{L}}_\xi([M \mid^s_{\xi[t]} N]) & = FV^{\mathtt{L}}_\xi(M[s]) \cup \{\xi\} & \text{if } \xi \notin BVar(t) \\
FV^{\mathtt{R}}_\xi([M \mid^s_{\xi[t]} N]) & = FV^{\mathtt{R}}_\xi(M[s]) \cup \{\xi\} & \text{if } \xi \notin BVar(t)
\end{array}$$

Intuitively, $FV^{\mathtt{L}}_\xi(e)$ (resp. $FV^{\mathtt{R}}_\xi(e)$) contains all the free variables of e except those that are on the right (resp. left) part of the sum terms rooted by the sum variable ξ. Thus, for example, $FV^{\mathtt{L}}_\xi(\lambda(x \mid_\xi y).[x \mid_\xi t]) = \emptyset$, $FV^{\mathtt{L}}_\xi(\lambda y.[x \mid_\xi t]) = \{x\}$, and $FV^{\mathtt{K}}_\xi(x[(\xi^x/\mathtt{L}).id]) = \{x\}$

We are finally ready to define the notion of *acceptable expression* which will avoid the example of creation of new free variables introduced before.

Definition 26 (Acceptable Expression) The raw expression e is said to be *acceptable* (or just called an *expression*) iff $Acc(e)$, where $Acc(\ )$ is the least congruence on expressions such that every variable and every constant is acceptable and also the following requirements hold:

$Acc(\psi[s])$	if $\psi \notin BVar(Q)$	$\forall(\xi^Q/\mathtt{K}) \in s$
$Acc(M[s])$	if $FV_\xi^{\mathtt{K}}(M) \cap BVar(Q) = \emptyset$	$\forall(\xi^Q/\mathtt{K}) \in s$
$Acc(s \circ t)$	if $FV_\xi^{\mathtt{K}}(s) \cap BVar(Q) = \emptyset$	$\forall(\xi^Q/\mathtt{K}) \in t$
$Acc(\lambda P.M)$	if $(FV_\xi^{\mathtt{R}}(M) \cap BVar(Q_1)) \cup (FV_\xi^{\mathtt{L}}(M) \cap BVar(Q_2)) = \emptyset$	$\forall(Q_1 \mid_\xi Q_2) \in P$
$Acc([M \mid_\xi^s N])$	if $Acc(M[s])$ and $Acc(N[s])$	
$Acc([M \mid_{\xi[t]}^s N])$	if $Acc(M[s])$ and $Acc(N[s])$ and $Acc(\xi[t])$	if $(\xi^Q/\mathtt{K}) \notin t$
$Acc([M \mid_{\xi[t]}^s N])$	if $Acc(M[s])$ and $Acc(\xi[t])$	if $(\xi^Q/\mathtt{L}) \in t$
$Acc([M \mid_{\mathtt{L}}^s N])$	if $Acc(M[s])$	
$Acc([M \mid_{\xi[t]}^s N])$	if $Acc(N[s])$ and $Acc(\xi[t])$	if $(\xi^Q/\mathtt{R}) \in t$
$Acc([M \mid_{\mathtt{R}}^s N])$	if $Acc(N[s])$	

Thus for example, the term $\lambda(x \mid_\xi y).[x \mid_\xi t]$ is acceptable while $\lambda(x \mid_\xi y).[x \mid_\psi t]$ is not acceptable. Indeed $FV_\xi^{\mathtt{R}}([x \mid_\psi t]) = \{x, t, \xi\}$ and $BVar(x) = \{x\}$ and thus by definition of $Acc()$ for abstractions we have that $\lambda(x \mid_\xi y).[x \mid_\psi t]$ is not acceptable. The reader may also remark that the terms U and $x[(\xi^x/\mathtt{L}).id]$ given in Example 3 are neither acceptable.

Remark 2. Let us suppose that:

- $[M_1 \mid_\xi M_2][s]$ is an acceptable term, then $\forall(\xi^P/\mathtt{K}) \in s,\ \xi \notin BVar(P)$
- $[M_1 \mid_{\xi[t_1]}^{t_2} M_2][s]$ is an acceptable term, then $\forall(\xi^P/\mathtt{K}) \in s,\ \xi \notin BVar(P)$

We have to show now that this new notion of acceptable expression recently introduced is correct to prevent creation of new free variables along reduction sequences, that is, reduction preserves acceptable expressions and free variables.

Lemma 21 For any expression e and any raw expression e', if $e \longrightarrow e'$ then $\forall \xi, \forall \mathtt{K},\ FV_\xi^{\mathtt{K}}(e') \subseteq FV_\xi^{\mathtt{K}}(e)$

Proof. We first remark that by α-conversion we can supposed that ξ is not bound in e. We then prove this statement by induction on the structure of expressions and by cases on the rule used.

$e = x$ The result holds since e has no reduct

$e = id$ The result holds since e has no reduct

$e = \mathtt{L}$ The result holds since e has no reduct

$e = \mathtt{R}$ The result holds since e has no reduct

$e = \xi$ The result holds since e has no reduct

$e = (x/M).s$ The result holds by induction hypothesis since e has no head reduction.

$e = (\xi^P/\mathtt{K}').s$ The result holds by induction hypothesis since e has no head reduction.

$e = s \circ t$ Then there is 6 possibilities

1. If $e' = s' \circ t$ with $s \longrightarrow s'$, then the result holds by induction hypothesis.
2. If $e' = s \circ t'$ with $t \longrightarrow t'$, then the result holds by induction hypothesis and remark 1.
3. If $s = id$ and $e' = t$, then the result obviously holds.

4. If $s = (\psi^Q/\mathtt{K}').u$ and $e' = (\psi^Q/\mathtt{K}').(u \circ t)$, then

$$\begin{array}{rcl} FV_\xi^{\mathtt{K}}(e) & = & (FV_\xi^{\mathtt{K}}((\psi^Q/\mathtt{K}').u) \backslash BVar(t)) \cup \; FV_\xi^{\mathtt{K}}(t) \\ & = & (FV_\xi^{\mathtt{K}}(u) \backslash BVar(t)) \; \cup \; FV_\xi^{\mathtt{K}}(t) \\ & = & FV_\xi^{\mathtt{K}}(u \circ t) \\ & = & FV_\xi^{\mathtt{K}}((\xi^Q/\mathtt{K}').(u \circ t)) \\ & = & FV_\xi^{\mathtt{K}}(e') \end{array}$$

5. If $s = (x/M).u$ and $e' = (x/M[t]).(u \circ t)$, then

$$\begin{array}{rcl} FV_\xi^{\mathtt{K}}(e) & = & FV_\xi^{\mathtt{K}}((x/M).u) \; \cup \; FV_\xi^{\mathtt{K}}(t) \\ & = & ((FV_\xi^{\mathtt{K}}(M) \cup FV_\xi^{\mathtt{K}}(u)) \backslash BVar(t)) \; \cup \; FV_\xi^{\mathtt{K}}(t) \\ & = & (FV_\xi^{\mathtt{K}}(M) \backslash BVar(t)) \; \cup \; ((FV_\xi^{\mathtt{K}}(u) \backslash BVar(t)) \cup FV_\xi^{\mathtt{K}}(t)) \\ & = & (FV_\xi^{\mathtt{K}}(M) \backslash BVar(t)) \; \cup \; FV_\xi^{\mathtt{K}}(u \circ t) \\ & = & FV_\xi^{\mathtt{K}}((x/M[t]).(u \circ t)) \\ & = & FV_\xi^{\mathtt{K}}(e') \end{array}$$

6. If $s = s_1 \circ s_2$ and $e' = s_1 \circ (s_2 \circ t)$, then

$$\begin{array}{rcl} FV_\xi^{\mathtt{K}}(e) & = & (FV_\xi^{\mathtt{K}}(s_1 \circ s_2) \backslash BVar(t)) \cup FV_\xi^{\mathtt{K}}(t) \\ & = & (((FV_\xi^{\mathtt{K}}(s_1) \backslash BVar(s_2)) \cup FV_\xi^{\mathtt{K}}(s_2)) \backslash BVar(t)) \cup FV_\xi^{\mathtt{K}}(t) \\ & = & (FV_\xi^{\mathtt{K}}(s_1) \backslash (BVar(s_2) \cup BVar(t))) \cup ((FV_\xi^{\mathtt{K}}(s_2) \backslash BVar(t)) \cup FV_\xi^{\mathtt{K}}(t)) \\ & = & (FV_\xi^{\mathtt{K}}(s_1) \backslash BVar(s_2 \circ t)) \cup FV_\xi^{\mathtt{K}}(s_2 \circ t) \\ & = & FV_\xi^{\mathtt{K}}(e') \end{array}$$

$e = \xi'[s]$ Then there is 5 possibilities:

1. If $e' = \xi'[s']$ with $s \longrightarrow s'$ then the result holds by induction hypothesis and remark 1.
2. If $s = id$ and $e' = \xi$ then the result obviously holds.
3. If $s = (\xi'^Q/\mathtt{K}').t$ and $e' = \mathtt{K}'$ then the result obviously holds.
4. If $s = (\psi^Q/\mathtt{K}').t$ with $\psi \neq \xi'$ and $e' = \xi'[t]$ then: First we remark that, since e is acceptable, we have

$$\forall (\psi'^{Q'}/\mathtt{K}) \in (\psi^Q/\mathtt{K}').t, \xi' \notin BVar(Q')$$

We deduce that: $\xi' \notin BVar(Q)$

$$\begin{array}{rcl} FV_\xi^{\mathtt{K}}(e) & = & (\{\xi'\} \backslash (\{\psi\} \cup BVar(Q) \cup BVar(t))) \; \cup FV_\xi^{\mathtt{K}}(t) \\ & = & (\{\xi'\} \backslash BVar(t)) \; \cup FV_\xi^{\mathtt{K}}(t) \text{ since } \xi' \neq \psi \text{ and } \xi \notin BVar(Q) \\ & = & FV_\xi^{\mathtt{K}}(e') \end{array}$$

5. If $s = (x/M).t$ and $e' = \xi'[t]$ then

$$\begin{array}{rcl} FV_\xi^{\mathtt{K}}(e) & = & (\xi' \backslash (BVar(t) \cup \{x\})) \; \cup FV_\xi^{\mathtt{K}}((x/M).t) \\ & = & (\xi' \backslash BVar(t)) \; \cup (FV_\xi^{\mathtt{K}}(M) \cup FV_\xi^{\mathtt{K}}(t)) \\ & \supset & (\xi' \backslash BVar(t)) \; \cup FV_\xi^{\mathtt{K}}(t) \\ & = & FV_\xi^{\mathtt{K}}(e') \end{array}$$

$e = (M\ N)$ Then there is 9 possibilities

1. If $e' = (M'\ N)$ and $M \longrightarrow M'$ then the result holds by induction hypothesis.
2. If $e' = (M\ N')$ and $N \longrightarrow N'$ then the result holds by induction hypothesis.
3. If $e = \lambda P.M_1$ and $e' = (\lambda P.M_1)[id]\ N$ then the result obviously holds.
4. If $e = (\lambda\langle P_1, P_2\rangle.M_1)[s]$, $N = \langle N_1, N_2\rangle$ and $e' = ((\lambda P_1.\lambda P_2.M_1)[s]\ N_1)\ N_2$ then we remark that by α-conversion we can suppose that $BVar(s) \cap (BVar(P_1) \cup BVar(P_2)) = \emptyset$ and then we have

$$\begin{aligned}
FV_\xi^{\mathtt{K}}(e) &= FV_\xi^{\mathtt{K}}((\lambda\langle P_1, P_2\rangle.M_1)[s]) \cup FV_\xi^{\mathtt{K}}(\langle N_1, N_2\rangle) \\
&= ((FV_\xi^{\mathtt{K}}(\lambda\langle P_1, P_2\rangle.M_1)\backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad (FV_\xi^{\mathtt{K}}(N_1) \cup FV_\xi^{\mathtt{K}}(N_2)) \\
&= (((FV_\xi^{\mathtt{K}}(M_1)\backslash(BVar(P_1) \cup BVar(P_2)))\backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad (FV_\xi^{\mathtt{K}}(N_1) \cup FV_\xi^{\mathtt{K}}(N_2)) \\
&= (((FV_\xi^{\mathtt{K}}(M_1)\backslash BVar(P_2))\backslash BVar(P_1))\backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad (FV_\xi^{\mathtt{K}}(N_1) \cup FV_\xi^{\mathtt{K}}(N_2)) \\
&= ((((FV_\xi^{\mathtt{K}}(\lambda P_2.M_1))\backslash BVar(P_1))\backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad (FV_\xi^{\mathtt{K}}(N_1) \cup FV_\xi^{\mathtt{K}}(N_2)) \\
&= (((FV_\xi^{\mathtt{K}}(\lambda P_1.\lambda P_2.M_1))\backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad (FV_\xi^{\mathtt{K}}(N_1) \cup FV_\xi^{\mathtt{K}}(N_2)) \\
&= ((FV_\xi^{\mathtt{K}}((\lambda P_1.\lambda P_2.M_1)[s]))\ \cup FV_\xi^{\mathtt{K}}(N_1)) \cup FV_\xi^{\mathtt{K}}(N_2) \\
&= (FV_\xi^{\mathtt{K}}((\lambda P_1.\lambda P_2.M_1)[s]\ N_1)) \cup FV_\xi^{\mathtt{K}}(N_2) \\
&= (FV_\xi^{\mathtt{K}}((\lambda P_1.\lambda P_2.M_1)[s]\ N_1)\ N_2) \\
&= FV_\xi^{\mathtt{K}}(e')
\end{aligned}$$

5. $e = (\lambda @(P_1, P_2).M_1)[s]$ and $e' = ((\lambda P_1.\lambda P_2.M_1)[s]\ N)\ N$ the proof is the same as in the previous case.
6. $e = (\lambda(P_1 \mid_\psi P_2).M_1)[s]$, $N = \mathtt{inl}(N_1)$ and $e' = (\lambda P_1.M_1)[(\psi^{P_2}/\mathtt{L}).s]\ N_1$ then, remarking that by α-conversion we can suppose that $BVar(s) \cap$

$(BVar(P_1) \cup BVar(P_2) \cup \{\psi\}) = \emptyset$, we have:

$$\begin{aligned}
FV_\xi^{\mathtt{K}}(e) &= FV_\xi^{\mathtt{K}}((\lambda(P_1 \mid_\psi P_2).M_1)[s]) \cup FV_\xi^{\mathtt{K}}(N_1) \\
&= ((FV_\xi^{\mathtt{K}}(\lambda(P_1 \mid_\psi P_2).M_1) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \ \cup FV_\xi^{\mathtt{K}}(N_1) \\
&= ((FV_\xi^{\mathtt{K}}(M_1) \backslash (\{\psi\} \cup BVar(P_1) \cup BVar(P_2)) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad FV_\xi^{\mathtt{K}}(N_1) \\
&= (((FV_\xi^{\mathtt{K}}(M_1) \backslash BVar(P_1)) \backslash (\{\psi\} \cup BVar(P_2) \cup BVar(s))) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad FV_\xi^{\mathtt{K}}(N_1) \\
&= ((FV_\xi^{\mathtt{K}}(\lambda P_1.M_1) \backslash (\{\psi\} \cup BVar(P_2) \cup BVar(s))) \cup FV_\xi^{\mathtt{K}}(s)) \cup \\
&\quad FV_\xi^{\mathtt{K}}(N_1) \\
&= FV_\xi^{\mathtt{K}}((\lambda P_1.M_1)[(\psi^{P_2}/\mathtt{L}).s]) \ \cup FV_\xi^{\mathtt{K}}(N_1) \\
&= FV_\xi^{\mathtt{K}}((\lambda P_1.M_1)[(\psi^{P_2}/\mathtt{L}).s] \ N_1) \\
&= FV_\xi^{\mathtt{K}}(e')
\end{aligned}$$

7. $e = (\lambda(P_1 \mid_\psi P_2).M_1)[s]$, $N = \mathtt{inr}(N_1)$ and $e' = (\lambda P_2.M_1)[(\psi^{P_1}/\mathtt{R}).s] \ N_1$ we reason as for the previous case.
8. $M = (\lambda x.M_1)[s]$ and $e' = M_1[(x/N).s]$ and then

$$\begin{aligned}
FV_\xi^{\mathtt{K}}(e) &= FV_\xi^{\mathtt{K}}((\lambda x.M_1)[s]) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&= ((FV_\xi^{\mathtt{K}}(\lambda x.M_1) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&= (((FV_\xi^{\mathtt{K}}(M_1) \backslash \{x\}) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&= (FV_\xi^{\mathtt{K}}(M_1) \backslash (BVar(s) \cup \{x\})) \cup FV_\xi^{\mathtt{K}}(s) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&= FV_\xi^{\mathtt{K}}(M_1[(x/N).s]) \\
&= FV_\xi^{\mathtt{K}}(e')
\end{aligned}$$

9. $e = (\lambda\_.M_1)[s]$ and $e' = M_1[s]$ and then

$$\begin{aligned}
FV_\xi^{\mathtt{K}}(e) &= FV_\xi^{\mathtt{K}}((\lambda_.M_1)[s]) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&= ((FV_\xi^{\mathtt{K}}(\lambda_.M_1) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&= (((FV_\xi^{\mathtt{K}}(M_1) \backslash \emptyset) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s)) \ \cup \ FV_\xi^{\mathtt{K}}(N) \\
&\supset (FV_\xi^{\mathtt{K}}(M_1) \backslash BVar(s)) \cup FV_\xi^{\mathtt{K}}(s) \\
&= FV_\xi^{\mathtt{K}}(M_1[s]) \\
&= FV_\xi^{\mathtt{K}}(e')
\end{aligned}$$

$e = \langle M_1, M_2 \rangle$ The proof is obvious by induction hypothesis since there is no head reduct for e.

$e = \mathtt{inl}(M_1)$ The proof is obvious by induction hypothesis since there is no head reduct for e.

$e = \mathtt{inr}(M_1)$ The proof is obvious by induction hypothesis since there is no head reduct for e.

$e = [M \mid_\psi N]$ The proof is obvious by induction hypothesis since there is no head reduct for e.

$e = [M_1 \mid^s_\Xi M_2]$ then there is 9 possibilities

1. If $e' = [M'_1 \mid^s_\Xi M_2]$ with $M_1 \longrightarrow M'_1$, then the result is obvious by induction hypothesis.
2. If $e' = [M_1 \mid^s_\Xi M'_2]$ with $M_2 \longrightarrow M'_2$ then the result obviously holds by induction hypothesis.
3. If $e' = [M_1 \mid^{s'}_\Xi M_2]$ with $s \longrightarrow s'$, then the result holds by induction hypothesis. and by Remark 1
4. If $e' = [M_1 \mid^s_{\Xi'} M_2]$ with $\Xi \longrightarrow \Xi'$, th result is obvious by induction hypothesis.
5. $\Xi = \mathtt{L}$ and $e' = M_1[s]$ then

$$\begin{array}{rcl} FV^{\mathtt{K}}_\xi(e) & = & ((FV^{\mathtt{K}}_\xi(M_1) \cup FV^{\mathtt{K}}_\xi(M_2))\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s) \\ & \supset & (FV^{\mathtt{K}}_\xi(M_1)\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s) \\ & = & FV^{\mathtt{K}}_\xi(e') \end{array}$$

6. If $\Xi = \mathtt{R}$ and $e' = M_2[s]$, we reason as for the previous case.
7. If $\mathtt{K} = \mathtt{L}$, $\Xi = \xi$ and $e' = [M_1[s] \mid_\xi M_2[s]]$ then

$$\begin{array}{rcl} FV^{\mathtt{L}}_\xi(e) & = & FV^{\mathtt{L}}_\xi(M_1[s]) \\ & = & FV^{\mathtt{L}}_\xi(e') \end{array}$$

8. If $\mathtt{K} = \mathtt{R}$, $\Xi = \xi$ and $e' = [M_1[s] \mid_\xi M_2[s]]$, we reason as in the previous case.
9. $\Xi = \psi$, $\psi \neq \xi$ and $e' = [M_1[s] \mid_\psi M_2[s]]$ then

$$\begin{array}{rcl} FV^{\mathtt{K}}_\xi(e) & = & FV^{\mathtt{K}}_\xi([M_1 \mid^s_\psi M_2]) \\ & = & ((FV^{\mathtt{K}}_\xi(M_1) \cup FV^{\mathtt{K}}_\xi(M_2))\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s) \cup \{\psi\} \\ & = & ((FV^{\mathtt{K}}_\xi(M_1)\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s)) \cup ((FV^{\mathtt{K}}_\xi(M_2)\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s)) \cup \{\psi\} \\ & = & FV^{\mathtt{K}}_\xi(M_1[s]) \cup FV^{\mathtt{K}}_\xi(M_2[s]) \cup \{\psi\} \\ & = & FV^{\mathtt{L}}_\xi(e') \end{array}$$

$e = \lambda P.M$ The proof is obvious by induction hypothesis since there is no head reduct for e.

$e = M[s]$ then there is 10 possibilities

1. $e' = M'[s]$ and $M \longrightarrow M'$ then the result obviously holds by induction hypothesis.
2. $e' = M[s']$ and $s \longrightarrow s'$ then the result obviously holds by induction hypothesis and remark 1.
3. $M = M_1\ M_2$ and $e' = M_1[s]\ M_2[s]$ then

$$\begin{array}{rcl} FV^{\mathtt{K}}_\xi(e) & = & (FV^{\mathtt{K}}_\xi((M_1\ M_2))\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s) \\ & = & ((FV^{\mathtt{K}}_\xi(M_1) \cup FV^{\mathtt{K}}_\xi(M_2))\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s) \\ & = & ((FV^{\mathtt{K}}_\xi(M_1))\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s)) \cup ((FV^{\mathtt{K}}_\xi(M_2)\backslash BVar(s)) \cup FV^{\mathtt{K}}_\xi(s)) \\ & = & FV^{\mathtt{K}}_\xi(M_1[s]) \cup FV^{\mathtt{K}}_\xi(M_1[s]) \\ & = & FV^{\mathtt{K}}_\xi(e') \end{array}$$

4. $M = \langle M_1, M_2\rangle$ and $e' = \langle M_1[s], M_2[s]\rangle$, we reason as when $M = M_1\ M_2$.
5. $M = \mathtt{inl}(M_1)$ and $e' = \mathtt{inl}(M_1[s])$, , we reason as when $M = M_1\ M_2$.
6. $M = \mathtt{inr}(M_1)$ and $e' = \mathtt{inr}(M_1[s])$, we reason as when $M = M_1\ M_2$.
7. $M = x$, $s = id$ and $e' = x$, the proof is obvious.
8. $M = x$, $s = (x/N).t$ and $e' = N$, the proof is obvious.
9. $M = x$, $s = (y/N).t$, $x \neq y$ and $e' = x[t]$, the proof is obvious.
10. $M = x$, $s = (\psi^Q/\mathtt{K}').t$ and $e' = x[t]$, the proof is obvious.

Lemma 22 (Correctness) If e is an expression and $e \longrightarrow\ e'$, then e' is an expression.

Proof. We reason by induction on the structure of e and then by case on the rule used to reduct e. The cases which are not treated in the following proof are obvious since in these cases either e has no reduct, either the reduction doesn't take place at the root of the expression and then the result holds by induction hypothesis and lemma 21.

Abs_id Then $e = \lambda P.M\ N$ and we know that $\lambda P.M$ and N are acceptable. Thus so is $(\lambda P.M)[id]$ and then the result holds.

Abs_pair Then we have $e = (\lambda\langle P_1, P_2\rangle.M)[s]\ \langle N_1, N_2\rangle$. By α-conversion we can suppose that $BVar(s) \cap (BVar(P_1) \cup BVar(P_2)) = \emptyset$ and then we know that

1. M, s, N_1 and N_2 are acceptable.
2. $(\lambda\langle P_1, P_2\rangle.M)[s]$ is acceptable

$$\Rightarrow \forall(\xi^Q/\mathtt{K}) \in s,\quad FV_\xi^{\mathtt{K}}(\lambda\langle P_1, P_2\rangle.M) \cap BVar(Q) = \emptyset$$
$$\Leftrightarrow \forall(\xi^Q/\mathtt{K}) \in s,\quad (FV_\xi^{\mathtt{K}}(M)\backslash(BVar(P_1) \cup BVar(P_2))) \cap BVar(Q) = \emptyset\ (*)$$

3. $\lambda\langle P_1, P_2\rangle.M$ is acceptable

$$\Rightarrow \forall(Q_1 \mid_\xi Q_2) \in \langle P_1, P_2\rangle, (FV_\xi^{\mathtt{R}}(M) \cap BVar(Q_1)) = \emptyset \text{ and } (FV_\xi^{\mathtt{L}}(M) \cap BVar(Q_2)) = \emptyset$$

Since N_2 is acceptable, we just have to show that $(\lambda P_1.\lambda P_2.M)[s]\ N_1$ is acceptable. Since N_1 is acceptable, we just have to show that $(\lambda P_1.\lambda P_2.M)[s]$ is acceptable. To obtain this result we have to:

- Show that $\forall(\xi^Q/\mathtt{K}) \in s,\ FV_\xi^{\mathtt{K}}(\lambda P_1.\lambda P_2.M) \cap BVar(Q) = \emptyset$. But $FV_\xi^{\mathtt{K}}(\lambda P_1.\lambda P_2.M) = FV_\xi^{\mathtt{K}}(M)\backslash(BVar(P_1) \cup BVar(P_2))$ and thus the result holds by $(*)$.
- Show that $\lambda P_1.\lambda P_2.M$ is acceptable. *i.e.* show that $\lambda P_2.M$ is acceptable and that $\forall(Q_1 \mid_\xi Q_2) \in P_1\ \ (FV_\xi^{\mathtt{R}}(\lambda P_2.M) \cap BVar(Q_1)) = \emptyset$ and $(FV_\xi^{\mathtt{L}}(\lambda P_2.M) \cap BVar(Q_2)) = \emptyset$. We don't show here the first point since the proof is the same as the following one. To show the second point, we remark that if $(Q_1 \mid_\xi Q_2) \in P_1$ then $(Q_1 \mid_\xi Q_2) \in \langle P_1, P_2\rangle$. Since $FV_\xi^{\mathtt{K}}(\lambda P_2.M) \subseteq FV_\xi^{\mathtt{K}}(M)$ then the result holds by point 3.

Abs_contr Same proof as in the *Abs_pair* case.

Abs_left Then we know that $e = (\lambda(P_1 \mid_\psi P_2).M)[s]\ \mathtt{inl}(N)$ and thus:

Hyp1.1 By α-conversion, we can suppose that $(BVar(P_1) \cup BVar(P_2) \cup \{\psi\}) \cap BVar(s) = \emptyset$.

Hyp1.2 M, N and s are acceptable.

Hyp1.3 $(\lambda(P_1 \mid_\psi P_2).M)[s]$ is acceptable

$$\Leftrightarrow \forall(\xi^Q/\mathtt{K}) \in s, \quad FV_\xi^{\mathtt{K}}(\lambda(P_1 \mid_\psi P_2).M) \cap BVar(Q) = \emptyset$$
$$\Leftrightarrow \forall(\xi^Q/\mathtt{K}) \in s, \quad (FV_\xi^{\mathtt{K}}(M) \backslash (BVar(P_1) \cup BVar(P_2) \cup \{\psi\})) \cap BVar(Q) = \emptyset$$

and by Hyp1.1 we have

$$\forall(\xi^Q/\mathtt{K}) \in s \;\; FV_\xi^{\mathtt{K}}(M) \cap BVar(Q) = \emptyset \qquad (1)$$

Hyp1.4 $\lambda(P_1 \mid_\psi P_2).M$ is acceptable $\Leftrightarrow$

$$\forall(Q_1 \mid_\xi Q_2) \in (P_1 \mid_\psi P_2), \; (FV_\xi^{\mathtt{R}}(M) \cap BVar(Q_1)) = \emptyset \qquad (2)$$
$$\forall(Q_1 \mid_\xi Q_2) \in (P_1 \mid_\psi P_2), \; FV_\xi^{\mathtt{L}}(M) \cap BVar(Q_2) = \emptyset \qquad (3)$$

Since N is acceptable, we have just to show that $(\lambda P_1.M)[(\psi^{P_2}/\mathtt{L}).s]$ is acceptable, that is:

- $\forall(\xi^Q/\mathtt{K}) \in (\psi^{P_2}/\mathtt{L}).s, \quad FV_\xi^{\mathtt{K}}(\lambda P_1.M) \cap BVar(Q) = \emptyset$. Let $(\xi^Q/\mathtt{K})$ be in $(\psi^{P_2}/\mathtt{L}).s$, then two sub-cases are possible:
  1. $(\xi^Q/\mathtt{K})$ is in s then we want $(FV_\xi^{\mathtt{K}}(M)\backslash BVar(P_1)) \cap BVar(Q) = \emptyset$. But, by equation (1), we know that $\forall(\xi^Q/\mathtt{K}) \in s, \quad FV_\xi^{\mathtt{K}}(M) \cap BVar(Q) = \emptyset$ Thus the result holds.
  2. $(\xi^Q/\mathtt{K}) = (\psi^{P_2}/\mathtt{L})$ then we want $(FV_\xi^{\mathtt{L}}(M)\backslash BVar(P_1)) \cap BVar(P_2) = \emptyset$ but we know that $FV_\xi^{\mathtt{L}}(M) \cap BVar(P_2) = \emptyset$ by equation (3) thus the result holds.
- $\lambda P_1.M$ is acceptable which is obvious since we know that
  1. M is acceptable.
  2. $\forall(Q_1 \mid_\xi Q_2) \in P_1 \quad FV_\xi^{\mathtt{R}}(M) \cap BVar(Q_1) = \emptyset$ and $FV_\xi^{\mathtt{L}}(M) \cap BVar(Q_2) = \emptyset$ which is trivial by equations (2) and (3).

Abs_right Same proof as in the *Abs_left* case.

Abs_var Then we know that $e = (\lambda x.M)[s]\ N$ and thus

- M, s and N are acceptable.
- $\forall(\xi^Q/\mathtt{K}) \in s \; FV_\xi^{\mathtt{K}}(\lambda x.M) \cap BVar(Q) = \emptyset$ and thus $\forall(\xi^Q/\mathtt{K}) \in s, \; (FV_\xi^{\mathtt{K}}(M)\backslash\{x\}) \cap BVar(Q) = \emptyset$ and since, by α-conversion, we can suppose that $x \notin BVar(s)$, $\forall(\xi^Q/\mathtt{K}) \in s, \quad FV_\xi^{\mathtt{K}}(M) \cap BVar(Q) = \emptyset$.

We now have to show that $M[(x/N).s]$ is acceptable. So we have to show that $(x/N).s$ is acceptable. But because N and s are acceptable $(x/N).s$ is acceptable. Since M is acceptable we just have to show that $\forall(\xi^Q/\mathtt{K}) \in (x/N).s \;\; FV_\xi^{\mathtt{K}}(M) \cap BVar(Q) = \emptyset$ which is obvious and thus the result hold.

Abs_wild The result is obvious.

Freeze Then we have $e = [M \mid_\psi N][s]$ and thus we know that:

- s, M and N are acceptable.
- $\forall(\xi^Q/\mathtt{K}) \in s, FV_\xi^{\mathtt{K}}([M \mid_\psi N]) \cap BVar(Q) = \emptyset \Leftrightarrow$

  $\forall(\xi^Q/\mathtt{K}) \in s$ if $\xi \neq \psi$ then

  $$(FV_\xi^{\mathtt{K}}(M) \cup FV_\xi^{\mathtt{K}}(N) \cup \{\psi\}) \cap BVar(Q) = \emptyset \qquad (1)$$
  $$\forall(\psi^Q/\mathtt{L}) \in s, \; FV_\xi^{\mathtt{L}}(M) \cap BVar(Q) = \emptyset \qquad (2)$$
  $$\forall(\psi^Q/\mathtt{R}) \in s, \; FV_\xi^{\mathtt{R}}(N) \cap BVar(Q) = \emptyset \qquad (3)$$

we have now to show that $[M \mid^s_{\psi[s]} N]$ is acceptable.
Since M, N and s are acceptable we have just to show that :

- $\psi[s]$ is acceptable, which is true by (1) and which implies that

$$\forall(\xi^Q/\mathtt{K}) \in s, \psi \notin BVar(Q)$$

- If $\psi \notin s$, we have $Acc(M[s])$ and $Acc(N[s])$.
- If $(\psi^Q/\mathtt{L}) \in s$, we have $Acc(M[s])$
- If $(\psi^Q/\mathtt{R}) \in s$, we have $Acc(N[s])$

In the three cases the result holds by hypothesis.

Left Then $e = [M \mid^s_{\mathtt{L}} N]$ and thus we know that M, N and s and $M[s]$ are acceptable and thus the result holds.

Right Then $e = [M \mid^s_{\mathtt{L}} N]$ and then we know that N,s and $N[s]$ are acceptable and thus the result holds.

Xi Then $e = [M \mid^s_{\xi} N]$ and then we know that M, N, s, $M[s]$ and $N[s]$ are all acceptable. Thus the result obviously hold.

Sub_app The result holds since $Acc()$ is a congruence.

Sub_left The result holds since $Acc()$ is a congruence.

Sub_right The result holds since $Acc()$ is a congruence.

Sub_pair The result holds since $Acc()$ is a congruence.

Sub_var$_1$ The result holds.

Sub_var$_2$ The result holds since $Acc()$ is a congruence.

Sub_var$_3$ The result holds since $Acc()$ is a congruence.

Sub_var$_4$ The result holds since $Acc()$ is a congruence.

Sub_sum_var$_1$ The result holds.

Sub_sum_var$_2$ The result holds.

Sub_sum_var$_3$ The result holds.

Sub_sum_var$_4$ The result holds.

Sub_clos Then we have $e = M[s][t]$ and we know that:

Hyp1.1 M, s, $M[s]$ and t are acceptable
Hyp1.2 $\forall(\xi^Q/\mathtt{K}) \in t,\ FV^{\mathtt{K}}_{\xi}(M[s]) \cap BVar(Q) = \emptyset$
Hyp1.3 $\forall(\xi^Q/\mathtt{K}) \in s,\ FV^{\mathtt{K}}_{\xi}(M) \cap BVar(Q) = \emptyset$

and we have to show that: $M[s \circ t]$ is acceptable *i.e.* that

Hyp2.1 M, s, t are acceptable
Hyp2.2 $s \circ t$ is acceptable *i.e.* $\forall(\xi^Q/\mathtt{K}) \in t,\ FV^{\mathtt{K}}_{\xi}(s) \cap BVar(Q) = \emptyset$
Hyp2.3 $\forall(\xi^Q/\mathtt{K}) \in s \circ t,\ FV^{\mathtt{K}}_{\xi}(M) \cap BVar(Q) = \emptyset$

The first point is trivial.

For the second point we have just to remark that $\forall \mathtt{K}, FV^{\mathtt{K}}_{\xi}(M[s]) = (FV^{\mathtt{K}}_{\xi}(M) \backslash BVar(s)) \cup FV^{\mathtt{K}}_{\xi}(s)$ and the result holds by hypothesis Hyp1.2.

For the third point we remark that $\forall(\xi^Q/\mathtt{K}) \in s \circ t, (\xi^Q/\mathtt{K}) \in s$ or $(\xi^Q/\mathtt{K}) \in t$ and then the result obviously holds by Hyp1.1 and Hyp1.2.

Sub_ass_env Then we have $e = (s \circ t) \circ u$ and we know that:

Hyp1.1 By α-conversion we van suppose that $BVar(t) \cap BVar(u) = \emptyset$
Hyp1.2 s, t and u are acceptable
Hyp1.3 $\forall(\xi^Q/\mathtt{K}) \in u,\ FV^{\mathtt{K}}_{\xi}(s \circ t) \cap BVar(Q) = \emptyset$
Hyp1.4 $\forall(\xi^Q/\mathtt{K}) \in t,\ FV^{\mathtt{K}}_{\xi}(s) \cap BVar(Q) = \emptyset$

and we have to show that: $s \circ (t \circ u)$ is acceptable *i.e.* that:

1. s, t and u are acceptable which is true by hypothesis.
2. $t \circ u$ is acceptable *i.e.* $\forall(\xi^Q/\mathtt{K}) \in u,\ FV_\xi^{\mathtt{K}}(t) \cap BVar(Q) = \emptyset$.
3. $\forall(\xi^Q/\mathtt{K}) \in t \circ u,\ FV_\xi^{\mathtt{K}}(s) \cap BVar(Q) = \emptyset$.

The second point holds by the fact that $FV_\xi^{\mathtt{K}}(s \circ t) = (FV_\xi^{\mathtt{K}}(s) \backslash BVar(t)) \cup FV_\xi^{\mathtt{K}}(t)$ and Hyp1.3

The third point holds by the fact that $\forall(\xi^Q/\mathtt{K}) \in t \circ u, (\xi^Q/\mathtt{K}) \in t$ or $(\xi^Q/\mathtt{K}) \in u$ and Hyp1.1

Sub_concat$_1$ Then $e = ((x/M).s) \circ t$ and then we know that:

M, s and t are acceptable

$\forall(\xi^Q/\mathtt{K}) \in t,\ FV_\xi^{\mathtt{K}}((x/M).s) \cap BVar(Q) = \emptyset$

$$\Leftrightarrow \forall(\xi^Q/\mathtt{K}) \in t,\ (FV_\xi^{\mathtt{K}}(M) \cup FV_\xi^{\mathtt{K}}(s)) \cap BVar(Q) = \emptyset \quad (1)$$

and we have to show that $(x/M[t]).(s \circ t)$ is acceptable *i.e.* that:

1. M, s and t are acceptable which is trivial
2. $M[t]$ is acceptable which is obvious by (1) and the fact that $FV_\xi^{\mathtt{K}}(M) \subseteq FV_\xi^{\mathtt{K}}((x/M).s)$
3. $s \circ t$ is acceptable which is also obvious for the same reasons.

Sub_concat$_2$ The proof is simpler than the *Sub_concat*$_1$ one.

Sub_id Obvious.

A basic, but in our case not trivial, property is preservation of free variables by reduction. This property holds by the following Lemma.

Lemma 23 For all expression e if $e \longrightarrow e'$ then $FV(e') \subseteq FV(e)$.

Proof. We prove this statement by induction on the structure of expressions and then by cases on the used rule.

All the cases were the reduction does not take place at the root of e are obvious by induction hypothesis. So we just make the proof in the cases where the reduction take place at the root of e.

Abs_id The result holds.

Abs_pair Then $e = (\lambda\langle P_1, P_2\rangle.M)[s]\ \langle N_1, N_2\rangle$ and $e' = ((\lambda P_1.\lambda P_2.M)[s]\ N_1)\ N_2$ and, by α-conversion, we can suppose that:

$$(BVar(P_1) \cup BVar(P_2)) \cap BVar(s) = \emptyset$$

and thus

$$\begin{aligned}
FV(e) &= FV((\lambda\langle P_1, P_2\rangle.M)[s]) \cup FV(\langle N_1, N_2\rangle) \\
&= (FV(\lambda\langle P_1, P_2\rangle.M) \backslash BVar(s)) \cup FV(s) \cup (FV(N_1) \cup FV(N_2)) \\
&= ((FV(M) \backslash (BVar(P_1) \cup BVar(P_2))) \backslash BVar(s)) \cup FV(s)\ \cup (FV(N_1) \cup FV(N_2)) \\
&= (((FV(M) \backslash BVar(P_2)) \backslash BVar(P_1)) \backslash BVar(s)) \cup FV(s)\ \cup FV(N_1) \cup FV(N_2) \\
&= ((FV(\lambda P_2.M) \backslash BVar(P_1)) \backslash BVar(s)) \cup FV(s)\ \cup FV(N_1) \cup FV(N_2) \\
&= (FV(\lambda P_1.\lambda P_2.M) \backslash BVar(s)) \cup FV(s)\ \cup FV(N_1) \cup FV(N_2) \\
&= FV((\lambda P_1.\lambda P_2.M)[s]) \cup FV(N_1) \cup FV(N_2) \\
&= FV((\lambda P_1.\lambda P_2.M)[s]\ N_1) \cup FV(N_2) \\
&= FV(e')
\end{aligned}$$

Abs_contr Same proof as in the *Abs_pair* case.

Abs_left Then $e = (\lambda(P_1 \mid_\xi P_2).M)[s]\ \texttt{inl}(N)$ and $e' = (\lambda P_1.M)[(\xi^{P_2}/\texttt{L}).s]\ N$ and and by α-conversion we can suppose that:

$$(BVar(P_1) \cup BVar(P_2) \cup \{\xi\}) \cap BVar(s) = \emptyset$$

and then we have:

$$\begin{aligned} FV(e) &= FV((\lambda(P_1 \mid_\xi P_2).M)[s]) \cup FV(\texttt{inl}(N)) \\ &= ((FV(\lambda(P_1 \mid_\xi P_2).M)\backslash BVar(s)) \cup FV(s)) \cup FV(N) \\ &= ((FV(M)\backslash(BVar(P_1) \cup BVar(P_2) \cup \{\xi\})\backslash BVar(s)) \cup FV(s)) \cup FV(N) \\ &= (((FV(M)\backslash BVar(P_1))\backslash(BVar(s) \cup BVar(P_2) \cup \{\xi\})) \cup FV(s)) \cup FV(N) \\ &= ((FV(\lambda P_1.M)\backslash BVar((\xi^{P_2}/\texttt{L}).s)) \cup FV((\xi^{P_2}/\texttt{L}).s)) \cup FV(N) \\ &= FV((\lambda P_1.M)[(\xi^{P_2}/\texttt{L}).s]) \cup FV(N) \\ &= FV(e') \end{aligned}$$

Abs_right The proof is the same as in the *Abs_left* case.

Abs_var Then $e = (\lambda x.M)[s]\ N$ and $e' = M[(x/N).s]$ and, by α-conversion, we can suppose that $x \notin BVar(s)$ and thus:

$$\begin{aligned} FV(e) &= FV((\lambda x.M)[s]) \cup FV(N) \\ &= ((FV(\lambda x.M)\backslash BVar(s)) \cup FV(s)) \cup FV(N) \\ &= (((FV(M)\backslash\{x\})\backslash BVar(s)) \cup FV(s)) \cup FV(N) \\ &= (FV(M)\backslash BVar((x/N).s)) \cup FV((x/N).s) \\ &= FV(e') \end{aligned}$$

Abs_wild The proof is almost the same as in the *Abs_var* case.

Freeze Then $e = [M \mid_\xi N][s]$ and $e' = [M \mid^s_{\xi[s]} N]$ and thus:

$$\begin{aligned} FV(e) &= (FV([M \mid_\xi N])\backslash BVar(s)) \cup FV(s) \\ &= ((FV(M) \cup FV(N) \cup \{\xi\})\backslash BVar(s)) \cup FV(s) \\ &= ((FV(M) \cup FV(N))\backslash BVar(s)) \cup FV(s) \cup (\{\xi\}\backslash BVar(s)) \\ &= ((FV(M) \cup FV(N))\backslash BVar(s)) \cup FV(s) \cup FV(\xi[s]) \\ &= FV(e') \end{aligned}$$

Left Then $e = [M \mid^s_{\texttt{L}} N]$ and $e' = M[s]$ and then we have:

$$\begin{aligned} FV(e) &= ((FV(M) \cup FV(N))\backslash BVar(s)) \cup FV(s) \cup FV(\texttt{L}) \\ &\supseteq (FV(M)\backslash BVar(s)) \cup FV(s) \\ &= FV(e') \end{aligned}$$

Right The proof is the same as in the *Left* case.

Xi Then $e = [M \mid^s_\xi N]$ and $e' = [M[s] \mid_\xi N[s]]$ and then we have

$$\begin{aligned} FV(e) &= ((FV(M) \cup FV(N))\backslash BVar(s)) \cup FV(s) \cup FV(\xi) \\ &= ((FV(M)\backslash BVar(s)) \cup FV(s)) \cup ((FV(N)\backslash BVar(s)) \cup FV(s)) \cup FV(\xi) \\ &= FV(M[s]) \cup FV(N[s]) \cup FV(\xi) \\ &= FV(e') \end{aligned}$$

Sub_app Then $e = (M\ N)[s]$ and $e' = M[s]\ N[s]$ and then we have:

$$\begin{aligned} FV(e) &= (FV(M\ N)\backslash BVar(s)) \cup FV(s) \\ &= ((FV(M)\backslash BVar(s)) \cup FV(s)) \cup ((FV(N)\backslash BVar(s)) \cup FV(s)) \\ &= FV(M[s]) \cup FV(N[s]) \\ &= FV(e') \end{aligned}$$

Sub_left The proof is the same as in the *Sub_app* case.

Sub_right The proof is the same as in the *Sub_app* case.

Sub_pair The proof is the same as in the *Sub_app* case.

Sub_var_1 The proof is obvious.

Sub_var_2 The proof is obvious.

Sub_var_3 Then $e = x[(\xi^P/\mathtt{K}).s]$, $e' = x[s]$ and the result holds since, by definition of expressions we have $x \notin BVar(P)$.

Sub_var_4 The proof is obvious.

$Sub\_sum\_var_1$ The proof is obvious.

$Sub\_sum\_var_2$ The proof is obvious.

$Sub\_sum\_var_3$ Then $e = \psi[(\xi^P/\mathtt{K}).s]$ with $\psi \neq \xi$, $e' = \xi[s]$ and the result holds since, by definition of expressions we have $\xi \notin BVar(P)$.

$Sub\_sum\_var_4$ The proof is obvious.

Sub_clos Then $e = M[s][t]$ and $e' = M[s \circ t]$ and by α-conversion we can suppose that $BVar(s) \cap BVar(t) = \emptyset$ and then we have:

$$\begin{aligned} FV(e) &= (FV(M[s])\backslash BVar(t)) \cup FV(t) \\ &= (((FV(M)\backslash BVar(s)) \cup FV(s))\backslash BVar(t)) \cup FV(t) \\ &= (FV(M)\backslash (BVar(s) \cup BVar(t))) \cup (FV(s)\backslash BVar(t)) \cup FV(t) \\ &= (FV(M)\backslash BVar(s \circ t)) \cup FV(s \circ t) \\ &= FV(e') \end{aligned}$$

Sub_ass_env Then $e = (u \circ s) \circ t$ and $e' = u \circ (s \circ t)$ and by α-conversion we can suppose that $BVar(s) \cap BVar(t) = \emptyset$ and then we have:

$$\begin{aligned} FV(e) &= (FV(u \circ s)\backslash BVar(t)) \cup FV(t) \\ &= (((FV(u)\backslash BVar(s)) \cup FV(s))\backslash BVar(t)) \cup FV(t) \\ &= (FV(u)\backslash (BVar(s) \cup BVar(t))) \cup (FV(s)\backslash BVar(t)) \cup FV(t) \\ &= (FV(u)\backslash BVar(s \circ t)) \cup FV(s \circ t) \\ &= FV(e') \end{aligned}$$

Sub_concat_1 Then $e = ((x/M).s) \circ t$ and $e' = (x/M[t]).(s \circ t)$ and then we have:

$$\begin{aligned} FV(e) &= (FV((x/M).s)\backslash BVar(t)) \cup FV(t) \\ &= ((FV(M) \cup FV(s))\backslash BVar(t)) \cup FV(t) \\ &= (FV(M)\backslash BVar(t)) \cup (FV(s)\backslash BVar(t)) \cup FV(t) \\ &= FV(M[t]) \cup FV(s \circ t) \\ &= FV(e') \end{aligned}$$

Sub_concat_2 The proof is almost the same as in the Sub_concat_1 case.

Sub_id The proof is obvious.

3 Confluence for $\boldsymbol{\lambda P_w}$

In this section we will show that $\longrightarrow_{\lambda P_w}$. First of all we will show that $\longrightarrow_{es}$ is confluent and strongly normalizing. Then we will define a new relation $\longrightarrow_{aux}$ on es-normal forms and we will show that $\longrightarrow_{aux}$ is confluent. Then by a technical lemma we will show the confluence of $\longrightarrow$.

Lemma 31 $\longrightarrow_{es}$ is strongly normalizing.

Proof. The following interpretation $I()$ shows strong normalization of $\longrightarrow_{es}$:

$$
\begin{array}{lll}
I(\mathtt{L} &) & = 0 \\
I(\mathtt{R} &) & = 0 \\
I(id &) & = 1 \\
I(x &) & = 0 \\
I(\xi &) & = 0 \\
I((x/M).s &) & = I(M) + I(s) + 1 \\
I((\xi/\mathtt{K}).s &) & = I(s) + 1 \\
I(s \circ t &) & = I(s) * I(t) + I(t) + 3 * I(s) + 1 \\
I(\xi[s] &) & = +I(s) \\
I(M[s] &) & = I(M) * I(s) + I(s) + 3 * I(M) + 1 \\
I(\mathtt{inl}(M) &) & = I(M) + 1 \\
I(\mathtt{inr}(M) &) & = I(M) + 1 \\
I(\langle M_1, M_2\rangle &) & = I(M_1) + I(M_2) + 1 \\
I(M_1\ M_2 &) & = I(M_1) + I(M_2) + 1 \\
I([M_1 \mid_\xi M_2] &) & = I(M_1) + I(M_2) + 2 \\
I([M_1 \mid^s_\Xi M_2] &) & = I(s) * (I(M_1) + I(M_2) + 2) + 3 * I(M_2) + I(\Xi) + 3 * I(M_1) + 5
\end{array}
$$

This interpretation has been found with C*i*ME [CMMU00].

Lemma 32 Let s and t be substitutions and ξ be a sum variable. The following statements are correct:

1. If $\xi[s] \longrightarrow^*_{es} \xi$ then $\xi[s \circ t] \longrightarrow^*_{es} \xi[t]$
2. If $\xi[s] \longrightarrow^*_{es} \mathtt{K}$ then $\xi[s \circ t] \longrightarrow^*_{es} \mathtt{K}$

Proof. We prove these two statements by induction on the structure of s and then by case on the possible one step reducts.

Base case If $s = id$ then the result is obvious.

Inductive case

– If $s = (x/M).s_1$, it is obvious that if $\xi[s] \longrightarrow^*_{es} \mathtt{T}$ then $\xi[s_1] \longrightarrow^*_{es} \mathtt{T}$ since $x \neq \xi$. Then we have the following derivation of $\xi[s \circ t]$:

$$\xi[s \circ t] \longrightarrow_{es} \xi[(x/M[t]).(s_1 \circ t)] \longrightarrow_{es} \xi[s_1 \circ t]$$

Now the final result holds applying the induction hypothesis.

– If $s = (\psi/\mathtt{K}').s_1$, two sub-cases are possible.

- If $\psi \neq \xi$ we reason as in the case when $s = (x/M).s_1$
- If $\psi = \xi$ then it is obvious that the case corresponds to the second point and that $\mathtt{K} = \mathtt{K}'$. Then we have the following derivation:

$$\xi[s \circ t] \longrightarrow_{es} \xi[(\xi/\mathtt{K}).(s_1 \circ t)] \longrightarrow_{es} \mathtt{K}$$

– If $s = s_1 \circ s_2$, we remark that $\xi[s \circ t] \longrightarrow_{es} \xi[s_1 \circ (s_2 \circ t)]$.
  1. If $\xi[s] \longrightarrow^*_{es} \xi$ then we obviously have that $\xi[s_1] \longrightarrow^*_{es} \xi$ and $\xi[s_2] \longrightarrow^*_{es} \xi$. Now applying induction hypothesis on s_1 we know that $\xi[s_1 \circ (s_2 \circ t)] \longrightarrow^*_{es} \xi[s_2 \circ t]$ and applying once again the induction hypothesis on s_2 the result holds.
  2. If $\xi[s] \longrightarrow^*_{es} \mathtt{K}$ we obviously have $\xi[s_1] \longrightarrow^*_{es} \mathtt{K}$ or $\xi[s_1] \longrightarrow^*_{es} \xi$ and $\xi[s_2] \longrightarrow^*_{es} \mathtt{K}$.
     - If $\xi[s_1] \longrightarrow^*_{es} \xi$, applying the induction hypothesis on s_1, we know that $\xi[s_1 \circ (s_2 \circ t)] \longrightarrow^*_{es} \xi[s_2 \circ t]$ and then applying once again the induction hypothesis on s_2 the result holds.
     - If $\xi[s_1] \longrightarrow^*_{es} \mathtt{K}$ the result holds applying induction hypothesis on s_1.

Lemma 33 $\longrightarrow_{es}$ is locally confluent.

Proof. There is only one not obvious critical pair. This critical pair is the critical pair obtained by applying one of the rules (*Freeze*) or (*Sub_clos*) on the term $M = [M_1 \mid_\xi M_2][s][t]$ to obtain $[M_1 \mid^s_{\xi[s]} M_2][t]$ in the first case and $[M_1 \mid_\xi M_2][s \circ t]$ in the second one.

– If $\xi[s] \longrightarrow^*_{es} \xi$, then we have the following derivation:

$$[M_1 \mid^s_{\xi[s]} M_2][t] \longrightarrow^*_{es} [M_1 \mid^s_\xi M_2][t] \longrightarrow_{es} [M_1[s] \mid_\xi M_2[s]][t] \longrightarrow_{es} [M_1[s] \mid^t_{\xi[t]} M_2[s]]$$

- If $\xi[t] \longrightarrow^*_{es} \xi$ then we have the following derivation:

$$[M_1[s] \mid^t_{\xi[t]} M_2[s]] \longrightarrow^*_{es} [M_1[s] \mid^t_\xi M_2[s]] \longrightarrow_{es} [M_1[s][t] \mid_\xi M_2[s][t]] \longrightarrow^*_{es} [M_1[s \circ t] \mid_\xi M_2[s \circ t]]$$

  By Lemma 32 we have $\xi[s \circ t] \longrightarrow^*_{es} \xi$ and then the following derivation is valid:

$$[M_1 \mid_\xi M_2][s \circ t] \longrightarrow_{es} [M_1 \mid^{s \circ t}_{\xi[s \circ t]} M_2] \longrightarrow^*_{es} [M_1 \mid^{s \circ t}_\xi M_2] \longrightarrow_{es} [M_1[s \circ t] \mid_\xi M_2[s \circ t]]$$

  and then in that case have close the diagram
- If $\xi[t] \longrightarrow^*_{es} \mathtt{L}$ then we have the following derivation:

$$[M_1[s] \mid^t_{\xi[t]} M_2[s]] \longrightarrow^*_{es} [M_1[s] \mid^t_{\mathtt{L}} M_2[s]] \longrightarrow_{es} M_1[s][t] \longrightarrow^*_{es} M_1[s \circ t]$$

  But in that case by Lemma 32 we have $\xi[s \circ t] \longrightarrow^*_{es} \mathtt{L}$ and then the following derivation is valid:

$$[M_1 \mid_\xi M_2][s \circ t] \longrightarrow_{es} [M_1 \mid^{s \circ t}_{\xi[s \circ t]} M_2] \longrightarrow^*_{es} [M_1 \mid^{s \circ t}_{\mathtt{L}} M_2] \longrightarrow_{es} M_1[s \circ t]$$

  and then in that case have close the diagram.
- If $\xi[t] \longrightarrow^*_{es} \mathtt{R}$ we reason as when $\xi[t] \longrightarrow^*_{es} \mathtt{L}$.

– If $\xi[s] \longrightarrow^*_{es}$ `L` then we have the following derivation:

$$[M_1 \mid^{s}_{\xi[s]} M_2][t] \longrightarrow^*_{es} [M_1 \mid^{s}_{\mathtt{L}} M_2][t] \longrightarrow_{es} M_1[s][t] \longrightarrow_{es} M_1[s \circ t]$$

By Lemma 32 we have $\xi[s \circ t] \longrightarrow^*_{es}$ `L` and then we have the following derivation:

$$[M_1 \mid_{\xi} M_2][s \circ t] \longrightarrow_{es} [M_1 \mid^{s \circ t}_{\xi[s \circ t]} M_2] \longrightarrow^*_{es} [M_1 \mid^{s \circ t}_{\mathtt{L}} M_2] \longrightarrow_{es} M_1[s \circ t]$$

and then in that case we have closed the diagram.

– If $\xi[s] \longrightarrow^*_{es}$ `R` we reason as when $\xi[s] \longrightarrow^*_{es}$ `L`.

Lemma 34 $\longrightarrow_{es}$ is confluent and strongly normalizing

Proof. By Lemma 31 and Lemma 33 and the Newman's Lemma.

Notation 35 For any expression e we will now note $es(e)$ its es-normal form.

Remark 3.

– If s is a substitution in es-normal form then $s \neq s_1 \circ s_2$
– If $M[s]$ is in es-normal form then $M = \lambda P.M_1$ and M_1 is in es-normal form.

Definition 31 We now define the relation $\longrightarrow_{aux}$ on es-normal forms as follows. Let M and N be two es-normal forms. We have $M \longrightarrow_{aux} N$ if and only if it exists M_1 such that $M \longrightarrow_P M_1$ and $es(M_1) = N$.

One can consider the reduction relation $\longrightarrow_{aux}$ as a definition of a calculus based on λP_w with an implicit treatment of substitutions.

Definition 32 The relation $\Rightarrow$ on all es-normal forms is defined to be the *least reflexive relation closed by contexts* and satisfying the following conditions:

$$\frac{M \Rightarrow M' \quad N \Rightarrow N'}{\lambda P.M\ N \Rightarrow (\lambda P.M')[id]\ N'}$$

$$\frac{M \Rightarrow M' \qquad N_1 \Rightarrow N_1' \qquad N_2 \Rightarrow N_2' \qquad s \Rightarrow s'}{(\lambda\langle P_1, P_2\rangle.M)[s]\ \langle N_1, N_2\rangle \Rightarrow ((\lambda P_1.\lambda P_2.M')[s']\ N_1')\ N_2'}$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N' \qquad s \Rightarrow s'}{(\lambda @(P_1, P_2).M)[s]\ N \Rightarrow ((\lambda P_1.\lambda P_2.M')[s']\ N')N'}$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N' \qquad s \Rightarrow s'}{(\lambda(P \mid_\xi Q).M)[s]\ \mathtt{inl}_B(N) \Rightarrow (\lambda P.M')[(\xi/\mathtt{L}).s']\ N'}$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N' \qquad s \Rightarrow s'}{(\lambda(P \mid_\xi Q).M)[s]\ \mathtt{inr}_A(N) \Rightarrow (\lambda P.M')[(\xi/\mathtt{R}).s']\ N'}$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N' \qquad s \Rightarrow s'}{(\lambda x.M)[s]\ N \Rightarrow es(M'[(x/N').s'])}$$

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N' \qquad s \Rightarrow s'}{(\lambda\_.M)[s]\ N \Rightarrow es(M'[s'])}$$

Remark 4. For every substitution s, if $s \Rightarrow s'$ then for every sum variable ξ, $es(\xi[s]) = es(\xi[s'])$.

Lemma 36 For every substitution s, if $s \Rightarrow s'$ then :

- for every term M, if $M \Rightarrow M'$ we have $es(M[s]) \Rightarrow es(M'[s'])$.
- for every substitution t, if $t \Rightarrow t'$ we have $es(t \circ s) \Rightarrow es(t' \circ s')$

Proof. We make the proof of the two statements together by induction on the lexicographic order (M or t, s) and then by cases.

We remark that, since we have $s \Rightarrow s'$ (resp. $M \Rightarrow M'$ and $t \Rightarrow t'$), s and s' (resp. M and M' and t and t') must be in es-normal forms. An then for example $es(s) = s$.

Base case

- If $M = x$ and $s = id$, then $M' = x$ and $s' = id$. Then $es(M[s]) = x \Rightarrow x = es(M'[s'])$.
- If $t = id$ then $t' = id$ and $es(t \circ s) = s \Rightarrow s' = es(t' \circ s')$ whatever s should be.

Inductive case

- If $M = x$ and $s = (x/N).s_1$, then $M' = x$ and $s' = (x/N').s_1'$ where $N \Rightarrow N'$ and $s_1 \Rightarrow s_1'$. Thus $es(M[s]) = N \Rightarrow N' = es(M'[s'])$.
- If $M = y$ and $s = (x/N).s_1$ with $y \neq x$, then $M' = y$ and $s' = (x/N').s_1'$ where $N \Rightarrow N'$ and $s_1 \Rightarrow s_1'$. $es(M[s]) = es(y[s_1])$ and by induction hypothesis $es(y[s_1]) \Rightarrow es(y[s_1']) = es(M'[s'])$.
- If $M = x$ and $s = (\xi^P/\mathtt{K}).s_1$, then $M' = x$ and $s' = (\xi^P/\mathtt{K}).s_1'$ where $s_1 \Rightarrow s_1'$. By induction hypothesis $es(M[s]) = es(M[s_1]) \Rightarrow es(M[s_1']) = es(M[s'])$.
- If $M = (\lambda P.N)[t_1]$ then $M' = (\lambda P.N')[t_1']$ where $N \Rightarrow N'$ and $t_1 \Rightarrow t_1'$ then, since $es(N) = N$ and $es(N') = N'$, $es(M[s]) = (\lambda P.N)[es(t_1 \circ s)]$ and $es(M') = (\lambda P.N')[es(t_1' \circ s')]$. By induction hypothesis we have $es(t_1 \circ s) \Rightarrow es(t_1' \circ s')$, whence the result since $\Rightarrow$ is closed under contexts.
- If $M = (N_1\ N_2)$ and $N_1 \Rightarrow N_1'$ and $N_2 \Rightarrow N_2'$ and $M' = (N_1'\ N_2')$. Then $es((N_1\ N_2)[s]) = es(N_1[s])\ es(N_2[s])$. By induction hypothesis we have $es(N_1[s]) \Rightarrow es(N_1'[s'])$ and $es(N_2[s]) \Rightarrow es(N_2'[s'])$, whence the result.
- If $M = (\lambda P.N_1)[t_1]\ N_2$ with $P \neq x$ and $P \neq \_$ and M we reason as in for $M = (N_1\ N_2)$. For example let us suppose that $P = \langle P_1, P_2\rangle$, $N_2 = \langle N_3, N_4\rangle$. If $N_1 \Rightarrow N_1'$, $N_2 \Rightarrow N_2'$, $N_3 \Rightarrow N_3'$, $t_1 \Rightarrow t_1'$ and $M' = (\lambda\langle P_1, P_2\rangle.N_1')[t_1']\ \langle N_3', N_4'\rangle$ then we have $es(M[s]) = (\lambda\langle P_1, P_2\rangle.N_1)[es(t_1 \circ s)]\ \langle es(N_3[s]), es(N_4[s])\rangle$. Now by hypothesis $N_1 \Rightarrow N_1'$. By induction hypothesis, $es(N_3[s]) \Rightarrow es(N_3'[s])$, $es(N_4[s]) \Rightarrow es(N_4'[s])$ and $es(t_1 \circ s) \Rightarrow es(t_1' \circ s)$ and then the result holds by definition of $\Rightarrow$.
- If $M = (\lambda x.N_1)[t_1]\ N_2$ and $M' = es(N_1'[(x/N_2').t_1'])$ where $N_1 \Rightarrow N_1'$, $N_2 \Rightarrow N_2'$ and $t_1 \Rightarrow t_1'$, then $es(M[s]) = (\lambda x.N_1)[es(t_1 \circ s)]\ es(N_2[s])$ and $es(M'[s']) = es(N_1'[(x/N_2'[s']).(t_1' \circ s')]) = es(N_1'[(x/es(N_2'[s'])).es(t_1' \circ s')])$ By induction hypothesis we have $es(N_2[s]) \Rightarrow es(N_2'[s'])$ and $es(t_1 \circ s) \Rightarrow es(t_1' \circ s')$, whence the result.

- If $M = (\lambda\_.N_1)[t_1]\ N_2$ we reason as in the previous case.
- If $M = [N_1 \ |_\xi\ N_2]$ and $[N_1 \ |_\xi\ N_2] \Rightarrow [N'_1 \ |_\xi\ N'_2]$ where $N_1 \Rightarrow N'_1$ and $N_2 \Rightarrow N'_2$ then three sub-cases are possible:
  1. If $es(\xi[s]) = \xi$ then $es([N_1 \ |_\xi\ N_2][s]) = [es(N_1[s]) \ |_\xi\ es(N_2[s])]$. By induction hypothesis we know that $es(N_1[s]) \Rightarrow es(N'_1[s'])$ and that $es(N_2[s]) \Rightarrow es(N'_2[s'])$. By remark 4 we know that $es(\xi[s']) = \xi$. The result then obviously holds.
  2. If $es(\xi[s]) = \mathtt{L}$ then $es([N_1 \ |_\xi\ N_2][s]) = es(N_1[s])$ and the result obviously holds as in the previous case.
  3. If $es(\xi[s]) = \mathtt{R}$ we reason as when $es(\xi[s]) = \mathtt{L}$.
- If $t = (x/N).t_1$ we reason as in the case of an application.
- If $t = (\xi/\mathtt{K}).t_1$ the result obviously holds.
- There is no other case since neither $[M_1 \ |^t_\Xi\ M_2]$, nor $M_1[t]$ nor $t_1 \circ t_2$ is an es-normal form.

Lemma 37 The relation $\longrightarrow_{aux}$ is confluent.

Proof. – We remark that we obviously have $\longrightarrow_{aux} \subseteq \Rightarrow \subseteq \longrightarrow^*_{aux}$ and then that $\longrightarrow^*_{aux} = \Rightarrow *$
- We will prove that $\Rightarrow$ has the diamond property. Thus $\Rightarrow$ is confluent and then $\longrightarrow_{aux}$ is confluent.

We prove that $\Rightarrow$ has the diamond property by induction on M or s and then by case.

- If none of the reductions takes place at the root of the considered expression the result trivially holds by induction hypothesis and by closure.
- We are left to the cases $M = (\lambda P.N_1)[s]\ N_2$ where one of the reductions "beta reduces" the term.
  If $P \neq x$ and $P \neq \_$, the result obviously holds by induction hypothesis.
  If $M = (\lambda\_.N_1)[s]\ N_2$ or $M = (\lambda x.N_1)[s]\ N_2$ we reason as follows. By Lemma 36 we know that whenever we have $M \Rightarrow M'$ and $s \Rightarrow s'$ we have $es(M[s]) \Rightarrow es(M'[s'])$. Then let's reason on the second case (which is the most general).
  - If $(\lambda x.N_1)[s]\ N_2 \Rightarrow es(N'_1[(x/N'_2).s'])$ and $(\lambda x.N_1)[s]\ N_2 \Rightarrow (\lambda x.N''_1)[s'']\ N''_2$ where $N_1 \Rightarrow N'_1$, $N_1 \Rightarrow N''_1$, $N_2 \Rightarrow N'_2$, $N_2 \Rightarrow N''_2$, $s \Rightarrow s'$ and $s \Rightarrow s''$. By induction hypothesis there exists N'''_1, N'''_2 and s''' such that $N'_1 \Rightarrow N'''_1$, $N''_1 \Rightarrow N'''_1$, $N'_2 \Rightarrow N'''_2$, $N''_2 \Rightarrow N'''_1$, $s' \Rightarrow s'''$, $s'' \Rightarrow s'''$. By definition of $\Rightarrow$ we have $(\lambda x.N''_1)[s'']\ N''_2 \Rightarrow es(N'''_1[(x/N'''_2).s'''])$ and $x \circ N'_2 s' \Rightarrow x \circ N'''_2 s'''$. Now by Lemma 36, we have $es(N'_1[(x/N'_2).s']) \Rightarrow es(N'''_1[(x/N'''_2).s'''])$ too.
  - Others cases are of the same form.

Lemma 38 If e and e' are two expressions such that $e \longrightarrow e'$ then $es(e) \longrightarrow^*_{aux} es(e')$.

Proof. The statement is obvious if $e \longrightarrow_{es} e'$. So we have just to check this statement if $e \longrightarrow_P e'$. We prove it by induction on $(\nu_\sigma(e), e)$ where ν_σ is the length of the longest $\longrightarrow_{es}$ derivation of e.

<u>Base case</u>

– If $e = x$, $e = \xi$ or $e = id$ the statement is obvious since there is no possible reduction.

Inductive case

– If the $\longrightarrow_P$ reduction does not take place at the root of e the result obviously holds by induction hypothesis.
– If $M = (N_1\ N_2)$ there are three cases :
 1. The reduction takes place in N_1 then $M' = N_1'\ N_2$ where $N_1 \longrightarrow N_1'$. By induction hypothesis we have $es(N_1) \longrightarrow^*_{aux} es(N_1')$. But $es(N_1\ N_2) = es(N_1)\ es(N_2)$, thus we deduce $es(N_1\ N_2) \longrightarrow^*_{aux} es(N_1'\ N_2)$.
 2. The reduction takes place in N_2. This case is symmetric to the previous one.
 3. The reduction takes place at the root we check all the possible cases.
  (a) If $N_1 = (\lambda\langle P_1, P_2\rangle.N_{11})[t]$ and $N_2 = \langle N_{21}, N_{22}\rangle$ then $M' = ((\lambda P_1.\lambda P_2.N_{11})[t]\ N_{21})\ N_{22}$. We have $es((\lambda\langle P_1, P_2\rangle.N_{11})[t]\ \langle N_{21}, N_{22}\rangle) = (\lambda\langle P_1, P_2\rangle.es(N_{11}))[es(t)]\ \langle es(N_{21}), es(N_{22})\rangle$ $\longrightarrow^*_{aux} ((\lambda P_1.\lambda P_2.es(N_{11}))[es(t)]\ es(N_{21}))\ es(N_{22}) = es(((\lambda P_1.\lambda P_2.N_{11})[t]\ N_{21})\ N_{22})$
  (b) For the cases of an application of one of the rules (Abs_contr), (Abs_left), (Abs_right), we proceed as in the previous case.
  (c) If $N_1 = (\lambda x.N_{11})[t]$ then $M' = N_{11}[(x/N_2).t]$. We have $es((\lambda x.N_{11})[t]\ N_2) = (\lambda x.es(N_{11}))[es(t)]\ es(N_2) \longrightarrow^*_{aux} es(es(N_{11})[(x/es(N_2)).es(t)])$. But $es(es(N_{11})[(x/es(N_2)).es(t)]) = es(N_{11}[(x/N_2).t]) = es(M')$ whence the result.
  (d) If $N_1 = (\lambda\_.N_{11})[t]$ then $M' = N_{11}[t]$. Then $es((\lambda\_.N_{11})[t]\ N_2) = (\lambda\_.es(N_{11}))[es(t)]\ es(N_2) \longrightarrow^*_{aux} es(es(N_{11})[es(t)]) = es(N_{11}[t]) = es(M')$

Lemma 39 (Interpretation Lemma) Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ be the union of two relations, $\mathcal{R}_1$ being confluent and strongly normalizing. We denote by $\mathcal{R}_1(a)$ the $\mathcal{R}_1$-normal form of a. Suppose that there is some relation $\mathcal{R}'$ on $\mathcal{R}_1$-normal forms satisfying:

$$\mathcal{R}' \subseteq \mathcal{R}^* \text{ and } (a \longrightarrow_{\mathcal{R}_2} b \Rightarrow \mathcal{R}_1(a) \longrightarrow^*_{\mathcal{R}'} \mathcal{R}_1(b))$$

Proof. This is the Hardin's interpretation Lemma. A proof can be found in [**?**].

Theorem 310 $\longrightarrow$ is confluent

Proof. By the interpretation Lemma and Lemmas 33, 36, 37 and 38.

4 A typing system for $\boldsymbol{\lambda P_w}$

As λ-calculus is strictly contained in λP_w, then λP_w is not strongly normalizing. In order to obtain strong normalization of λP_w we define a typing system which is capable of associating types to terms, sum terms and substitutions in a given environment. While typing systems have been already studied for calculi with explicit substitutions [DG01] and also for calculi with patterns [KPT96, CK], no formalism in the literature exists to correctly type explicit choice sum terms.

The typing system that we present in this section is shown to have the *subject reduction* property, that is, typing is preserved under reduction. The notion of acceptable expression (Definition 26) is essential to guarantee such a property.

4.1 Preliminaries

We restrict now our attention to a special kind of patterns called *acceptable*. For that, we say that a pattern P is *linear* if and only if every variable occurs at most once in P. We define a *type environment* to be a pair $\Phi;\Gamma$ such that Φ is a *sum environment* defined as a set of pairs of the form $\xi : \mathtt{K}$ and Γ is a *pattern environment* defined as a set of *typed patterns*, which are pairs of the form $P:A$. We say that a type environment $\Phi;\Gamma$ is *linear* if every variable occurs at most once in $\Phi;\Gamma$.

Definition 41 (Acceptable Patterns and Environments) The set of *acceptable patterns of type* A, denoted by $\mathcal{AP}(A)$, is defined to be the smallest set of *linear* patterns verifying the following properties: $\_ \in \mathcal{AP}(A)$; $x \in \mathcal{AP}(A)$ for any variable x; $@(P,Q) \in \mathcal{AP}(A)$ if $P \in \mathcal{AP}(A)$ and $Q \in \mathcal{AP}(A)$; $\langle P,Q\rangle \in \mathcal{AP}(B \times C)$ and $(P \mid_\xi Q) \in \mathcal{AP}(B+C)$ if $P \in \mathcal{AP}(B)$ and $Q \in \mathcal{AP}(C)$. The role of the notion of "acceptable patterns" is to prevent the $(wildcard)$ typing rule (corresponding to (*weakening*) in logic) to introduce meaningless pattern expressions. This notion extends naturally to environments by defining $\Phi;\Gamma$ to be *acceptable* if and only if each pattern appearing in $\Phi;\Gamma$ is acceptable.

Thus for example the pattern $\langle x,y\rangle$ is linear but *is not* in $\mathcal{AP}(A+B)$ since a pair pattern is not compatible with a sum type.

We now introduce the typing rules for terms and substitutions (resp. for sum terms) in Figure 2 (resp. Figure 3) assuming that all the patterns and type environments appearing in these rules are acceptable.

In the rules $(Case_1)$ and $(Frozencase_1)$, ξ is a fresh sum variable and $(P \mid_\xi Q)$ is linear. In the rule $(Wildcard)$, $\Phi; P : A, \Gamma$ has to be linear. In the rule $(Proj_1)$, all the x_i's are distinct usual variables. In the rules $(Proj_2)$ and $(Nproj)$, all the ξ_i's are distinct sum variables. In the rule $(Nproj)$ ξ does not appear in Γ. In the rule (*App*), we require that N does not contain free sum variables. In the rules (*Sub_term*), $(Frozencase_1)$ and $(Frozencase_2)$ we require that s does not contain free sum variables. In the rule (Sub_cons_1) we require that M does not contain free sum variables. In the rules (Sub_cons_1), (Sub_cons_2), all contexts have to be linear.

We say that the term M (resp. the substitution s and the sum term Ξ) *has type* A (resp. *co-environment* $\Phi';\Gamma'$ and *sum type* $\mathtt{T}$) in a type environment $\Phi;\Gamma$ if and only if there is a type derivation ending with $\Phi;\Gamma \vdash M : A$ (resp. $\Phi;\Gamma \vdash s \triangleright \Phi';\Gamma'$ and $\Phi;\Gamma \vdash \Xi \rightsquigarrow \mathtt{T}$). We say that the term M (resp. the substitution s and the sum term Ξ) is *well-typed* in $\Phi;\Gamma$ if and only if there is a type A such that M has type A in $\Phi;\Gamma$ (resp there is an environment $\Phi';\Gamma'$ such that s has is co-environment $\Phi';\Gamma'$ in $\Phi;\Gamma$ and there is a sum type $\mathtt{T}$ such that Ξ has sum type $\mathtt{T}$ in $\Phi;\Gamma$). We will make an abuse of notation by writing

$$\frac{}{\Phi; x_1\!:\!A_1, \ldots, x_n\!:\!A_n \vdash x_i\!:\!A_i}\ (Proj_1)$$

$$\frac{\Phi;\Gamma \vdash M\!:\!A}{\Phi;\Gamma \vdash \mathtt{inl}_B(M)\!:\!A+B}\ (+Right_1) \qquad \frac{\Phi;\Gamma \vdash M\!:\!B}{\Phi;\Gamma \vdash \mathtt{inr}_A(M)\!:\!A+B}\ (+Right_2)$$

$$\frac{\Phi;\Gamma \vdash \xi \rightsquigarrow \xi \qquad \Phi; P\!:\!B,\Gamma \vdash M\!:\!A \qquad \Phi; Q\!:\!C,\Gamma \vdash N\!:\!A}{\Phi; (P \mid_\xi Q)\!:\!B+C,\Gamma \vdash [M \mid_\xi N]\!:\!A}\ (Case_1)$$

$$\frac{\Phi;\Gamma \vdash \xi \rightsquigarrow \mathtt{K} \qquad \Phi;\Gamma \vdash M\!:\!A \qquad \Phi;\Gamma \vdash N\!:\!A}{\Phi;\Gamma \vdash [M \mid_\xi N]\!:\!A}\ (Case_2)$$

$$\frac{\Phi;\Gamma \vdash \Xi \rightsquigarrow \xi \qquad \Phi; P\!:\!A;\Gamma \vdash M[s]\!:\!C \qquad \Phi; Q\!:\!B;\Gamma \vdash N[s]\!:\!C}{\Phi; (P \mid_\xi Q)\!:\!A+B,\Gamma \vdash [M \mid^s_\Xi N]\!:\!C}\ (Frozencase_1)$$

$$\frac{\Phi;\Gamma \vdash \Xi \rightsquigarrow \mathtt{K} \qquad \Phi;\Gamma \vdash M[s]\!:\!A \qquad \Phi;\Gamma \vdash N[s]\!:\!A}{\Phi;\Gamma \vdash [M \mid^s_\Xi N]\!:\!A}\ (Frozencase_2)$$

$$\frac{\Phi; P\!:\!A, Q\!:\!B,\Gamma \vdash M\!:\!C}{\Phi; \langle P,Q\rangle\!:\!A\times B,\Gamma \vdash M\!:\!C}\ (\times Left) \qquad \frac{\Phi;\Gamma \vdash M\!:\!A \qquad \Phi;\Gamma \vdash N\!:\!B}{\Phi;\Gamma \vdash \langle M,N\rangle\!:\!A\times B}\ (\times Right)$$

$$\frac{\Phi; P\!:\!A,\Gamma \vdash M\!:\!B}{\Phi;\Gamma \vdash \lambda P\!:\!A.M\!:\!A\rightarrow B}\ (\rightarrow Right) \qquad \frac{\Phi;\Gamma \vdash M\!:\!A\rightarrow B \qquad \Phi;\Gamma \vdash N\!:\!A}{\Phi;\Gamma \vdash (MN)\!:\!B}\ (App)$$

$$\frac{\Phi; P\!:\!A, Q\!:\!A,\Gamma \vdash M\!:\!B}{\Phi; @(P,Q)\!:\!A,\Gamma \vdash M\!:\!B}\ (Layered) \qquad \frac{\Phi;\Gamma \vdash M\!:\!B}{\Phi; P\!:\!A,\Gamma \vdash M\!:\!B}\ (Wildcard)$$

$$\frac{}{\Phi;\Gamma \vdash id \triangleright \Phi;\Gamma}\ (Sub\_axiom) \qquad \frac{\Phi;\Gamma \vdash t \triangleright \Phi';\Gamma' \qquad \Phi';\Gamma' \vdash s \triangleright \Phi'';\Gamma''}{\Phi;\Gamma \vdash s\circ t \triangleright \Phi'';\Gamma''}\ (Sub\_concat)$$

$$\frac{\Phi;\Gamma \vdash M\!:\!A \qquad \Phi;\Gamma \vdash s \triangleright \Phi';\Gamma'}{\Phi;\Gamma \vdash (x/M).s \triangleright \Phi'; x:A,\Gamma'}\ (Sub\_cons_1) \frac{\Phi;\Gamma \vdash s \triangleright \Phi';\Gamma'}{\Phi;\Gamma \vdash (\xi^{P:A}/\mathtt{K}).s \triangleright \xi\!:\!\mathtt{K},\Phi'; P\!:\!A,\Gamma'}\ (Sub\_cons_2)$$

$$\frac{\Phi;\Gamma \vdash s \triangleright \Phi';\Gamma' \qquad \Phi';\Gamma' \vdash M\!:\!A}{\Phi;\Gamma \vdash M[s]\!:\!A}\ (Sub\_term)$$

Fig. 2. Typing Rules for Terms and Substitutions

$$\frac{}{\xi_1 : \mathtt{K}_1, \ldots, \xi_m : \mathtt{K}_m; \Gamma \vdash \xi_j \rightsquigarrow \mathtt{K}_j} \; (Proj_2) \qquad \frac{\text{if } \forall i, \xi \neq \xi_i}{\xi_1 : \mathtt{K}_1, \ldots, \xi_m : \mathtt{K}_m; \Gamma \vdash \xi \rightsquigarrow \xi} \; (Nproj)$$

$$\frac{}{\Phi; \Gamma \vdash \mathtt{L} \rightsquigarrow \mathtt{L}} \; (\mathtt{L}) \qquad \frac{}{\Phi; \Gamma \vdash \mathtt{R} \rightsquigarrow \mathtt{R}} \; (\mathtt{R})$$

$$\frac{\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma' \qquad \Phi'; \Gamma' \vdash \xi \rightsquigarrow \mathtt{T}}{\Phi; \Gamma \vdash \xi[s] \rightsquigarrow \mathtt{T}} \; (Sub\_sum)$$

Fig. 3. Typing Rules for Sum Terms

$\Phi; \Gamma \vdash M : A$ (resp. $\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma'$ and $\Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{T}$) to indicate that M (resp. s and Ξ) has type A in $\Phi; \Gamma$.

First of all, we remark that for any substitution s, the co-environment of s contains its typing environment. This observation can be formalized as follows:

Remark 5. If $\Psi; \Delta \vdash s \triangleright \Phi; \Gamma$ then there exists Ψ' and Δ' such that $\Phi = \Psi\Psi'$ and $\Gamma = \Delta\Delta'$.

Also note that for any well-typed expression e in $\Phi; \Gamma$ all its free variables appear in $\Phi; \Gamma$.

An important property used in the subject reduction proof (Theorem 412) states that if an expression e is well-typed in a typing environment $\Phi; \Gamma$, then it is also well-typed in any "reasonable" typing environment containing $\Phi; \Gamma$.

Lemma 41 (Weakening for Environments)

- If $\Phi; \Gamma \vdash M : A$ then for all acceptable $\Psi; \Delta$ such that $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup FSV(M)) = \emptyset$, then $\Psi\Phi; \Gamma\Delta \vdash M : A$.
- If $\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma'$, then for all acceptable $\Psi; \Delta$ such that $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup BVar(s) \cup FSV(s)) = \emptyset$, then $\Psi\Phi; \Delta\Gamma \vdash s \triangleright \Psi\Phi'; \Delta\Gamma'$.
- If $\Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{K}$, then for all acceptable $\Psi; \Delta$ such that $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cap FSV(\Xi)) = \emptyset$, then $\Psi\Phi; \Delta\Gamma \vdash \Xi \rightsquigarrow \mathtt{K}$.

Proof. We prove these three statements by induction on $(|\Psi| + |\Delta|, h)$ where $|\Delta|$ is the number of patterns appearing in Δ, $|\Psi|$ is the number of sum variables appearing in Ψ and h is the height of the considered proof. Base case If $h = 0$ and $\Psi; \Delta = \emptyset; \emptyset$. The three statements trivially hold.

Inductive case By case on the last typing rule used in $\mathcal{P}$.

$(Proj_1)$ Then M is a variable x. Two cases are now possible:

$\Delta = \emptyset)$ By hypothesis (Φ, Γ) and Ψ do not share variables so that we can apply the rule $(Proj_1)$ rule and the result holds.

$\Delta = P\!:\!B, \Delta'$) By induction hypothesis, $\Psi\Phi; \Delta'\Gamma \vdash M : A$ is derivable and thus, since $\Psi\Phi, P\!:\!B\Delta\Gamma$ is linear, by applying the rule ($Wildcard$) the result holds.

($Proj_2$) Then Ξ is a sum variable ξ and $\mathtt{K} \in \{\mathtt{L};\mathtt{R}\}$. By hypothesis ξ does not appear in $\Psi; \Delta\Gamma$. Thus the rule ($Proj_2$) is applicable and the result holds.

($Nproj$) We reason as in the case of an application of the rule ($Proj_2$).

(L) or (R) This case is trivial by the conditions imposed on the variables of $\Psi; \Delta$.

($+Right_1$) Then $M = \mathtt{inl}_{A_2}(M_1)$ and $A = A_1 + A_2$. By induction hypothesis there is a proof of $\Psi\Phi; \Delta\Gamma \vdash M_1 : A_1$. Thus the result hold by an application of the rule ($+Right1$)

Other rules for terms) By α-conversion and induction hypothesis.

(Sub_axiom) This case is trivial by the conditions imposed on the variables of $\Psi; \Delta$.

(Sub_cons_1) Then we have: $s = (x/M).s'$ is derivable and $\Phi; \Gamma \vdash s \triangleright \Phi'; x : A, \Gamma'$. Since $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup BVar(s) \cup FSV(s)) = \emptyset$ we have in particular $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup FSV(M)) = \emptyset$ and $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup BVar(s') \cup FSV(s')) = \emptyset$. By induction hypothesis we have $\Psi\Phi; \Delta\Gamma \vdash M : A$ and $\Psi\Phi; \Delta\Gamma \vdash s' \triangleright \Psi\Phi'; \Delta\Gamma'$. Ant thus the result holds applying the rule (Sub_cons_1).

(Sub_cons_2) Then we have: $s = (\xi^{P:A}/\mathtt{K}).s'$ and $\Phi; \Gamma \vdash s \triangleright \xi\!:\!\mathtt{K}, \Phi'; P\!:\!A, \Gamma'$ are derivable. Since $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup BVar(s) \cup FSV(s)) = \emptyset$ we have in particular $BVar(\Psi; \Delta) \cap (BVar(\Phi; \Gamma) \cup BVar(s') \cup FSV(s')) = \emptyset$. $\Psi\Phi; \Delta\Gamma \vdash s' \triangleright \Psi\Phi'; \Delta\Gamma'$ and thus since neither ξ, neither any variable appearing in P can't appear in $\Psi\Phi'; \Delta\Gamma'$, the result holds.

Other rules for substitution) By α-conversion and induction hypothesis.

Remark 6. Let s be such that $\Phi; \Gamma \vdash s \triangleright \Phi; (P \mid_\xi Q) : A + B, \Gamma'$ is derivable. Then the following statements hold:

- If there exists a term $M = [M_1 \mid_\xi M_2]$ such that $M[s]$ is an acceptable term, then $(P \mid_\xi Q)\!:\!A + B \in \Gamma$
- If there exists a term $M = [M_1 \mid^t_{\xi[u]} M_2]$ such that $M[s]$ is an acceptable term, then $(P \mid_\xi Q)\!:\!A + B \in \Gamma$

Proof. Let us suppose that $(P \mid_\xi Q) \notin \Gamma$. Then it is easy to show by induction on the structure of s that there exist ψ and $\mathtt{K}$ such that $(\psi^{(P|_\xi Q):B+C}/\mathtt{K}) \in s$. Then by remark 2 $M[s]$ is not a term.

4.2 Subject reduction for λP_w

One of the most important properties of a *typed* calculus is subject reduction which ensures that the types are preserved during reduction. This section is dedicated to show that λP_w enjoys this property.

The proof of subject reduction is strongly based on the possibility of deconstructing patterns into more simpler ones via the $Dec(\,)$ operation which is defined as follows:

Definition 42 Given a typed pattern $P : A$, we define its *deconstruction* as follows:

$$\begin{array}{lll} Dec(\_ : A) & = \_ : A & Dec(x : A) = x : A \\ Dec((P_1 \mid_\xi P_2) : A_1 + A_2) & = (P_1 \mid_\xi P_2) : A_1 + A_2 & \\ Dec(\langle P_1, P_2 \rangle : A_1 \times A_2) & = Dec(P_1 : A_1), Dec(P_2 : A_2) & \\ Dec(@(P_1, P_2) : A) & = Dec(P_1 : A), Dec(P_2 : A) & \end{array}$$

This notion extends naturally to a pattern environment $\Gamma = P_1 : A_1, \ldots, P_n : A_n$ by defining $Dec(\Gamma)$ as $Dec(P_1 : A_1), \ldots, Dec(P_n : A_n)$.

The typing system enjoys the property that any well-typed expression in a typing environment is also a well-typed expression in the deconstructed environment.

Lemma 42 Let Γ_1, Γ_2 be a partition of Γ in two disjoint subsets. Then we have

- $\Phi; \Gamma \vdash M : A$ if and only if $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash M : A$.
- $\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma\Gamma'$ if and only if $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash s \triangleright \Phi'; \Gamma_1 Dec(\Gamma_2)\Gamma'$.
- $\Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{K}$ if and only if $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash \Xi \rightsquigarrow \mathtt{K}$.

Proof. – The *only if* part is proved by induction on the height h of the proof of the statement.
  - If $h = 0$, we have an axiom and we make the proof by case on the axiom
    * If the axiom is $(Proj_1)$, then $\Gamma = Dec(\Gamma)$, thus the result follows.
    * If the axiom is $(Proj_2)$,$(Nproj)$, (Sub_axiom), $(\mathtt{L})$ or $(\mathtt{R})$, then $\Gamma_1 Dec(\Gamma_2)$ is linear so that the property holds.
  - If $h > 0$, we make the proof by case on the last rule used.

$(+Right1)$ We have a proof $\mathcal{P}$ ending in

$$\frac{\Phi; \Gamma \vdash M : A}{\Phi; \Gamma \vdash \mathtt{inl}_B(M) : A + B}$$

By induction hypothesis we have a proof $\mathcal{P}_1$ of $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash M : A$ and thus applying the rule $(+Right1)$ the result follows.

$(+Right2)$ *c.f.* $(+Right1)$.

$(Case_1)$ Then $M = [M_1 \mid_\xi M_2]$ and we have a proof $\mathcal{P}$ ending in:

$$\frac{\Phi; \Gamma \vdash \xi \rightsquigarrow \xi \qquad \Phi; P : B, \Gamma \vdash M_1 : A \qquad \Phi; Q : C, \Gamma \vdash M_2 : A}{\Phi; (P \mid_\xi Q) : B + C, \Gamma \vdash [M_1 \mid_\xi M_2] : A}$$

Now, without lost of generality we can suppose that $(P \mid_\xi Q) \in \Gamma_1$ since $Dec((P \mid_\xi Q)) = (P \mid_\xi Q)$ and then the result holds by induction hypothesis.

$(Case_2)$ Then $M = [M_1 \mid_\xi M_2]$ and we have a proof $\mathcal{P}$ ending in:

$$\frac{\Phi; \Gamma \vdash \xi \rightsquigarrow \mathtt{K} \qquad \Phi; \Gamma \vdash M_1 : A \qquad \Phi; \Gamma \vdash M_2 : A}{\Phi; \Gamma \vdash [M_1 \mid_\xi M_2] : A}$$

By induction hypothesis on sum terms we have a proof $\mathcal{P}_1$ of $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash \xi \rightsquigarrow \mathtt{K}$. By induction hypothesis on terms we have a proofs $\mathcal{P}_2$ and $\mathcal{P}_3$ of respectively $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash M_1 : A$ and $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash M_2 : A$. Thus the result holds applying rule $Case_2$.

$(Frozencase_1)$ or $(Frozencase_2)$ The result trivially holds by induction hypothesis.

$(\times Right)$, $(\rightarrow Right)$ or (App) The result trivially holds by induction hypothesis.

$(\times Left)$ In this case $\Gamma = \langle P_1, P_2 \rangle : B_1 \times B_2, \Gamma'$ and $\mathcal{P}$ ends in:

$$\frac{\Phi; P_1 : B_1, P_2 : B_2, \Gamma' \vdash M : A}{\Phi; \Gamma \vdash M : A}$$

Now two cases are possible:

1. If $\langle P_1, P_2 \rangle : B_1 \times B_2 \in \Gamma_1$, then by induction hypothesis, there is a proof of $\Phi; P_1 : B_1, P_2 : B_2, \Gamma'_1 Dec(\Gamma_2) \vdash M : A$ where $\Gamma'_1 = \Gamma_1 - \{\langle P_1, P_2 \rangle : B_1 \times B_2\}$ and we can conclude applying the rule $(\times Left)$.
2. If $\langle P_1, P_2 \rangle : B_1 \times B_2 \in \Gamma_2$, we conclude by induction hypothesis since $Dec(\langle P_1, P_2 \rangle : A \times B, \Delta) = Dec(P_1 : A, P_2 : B, \Delta)$.

$(Layered)$ We reason as in the $(\times Left)$ case.

$(Wildcard)$ Then $\Gamma = P : B, \Gamma'$ and $\mathcal{P}$ ends in

$$\frac{\Phi; \Gamma' \vdash M : A}{\Phi; P : B, \Gamma' \vdash M : A}$$

Once again two cases are possible:

1. If $P : B \in \Gamma_1$ we reason as in the similar case of the $(\times Left)$ case.
2. If $P : B \in \Gamma_2$, by induction hypothesis we have a proof of $\Phi; \Gamma_1 Dec(\Gamma'_2) \vdash M : A$ where $\Gamma'_2 = \Gamma_2 - \{P : B\}$. Now we can conclude by applying the rule $(Wildcard)$ as many times as it is necessary to introduce all the patterns in $Dec(P)$.

(Sub_cons_2) Then $s = (\xi^{P.A}/\mathtt{K}).s_1$ and by induction hypothesis we have a proof of $\Phi; \Gamma_1 Dec(\Gamma_2) \vdash s' \triangleright \Phi''; \Gamma_1 Dec(\Gamma_2)\Gamma''$ with $\Phi' = \xi : \mathtt{K}, \Phi''$ and $\Gamma' = P : A, \Gamma''$. Thus, since the rule (Sub_cons_2) is applicable, the result holds.

$*$ For all other rules the result trivially holds by induction hypothesis.

– The if part can be done by induction on the height of the proof $\mathcal{P}$ using the fact that it is sufficient to apply the rules $(\times Left)$, $(Layered)$ and $(Wildcard)$ to "reconstruct" Γ_2 from $Dec(\Gamma_2)$.

Corollary 1.

– $\Phi; \Gamma \vdash M : A$ *if and only if* $\Phi; Dec(\Gamma) \vdash M : A$.
– $\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma\Gamma'$ *if and only if* $\Phi; Dec(\Gamma) \vdash s \triangleright \Phi'; Dec(\Gamma)\Gamma'$.
– $\Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{K}$ *if and only if* $\Phi; Dec(\Gamma) \vdash \Xi \rightsquigarrow \mathtt{K}$.

Lemma 43 Let $R \in \{(\times Right), (\rightarrow Right), (App), (+Right1), (+Right2)\}$ and let $W \in \{(\times Left), (Layered), (Wildcard)\}$. If $\Phi; \Gamma \vdash M : A$ has a proof ending with R followed by W, then there is a proof of $\Phi; \Gamma \vdash M : A$ ending with W followed by R.

Proof. We prove these statements by cases on W and R.

$W = (\times Left)$. In this case $\Gamma = \langle P_1, P_2 \rangle : B \times C, \Gamma'$

$R = (+Right1)$. In this case $A = A_1 + A_2$ and $M = \texttt{inl}_{A_2}(M_1)$

We have a proof ending with

$$\dfrac{\dfrac{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash M_1 : A_1}{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash \texttt{inl}_{A_2}(M_1) : A_1 + A_2}\, R}{\Phi; \Gamma \vdash \texttt{inl}_{A_2}(M_1) : A_1 + A_2}\, W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash M_1 : A_1}{\Phi; \Gamma \vdash M_1 : A_1}\, W}{\Phi; \Gamma \vdash \texttt{inl}_{A_2}(M_1) : A_1 + A_2}\, R$$

$R = (+Right2)$. This case is similar to the previous one.

$R = (\times Right)$. In this case $A = A_1 \times A_2$ and $M = \langle M_1, M_2 \rangle$

We have a proof ending with:

$$\dfrac{\dfrac{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash M_1 : A_1 \qquad \Phi; P_1 : B, P_2 : C, \Gamma' \vdash M_2 : A_2}{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}\, R}{\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}\, W$$

Thus we also have the following proof

$$\dfrac{\dfrac{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash M_1 : A_1}{\Phi; \Gamma \vdash M_1 : A_1}\, W \qquad \dfrac{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash M_2 : A_2}{\Phi; \Gamma \vdash M_2 : A_2}\, W}{\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}\, R$$

$R = (\rightarrow Right)$. In this case $A = A_1 \rightarrow A_2$ and $M = \lambda Q : A_1 . M_1$

We have a proof ending with:

$$\dfrac{\dfrac{\Phi; P_1 : B, P_2 : C, Q : A_1, \Gamma' \vdash M_1 : A_2}{\Phi; P_1 : B, P_2 : C, \Gamma' \vdash \lambda Q : A_1 . M_1 : A_1 \rightarrow A_2}\, R}{\Phi; \Gamma \vdash \lambda Q : A_1 . M_1 : A_1 \rightarrow A_2}\, W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; P_1 : B, P_2 : C, Q : A_1, \Gamma' \vdash M_1 : A_2}{\Phi; Q : A_1, \Gamma \vdash M_1 : A_2}\, W}{\Phi; \Gamma \vdash \lambda Q : A_1 . M_1 : A_1 \rightarrow A_2}\, R$$

which is the result

$R = (App)$. In this case $M = (M_1 M_2)$

We have a proof ending with:

$$\dfrac{\dfrac{\Phi; P_1 : B; P_2 : C, \Gamma' \vdash M_1 : D \rightarrow A \qquad \Phi; P_1 : B; P_2 : C, \Gamma' \vdash M_2 : D}{\Phi; P_1 : B; P_2 : C, \Gamma' \vdash (M_1 M_2) : A}\, R}{\Phi; \Gamma \vdash (M_1 M_2) : A}\, W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; P_1 : B; P_2 : C, \Gamma' \vdash M_1 : D \to A}{\Phi; \Gamma \vdash M_1 : D \to A}\ W \qquad \dfrac{\Phi; P_1 : B; P_2 : C, \Gamma' \vdash M_2 : D}{\Phi; \Gamma \vdash M_2 : D}\ W}{\Phi; \Gamma \vdash (M_1 M_2) : A}\ R$$

which is the result

$W = (Layered)$. In this case $\Gamma = @(P_1, P_2) : B, \Gamma'$ and the proof proceeds very similarly to the case $W = (\times Left)$ so we do not give more details.

$W = (Wildcard)$. In this case $\Gamma = P : B, \Gamma'$

$R = (+Right1)$. In this case $A = A_1 + A_2$ and $M = \mathtt{inl}_{A_2}(M_1)$
We have a proof ending with:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash M_1 : A_1}{\Phi; \Gamma' \vdash \mathtt{inl}_{A_2}(M_1) : A_1 + A_2}\ R}{\Phi; \Gamma \vdash \mathtt{inl}_{A_2}(M_1) : A_1 + A_2}\ W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash M_1 : A_1}{\Phi; \Gamma \vdash M_1 : A_1}\ W}{\Phi; \Gamma \vdash \mathtt{inl}_{A_2}(M_1) : A_1 + A_2}\ R$$

which is the result

$R = (+Right2)$. This case is similar to the previous one.

$R = (\times Right)$. In this case $A = A_1 \times A_2$ and $M = \langle M_1, M_2 \rangle$
We have a proof ending with:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash M_1 : A_1 \qquad \Phi; \Gamma' \vdash M_2 : A_2}{\Phi; \Gamma' \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}\ R}{\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}\ W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash M_1 : A_1}{\Phi; \Gamma \vdash M_1 : A_1}\ W \qquad \dfrac{\Phi; \Gamma' \vdash M_2 : A_2}{\Phi; \Gamma \vdash M_2 : A_2}\ W}{\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}\ R$$

which is the result

$R = (\to Right)$. In this case $A = A_1 \to A_2$ and $M = \lambda Q : A_1 . M_1$
We have a proof ending with:

$$\dfrac{\dfrac{\Phi; Q : A_1, \Gamma' \vdash M_1 : A_2}{\Phi; \Gamma' \vdash \lambda Q : A_1 . M_1 : A_1 \to A_2}\ R}{\Phi; \Gamma \vdash \lambda Q : A_1 . M_1 : A_1 \to A_2}\ W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; Q : A_1, \Gamma' \vdash M_1 : A_2}{\Phi; Q : A_1, \Gamma \vdash M_1 : A_2}\ W}{\Phi; \Gamma \vdash \lambda Q : A_1 . M_1 : A_1 \to A_2}\ R$$

which is the result

$R = (App)$. In this case $M = (M_1 M_2)$

We have a proof ending with:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash M_1 : C \to A \qquad \Phi; \Gamma' \vdash M_2 : C}{\Phi; \Gamma' \vdash (M_1 M_2) : A}\,R}{\Phi; \Gamma \vdash (M_1 M_2) : A}\,W$$

Thus we also have the following proof:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash M_1 : C \to A}{\Phi; \Gamma \vdash M_1 : C \to A}\,W \qquad \dfrac{\Phi; \Gamma' \vdash M_2 : C}{\Phi; \Gamma \vdash M_2 : C}\,W}{\Phi; \Gamma \vdash (M_1 M_2) : A}\,R$$

which is the result

Now we will give some Lemmas that give the end of typing judgment given an expression.

Lemma 44 If $\Phi; \Gamma \vdash (M_1\ M_2) : A$ (resp. $\Phi; \Gamma \vdash \lambda P : A.M : A \to B$, $\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A \times B$, $\Phi; \Gamma \vdash \mathtt{inl}_B(M_1) : A + B$ or $\Phi; \Gamma \vdash \mathtt{inr}_A(M_1) : A + B$), there is a proof of it ending with the rule (App) (resp. $(\to Right)$, $(\times Right)$, $(+Right_1)$, $(+Right_2)$).

Proof. We only prove the first statement. The other ones are similar. We reason by induction on the height h of a proof $\mathcal{P}$ of $\Phi; \Gamma \vdash (M_1\ M_2) : A$. Base case If $h = 0$, the property vacuously holds.

Inductive case If $h > 0$, then we proceed by cases on the last rule used in the proof $\mathcal{P}$. The only possible cases are:

- If the last rule is (App) the property trivially holds.
- If the last rule used is $W = \{(\times Left), (Layered), (Wildcard)\}$:

$$\dfrac{\Pi}{\Phi, \Gamma \vdash (M_1 M_2) : A}\,W$$

  then we know by induction hypothesis that there is a proof of Π which ends with the application of the rule (App). We can then apply Lemma 43 to get the desired result.

Lemma 45 If $\Phi; \Gamma \vdash [M_1 \mid_\xi M_2] : A$, then $\Phi; Dec(\Gamma) \vdash [M_1 \mid_\xi M_2] : A$ has a proof ending with $(Case_1)$ or $(Case_2)$.

Proof. First of all we know that $\Phi; Dec(\Gamma) \vdash [M_1 \mid_\xi M_2] : A$ has a proof $\mathcal{P}$ by Corollary 1.

Let h be the height of the proof $\mathcal{P}$. We prove the statement by induction on h. Base case If $h = 0$ the property vacuously holds.

Inductive case If $h > 0$, we proceed by cases on the last rule used in $\mathcal{P}$. We remark that the last rule used in $\mathcal{P}$ cannot be $(\times Left)$ nor $(Layered)$ because $Dec(\Gamma)$ doesn't contain product nor layered pattern. The only possible cases are:

– If the last rule used is $(Wildcard)$ then there is a proof of $\Phi; Dec(\Gamma) \vdash [M_1 \mid_\xi M_2] : A$ ending as follows:

$$\frac{\Phi; \Gamma' \vdash [M_1 \mid_\xi M_2] : A}{\Phi; R : B, \Gamma' \vdash [M_1 \mid_\xi M_2] : A}$$

where $Dec(\Gamma) = R : B, \Gamma'$.
By induction hypothesis two sub-cases are possible:

1. If there is a proof of $\Phi; \Gamma' \vdash [M_1 \mid_\xi M_2] : A$ ending with $(Case_2)$ then we have a proof having the form:

$$\dfrac{\dfrac{\Phi; \Gamma' \vdash \Xi \rightsquigarrow \kappa \quad \Phi; \Gamma' \vdash M_1 : A \quad \Phi; \Gamma' \vdash M_2 : A}{\Phi; \Gamma' \vdash [M_1 \mid_\xi M_2] : A}\,(Case_2)}{\Phi; R : B, \Gamma' \vdash [M_1 \mid_\xi M_2] : A}\,(Wildcard)$$

Since we can suppose that all the variables that occur in R don't occur anywhere else in this proof, we can conclude using Lemma 41 and the rule $(Case_2)$.

2. If the last rule used is the rule $(Case_1)$. By linearity of $R : B, \Gamma'$, R could not have ξ in its set $BVar()$ and then we can also conclude by Lemma 41

– If the last rule used is $(Case_1)$ or $(Case_2)$ the result trivially holds.

Lemma 46 If $\Phi; \Gamma \vdash [M_1 \mid^s_\Xi M_2] : A$, then $\Phi; Dec(\Gamma) \vdash [M_1 \mid^s_\Xi M_2] : A$ has a proof ending with $(Frozencase_1)$ or $(Frozencase_2)$.

Proof. First of all we know that $\Phi; Dec(\Gamma) \vdash [M_1 \mid^s_\Xi M_2] : A$ has a proof $\mathcal{P}$ by Corollary 1. Let h be the height of the proof $\mathcal{P}$. We prove the statement by induction on h. Base case If $h = 0$ the property vacuously holds.

Inductive case If $h > 0$, we proceed by cases on the last rule used in $\mathcal{P}$. We remark that the last rule used in $\mathcal{P}$ cannot be $(\times Left)$ nor $(Layered)$ because $Dec(\Gamma)$ doesn't contain product or layered pattern. The only possible cases are:

– If the last rule used is $(Wildcard)$ then there is a proof of $\Phi; Dec(\Gamma) \vdash [M_1 \mid^s_\Xi M_2] : A$ ending as follows:

$$\frac{\Phi; \Gamma' \vdash [M_1 \mid^s_\Xi M_2] : A}{\Phi; R : B, \Gamma' \vdash [M_1 \mid^s_\Xi M_2] : A}$$

where $Dec(\Gamma) = R : B, \Gamma'$.
By induction hypothesis two sub-cases are possible:

1. If there is a proof of $\Phi; \Gamma' \vdash [M_1 \mid^s_\Xi M_2] : A$ ending with $(Frozencase_1)$ then we have a proof having the form:

$$\dfrac{\dfrac{\Phi; \Gamma'' \vdash \Xi \rightsquigarrow \xi \quad \Phi; P : C, \Gamma'' \vdash M_1[s] : A \quad \Phi; Q : D, \Gamma'' \vdash M_2[s] : A}{\Phi; (P \mid_\xi Q) : C + D, \Gamma'' \vdash [M_1 \mid^s_\Xi M_2] : A}\,(Frozencase_1)}{\Phi; R : B, (P \mid_\xi Q) : C + D, \Gamma'' \vdash [M_1 \mid^s_\Xi M_2] : A}\,(Wildcard)$$

where $\Gamma' = (P \mid_\xi Q) : C + D, \Gamma''$.
By linearity of $R : B, (P \mid_\xi Q) : C + D, \Gamma''$, $BVar(R)$ could not contain ξ and by Lemma 41, we know that we a proof for $\Phi; R : B, \Gamma'' \vdash \Xi \rightsquigarrow \xi$. Now we can conclude applying the rule $(Wildcard)$ to the proof of $\Phi; P : C, \Gamma'' \vdash M_1[s] : A$ and $\Phi; P : C, \Gamma'' \vdash M_2[s] : A$ and then the rule $(Frozencase_1)$.

2. If there is a proof of $\Phi; \Gamma' \vdash [M_1 \mid^s_\Xi M_2] : A$ ending with $(Frozencase_2)$ then we have a proof having the form:

$$\cfrac{\cfrac{\Phi; \Gamma' \vdash \Xi \rightsquigarrow \mathtt{K} \quad \Phi; \Gamma' \vdash M_1[s] : A \quad \Phi; \Gamma' \vdash M_2[s] : A}{\Phi; \Gamma' \vdash [M_1 \mid^s_\Xi M_2] : A}\ (Frozencase_2)}{\Phi; R : B, \Gamma' \vdash [M_1 \mid^s_\Xi M_2] : A}\ (Wildcard)$$

Since we can suppose that all the variables that occur in R don't occur anywhere else in this proof, we can conclude using Lemma 41 and the rule $(Frozencase_2)$.

– If the last rule used is $(Frozencase_1)$ or $(Frozencase_2)$ the result trivially holds.

Lemma 47 If $\Psi; \Delta \vdash (x/M).s \triangleright \Phi; \Gamma$ (resp. $\Psi; \Delta \vdash (\xi^{P,A}/\mathtt{K}).s \triangleright \Phi; \Gamma$ or $\Psi; \Delta \vdash s \circ t \triangleright \Phi; \Gamma$) then any proof of this judgment ends with the rule (Sub_cons_1) (resp. (Sub_cons_2) or (Sub_concat)).

Proof. Only one rule can be applied to these forms of term

Lemma 48 If $\Phi; \Gamma \vdash M[s] : A$, then there is a proof of $\Phi; Dec(\Gamma) \vdash M[s] : A$ ending with the rule (Sub_term).

Proof. By Lemma 1 we have a proof $\mathcal{P}$ of $\Phi; Dec(\Gamma) \vdash M[s] : A$ of height h. We proceed by induction on h. Base case If $h = 0$ the property vacuously holds.

Inductive case If $h > 0$, then we proceed by cases on the last rule used in $\mathcal{P}$. Since $Dec(\Gamma)$ does not contain product or layered pattern, the only possible cases are:

– If the last rule used is (Sub_term), then the statement trivially holds.
– If the last rule used is $(Wildcard)$. Then, by induction hypothesis there is a proof of $\Phi; Dec(\Gamma) \vdash M[s] : A$ ending in:

$$\cfrac{\cfrac{\Phi; \Gamma' \vdash s \triangleright \Phi'; \Gamma'\Gamma'' \quad \Phi'; \Gamma'\Gamma'' \vdash M : A}{\Phi; \Gamma' \vdash M[s] : a}}{\Phi; P : B, \Gamma' \vdash M[s] : A}$$

where $Dec(\Gamma) = P : B, \Gamma'$. Now by α-conversion we can suppose that all the variables appearing in P are fresh in $M[s]$. Then by Lemma 41, we have proofs for: $\Phi; Dec(\Gamma) \vdash s \triangleright \Phi'; Dec(\Gamma)\Gamma''$ and for $\Phi'; Dec(\Gamma)\Gamma'' \vdash M : A$ and the applying the rule (Sub_term) the result holds.

Remark 7. if $\Phi; \Gamma \vdash \Xi \rightsquigarrow \xi$ then ξ does not appear in Γ nor Φ.

Proof. By induction on Ξ and then by case on the last typing rule used.

Lemma 49
If $\Phi; \Gamma \vdash \Xi \rightsquigarrow \xi$ then if $\mathtt{K}$ is a constant we have $\xi : \mathtt{K}, \Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{K}$.

Proof. We prove this statement by induction on Ξ.

Base case If $\Xi = \xi$, then the only typing rule that applies is $(Nproj)$. The result obviously holds since $\xi:\mathtt{K}, \Phi; \Gamma$ is linear.

Inductive case If $\Xi = \xi[s]$, then the only typing rule that applies to $\xi[s]$ is the rule (Sub_sum). Thus any proof of $\Phi; \Gamma \vdash \xi[s] \leadsto \xi$ ends with:

$$\frac{\Phi; \Gamma \vdash s \,\triangleright\, \Phi'; \Gamma' \quad \Phi'; \Gamma' \vdash \xi \leadsto \xi}{\Phi; \Gamma \vdash \xi[s] \leadsto \xi}$$

By remark 7, ξ does not appear in $\Phi'; \Gamma'$. Moreover by Remark 5, ξ does not appear in s nor in $\Phi; \Gamma$ and then we have by Lemma 41 $\xi:\mathtt{K}, \Phi; \Gamma \vdash s \triangleright \xi:\mathtt{K}, \Phi'; \Gamma'$. Thus the result obviously holds.

The deconstruction operation given in Definition 42 can be used to simplify pair and contraction patterns, but there is no operation to simplify sum patterns. There is however a property of typing derivations which allows us to decompose sum patterns appearing on the left hand side of sequents as follows:

Lemma 410 Let $\mathtt{K}$ and $\mathtt{K}'$ be $\mathtt{L}$ or $\mathtt{R}$, then

- If $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash M : C$ then so is $\xi:\mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash M : C$.
- If $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash s \,\triangleright\, \Phi'; (P \mid_\xi Q) : A + B, \Gamma'$ then so is $\xi:\mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash s \,\triangleright\, \xi:\mathtt{K}, \Phi'; P : A, Q : B, \Gamma'$.
- If $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash \Xi \leadsto \mathtt{K}'$ then $\xi:\mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash \Xi \leadsto \mathtt{K}'$.
- If $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash \Xi \leadsto \psi$ then $\xi:\mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash \Xi \leadsto \psi$.

Proof. We only prove the case where $\xi:\mathtt{L}$, the other being similar.

We prove this statement by induction on the structure of the term M or of the substitution s or the sum term Ξ.

Base case

- If M is a variable x, then we have a proof of $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash x : C$. Remark first that, since $(P \mid_\xi Q)$ is not a variable, any proof of $\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash x : C$ must end with an application of the rule (Wildcard). We show by induction on $Dec(\Gamma)$ that there exists a proof of this judgment ending with an application of this rule introducing $(P \mid_\xi Q)$.
  - If $Dec(\Gamma) = \emptyset$, we must have a proof of $\Phi; (P \mid_\xi Q) : A + B \vdash x : C$ which is impossible.
  - If $Dec(\Gamma) = R : D, \Gamma'$ then we remark that the only rule that should be apply is $(Wildcard)$. Now we can suppose that R is introduced by the last application of $(Wildcard)$ and then the result obviously hold by induction hypothesis..

  Then we have a proof of the form:

  $$\frac{\Phi; Dec(\Gamma) \vdash x : C}{\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash x : C} \ (\text{Wildcard})$$

  Then, because the rule $(Wildcard)$ is applicable, we know that ξ does not occurs in $\Phi; Dec(\Gamma)$. Thus by Lemma 41 we have $\xi : \mathtt{K}, \Phi; Dec(\Gamma) \vdash x : C$, an then applying twice the rule $(Wildcard)$ we have $\xi : \mathtt{K}, \Phi; P : A, Q : B, Dec(\Gamma) \vdash x : C$. Now the result holds for Γ by the Corollary 1.

– If the substitution s is id. Then we have a proof of the form $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash s \triangleright \Phi; (P \mid_\xi Q) : A + B, \Gamma$. Thus $\Phi; (P \mid_\xi Q) : A + B, \Gamma$ is linear. So $\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma$ is also linear. The results follows.
– If the sum term Ξ is $\mathtt{L}$, then $\mathtt{T} = \mathtt{L}$ and the only way to prove $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash \mathtt{L} \rightsquigarrow \mathtt{L}$ is to apply the rule ($\mathtt{L}$). Now since $\Phi; (P \mid_\xi Q) : A + B, \Gamma$ is linear, so is $\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma$ and the result holds applying the rule ($\mathtt{L}$).
– If the sum term Ξ is $\mathtt{R}$ we reason as when $\Xi = \mathtt{L}$.
– If the sum term Ξ is a sum variable ψ, we first remark that $\psi \neq \xi$. The only two way to obtain a proof of $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash \Xi \rightsquigarrow \mathtt{T}$ is to apply one of the rule ($Proj_2$) or ($Nproj$). In this two cases $\psi \neq \xi$. Now we can reason as when $\Xi = \mathtt{L}$.

Inductive case
By case on M or s:

$M \equiv (M_1\ M_2)$: Then by Lemma 44, we have a proof ending with:

$$\frac{\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash M_1 : C \to D \qquad \Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash M_2 : C}{\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash M_1\ M_2 : D}\ (\to \text{Left})$$

By induction hypothesis we have a proof of $\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash M_1 : C \to D$ and a proof of $\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash M_2 : C$. Thus we have:

$$\frac{\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash M_1 : C \to D \qquad \xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash M_2 : C}{\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash M_1\ M_2 : D}\ (\to \text{Left})$$

$M \equiv \langle M_1, M_2 \rangle$, $M \equiv \mathtt{inl}_D(M_1)$, $M \equiv \mathtt{inr}_C(M_1)$ or $\lambda R : C.M_1$: By induction hypothesis and Lemma 44.

$M[s]$: Then by Lemma 48, we have a proof of $\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash M[s] : C$ ending in:

$$\frac{\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash s \triangleright \Phi'; (P \mid_\xi Q) : A + B, \Gamma' \qquad \Phi'; (P \mid_\xi Q) : A + B, \Gamma' \vdash M : C}{\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash M[s] : C}$$

By induction hypothesis we have a proof of $\xi : \mathtt{L}, \Phi; P : A, Q : B, Dec(\Gamma) \vdash s \triangleright \xi : \mathtt{L}, \Phi'; P : A, Q : B, \Gamma'$ and a proof of $\xi : \mathtt{L}, \Phi'; P : A, Q : B, \Gamma' \vdash M : C$. Thus, applying the rule (Sub_term), the result follows.

$M \equiv [M_1 \mid_\psi M_2]$: Two sub-cases are possible.

1. If $\psi = \xi$: Then, by Lemma 45, we have a proof ending with:

$$\frac{\Phi; Dec(\Gamma) \vdash \xi \rightsquigarrow \xi \qquad \Phi; P : A, Dec(\Gamma) \vdash M_1 : C \qquad \Phi; Q : B, Dec(\Gamma) \vdash M_2 : C}{\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash [M_1 \mid_\xi M_2] : C}\ (Case_1)$$

   We cannot have an application of rule ($Case_2$) since $\Phi; Dec(\Gamma) \vdash \xi \rightsquigarrow \xi$
   By Lemma 49 we have a proof of $\xi : L, \Phi; Dec(\Gamma) \vdash \xi \rightsquigarrow L$.
   By Lemma 41 we also have proofs of $\xi : L, \Phi; P : A, Q : B; Dec(\Gamma) \vdash M_1 : C$ and $\xi : L, \Phi; P : A, Q : B; Dec(\Gamma) \vdash M_2 : C$.
   Thus the result holds applying the rule ($Case_2$)
2. If $\psi \neq \xi$: *c.f.* others cases. The proof is of the same type

$M \equiv [M_1 \mid^{s}_{\Xi} M_2]$ In this case again two sub-cases are possible by Lemma 46.

1. If we have a proof $\mathcal{P}$ of $\Phi; (P \mid_\xi Q) : A + B, Dec(\Gamma) \vdash M : C$ ending with the rule $(frozencase_1)$, Two sub cases are now possible:
   - $\mathcal{P}$ ends in:

$$\frac{\Phi; (P \mid_\xi Q) : A + B, \Gamma' \vdash \Xi \leadsto \psi \quad \Phi; (P \mid_\xi Q) : A + B, R_1 : D, \Gamma' \vdash M_1[s] : C \quad \Phi; (P \mid_\xi Q) : A + B, R_2 : E, \Gamma' \vdash M_2[s] : C}{\Phi; (P \mid_\xi Q) : A + B, (R_1 \mid_\xi R_2) : D + E, \Gamma' \vdash [M_1 \mid^{s}_{\Xi} M_2] : C}$$

   where $Dec(\Gamma) = (R_1 \mid_\xi R_2) : D + E, \Gamma'$.
   Then by remark 7 we know that $\psi \neq \xi$. And then by induction hypothesis we have a proof of $\xi : \mathtt{L}, \Phi; P : A, Q : B, \Gamma \vdash \Xi \leadsto \psi$ Then we can conclude using Lemma 41 and applying rule $(frozencase_1)$.
   - $\mathcal{P}$ ends in:

$$\frac{\Phi; \Gamma \vdash \Xi \leadsto \xi \quad \Phi; P : A, \Gamma' \vdash M_1[s] : C \quad \Phi; Q : B, \Gamma \vdash M_2[s] : C}{\Phi; (P \mid_\xi Q) : A + B, \Gamma' \vdash [M_1 \mid^{s}_{\Xi} M_2] : C}$$

   By Lemma 49 we have a proof of $\xi : \mathtt{L}, \Phi; \Gamma \vdash \Xi \leadsto \mathtt{L}$ and then we can conclude using Lemma 41 and applying rule $(frozencase_2)$.
2. If we have a proof of $\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash M : C$ ending with the rule $(frozencase_2)$, by induction hypothesis, we have a proof of $\xi : \mathtt{K}, \Phi; P : A, Q : B, \Gamma \vdash \Xi \leadsto \mathtt{K}$. And then we conclude by induction hypothesis and rule $(frozencase_2)$.

$s = (x/M).s_1$: By induction hypothesis.

$s = (\psi^{RC}/\mathtt{K}).s_1$: Then by Lemma 47 and Remark 5, we have a proof ending with:

$$\frac{\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash s_1 \rhd \Phi'; (P \mid_\xi Q) : A + B, \Gamma'}{\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash s \rhd \psi : \mathtt{K}, \Phi'; R : C, (P \mid_\xi Q) : A + B, \Gamma'}$$

First remark that we must have $\xi \neq \psi$ since $\psi : \mathtt{K}, \Phi'; R : C, (P \mid_\xi Q) : A + B, \Gamma'$ is linear.
Thus, by induction hypothesis, we have a proof of $\xi : \mathtt{L}, \Phi; P : A, Q : B, \Gamma \vdash s_1 \rhd \xi : \mathtt{L}, \Phi'; P : A, Q : B, \Gamma'$.
Thus, applying the rule (Sub_cons_2), the result follows.

$s = s_1 \circ s_2$: The by Lemma 47, we have a proof ending with:

$$\frac{\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash s_1 \rhd \Phi''; (P \mid_\xi Q) : A + B, \Gamma'' \qquad \Phi''; (P \mid_\xi Q) : A + B, \Gamma'' \vdash s_2 \rhd \Phi'; (P \mid_\xi Q) : A + B, \Gamma'}{\Phi; (P \mid_\xi Q) : A + B, \Gamma \vdash s \rhd \Phi'; (P \mid_\xi Q) : A + B, \Gamma'}$$

By induction hypothesis, we have proofs of $\xi : \mathtt{L}, \Phi; P : A, Q : B, \Gamma \vdash s_1 \rhd \xi : \mathtt{L}, \Phi''; P : A, Q : B, \Gamma''$ and $\xi : \mathtt{L}, \Phi''; P : A, Q : B, \Gamma'' \vdash s_2 \rhd \xi : \mathtt{L}, \Phi'; P : A, Q : B, \Gamma'$.
Thus,applying the rule (Sub_concat), the result follows

$\Xi = \psi[s]$ The result trivially holds by induction hypothesis.

Lemma 411

- If $\Phi; P : B, \Gamma \vdash M : A$ and $FV(M) \cap BVar(P) = \emptyset$ then $\Phi; \Gamma \vdash M : A$.
- If $\Phi; P : B, \Gamma \vdash s \rhd \Phi'; P : B, \Gamma'$ and $FV(s) \cap BVar(s)$ then $\Phi; \Gamma \vdash s \rhd \Phi'; \Gamma'$.
- If $\Phi; P : B, \Gamma \vdash \Xi \leadsto \mathtt{K}$ and $FV(\Xi) \cap BVar(P) = \emptyset$ then $\Phi; \Gamma \vdash \Xi \cap \mathtt{K}$.

– $\xi:\mathtt{K}, \Phi; \Gamma \vdash M : A$ and $\xi \notin FV(M)$ then $\Phi; \Gamma \vdash M \vdash A$.
– $\xi:\mathtt{K}, \Phi; \Gamma \vdash s \rhd \xi:\mathtt{K}, \Phi'; \Gamma'$ and $\xi \notin FV(s)$ then $\Phi; \Gamma \vdash s \rhd \Phi'; \Gamma'$.
– $\xi:\mathtt{K}, \Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{K}'$ and $\xi \notin FV(\Xi)$ then $\Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{K}'$.

Proof. By induction on the known proof and then by case on the last used rule.

We can now state the following result:

Theorem 412 (Subject reduction)

– If $\Phi; \Gamma \vdash M : A$ and $M \longrightarrow M'$ then $\Phi; \Gamma \vdash M' : A$.
– If $\Phi; \Gamma \vdash s \rhd \Phi'; \Gamma'$ and $s \longrightarrow s'$ then $\Phi; \Gamma \vdash s' \rhd \Phi'; \Gamma'$.
– If $\Phi; \Gamma \vdash \Xi \rightsquigarrow \mathtt{T}$ (with $\mathtt{T} \in \{\mathtt{L}, \mathtt{R}, \xi\}$) and $\Xi \longrightarrow \Xi'$ then $\Phi; \Gamma \vdash \Xi' \rightsquigarrow \mathtt{T}$.

Proof. We prove these three properties by induction on the structure of the considered expression.

Base case For all axioms the result the property vacuously holds.

Inductive case

– If $\Phi; \Gamma \vdash (x/M).s_1 \rhd \Phi'; \Gamma'$, then the only rule than can have been applied to prove $\Phi; \Gamma \vdash (x/M).s_1 \rhd \Phi'; \Gamma'$ is the rule (Sub_cons$_1$) and then we have:

$$\frac{\Phi; \Gamma \vdash M : A \qquad \Phi; \Gamma \vdash s \rhd \Phi'; \Gamma''}{\Phi; \Gamma \vdash (x/M).s_1 \rhd \Phi'; x : A, \Gamma''}$$

with $\Gamma' = x : A, \Gamma''$.

There is now two possibilities:

1. $M \longrightarrow M'$. By induction hypothesis we know that $\Phi; \Gamma \vdash M' : A$ and then applying the rule (Sub_cons$_1$) that $\Phi; \Gamma \vdash (x/M').s_1 \rhd \Phi'; \Gamma'$.
2. $s_1 \longrightarrow s_1'$. By induction hypothesis we know that $\Phi; \Gamma \vdash s_1' \rhd \Phi'; \Gamma''$ and then applying the rule (Sub_cons$_1$) that $\Phi; \Gamma \vdash (x/M).s_1' \rhd \Phi'; \Gamma'$.

– If $\Phi; \Gamma \vdash (\xi^{P:A}/\mathtt{K}).s_1 \rhd \Phi'; \Gamma'$. Then the only rule than can have been applied to prove $\Phi; \Gamma \vdash (\xi/\mathtt{K}).s_1 \rhd \Phi'; \Gamma'$ is the rule (Sub_cons$_2$) and then we have:

$$\frac{\Phi; \Gamma \vdash s \rhd \Phi''; \Gamma''}{\Phi; \Gamma \vdash (\xi^{P:A}/\mathtt{K}).s_1 \rhd \xi : \mathtt{K}, \Phi''; P : A, \Gamma''}$$

with $\Phi' = \xi : \mathtt{K}, \Phi''$ and $\Gamma' = P : A, \Gamma''$.

There is only one way to reduce this substitution: $s_1 \longrightarrow s_1'$. By induction hypothesis we know that $\Phi; \Gamma \vdash s_1' \rhd \Phi''; \Gamma''$ and then applying the rule (Sub_cons$_2$) that $\Phi; \Gamma \vdash (\xi/\mathtt{K}).s_1' \rhd \Phi'; \Gamma'$.

– If $\Phi; \Gamma \vdash s_1 \circ s_2 \rhd \Phi'; \Gamma'$, then the only rule than can have been applied to prove $\Phi; \Gamma \vdash s_1 \circ s_2 \rhd \Phi'; \Gamma'$ is the rule (Sub_concat) and then we have:

$$\frac{\Phi; \Gamma \vdash s_2 \rhd \Phi''; \Gamma'' \qquad \Phi''; \Gamma'' \vdash s_1 \rhd \Phi'; \Gamma'}{\Phi; \Gamma \vdash s_1 \circ s_2 \rhd \Phi'; \Gamma'}$$

There is now six possibilities:

1. $s_1 \longrightarrow s_1'$. By induction hypothesis we know that $\Phi;\Gamma \vdash s_1' \rhd \Phi'';\Gamma''$ and then applying the rule (Sub_concat) that $\Phi;\Gamma \vdash s_1' \circ s_2 \rhd \Phi';\Gamma'$.
2. $s_2 \longrightarrow s_2'$. By induction hypothesis we know that $\Phi'';\Gamma'' \vdash s_2' \rhd \Phi';\Gamma'$ and then applying the rule (Sub_concat) that $\Phi;\Gamma \vdash s_1 \circ s_2' \rhd \Phi';\Gamma'$.
3. if $s_1 = (x/M).s_1'$ and $s_1 \circ s_2 \longrightarrow (x/M[s_2]).(s_1' \circ s_2)$. Then by Lemma 47, $\Gamma' = x:A,\Gamma'''$ and we have a proof of $\Phi'';\Gamma'' \vdash (x/M).s_1' \rhd \Phi';\Gamma'$ ending with

$$\frac{\Phi'';\Gamma'' \vdash M:A \qquad \Phi'';\Gamma'' \vdash s_1' \rhd \Phi';\Gamma'''}{\Phi'';\Gamma'' \vdash (x/M).s_1' \rhd \Phi';x:A,\Gamma'''}$$

So we have a proof of:

$$\frac{\dfrac{\Phi;\Gamma \vdash s_2 \rhd \Phi'';\Gamma'' \qquad \Phi'';\Gamma'' \vdash M:A}{\Phi;\Gamma \vdash M[s_2]:A} \qquad \dfrac{\Phi;\Gamma \vdash s_2 \rhd \Phi'';\Gamma'' \qquad \Phi'';\Gamma'' \vdash s_1' \rhd \Phi';\Gamma'''}{\Phi;\Gamma \vdash s_1' \circ s_2 \rhd \Phi';\Gamma'''}}{\Phi;\Gamma \vdash (x/M[s_2]).(s_1' \circ s_2) \rhd \Phi';\Gamma'}$$

4. If $s_1 = (\xi/\mathtt{K}).s_1'$ and $s_1 \circ s_2 \longrightarrow (\xi^{P.A}/\mathtt{K}).s_1 \circ s_2$ the proof is of the same type.
5. If $s_1 = id$ and $id \circ s_2 \longrightarrow s_2$, the statement trivially holds
6. If $s_1 = t_1 \circ t_2$ and $(t_1 \circ t_2) \circ s_2 \longrightarrow t_1 \circ (t_2 \circ s_2)$ then by Lemma 47 we have a proof of $\Phi'';\Gamma'' \vdash t_1 \circ t_2 \rhd \Phi';\Gamma'$ ending with:

$$\frac{\Phi'';\Gamma'' \vdash t_2 \rhd \Phi''';\Gamma''' \qquad \Phi''';\Gamma''' \vdash t_1 \rhd \Phi';\Gamma'}{\Phi'';\Gamma'' \vdash t_1 \circ t_2 \Phi';\Gamma'}$$

And then we have a proof of:

$$\frac{\dfrac{\Phi;\Gamma \vdash s_2 \rhd \Phi'';\Gamma'' \qquad \Phi'';\Gamma'' \vdash t_2 \rhd \Phi''';\Gamma'''}{\Phi;\Gamma \vdash t_2 \circ s_2 \rhd \Phi''';\Gamma'''} \qquad \Phi''';\Gamma''' \vdash t_1 \rhd \Phi';\Gamma'}{\Phi;\Gamma \vdash t_1 \circ (t_2 \circ s_2) \rhd \Phi';\Gamma'}$$

– If $\Phi;\Gamma \vdash \xi[s] \rightsquigarrow \mathtt{K}$, by Corollary 1, we can suppose that $\Gamma = Dec(\Gamma)$. The only rule that can have been applied to prove this statement is the rule (Sub_sum) and then we have:

$$\frac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma' \quad \Phi';\Gamma' \vdash \xi \rightsquigarrow \mathtt{K}}{\Phi;\Gamma \vdash \xi[s] \rightsquigarrow \mathtt{K}}$$

There is now 4 possibilities:
1. $\xi[s] \longrightarrow \xi[s']$ with $s \longrightarrow s'$. By induction hypothesis we know that $\Phi;\Gamma \vdash s' \rhd \Phi';\Gamma'$ and then the result holds applying the rule (Sub_sum)
2. $s = (x/M).t$ and $\xi[(x/M).t] \longrightarrow \xi[t]$ then we have $\Gamma' = x:A,\Gamma''$ and a proof of $\Phi;\Gamma \vdash (x/M).t \rhd \Phi;x:A,\Gamma''$ ending with:

$$\frac{\Phi;\Gamma \vdash M:A \quad \Phi;\Gamma \vdash t \rhd \Phi';\Gamma''}{\Phi;\Gamma \vdash (x/M).t \rhd \Phi;x:A,\Gamma''}$$

Now by Lemma 411 since $x \notin FV(\xi)$ we know that $\Phi';\Gamma'' \vdash \xi \rightsquigarrow \mathtt{K}$ and then we have a proof ending with:

$$\frac{\Phi;\Gamma \vdash t \rhd \Phi';\Gamma'' \quad \Phi';\Gamma'' \vdash \xi \rightsquigarrow \mathtt{K}}{\Phi;\Gamma \vdash \xi[t] \rightsquigarrow \mathtt{K}}$$

3. $s = (\psi^{P.A}/\mathtt{K}').t$ with $\xi \neq \psi$ and $\xi[(\psi^{P.A}/\mathtt{K}).t] \longrightarrow \xi[t]$ we reason as for previous cases.
4. $s = (\xi^{P.A}/\mathtt{K}).t$ and $\xi[(\xi^{P.A}/\mathtt{K}).t] \longrightarrow \mathtt{K}$ then $\mathtt{K} \in \{\mathtt{L}, \mathtt{R}\}$ and since $\Phi; \Gamma$ is acceptable we can apply the rule (L) or the (R) to get the result.

– If $\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2$, then by lemma 44 there is a proof of it ending with the rule ($\times$Right) and then we have:

$$\frac{\Phi; \Gamma \vdash M_1 : A_1 \qquad \Phi; \Gamma \vdash M_2 : A_2}{\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2}$$

There is now two possibilities:

1. $M_1 \longrightarrow M_1'$. By induction hypothesis we know that $\Phi; \Gamma \vdash M_1' : A_1$ and then applying the rule ($\times$Right) that $\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2$.
2. $M_2 \longrightarrow M_2'$. By induction hypothesis we know that $\Phi; \Gamma \vdash M_2 : A_2$ and then applying the rule ($\times$Right) that $\Phi; \Gamma \vdash \langle M_1, M_2 \rangle : A_1 \times A_2$.

– If $\Phi; \Gamma \vdash \lambda P.M_1 : A \to B$, Then by lemma 44 there is a proof of it ending with the rule ($\to$Right) and then we have:

$$\frac{\Phi; P : A, \Gamma \vdash M_1 : B}{\Phi; \Gamma \vdash \lambda P.M_1 : A \to B}$$

There is now only one possibility: $M_1 \longrightarrow M_1'$. By induction hypothesis we know that $\Phi; P : A, \Gamma \vdash M_1' : B$ and then applying the rule ($\to$Right) that $\Phi; \Gamma \vdash \lambda P.M_1' : A \to B$.

– If $\Phi; \Gamma \vdash \mathtt{inl}_B(M_1) : A + B$, then by lemma 44 there is a proof of it ending with the rule (Right) and then we have:

$$\frac{\Phi; \Gamma \vdash M_1 : A}{\Phi; \Gamma \vdash \mathtt{inl}_B(M_1) : A + B}$$

There is now only one possibility: $M_1 \longrightarrow M_1'$. By induction hypothesis we know that $\Phi; \Gamma \vdash M_1' : A$ and then applying the rule (Right) that $\Phi; \Gamma \vdash \mathtt{inl}_B(M_1') : A + B$.

– If $\Phi; \Gamma \vdash \mathtt{inr}_A(M_1) : A + B$. Then by lemma 44 there is a proof of it ending with the rule (Left) and then we have:

$$\frac{\Phi; \Gamma \vdash M_1 : B}{\Phi; \Gamma \vdash \mathtt{inr}_A(M_1) : A + B}$$

There is now only one possibility: $M_1 \longrightarrow M_1'$. By induction hypothesis we know that $\Phi; \Gamma \vdash M_1' : B$ and then applying the rule (Left) that $\Phi; \Gamma \vdash \mathtt{inr}_A(M_1') : A + B$.

– If $\Phi; \Gamma \vdash [M_1 \mid_\xi M_2] : A$ then by Lemma 45 there is a proof of it ending with one of the rules $(case_1)$ or $(case_2)$. We reason as for previous cases.

– If $\Phi; \Gamma \vdash [M_1 \mid^s_\Xi M_2] : A$ then by Lemma 46 we can suppose that $\Gamma = Dec(\Gamma)$. Now 8 solutions are possible:

1. if $[M_1 \mid^s_\Xi M_2] \longrightarrow [M_1' \mid^s_\Xi M_2]$ with $M_1 \longrightarrow M_1'$ we can conclude by induction hypothesis.

2. if $[M_1 \mid^s_\Xi M_2] \longrightarrow [M_1 \mid^s_\Xi M_2']$ with $M_1 \longrightarrow M_2$ we can conclude by induction hypothesis.
3. if $[M_1 \mid^s_\Xi M_2] \longrightarrow [M_1 \mid^s_\Xi M_2']$ with $M_2 \longrightarrow M_2'$ we can conclude by induction hypothesis.
4. if $[M_1 \mid^s_\Xi M_2] \longrightarrow [M_1 \mid^{s'}_\Xi M_2]$ with $s \longrightarrow s'$ we can conclude by induction hypothesis.
5. if $[M_1 \mid^s_\Xi M_2] \longrightarrow [M_1 \mid^s_{\Xi'} M_2]$ with $\Xi \longrightarrow \Xi'$ we can conclude by induction hypothesis.
6. If $\Xi = \mathtt{L}$ and $[M_1 \mid^s_\mathtt{L} M_2] \longrightarrow M[s]$. We remark that by Lemma 46 we have a proof of $\Phi; \Gamma \vdash [M_1 \mid^s_\mathtt{L} M_2] : A$ ending with

$$\frac{\Phi; \Gamma \vdash \mathtt{L} \rightsquigarrow \mathtt{L} \quad \Phi; \Gamma \vdash M_1[s] : A \quad \Phi; \Gamma \vdash M_2[s] : A}{\Phi; \Gamma \vdash [M_1 \mid^s_\mathtt{L} M_2] : A}$$

and then the result holds
7. If $\Xi = \mathtt{R}$ and $[M_1 \mid^s_\mathtt{R} M_2] \longrightarrow M[s]$. We reason as when $\Xi = \mathtt{L}$
8. if $\Xi = \xi$ and $[M_1 \mid^s_\xi M_2] \longrightarrow [M_1[s] \mid_\xi M_2[s]]$, we remark that by Lemma 46 we have a proof of $\Phi; \Gamma \vdash [M_1 \mid^s_\xi M_2] : A$ ending with

$$\frac{\Phi; \Gamma' \vdash \xi \rightsquigarrow \xi \quad \Phi; P : B, \Gamma' \vdash M_1[s] : A \quad \Phi; Q : B, \Gamma' \vdash M_2[s] : A}{\Phi; (P \mid_\xi Q) : B + C, \Gamma' \vdash [M_1 \mid^s_\xi M_2] : A}$$

where $\Gamma = (P \mid_\xi Q) : B + C, \Gamma'$ and then we obviously have a proof ending in:

$$\frac{\Phi; \Gamma' \vdash \xi \rightsquigarrow \xi \quad \Phi; P : B, \Gamma' \vdash M_1[s] : A \quad \Phi; Q : B, \Gamma' \vdash M_2[s] : A}{\Phi; (P \mid_\xi Q) : B + C, \Gamma' \vdash [M_1[s] \mid_\xi M_2[s]] : A}$$

– If $\Phi; \Gamma \vdash M[s] : A$ then by Corollary 48 we can suppose that $\Gamma = Dec(\Gamma)$ without lost of generality. There is a proof of $\Phi; \Gamma \vdash M[s] : A$ ending in:

$$\frac{\Phi; \Gamma \vdash s \rhd \Phi'; \Gamma\Gamma' \quad \Phi'; \Gamma\Gamma' \vdash M : A}{\Phi; \Gamma \vdash M[s] : A}$$

We reason now by cases on the reduction:
1. If $M \longrightarrow M'$ then by induction hypothesis $\Phi'; \Gamma\Gamma' \vdash M' : A$ and so the result holds applying the rule (Sub_term).
2. If $s \longrightarrow s'$ then by induction hypothesis $\Phi; \Gamma \vdash s' \rhd \Phi'; \Gamma\Gamma'$ and so the result holds.
3. If $M = M_1\ M_2$ and $M[s] \longrightarrow M_1[s]\ M_2[s]$ then there is a proof of $\Phi'; \Gamma\Gamma' \vdash M : A$ ending in

$$\frac{\Phi'; \Gamma\Gamma' \vdash M_1 : C \to A \quad \Phi'; \Gamma\Gamma' \vdash M_2 : C}{\Phi'; \Gamma\Gamma' \vdash M_1\ M_2 : A}$$

Thus, since we know that $FV(M_1)$ and $FV(M_2)$ don't contain sum variable (because $FV(M_1\ M_2)$ don't, we have a proof of $\Phi; \Gamma \vdash M_1[s] : C \to A$ ending in:

$$\frac{\Phi; \Gamma \vdash s \rhd \Phi'; \Gamma\Gamma' \quad \Phi'; \Gamma\Gamma' \vdash M_1 : C \to A}{\Phi; \Gamma \vdash M_1[s] : c \to A}$$

and a proof of $\Phi;\Gamma \vdash M_2[s]:C$ ending in:

$$\frac{\Phi;\Gamma \vdash s \triangleright \Phi';\Gamma\Gamma' \quad \Phi';\Gamma\Gamma' \vdash M_2:C}{\Phi;\Gamma \vdash M_2[s]:C}$$

and thus we have a prof of $\Phi;\Gamma \vdash (M_1[s]\ M_2[s]):A$ ending in

$$\frac{\Phi;\Gamma \vdash M_1[s]:c \to A \quad \Phi;\Gamma \vdash M_2:C}{\Phi;\Gamma \vdash (M_1[s]\ M_2[s]):A}$$

4. If $M = \mathtt{inl}_C(M_1)$, $M = \mathtt{inr}_C(M_1)$ or $M = \langle M_1, M_2\rangle$ we reason as in the case of $M = M_1\ M_2$
5. If $M = [M_1 \mid_\xi M_2]$ and $M[s] \longrightarrow\ [M_1 \mid^s_{\xi[s]} M_2]$. By Lemma 45 there are two possibilities :
   - There is a proof of $\Phi';\Gamma\Gamma' \vdash [M_1 \mid_\xi M_2] : A$ ending with the rule $(case_1)$.
     We remark that then there is a proof of $\Phi';\Gamma\Gamma' \vdash [M_1 \mid_\xi M_2] : A$ ending in:

$$\frac{\Phi';\Gamma''' \vdash \xi \rightsquigarrow \xi \quad \Phi';P:B,\Gamma''' \vdash M_1:A \quad \Phi';Q:C,\Gamma''' \vdash M_22:A}{\Phi';(P \mid_\xi Q):B+C,\Gamma''' \vdash [M_1 \mid_\xi M_2]:A}$$

Where $(P \mid_\xi Q):B+C,\Gamma''' = \Gamma\Gamma'$.
By Remark 6, we know that $(P \mid_\xi Q) \in \Gamma$.
We have the following proof:

$$\frac{\Phi';\Gamma''\Gamma' \vdash \xi \rightsquigarrow \xi \quad \Phi';P:B,\Gamma''\Gamma' \vdash M_1:A \quad \Phi';Q:C,\Gamma''\Gamma' \vdash M_2:A}{\Phi';(P \mid_\xi Q):B+C,\Gamma''\Gamma' \vdash [M_1 \mid_\xi M_2]:A}$$

where $\Gamma = (P \mid_\xi Q):B+C,\Gamma''$.
We remark that $FV(s) \cap BVar((P \mid_\xi Q)) = \emptyset$. Let us suppose that $FV(s) \cap BVar((P \mid_\xi Q)) \neq \emptyset$ then obviously there should exists a minimal sub-term M of s such that $FV(M) \cap BVar((P \mid_\xi Q)) \neq \emptyset$. In that case in particular $\xi \in FV(M)$ and then $\xi \in FV(s)$ contradicting the fact that $FV(s)$ does not contain sum variables. Thus by Lemma 411 we have a proofs of $\Phi;\Gamma'' \vdash s \triangleright \Phi';\Gamma''\Gamma'$, $\Phi;P:B,\Gamma'' \vdash s \triangleright \Phi';P:B,\Gamma''\Gamma'$ and $\Phi;Q:C,\Gamma'' \vdash s \triangleright \Phi';Q:C,\Gamma''\Gamma'$. Then, we have the following proofs:

$$\frac{\Phi;\Gamma'' \vdash s \triangleright \Phi';\Gamma''\Gamma' \quad \Phi';\Gamma''\Gamma' \vdash \xi \rightsquigarrow \xi}{\Phi;\Gamma'' \vdash \xi[s] \rightsquigarrow \xi}$$

$$\frac{\Phi;P:B,\Gamma'' \vdash s \triangleright \Phi';P:B,\Gamma''\Gamma' \quad \Phi';P:B,\Gamma''\Gamma' \vdash M_1:A}{\Phi;P:B,\Gamma'' \vdash M_1[s]:A}$$

and

$$\frac{\Phi;Q:C,\Gamma'' \vdash s \triangleright \Phi';Q:C,\Gamma''\Gamma' \quad \Phi';Q:C,\Gamma''\Gamma' \vdash M_2:A}{\Phi;Q:C,\Gamma'' \vdash M_2[s]:A}$$

Thus the result holds applying the rule $(frozencase_1)$.

- There is a proof of $\Phi'; \Gamma\Gamma' \vdash [M_1 \mid_\xi M_2] : A$ ending with the rule $(case_2)$. Then we have a proof ending in:

$$\frac{\Phi'; \Gamma' \vdash \xi \rightsquigarrow \mathtt{K} \quad \Phi'; \Gamma' \vdash M_1 : A \quad \Phi'; \Gamma' \vdash M_2 : A}{\Phi' \Gamma' \vdash [M_1 \mid_\xi M_2] : A}$$

  And then we a proof of $\Phi; \Gamma \vdash [M_1 \mid^s_{\xi[s]} M_2] : A$ ending in:

$$\frac{\dfrac{\Phi; \Gamma s \triangleright \Phi'; \Gamma' \quad \Phi'; \Gamma' \vdash \xi \rightsquigarrow \mathtt{K}}{\Phi; \Gamma \vdash \xi[s] \rightsquigarrow \mathtt{K}} \quad \dfrac{\Phi; \Gamma s \triangleright \Phi'; \Gamma' \quad \Phi'; \Gamma' \vdash M_1 : A}{\Phi; \Gamma \vdash M_1[s] : A} \quad \dfrac{\Phi; \Gamma s \triangleright \Phi'; \Gamma' \quad \Phi'; \Gamma' \vdash M_2 : A}{\Phi; \Gamma \vdash M_2[s] : A}}{\Phi; \Gamma \vdash [M_1 \mid^s_\Xi M_2] : A}$$

- There is a proof of $\Phi; \Gamma \vdash [M_1 \mid_\xi M_2][s] \rightsquigarrow A$ ending with the rule $(+LEFT)$ and then the rule (Sub_term). Then, there is a proof of $M = [M_1 \mid_\xi M_2]$ ending in:

$$\frac{\Phi'; \Gamma'' \vdash \Xi \rightsquigarrow \xi \quad \Phi'; P : B, \Gamma'' \vdash M_1 : A \quad \Phi; Q : C, \Gamma'' \vdash M_2 : A}{\Phi'; (P \mid_\xi Q) : B + C, \Gamma'' \vdash [M_1 \mid_\xi M_2] : A}$$

  where $\Gamma' = (P \mid_\xi Q) : B + C, \Gamma''$ And then we reason as in the previous cases.

6. If $M = x$ then 5 sub-cases are possibles
   (a) If $s = id$ and $x[id] \longrightarrow x$, the result trivially holds.
   (b) If $s = (x/N).t$ and $x[(x/N).t] \longrightarrow N$ then there is a proof of $\Phi; \Gamma \vdash x[((x/N).t)] : A$ ending with:

$$\frac{\dfrac{\Phi; \Gamma \vdash N : A \quad \Phi; \Gamma \vdash t : \Phi'; \Gamma\Gamma'}{\Phi; \Gamma \vdash ((x/N).t) \triangleright \Phi'; x : A, \Gamma\Gamma'} \quad \Phi'; x : A, \Gamma\Gamma' \vdash x : A}{\Phi; \Gamma \vdash x[(x/N).t] : A}$$

   And thus we have a proof of $\Phi; \Gamma \vdash N : A$.
   (c) If $s = (y/N).t$ with $x \neq y$ and $x[(y/N).t] \longrightarrow x[t]$ then there is a proof of $\Phi; \Gamma \vdash x[((y/N).t)] : A$ ending in:

$$\frac{\dfrac{\Phi; \Gamma \vdash N : B \quad \Phi; \Gamma \vdash t : \Phi'; \Gamma'\Gamma}{\Phi; \Gamma \vdash ((y/N).t) \triangleright \Phi'; y : B, \Gamma'\Gamma} \quad \Phi'; y : B, \Gamma\Gamma' \vdash x : A}{\Phi; \Gamma \vdash x[(y/N).t] : A}$$

   By Lemma 411 there is a proof of $\Phi'; \Gamma\Gamma' \vdash x : A$ and thus the result holds applying the rule (Sub_term).
   (d) If $s = (\xi^{P:A}/\mathtt{K}).t$, $M = x$, $x \notin P$ and $x[s] \longrightarrow x[t]$ we reason as in the previous case.
   (e) If $s = (\xi^{P:A}/\mathtt{K}).t$, $M = x$, $x \in P$ and $x[s] \longrightarrow \bullet$ then the result obviously holds.
7. If $M = M_1[t]$ and $M[s] \longrightarrow M_1[t \circ s]$, then we have a proof of $\Phi; \Gamma \vdash (M_1[t])[s] : A$ ending in

$$\frac{\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma'\Gamma \quad \dfrac{\Phi'; \Gamma'\Gamma \vdash t \triangleright \Phi''; \Gamma'' \quad \Phi''; \Gamma'' \vdash M_1 : A}{\Phi'; \Gamma'\Gamma \vdash M_1[t] : A}}{\Phi; \Gamma \vdash (M_1[t])[s] : A}$$

   Then we have a proof of $\Phi; \Gamma \vdash t \circ s \triangleright \Phi''; \Gamma''$ ending in:

$$\frac{\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma'\Gamma \quad \Phi'; \Gamma'\Gamma \vdash \Phi''; \Gamma''}{\Phi; \Gamma \vdash t \circ s \triangleright \Phi''; \Gamma''}$$

   And thus applying the rule (Sub_term), the result holds.

– If $\Phi;\Gamma \vdash M_1\ M_2 : A$, then by lemma 44 there is a proof of it ending with the rule ($\rightarrow$ Left) and then we have:

$$\frac{\Phi;\Gamma \vdash M_1 : B \rightarrow A \qquad \Phi;\Gamma \vdash M_2 : B}{\Phi;\Gamma \vdash M_1\ M_2 : A}$$

We now reason by case:

- If $M_1 \longrightarrow M_1'$, by induction hypothesis we know that $\Phi;\Gamma \vdash M_1' : B \rightarrow A$ and then applying the rule ($\rightarrow$Left) that $\Phi;\Gamma \vdash M_1'\ M_2 : A$.
- If $M_2 \longrightarrow M_2'$, by induction hypothesis we know that $\Phi;\Gamma \vdash M_2' : B$ and then applying the rule ($\rightarrow$Left) that $\Phi;\Gamma \vdash M_1\ M_2' : A$.
- If $M_1 = \lambda P.N$ and $(M_1\ M_2) \longrightarrow (\lambda P.N)[id]\ M_2$ then we have a proof of $\Phi;\Gamma \vdash (\lambda P.N)[id]\ M_2 : A$ ending in:

$$\frac{\dfrac{\Phi;\Gamma \vdash id \rhd \Phi;\Gamma \quad \Phi;\Gamma \vdash M_1 : B \rightarrow A}{\Phi;\Gamma \vdash (\lambda P.N)[id] : B \rightarrow A} \qquad \Phi;\Gamma \vdash M_2 : A}{\Phi;\Gamma \vdash (\lambda P.N)[id]\ M_2 : A}$$

- If $M_1 = (\lambda\langle P_1, P_2\rangle : A_1 \times A_2.N_1)[s]$ and $M_2 = \langle N_2, N_3\rangle$ and $M_1\ M_2 \longrightarrow ((\lambda\ P_1 : A_1.\ \lambda\ P_2 : A_2.\ N_1)\ [\ s\ ]\ \ N_2)\ N_3$. Then $B = A_1 \times A_2$ and we have seen that we have a proof of $\Phi;\Gamma : (\lambda\langle P_1, P_2\rangle : A_1 \times A_2.N_1)[s]$. By Corollary 1 we can suppose that $\Gamma = Dec(\Gamma)$. By Lemma 48 we then have a proof of it of the form:

$$\frac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma' \qquad \Phi';\Gamma' \vdash \lambda\langle P_1, P_2\rangle : A_1 \times A_2.N_1 : (A_1 \times A_2) \rightarrow A}{\Phi;\Gamma \vdash (\lambda\langle P_1, P_2\rangle : A_1 \times A_2.N_1)[s] : (A_1 \times A_2) \rightarrow A}$$

By Lemma 44 again we have a proof of $\Phi;\Gamma \vdash \langle N_2, N_3\rangle : A_2 \times A_3$ of the form:

$$\frac{\Phi;\Gamma \vdash N_2 : A_1 \qquad \Phi;\Gamma \vdash N_3 : A_2}{\Phi;\Gamma \vdash \langle N_2, N_3\rangle : A_1 \times A_3}$$

By Lemma 44 we have a proof of $\Phi';\Gamma' \vdash \lambda\langle P_1, P_2\rangle : A_1 \times A_2.N_1 : (A_1 \times A_2) \rightarrow A$ of the form:

$$\frac{\Phi';\langle P_1, P_2\rangle : A_1 \times A_2, \Gamma' \vdash N_1 : A}{\Phi';\Gamma' \vdash \lambda\langle P_1, P_2\rangle : A_1 \times A_2.N_1 : (A_1 \times A_2) \rightarrow A}$$

By Corollary 1 we then have a proof of $\Phi'; P_1 : A_1, P_2 : A_2, \Gamma' : N_1$. So we have a proof of the form:

$$\frac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma' \qquad \dfrac{\dfrac{\Phi'; P_1 : A_1, P_2 : A_2, \Gamma' \vdash N_1 : A}{\Phi'; P_1 : A_1, \Gamma' \vdash \lambda P_2 : A_2.N_1 : A_1 \rightarrow A}}{\Phi';\Gamma' \vdash \lambda P_1 : A_1.\lambda P_2 : A_2.N_1 : A_1 \rightarrow A_2 \rightarrow A}}{\Phi;\Gamma \vdash (\lambda P_1 : A_1.\lambda P_2 : A_2.N_1)[s] : A_1 \rightarrow A_2 \rightarrow A}$$

And then we have a proof of:

$$\frac{\dfrac{\Phi;\Gamma \vdash (\lambda P_1 : A_1.\lambda P_2 : A_2.N_1)[s] : A_1 \rightarrow A_2 \rightarrow A \qquad \Phi;\Gamma \vdash N_2 : A_1}{\Phi;\Gamma \vdash (\lambda P_1 : A_1.\lambda P_2 : A_2.N_1)[s]\ N_2 : A_2 \rightarrow A} \qquad \Phi;\Gamma \vdash N_3 : A_2}{\Phi;\Gamma \vdash ((\lambda P_1 : A_1.\lambda P_2 : A_2.N_1)[s]\ N_2)\ N_3 : A}$$

- If $M_1 = (\lambda @(P_1,P_2)\!:\!B.N)[s]$ and $M_1\ M_2 \longrightarrow\ (\lambda P_1\!:\!B.\lambda P_2\!:\!B.N[s]\ N)\ N$. By Corollary 1 we can suppose that $\Gamma = Dec(\Gamma)$. By Lemma 48 we have a proof of $\Gamma;\Phi \vdash (\lambda @(P_1,P_2)\!:\!B.N)[s]\!:\!B \to A$ ending with:

$$\frac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma' \qquad \Phi';\Gamma' \vdash \lambda @(P_1,P_2)\!:\!B.N\!:\!B \to A}{\Phi;\Gamma \vdash (\lambda @(P_1,P_2)\!:\!B.N)[s]\!:\!B \to A}$$

By Lemma 44 we have a proof of $\Phi';\Gamma' \vdash \lambda @(P_1,P_2) : B.N : B \to A$ ending with:

$$\frac{\Phi';@(P_1,P_2)\!:\!B,\Gamma' \vdash N\!:\!A}{\Phi';\Gamma' \vdash \lambda @(P_1,P_2)\!:\!B.N\!:\!B \to A}$$

By Corollary 1 we then have a proof of $\Phi';P_1\!:\!B,P_2\!:\!B,\Gamma' \vdash N\!:\!A$. So we have a proof of:

$$\frac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma' \qquad \dfrac{\dfrac{\Phi';P_1\!:\!B,P_2\!:\!B,\Gamma' \vdash N\!:\!A}{\Phi';P_1\!:\!B,\Gamma' \vdash \lambda P_2\!:\!B.B\!:\!B \to A}}{\Phi';\Gamma' \vdash \lambda P_1\!:\!B.\lambda P_2\!:\!B.N\!:\!B \to B \to A}}{\Phi;\Gamma \vdash (\lambda P_1\!:\!B.\lambda P_2\!:\!B.N_1)[s]\!:\!B \to B \to A}$$

And then we have a proof of:

$$\frac{\dfrac{\Phi;\Gamma \vdash (\lambda P_1\!:\!B.\lambda P_2\!:\!B.N)[s]\!:\!B \to B \to A \qquad \Phi;\Gamma \vdash M_2\!:\!B}{\Phi;\Gamma \vdash (\lambda P_1\!:\!B.\lambda P_2\!:\!B.N)[s]\ N\!:\!B \to A} \qquad \Phi;\Gamma \vdash M_2\!:\!B}{\Phi;\Gamma \vdash ((\lambda P_1\!:\!B.\lambda P_2\!:\!B.N)[s]\ M_2)\ M_2\!:\!A}$$

- If $M_1 = (\lambda(P_1\ |_\xi\ P_2) : B_1 + B_2.N_1)[s]$ and $M_2 = \mathtt{inl}_{B_2}(N_2)$ and $M_1\mathrm{M}_2 \longrightarrow\ (\lambda P_1 : A.N_1)[(\xi^{P_2:B_2}/\mathtt{L}).s]\ N_2$. By Corollary 1, we can suppose that $\Gamma = Dec(\Gamma)$. By Lemma 48, we have a proof of $\Gamma;\Phi \vdash (\lambda(P_1\ |_\xi\ P_2)\!:\!B_1 + B_2.N_1)[s]\!:\!(B_1+B_2) \to A$ ending with:

$$\frac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma' \qquad \Phi';\Gamma' \vdash \lambda(P_1\ |_\xi\ P_2)\!:\!B_1+B_2.N_1\!:\!(B_1+B_2) \to A}{\Phi;\Gamma \vdash (\lambda(P_1\ |_\xi\ P_2)\!:\!B.N_1)[s]\!:\!B \to A}$$

By Lemma 44 we have a proof of $\Phi';\Gamma' \vdash \lambda(P_1\ |_\xi\ P_2) : B_1 + B_2.N_1 : (B_1+B_2) \to A$ ending with:

$$\frac{\Phi';(P_1\ |_\xi\ P_2)\!:\!B_1+B_2,\Gamma' \vdash N_1\!:\!A}{\Phi';\Gamma' \vdash \lambda(P_1\ |_\xi\ P_2)\!:\!B_1+N_2.N_1\!:\!(B_1+B_2) \to A}$$

By Lemma 44 again we have a proof of $\Phi;\Gamma \vdash \mathtt{inl}_{B_2}(N_2)$ ending with:

$$\frac{\Phi;\Gamma \vdash N_2\!:\!B_1}{\Phi;\Gamma \vdash \mathtt{inl}_{B_2}(N_2)\!:\!B_1+B_2}$$

Now by Lemma 410 we have a proof of $\xi\!:\!\mathtt{L},\Phi';P_1\!:\!B_1,Q_2\!:\!B_2;\Gamma' \vdash N_1 : A$. So we have a proof of:

$$\frac{\dfrac{\Phi;\Gamma \vdash s \rhd \Phi';\Gamma'}{\Phi;\Gamma \vdash (\xi^{P_2:B_2}/\mathtt{L}).s \vdash \xi\!:\!\mathtt{L},\Phi';P_2\!:\!B_2,\Gamma'} \qquad \dfrac{\xi\!:\!\mathtt{L},\Phi';P_1\!:\!B_1,P_2\!:\!B_2,\Gamma' \vdash N_1\!:\!A}{\xi\!:\!\mathtt{L},\Phi';P_2\!:\!B_2,\Gamma' \vdash \lambda P_1\!:\!B_1.N_1\!:\!B_1 \to A}}{\Phi;\Gamma \vdash (\lambda P_1\!:\!B_1.N_1)[(\xi^{P_2:B_2}/\mathtt{L}).s]\!:\!B_1 \to A}$$

And then we have a proof of:

$$\frac{\Phi;\Gamma \vdash (\lambda P_1\!:\!B_1.N_1)[(\xi^{P_2:B_2}/\mathtt{L}).s]\!:\!B_1 \to A \qquad \Phi;\Gamma \vdash N_2\!:\!B_1}{\Phi;\Gamma \rhd \Phi;\Gamma \vdash (\lambda P_1\!:\!B_1.N_1)[(\xi^{P_2:B_2}/\mathtt{L}).s]\ N_2\!:\!A}$$

- If $M_1 = (\lambda(P_1 \mid_\xi P_2) : B_1 + B_2.N_1)[s]$ and $M_2 = \mathtt{inr}_{B_1}(N_2)$ the proof is of the same type.
- If $M_1 = (\lambda x : B.N)[s]$ and $M_1\ M_2 \longrightarrow\ N[(x/M_2).s]$. By Corollary 1, we can suppose that $\Gamma = Dec(\Gamma)$. By Lemma 48 we have a proof of $\Phi; \Gamma \vdash (\lambda x : B.N)[s] : B \to A$ ending with:

$$\frac{\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma' \qquad \Phi'; \Gamma' \vdash \lambda x : B.N : B \to A}{\Phi; \Gamma \vdash (\lambda x : B.N)[s] : B \to A}$$

  By Lemma 44 again we have a proof of $\Phi'; \Gamma' \vdash \lambda x : B.N : B \to A$ ending with:

$$\frac{\Phi'; x : B, \Gamma' \vdash N : A}{\Phi'; \Gamma' \vdash \lambda x : B.N : B \to A}$$

  So we have a proof of:

$$\frac{\dfrac{\Phi; \Gamma \vdash M_2 : B \qquad \Phi; \Gamma \vdash s \Phi'; \Gamma'}{\Phi; \Gamma \vdash (x/M_2).s \triangleright \Phi'; x : B, \Gamma'} \qquad \Phi'; x : B, \Gamma' \vdash N : A}{\Phi; \Gamma \vdash N[(x/M_2).s] : A}$$

- If $M_1 = (\lambda\_ : B.N)[s]$ and $M_1\ M_2 \longrightarrow\ N[s]$, then by Corollary 1, we can suppose that $\Gamma = Dec(\Gamma)$ and by Lemma 48 we have a proof of $\Phi; \Gamma \vdash (\lambda\_ : B.N)[s] \vdash B \to A$ ending with:

$$\frac{\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma' \qquad \dfrac{\dfrac{\Phi'; \Gamma' \vdash N : A}{\Phi'; \_ : B, \Gamma' \vdash N : A}}{\Phi'; \Gamma' \vdash \lambda\_ : B.N : B \to A}}{\Phi; \Gamma \vdash (\lambda\_ : B.N)[s] \vdash B \to A}$$

  And then we have a proof of :

$$\frac{\Phi; \Gamma \vdash s \triangleright \Phi'; \Gamma' \qquad \Phi'; \Gamma' \vdash N : A}{\Phi; \Gamma \vdash N[s] : A}$$

5 A proof of strong normalization of $\lambda_{\sigma w}$

We will now make a pause in the this paper. In that pause we will show the strong normalization of the weak named version of the very well-known λ_σ calculus [ACCL90] which we will note $\lambda_{\sigma w}$[**?**].

The reason why we give here this proof is that we will use the same technique to show λP_w strong normalization and that its presentation is simpler for $\lambda_{\sigma w}$ that for λP_w. This proof is an adaptation of a proof due to Ritter [Rit94] for a restricted version of a de Bruijn version of λ_σ where substitutions can only cross the leftmost outermost lambda. The main differences with Ritter's proof is that Ritter used a calculus modulo a certain set of equations. As far as we know this proof is the first proof of typed $\lambda_{\sigma w}$ in its named version.

The scheme of the proof is the following one:

1. We (re-)define $\lambda_{\sigma w}$ and it's typing system and remark that this version of $\lambda_{\sigma w}$ has the subject reduction property.
2. We define a new calculus ($\lambda_{\sigma w/\equiv}$) by replacing a strongly normalizing subset of the reduction rules of $\lambda_{\sigma w}$ by the corresponding equations on $\lambda_{\sigma w}$-terms.
3. We show the strong normalization for $\lambda_{\sigma w/\equiv}$ well typed terms.
4. By a technical Lemma we deduce the strong normalization of well typed $\lambda_{\sigma w}$ terms.

5.1 The calculus $\boldsymbol{\lambda_{\sigma w}}$

Syntax Types

$$\begin{array}{lll} A ::= \iota & & \text{Base types} \\ \quad \mid A \to A & & \text{Functional types} \end{array}$$

Substitutions

$$\begin{array}{ll} s ::= id & \text{Identity} \\ \quad \mid (x/M).s & \text{Cons} \\ \quad \mid s \circ t & \text{Concatenation} \end{array}$$

Terms

$$\begin{array}{ll} M ::= x & \text{Variable} \\ \quad \mid (M\ M) & \text{Application} \\ \quad \mid \lambda x{:}A.M & \text{Abstraction} \\ \quad \mid M[s] & \text{Substitution} \end{array}$$

We may omit the parenthesis in an application term if they are clear from the context.

Expressions ($\lambda_{\sigma w}$-terms)

$$\begin{array}{ll} e ::= M & \text{Terms} \\ \quad \mid s & \text{Substitutions} \end{array}$$

Reduction rules

$$\begin{array}{llll}
(\lambda x.M)[s]\ N & \longrightarrow & M[(x/N).s] & (Abs_var_1) \\
\lambda x.M\ N & \longrightarrow & (\lambda x.M)[id]\ N & (Abs_var_2) \\
(M\ N)[s] & \longrightarrow & M[s]N[s] & (Sub_app) \\
(s \circ t) \circ u & \longrightarrow & s \circ (t \circ u) & (Sub_ass_env) \\
x[id] & \longrightarrow & x & (Sub_var_1) \\
x[(x/N).s] & \longrightarrow & N & (Sub_var_2) \\
y[(x/N).s] & \longrightarrow & y[s] \text{ if } y \neq x & (subvarthree) \\
M[s][t] & \longrightarrow & M[s \circ t] & (Sub_clos) \\
((x/M).s) \circ t & \longrightarrow & (x/M[t]).(s \circ t) & (Sub_concat) \\
id \circ s & \longrightarrow & s & (Sub_id)
\end{array}$$

Notation 51 For the rest of this section we will note this reduction relation $\longrightarrow_{\lambda_{\sigma w}}$ when a simple $\longrightarrow$ may be ambiguous.

Typing rules

Definition 51 (Environment) An *environment* is a set of pairs of the form $x : A$ where x is a variable and A is a type.

$$\frac{\text{the } x_i \text{ are distinct variables}}{x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i} \quad \text{(Proj)}$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M\ N) : B} \quad \text{(App)}$$

$$\frac{x : A, \Gamma \vdash M : B}{\Gamma \vdash \lambda x : A.M : A \to B} \quad (\to \text{Right})$$

$$\frac{\Gamma \vdash s \triangleright \Gamma' \qquad \Gamma' \vdash M : A}{\Gamma \vdash M[s] : A} \quad \text{(Sub)}$$

$$\frac{}{\Gamma \vdash id \triangleright \Gamma} \quad \text{(Sub\_axiom)}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash s \triangleright \Gamma'}{\Gamma \vdash (x/M).s \triangleright x : A, \Gamma'} \quad \text{(Sub\_cons)}$$

$$\frac{\Gamma \vdash t \triangleright \Gamma' \qquad \Gamma' \vdash s \triangleright \Gamma''}{\Gamma \vdash s \circ t \triangleright \Gamma''} \quad \text{(Sub\_concat)}$$

A term M (resp. a substitution s) is said to be *well-typed* if there exist a type A (resp. an environment Δ) and an environment Γ such that $\Gamma \vdash M : A$ (resp. $\Gamma \vdash s \triangleright \Delta$) can be derived from the previous set of typing rules. A term M is said to be *of type* A if there is an environment Γ such that $\Gamma \vdash M : A$ can be derived from the previous set of typing rules.

Remark 8. We remark that the calculus $\longrightarrow_{\lambda_{\sigma w}}$ is a sub-calculus of $\lambda P_{\sigma w}$, in the sense that if we restrict the syntax of $\lambda P_{\sigma w}$ to that of $\lambda_{\sigma w}$, then

- $a \longrightarrow_{\lambda_{\sigma w}} b$ if and only if $a \longrightarrow_{\lambda P_{\sigma w}} b$ and
- $\Gamma \vdash a : A$ in $\lambda_{\sigma w}$ if and only if $\Gamma \vdash a : A$ in $\lambda P_{\sigma w}$.

And then, from Lemma 412 we have the subject reduction property for $\lambda_{\sigma w}$.

Lemma 52

1. If $\Gamma \vdash M : A$, then $\Gamma, x : B \vdash M : A$.
2. If $\Gamma \vdash s \triangleright \Delta$ and $\Gamma \vdash s \triangleright \Delta'$, then $\Delta = \Delta'$.

Definition 52 (Void substitutions) The set of *void substitutions* is defined to be the least set of substitutions stable by concatenation and containing id

Lemma 53 If $\Delta \vdash s \triangleright \emptyset$, then $\Delta = \emptyset$ and s is a void substitution.

Proof. By induction on the height of the typing derivation of $\Delta \vdash s \triangleright \emptyset$.

Definition 53 For any strongly normalizing expression e we note $\nu(e)$ the length of the longest reduction sequence starting at e.

5.2 Definition of $\boldsymbol{\lambda_{\sigma w/\equiv}}$

Definition 54 The congruence $\equiv$ on $\lambda_{\sigma w}$-terms is defined to be the least reflexive, symmetric and transitive relation closed under contexts and substitutions and containing the axioms

$$(M\ N)[s] =_{\text{Sub\_app}} M[s]\ N[s] \tag{4}$$
$$(s \circ t) \circ u =_{\text{Sub\_ass\_env}} s \circ (t \circ u) \tag{5}$$
$$M[s][t] =_{\text{Sub\_clos}} M[s \circ t] \tag{6}$$

We will consider in this section the reduction system $\lambda_{\sigma w/\equiv}$, where $a \longrightarrow_{\lambda_{\sigma w/\equiv}} b$ if and only if there exist a', b' such that

$$a \equiv a' \longrightarrow_{\mathcal{R}} b' \equiv b, \text{where } \mathcal{R} = \lambda_{\sigma w} \setminus \{\text{Sub\_app,Sub\_ass\_env,Sub\_clos}\}$$

This definition can also be interpreted as a reduction relation on `equivalence classes`, i.e., $[a] \longrightarrow_{\lambda_{\sigma w/\equiv}} [b]$ if and only if $a \equiv a' \longrightarrow_{\mathcal{R}} b' \equiv b$. We may use both interpretations according to the context.

Remark 9. By subject reduction 412, there is no problem to define well-typed $\lambda_{\sigma w/\equiv}$-terms.

5.3 Strong normalization for $\boldsymbol{\lambda_{\sigma w/\equiv}}$

For the rest of this section we will denote by $\longrightarrow$ the $\lambda_{\sigma w/\equiv}$ reduction relation.

Lemma 54 If s is a void substitution, then s is $\longrightarrow$ -strongly normalizing.

Proof. By equation (5), there is only two possibilities:

- $s = id$, then s is in $\longrightarrow$ -normal form so it is $\longrightarrow$ -strongly normalizing.
- $s = id \circ s'$ where s' is a void substitution. Then it is easy to show that s is strongly normalizing by induction on the number of concat in s.

Definition 55 (Neutral terms and substitutions)

- A term M is *neutral* if and only if it is neither of the form $(\lambda x.N)[s]$ nor $\lambda x.N$.
- A substitution s is *neutral* if and only if it is not of the form $(x/M).t$.

Definition 56 (Reducible terms and substitutions) The set of *reducible terms for a type in an environment Γ* is defined by induction on types as follows:

$[\![\iota]\!]_{\Gamma} =_{def} \{M \mid \Gamma \vdash M : \iota$ and M is strongly normalizing$\}$.
$[\![A \to B]\!]_{\Gamma} =_{def} \{M \mid \Gamma \vdash M : A \to B$ and $\forall \Delta$ such that $\forall N \in [\![A]\!]_{\Gamma\Delta}, (M\ N) \in [\![B]\!]_{\Gamma\Delta}\}$

The set of *reducible substitutions for an environment* Γ *in an environment* Δ is defined as follows:

$$[\![\Gamma]\!]_\Delta =_{def} \{s \mid \Delta \vdash s \rhd \Gamma \text{ and } \forall (x:A) \in \Gamma, x[s] \in [\![B]\!]_\Delta\}$$

Remark 10. We remark that, by definition, any substitution s such that $\Delta \vdash s \rhd \emptyset$ is in $[\![\emptyset]\!]_\Delta$.

Corollary 2. *Void substitutions are in* $[\![\emptyset]\!]_\emptyset$.

Notation 55

- For any term M we say that M is reducible if there is a type A and a environment Γ such that $M \in [\![A]\!]_\Gamma$.
- For any substitution s we say that s is reducible if there are two environments Γ and Δ such that $s \in [\![\Gamma]\!]_\Delta$.

Remark 11. In the rest of this section we may omit environments if they are clear from the context or if they are not necessary in the statements.

Lemma 56

1. If $M \in [\![C]\!]_\Gamma$, then M is strongly normalizing.
2. If $\Gamma \vdash (xM_1 \dots M_n) : C$ and $M_1 \dots M_n$ are strongly normalizing, then $(xM_1 \dots M_n) \in [\![C]\!]_\Gamma$.
3. If $M \in [\![C]\!]_\Gamma$ and $M \longrightarrow M'$ then $M' \in [\![C]\!]_\Gamma$.
4. If M is a neutral of type C and all its one-step reducts are reducible expressions, then M is reducible.

Proof. We first show the four properties for terms by induction on the type C.
Base case ι:

1. By definition of $[\![\iota]\!]_\Gamma$.
2. Since $M_1 \dots M_n$ are strongly normalizing, then the only reduction sequences starting at $(xM_1 \dots M_n)$ proceed independently in the terms $M_i's$, and all these reduction sequences terminate. As a consequence, the term $(xM_1 \dots M_n)$ is strongly normalizing and thus by definition $(xM_1 \dots M_n) \in [\![\iota]\!]_\Gamma$.
3. Let M be in $[\![\iota]\!]_\Gamma$. By definition $\Gamma \vdash M : \iota$ and M is strongly normalizing. Then, any M' such that $M \longrightarrow M'$ is also strongly normalizing. By subject reduction (Theorem 412) $\Gamma \vdash M' : \iota$ and thus by definition M' is in $[\![\iota]\!]_\Gamma$.
4. Let M be a well-typed (*i.e.* $\Gamma \vdash M : \iota$) and neutral term such that all its one-step reducts are in $[\![\iota]\!]_\Gamma$. By definition all these one-steps reducts are strongly normalizing so that M is strongly normalizing and thus by definition M is in $[\![\iota]\!]_\Gamma$.

Inductive case

1. Let M be in $[\![A \to B]\!]_\Gamma$. By induction hypothesis (property 2 with $n = 0$) there is a fresh variable x of type A in $[\![A]\!]_{\Gamma,xA}$ so that by definition of $[\![A \to B]\!]_\Gamma$, the term $(M\ x)$ is in $[\![B]\!]_{\Gamma,xA}$. By induction hypothesis (property 1) on B, $(M\ x)$ is strongly normalizing and then so is M.
2. Let $(x\ M_1\ \dots\ M_n)$ be a term such that $\Gamma \vdash (x\ M_1\ \dots\ M_n) : A \to B$ and $M_1, \dots, M_n$ are strongly normalizing. Let N be *any* term in $[\![A]\!]_{\Gamma\Delta}$. By induction hypothesis (property 1) we know that N is strongly normalizing, and by Lemma 412 $\Gamma\Delta \vdash (xM_1 \dots M_n N) : B$. As a consequence, by induction hypothesis (property 2) we have that $(xM_1 \dots M_n N)$ is in $[\![B]\!]_{\Gamma\Delta}$, and thus by definition $(xM_1 \dots M_n)$ is in $[\![A \to B]\!]_\Gamma$.
3. Let M be in $[\![A \to B]\!]_\Gamma$ and let consider the reduction step $M \longrightarrow M'$. By subject reduction (Theorem 412) $\Gamma \vdash M' : A \to B$. Take any term $N \in [\![A]\!]_{\Gamma\Delta}$. By definition of $[\![A \to B]\!]_\Gamma$, $(MN) \in [\![B]\!]_{\Gamma\Delta}$. Since $(MN) \longrightarrow (M'N)$, then by induction hypothesis (property 3) on B, we have that $(M'N) \in [\![B]\!]_{\Gamma\Delta}$. And then $M' \in [\![A \to B]\!]_\Gamma$.
4. Let M be a neutral term such that $\Gamma \vdash M : A \to B$ and all its one-step reducts are in $[\![A \to B]\!]_\Gamma$. Let N be in $[\![A]\!]_{\Gamma\Delta}$. We have to show that $(MN) \in [\![B]\!]_{\Gamma\Delta}$. Since (MN) is neutral and $\Gamma\Delta \vdash (M\ N) : B$, then by induction hypothesis (property 4) it is sufficient to show that all its one-step reducts are in $[\![B]\!]_{\Gamma\Delta}$. By induction hypothesis (property 1) we know that N is strongly normalizing, so we can reason by induction on $\nu(N)$ as follows:
   The one-steps reducts of $(M\ N)$ are:
   - $(M\ N')$, with $N \longrightarrow N'$. Then $\nu(N') < \nu(N)$ and the property holds by induction hypothesis.
   - $(M'\ N)$, with $M \longrightarrow M'$. Since M' is in $[\![A \to B]\!]_\Gamma$ by hypothesis, then $(M'\ N)$ is in $[\![B]\!]_{\Gamma\Delta}$ by definition.
   - There is no other possible case since M is neutral.

   We can then conclude that $M \in [\![A \to B]\!]_\Gamma$.

Lemma 57

1. If $s \in [\![\Gamma]\!]_\Delta$, then s is strongly normalizing.
2. If $s \in [\![\Gamma]\!]_\Delta$ and $s \longrightarrow s'$ then $s' \in [\![\Gamma]\!]_\Delta$.
3. If s is a neutral substitution such that $\Delta \vdash s \triangleright \Gamma$ and all its one-step reducts are reducible expressions, then $s \in [\![\Gamma]\!]_\Delta$.

Proof. We prove the properties by cases on Γ.

- Let us suppose $\Gamma = \emptyset$. Then by Lemma 53, Δ is also empty and the only substitutions in $[\![\emptyset]\!]_\emptyset$ are void substitutions. By Lemma 54 void substitutions are strongly normalizing so that Property 1 holds.
  Now, if $s \in [\![\emptyset]\!]_\emptyset$ and $s \longrightarrow s'$, by the subject reduction property (Theorem 412) $\emptyset \vdash s' \triangleright \emptyset$ so that by Remark 10 $s' \in [\![\emptyset]\!]_\emptyset$ and thus Property 2 also holds.
  To show the third property suppose that s is neutral and $\emptyset \vdash s \triangleright \emptyset$. Then by definition $s \in [\![\emptyset]\!]_\emptyset$ and thus Property 3 also holds.
- Let us now suppose $\Gamma \neq \emptyset$.

1. Let s be in $[\![\Gamma]\!]_\Delta$. Take $(x\!:\!A) \in \Gamma$. Then, by definition of $[\![\Gamma]\!]_\Delta$, the term $x[s]$ is in $[\![A]\!]_\Delta$. Since the properties hold for terms, then $x[s]$ is strongly normalizing and then s is strongly normalizing.
2. Let s be in $[\![\Gamma]\!]_\Delta$. Then $\Delta \vdash s \rhd \Gamma$. Take $(x\!:\!A) \in \Gamma$ and let s' such that $s \longrightarrow s'$. By definition of $[\![\Gamma]\!]_\Delta$, $x[s] \in [\![A]\!]_\Delta$ and since the properties hold for terms, then $x[s'] \in [\![A]\!]_\Delta$. By subject reduction (Theorem 412) $\Delta \vdash s' \rhd \Gamma$ so that by definition of $[\![\Gamma]\!]_\Delta$, $s' \in [\![\Gamma]\!]_\Delta$.
3. Let s be a well-typed (*i.e.* $\Delta \vdash s \rhd \Gamma$) and neutral substitution and let us suppose that all the one-step reducts of s are in $[\![\Gamma]\!]_\Delta$. Take $(x : A)$ in Γ. Since s is neutral, $x[s]$ may either reduce to $x[s']$ with $s \longrightarrow s'$, or to x (if $s = id$). In the first case we have that $\Delta \vdash s' \rhd \Gamma$ holds by subject reduction (Theorem 412) and s' is reducible by hypothesis, so that $x[s'] \in [\![A]\!]_\Delta$ by definition and we are done. In the second case, we have to show that $x \in [\![A]\!]_\Delta$, but Δ must be equal to Γ since $\Delta \vdash s \rhd \Gamma$ so that $x \in [\![A]\!]_\Gamma$ holds by Lemma 56 (property 2). As a consequence $s \in [\![\Gamma]\!]_\Delta$.

Now we will state (and prove) some Lemmas which prove the reducibility of certain expressions given the reducibility of some of their reducts.

Lemma 58 Let Δ be a environment. Let M be in $[\![A]\!]_\Delta$ and s be in $[\![\Gamma]\!]_\Delta$. If x is a fresh variable, then $t = (x/M).s \in [\![x\!:\!A, \Gamma]\!]_\Delta$.

Proof. First of all we remark that $\Delta \vdash t \rhd x\!:\!A, \Gamma$. To prove that t is in $[\![x\!:\!A, \Gamma]\!]_\Delta$ it is sufficient to prove that $\forall (y\!:\!B) \in (x\!:\!A, \Gamma)$ we have that $y[t]$ is in $[\![B]\!]_\Delta$.

Take $(y : B) \in (x : A, \Gamma)$. Then $y[t]$ is neutral. Now, if we show that all its one-step reducts are in $[\![B]\!]_\Delta$ we may conclude that $y[t]$ is in $[\![B]\!]_\Delta$ by property 4 of Lemma 56. Now, since M and s are respectively in $[\![A]\!]_\Delta$ and $[\![\Gamma]\!]_\Delta$, then they are strongly normalizing by Lemma 56 and 57 and thus we may proceed by induction on $\nu(M) + \nu(s)$.

Now, the one-step reducts of $y[t]$ are:

– $y[(x/M').s]$ with $M \longrightarrow M'$. Then we conclude by induction hypothesis.
– $y[(x/M).s']$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
– M if $x = y$. Then we conclude by hypothesis.
– $y[s]$ if $x \neq y$. Then we conclude by hypothesis.

Lemma 59 If $s \circ t \in [\![\Gamma]\!]_\Delta$, $M[t] \in [\![A]\!]_\Delta$, and if x is a fresh variable, then $u = ((x/M).s) \circ t$ is in $[\![(x\!:\!A, \Gamma)]\!]_\Delta$.

Proof. First of all we have to prove that u is well typed in Δ. But $\Delta \vdash s \circ t \rhd \Gamma$ implies that there exists Γ' such that $\Delta \vdash t \rhd \Gamma'$, $\Gamma' \vdash s \rhd \Gamma$, and $\Delta \vdash M[t]\!:\!A$. On the other hand, Lemma 52 implies $\Delta \vdash t \rhd \Gamma'$ and $\Gamma' \vdash M\!:\!A$, so that

$$\frac{\Delta \vdash t \rhd \Gamma' \qquad \dfrac{\Gamma' \vdash M\!:\!A \qquad \Gamma' \vdash s \rhd \Gamma}{\Gamma' \vdash (x/M).s \rhd x\!:\!A, \Gamma}}{\Delta \vdash ((x/M).s) \circ t \rhd x\!:\!A, \Gamma}$$

It suffices to prove that for all $(y:B) \in (x:A,\Gamma)$, $y[u]$ is in $[\![B]\!]_\Delta$. Let $y:B$ be in $(x:A,\Gamma)$. Since the expressions $M[t]$ and $s \circ t$ are strongly normalizing by Lemma 56 and 57, we can prove the property by induction on $\nu(M[t])+\nu(s\circ t)$.

Now, the term $y[u]$ is neutral so it is sufficient to prove that all the one-step reducts of $y[u]$ are in $[\![B]\!]_\Delta$. This reducts are:

- $y[((x/M').s)\circ t]$ with $M \longrightarrow M'$. Then we conclude by induction hypothesis.
- $y[((x/M).s')\circ t]$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
- $y[((x/M).s)\circ t']$ with $t \longrightarrow t'$. Then we conclude by induction hypothesis.
- $y[(x/M[t]).(s\circ t)]$. Since $M[t]$ and $s\circ t$ are respectively in $[\![A]\!]_\Delta$ and $[\![\Gamma]\!]_\Delta$ by hypothesis, then the substitution $(x/M[t]).(s\circ t)$ is in $[\![x:A,\Gamma]\!]_\Delta$ by Lemma 58, and thus $y[(x/M[t]).(s\circ t)]$ is in $[\![B]\!]_\Delta$.

Lemma 510 Let Δ and Γ be environments. Let $R = (\lambda x:A.M)[s]$ be a term such that $\Delta \vdash R:A \to B$ and $s \in [\![\Gamma]\!]_\Delta$. If for all Δ' and for all $N \in [\![A]\!]_{\Delta\Delta'}$ we have $M[(x/N).s] \in [\![B]\!]_{\Delta\Delta'}$, then $R \in [\![A \to B]\!]_\Delta$

Proof. By definition of reducibility of R, it suffices to show that for all $N \in [\![A]\!]_{\Delta\Delta'}$, $(R\ N) \in [\![B]\!]_{\Delta\Delta'}$.

Let N be in $[\![A]\!]_{\Delta\Delta'}$. Since $M[(x/N).s] \in [\![B]\!]_{\Delta\Delta'}$ then it is strongly normalizing, so we can reason by induction on $\nu(M)+\nu(N)+\nu(s)$. Now, $(R\ N)$ is neutral, and thus it is sufficient to prove that all the one-step reducts of $(R\ N)$ are in $[\![B]\!]_{\Delta\Delta'}$.

Now, $(R\ N)$ can only reduce to:

- $(\lambda x.M')[s]\ N$ with $M \longrightarrow M'$. We conclude by induction hypothesis.
- $(\lambda x.M)[s']\ N$ with $s \longrightarrow s'$. We conclude by induction hypothesis.
- $(\lambda x.M)[s]\ N'$ with $N \longrightarrow N'$. We conclude by induction hypothesis.
- $M[(x/N).s]$ which is in $[\![B]\!]_{\Delta\Delta'}$ by hypothesis.

We are now almost ready to prove that all well-typed expressions are reducible.

Theorem 511 Let Δ and Γ be valid environments and s be a substitution in $[\![\Gamma]\!]_\Delta$.

- For every substitution t such that $\Gamma \vdash t \rhd \Gamma'$ for some valid environment Γ', $t\circ s$ is in $[\![\Gamma']\!]_\Delta$.
- For all term M such that $\Gamma \vdash M:A$ for some type A, $M[s]$ is in $[\![A]\!]_\Delta$.

Proof. By induction on the structure of the term M or the substitution t.

- If $e = id$, then $id\circ s$ is neutral, so that we show by induction on $\nu(s)$ (since s is strongly normalizing by Lemma 57) that all the one-step reducts of $id\circ s$ are in $[\![\Gamma]\!]_\Delta$. These reducts are:
  - $id\circ s'$ with $s \longrightarrow s'$. The property holds by induction hypothesis.
  - s. The property holds by hypothesis.

- If $e = x$, then Γ is not empty since x must be typed in Γ. Moreover, $x : A$ is in Γ and thus $x[s]$ in $[\![A]\!]_\Delta$ by hypothesis.
- If $e = (x/M).v$, then by induction hypothesis $M[s]$ and $v \circ s$ are respectively in $[\![A]\!]_\Delta$ and $[\![\Gamma']\!]_\Delta$ for a type A and an environment Γ'. Thus, by Lemma 59 $((x/M).v) \circ s$ is in $[\![x : A, \Gamma']\!]_\Delta$.
- If $e = v \circ u$, then there exist two environments Γ' and Γ'' such that $\Gamma \vdash u \rhd \Gamma''$ and $\Gamma'' \vdash v \rhd \Gamma'$. By application of induction hypothesis we obtain consequently $u \circ s$ in $[\![\Gamma'']\!]_\Delta$ and $v \circ (u \circ s)$ in $[\![\Gamma']\!]_\Delta$. Since $v \circ (u \circ s) =_{Sub\_ass\_env} (v \circ u) \circ s$ then the property holds.
- If $e = (MN)$, then by induction hypothesis $M[s]$ and $N[s]$ are respectively in $[\![B \rightarrow A]\!]_\Delta$ and $[\![B]\!]_\Delta$ for some type B. By definition of reducibility $(M[s]N[s])$ is in $[\![A]\!]_\Delta$. But $(M[s]N[s]) =_{Sub\_app} (MN)[s]$, so $(MN)[s]$ is in $[\![B]\!]_\Delta$.
- If $e = \lambda x.M$ then by Lemma 58, $(x/N).s$ is in $[\![x : A, \Gamma]\!]_{\Delta\Delta'}$ for all Δ' and all N in $[\![A]\!]_{\Delta\Delta'}$. Without lost of generality we can assume that x does not appear in Δ'. Thus we have $\Delta\Delta' \vdash s \rhd \Gamma\Delta'$ by Lemma 41. By induction hypothesis $M[(x/N).s]$ is $[\![B]\!]_{\Delta\Delta'}$ and then by Lemma 510, $(\lambda x.M)[s]$ is also in $[\![B]\!]_\Delta$.
- If $e = M[t]$, then by induction hypothesis $t \circ s$ is in $[\![\Gamma']\!]_\Delta$ for some Γ' and thus by induction hypothesis again $M[t \circ s]$ is in $[\![A]\!]_\Delta$. Since $M[t \circ s] =_{Sub\_clos} M[t][s]$ then we are done.

We can now claim the next theorem.

Theorem 512 $\longrightarrow_{\lambda_{\sigma w/\equiv}}$ is strongly normalizing.

Proof. First of all we remark that for any environment Γ the substitution id is in $[\![\Gamma]\!]_\Gamma$: for that it is sufficient to prove that for every $x : A$ in Γ, $x[id]$ is in $[\![A]\!]_\Gamma$ for a type A. But $x[id]$ is a neutral term so it is sufficient to prove that all the one-steps reducts of $x[id]$ are in $[\![A]\!]_\Gamma$. But the only possible reduct of $x[id]$ is x and x is neutral, so to show that x is in $[\![A]\!]_\Gamma$ it is sufficient to show that all its one-step reducts are in $[\![A]\!]_\Gamma$. As x is in normal form then Property 4 of Lemma 56 gives us that x is reducible.

Now, since id is reducible, then by Theorem 511 the term $M[id]$ is reducible for all well-typed term M. Property 1 of Lemma 56 allows us to conclude that $M[id]$ is strongly normalizing and thus that M is strongly normalizing.

5.4 Strong normalization for $\boldsymbol{\lambda_{\sigma w}}$

We are now able to deduce the strong normalization of $\lambda_{\sigma w}$ from that of $\lambda_{\sigma w/\equiv}$. For that we will use the following abstract Lemma:

First of all we introduce an abstract Lemma:

Lemma 513 Let $A = \langle \mathcal{O}, R_1 \cup R_2 \rangle$ be an abstract reduction system such that:

- R_2 is strongly normalizing;

– there exists a reduction system $S = \langle \mathcal{O}', R' \rangle$ and translation $\mathcal{T}$ from $\mathcal{O}$ to $\mathcal{O}'$ such that:
  - $a \longrightarrow_{R_1} b$ implies $\mathcal{T}(a) \longrightarrow_{R'} \mathcal{T}(b)$,
  - $a \longrightarrow_{R_2} b$ implies $\mathcal{T}(a) = \mathcal{T}(b)$.

Then for all term $a \in \mathcal{O}$ such that $\mathcal{T}(a)$ is R'-strongly normalizing we have that a is $(R_1 \cup R_2)$-strongly normalizing.

Proof. We proceed by contradiction. Let $a \in \mathcal{O}$ be a term such that a is not $(R_1 \cup R_2)$-strongly normalizing. Since $\mathcal{T}(a)$ is R'-strongly normalizing we remark that any infinite reduction of a contains infinitely many steps of R_1: if it only contains finitely many steps of R_1, then it would have to contain infinitely many steps of R_2 contradicting the R_2 strong normalization. So, an infinite sequence of reductions starting with a may be written as:

$$a \longrightarrow^*{}_{R_2}\ a_1 \longrightarrow^+{}_{R_1}\ a_2 \longrightarrow^*{}_{R_2}\ a_3 \longrightarrow^+{}_{R_1}\ a_4 \ldots$$

But such a sequence translates in S to the following one:

$$\mathcal{T}(a) = \mathcal{T}(a_1) \longrightarrow^+{}_{R'}\ \mathcal{T}(a_2) = \mathcal{T}(a_3) \longrightarrow^+{}_{R'}\ \mathcal{T}(a_4) \longrightarrow^+{}_{R'}\ \ldots$$

contradicting the R'-strong normalization of $\mathcal{T}(a)$.

We can now conclude the following:

Theorem 514 $\lambda_{\sigma w}$ is strongly normalizing on well typed terms.

Proof. We note here R_2 the reduction relation generated by the three rules (Sub_app), (Sub_ass_env) and (Sub_clos) of $\lambda_{\sigma w}$, R_1 the reduction relation generated by the other rules of $\lambda_{\sigma w}$ and $\mathcal{O}$ the set of well typed terms of $\lambda_{\sigma w}$. We also note $\mathcal{O}'$ the set of $\equiv$-equivalence classes of well-typed terms $\lambda_{\sigma w}$ and R' the reduction relation $\lambda_{\sigma w/_\equiv}$. We define $\mathcal{T}$ to be the canonical projection from $\mathcal{O}$ to $\mathcal{O}'$ (that is $\mathcal{T}(a) = [a]$). This translation $\mathcal{T}$ clearly satisfies the second condition of Lemma 513. Now, we know by Theorem 512 that $\lambda_{\sigma w/_\equiv}$ is strongly normalizing on $\lambda_{\sigma w}$ well-typed terms, that is, that $\lambda_{\sigma w/_\equiv}$ is strongly normalizing on well-typed $\equiv$-equivalence classes. By application of Lemma 513 we get that well-typed terms of $\lambda_{\sigma w}$ are $\lambda_{\sigma w}$-strongly normalizing.

6 Strong normalization for $\boldsymbol{\lambda P_w}$

This section is devoted to the proof of strong normalization of well-typed λP_w-terms. We will use the same technique that we have used in Section 5. The scheme of our proof can be summarized as follows:

– We first define a calculus *modulo* an equational theory, noted $\lambda P_{w/_\equiv}$.
– We then define the notion of *reducible term* and *reducible substitution* for $\lambda P_{w/_\equiv}$-expressions. We show that any reducible term (resp. substitution) is strongly normalizing.

- We show that the term $M[s]$ and the substitution $t \circ s$ are reducible for any reducible substitution s, well-typed term M and well-typed substitution t.
- We use the previous point to show that any well-typed $\lambda P_{w/\equiv}$-expression is reducible and thus strongly normalizing.
- Finally, we deduce strong normalization for λP_w-expressions from strong normalization for $\lambda P_{w/\equiv}$-expressions.

Definition 61 The set VS of void substitutions is defined to be the smallest set of substitutions stable by concatenation such that:

- $id \in VS$
- $(\xi^{P:A}/\mathtt{K}).s \in VS$ if and only if $s \in VS$ and $BVar(P) = \emptyset$

Void substitutions enjoy the following properties:

Remark 12.

- Let s be a void substitution such that $s \longrightarrow s'$, then s' is a void substitution.
- Any void substitution is strongly normalizing.

Lemma 61 Let s be a substitution such that $\Psi; \Delta \vdash s \triangleright \Phi; \emptyset$, then we have:

- $\Delta = \emptyset$
- s is a void substitution

Proof. The first point holds by Remark 5 and the second one by contradiction.

6.1 Definition of $\lambda P_{w/\equiv}$

We can now define the notion of $\lambda P_{w/\equiv}$-reduction modulo an equational theory.

Definition 62 The congruence $\equiv$ on λP_w-terms is defined by:

$$(M\ N)[s] =_{\text{Sub\_app}} M[s]\ N[s] \qquad (s \circ t) \circ u =_{\text{Sub\_ass\_env}} s \circ (t \circ u)$$

$$M[s][t] =_{\text{Sub\_clos}} M[s \circ t]$$

We will consider the reduction system $\lambda P_{w/\equiv}$, where $a \longrightarrow_{\lambda P_{w/\equiv}} b$ if and only if there exist a', b' such that $a \equiv a' \longrightarrow_{\mathcal{R}} b' \equiv b$, where $\mathcal{R} = \lambda P_w \setminus \{\text{Sub\_app}, \text{Sub\_ass\_env}, \text{Sub\_clos}\}$. This definition can also be interpreted as a reduction on *equivalence classes*, i.e., $[a] \longrightarrow_{\lambda P_{w/\equiv}} [b]$ if and only if $a' \longrightarrow_{\mathcal{R}} b'$, for $a' \in [a]$ and $b' \in [b]$. We may use indistinctly both interpretations according to the context.

We remark that by subject reduction (Theorem 412), the notion of well-typed $\lambda P_{w/\equiv}$-terms is well-defined.

6.2 Strong normalization for $\lambda P_{w/\equiv}$

We are now able to introduce the notion of reducible expressions, which makes use of the following concept.

Definition 63 (Neutral terms and substitutions)

- A term is neutral if and only if it is neither of the forms $(\lambda x.N)[s]$, $\mathtt{inr}_A(M)$, $\mathtt{inl}_B(M)$ nor $\langle M_1, M_2\rangle$.
- A substitution s is *neutral* if and only if it is not of the form $(x/M).t$.

Notation 62 Let M be a term.

- $\mathcal{P}^1_{A\times B}(M)$ denotes the term $(\lambda\langle x, \_\rangle : A \times B.x)[id]\ M$ where x is a fresh variable.
- $\mathcal{P}^2_{A\times B}(M)$ denotes the term $(\lambda\langle \_, x\rangle : A \times B.x)[id]\ M$ where x is a fresh variable.
- $\mathcal{S}_{A+B}(M)$ denotes the term $(\lambda(x \mid_\xi y) : A + B.[\langle x, w_2\rangle \mid_\xi \langle w_1, y\rangle])[id]\ M$ where x, y, w_1, w_2 and ξ are fresh variables.

Definition 64 (Reducible terms and substitutions) The set of *reducible terms* for a given type in an environment $\Phi;\Gamma$ is defined by induction on types as follows:

$[\![\iota]\!]_{\Phi;\Gamma} =_{def} \{M \mid \Phi;\Gamma \vdash M : \iota$ and M is strongly normalizing$\}$.
$[\![A \to B]\!]_{\Phi;\Gamma} =_{def} \{M \mid \Phi;\Gamma \vdash M : A \to B$ and $\forall N \in [\![A]\!]_{\Phi;\Gamma\Delta}, (M\ N) \in [\![B]\!]_{\Phi;\Gamma\Delta}\}$ where Δ satisfies the condition of the point 1 of Lemma 41.
$[\![A \times B]\!]_{\Phi;\Gamma} =_{def} \{M \mid \Phi;\Gamma \vdash M : A \times B, \mathcal{P}^1_{A\times B}(M) \in [\![A]\!]_{\Phi;\Gamma}$ and $\mathcal{P}^2_{A\times B}(M) \in [\![B]\!]_{\Phi;\Gamma}\}$
$[\![A{+}B]\!]_{\Phi;\Gamma} =_{def} \{M \mid \Phi;\Gamma \vdash M : A{+}B$ and $\forall$ fresh variables $w_1, w_2, \mathcal{S}_{A+B}(M) \in [\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}\}$

The set of *reducible substitutions for an environment $\Phi;\Gamma$ in an environment $\Psi;\Delta$* is defined as follows:

$$[\![\Phi;\Gamma]\!]_{\Psi;\Delta} =_{def} \{s \mid \Psi;\Delta \vdash s \rhd \Phi;\Gamma \text{ and } \forall (x : A) \in \Gamma, x[s] \in [\![A]\!]_{\Psi;\Delta}\}$$

Reducible terms enjoy the following expected properties:

Lemma 63 For every type C the following statements hold:

1. If $M \in [\![C]\!]_{\Phi;\Gamma}$, then M is strongly normalizing.
2. If $\Phi;\Gamma \vdash (xM_1 \dots M_n) : C$ and $M_1 \dots M_n$ are strongly normalizing, then $(xM_1 \dots M_n) \in [\![C]\!]_{\Phi;\Gamma}$.
3. If $M \in [\![C]\!]_{\Phi;\Gamma}$ and $M \longrightarrow M'$ then $M' \in [\![C]\!]_{\Phi;\Gamma}$.
4. If M is a neutral of type C and all its one-step reducts are reducible expressions, then M is reducible.

Proof. We prove these statement by induction on the type C.

Base case ι:

1. By definition of $[\![\iota]\!]_{\Phi;\Gamma}$.
2. Since $M_1 \dots M_n$ are strongly normalizing, then the only reduction sequences starting at $(x\ M_1\ \dots\ M_n)$ proceed independently in the terms $M_i's$, and all these reduction sequences terminate. As a consequence, the term $(x\ M_1\ \dots\ M_n)$ is strongly normalizing and thus by definition $(x\ M_1\ \dots\ M_n) \in [\![\iota]\!]_{\Phi;\Gamma}$.
3. Let $M \in [\![\iota]\!]_{\Phi;\Gamma}$. By definition, $\Phi;\Gamma \vdash M : \iota$ and M is strongly normalizing. Then, any M' such that $M \longrightarrow M'$ is also strongly normalizing. By subject reduction (Theorem 412) $\Phi;\Gamma \vdash M' : \iota$ and thus by definition $M' \in [\![\iota]\!]_{\Phi;\Gamma}$.
4. Let M be a neutral and well-typed term in $\Phi;\Gamma$ (*i.e.* $\Phi;\Gamma \vdash M : \iota$) such that all its one-step reducts are in $[\![\iota]\!]_{\Phi;\Gamma}$. By definition all these one-steps reducts are strongly normalizing so that M is strongly normalizing and thus by definition $M \in [\![\iota]\!]_{\Phi;\Gamma}$.

Inductive case

$[\![A \to B]\!]$

1. Let M be in $[\![A \to B]\!]_{\Phi;\Gamma}$. By induction hypothesis (Point 2 with $n = 0$) there is a fresh variable x in $[\![A]\!]_{\Phi;\Gamma,xA}$. By definition the term $(M\ x)$ is in $[\![B]\!]_{\Phi;\Gamma,xA}$. By induction hypothesis on B, $(M\ x)$ is strongly normalizing and then so is M.
2. Let $(x\ M_1\ \dots\ M_n)$ be a term such that $\Phi;\Gamma \vdash (x\ M_1\ \dots\ M_n) : A \to B$ and $M_1 \dots M_n$ are strongly normalizing. Let N be *any* term in $[\![A]\!]_{\Phi;\Delta\Gamma}$. From the induction hypothesis (Point 1) we have that N is strongly normalizing, and by Lemma 41 $\Phi;\Delta\Gamma \vdash (xM_1 \dots M_n N) : B$. As a consequence, by induction hypothesis (Point 2) we have that $(xM_1 \dots M_n N)$ is in $[\![B]\!]_{\Phi;\Delta\Gamma}$, and thus by definition $(xM_1 \dots M_n)$ is in $[\![A \to B]\!]_{\Phi;\Gamma}$.
3. Let $M \in [\![A \to B]\!]_{\Phi;\Gamma}$ and let consider the reduction step $M \longrightarrow M'$. By subject reduction (Theorem 412) $\Phi;\Gamma \vdash M' : A \to B$. Take any term $N \in [\![A]\!]_{\Phi;\Delta\Gamma}$. By definition $(MN) \in [\![B]\!]_{\Phi;\Delta\Gamma}$. Since $(MN) \longrightarrow (M'N)$, then by induction hypothesis on B, we have that $(M'N) \in [\![B]\!]_{\Phi;\Delta\Gamma}$ and thus by definition $M' \in [\![A \to B]\!]_{\Phi;\Gamma}$.
4. Let M be a neutral term such that $\Phi;\Gamma \vdash M : A \to B$ and all its one-step reducts are in $[\![A \to B]\!]_{\Phi;\Gamma}$. Let N be a term in $[\![A]\!]_{\Phi;\Delta\Gamma}$. We have to show that $(M\ N) \in [\![B]\!]_{\Phi;\Delta\Gamma}$. Since $(M\ N)$ is neutral and $\Phi;\Delta,\Gamma \vdash (M\ N) : B$ (Lemma 41), then by induction hypothesis it is sufficient to show that all its one-step reducts are in $[\![B]\!]_{\Phi;\Delta\Gamma}$. By induction hypothesis (Point 1) we know that N is strongly normalizing, so we can reason by induction on $\nu(N)$ as follows:
   The one-steps reducts of $(M\ N)$ are:
   - $(M\ N')$, with $N \longrightarrow N'$. Then $\nu(N') < \nu(N)$ and the property holds by induction hypothesis.
   - $(M'\ N)$, with $M \longrightarrow M'$. By hypothesis M' is a reducible term in $[\![A \to B]\!]_{\Phi;\Gamma}$, so that $(M'\ N)$ is in $[\![B]\!]_{\Phi;\Delta\Gamma}$ by definition.

- There is no other possible case since M is neutral.

We can then conclude that $M \in [\![A \to B]\!]_{\Phi;\Gamma}$.

$[\![A \times B]\!]$

1. Let M be in $[\![A \times B]\!]_{\Phi;\Gamma}$. By definition we have that $((\lambda\langle x, \_\rangle.x)[id]\ M) \in [\![A]\!]_{\Phi;\Gamma}$. By induction hypothesis (Point 1) on A we have that $\mathcal{P}^1_{A\times B}(M)$ is strongly normalizing and thus so is M.
2. Let $M = x\ M_1\ \ldots\ M_n$ be a term such that $\Phi;\Gamma \vdash x\ M_1\ \ldots\ M_n : A \times B$ where the M_i's are strongly normalizing. We have to show that $\mathcal{P}^1_{A\times B}(M)$ is in $[\![A]\!]_{\Phi;\Gamma}$ and $\mathcal{P}^2_{A\times B}(M)$ is in $[\![B]\!]_{\Phi;\Gamma}$. We show the first statement, leaving the second one which is similar. For that we reason by induction on $\sigma = \sum_i \nu(M_i)$.
   - If $\sigma = 0$ then $\mathcal{P}^1_{A\times B}(M)$ is a neutral term in normal form. By induction hypothesis (Point 4) on A, the term $\mathcal{P}^1_{A\times B}(M)$ is then reducible.
   - If $\sigma > 0$ then $\mathcal{P}^1_{A\times B}(M)$ can only reduce to $\mathcal{P}^1_{A\times B}(M')$ where $M' = x\ M_1 \ldots M_{i-1} M'_i M_{i+1} \ldots M_n$ and $M_i \longrightarrow M'_i$. By induction hypothesis on σ the term $\mathcal{P}^1_{A\times B}(M')$ is then reducible, and by induction hypothesis (Point 4) on A the term $\mathcal{P}^1_{A\times B}(M)$ is reducible.

   Since $\mathcal{P}^1_{A\times B}(M)$ and $\mathcal{P}^2_{A\times B}(M)$ are reducible, then M is reducible by definition.
3. Let M be in $[\![A \times B]\!]_{\Phi;\Gamma}$ and M' such that $M \longrightarrow M'$. We have to show that $\mathcal{P}^1_{A\times B}(M')$ is in $[\![A]\!]_{\Phi;\Gamma}$ and that $\mathcal{P}^2_{A\times B}(M')$ is in $[\![B]\!]_{\Phi;\Gamma}$. We show the first statement (the second one begin similar). By definition we know that $\mathcal{P}^1_{A\times B}(M)$ is in $[\![A]\!]_{\Phi;\Gamma}$. By induction hypothesis (Point 3) on A the result holds since $\mathcal{P}^1_{A\times B}(M) \longrightarrow \mathcal{P}^1_{A\times B}(M')$.
4. Let M be a neutral term such that $\Phi;\Gamma \vdash M : A \times B$ and such that all its one-step reducts are in $[\![A \times B]\!]_{\Phi;\Gamma}$. We show that the neutral term $\mathcal{P}^1_{A\times B}(M)$ is in $[\![A]\!]_{\Phi;\Gamma}$ (the other case being similar). Since M is neutral $\mathcal{P}^1_{A\times B}(M)$ can only reduce to $\mathcal{P}^1_{A\times B}(M)'$, where $M \longrightarrow M'$. Since M' is reducible by hypothesis, then $\mathcal{P}^1_{A\times B}(M')$ is reducible by definition and we can conclude by induction hypothesis (Point 4) on A that $\mathcal{P}^1_{A\times B}(M)$ is reducible, so that M is reducible too.

$[\![A + B]\!]$

1. Let M be in $[\![A+B]\!]_{\Phi;\Gamma}$. By definition of $[\![A+B]\!]_{\Phi;\Gamma}$ we have $\mathcal{S}_{A+B}(M) \in [\![A\times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$. By the previous point any term in $[\![A\times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$ is strongly normalizing, thus $\mathcal{S}_{A+B}(M)$ is strongly normalizing and thus so is M.
2. Let $M = x\ M_1\ \ldots\ M_n$ be a term such that $\Phi;\Gamma \vdash x\ M_1\ \ldots\ M_n : A + B$ where the M_i's are strongly normalizing. We have to show that $\mathcal{S}_{A+B}(M)$ is in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$.

   We show this statement by induction on $\sigma = \sum_i \nu(M_i)$
   - If $\sigma = 0$ then $\mathcal{S}_{A+B}(M)$ is a neutral normal form. Since Point 4 holds for $A \times B$, then $\mathcal{S}_{A+B}(M)$ is reducible.
   - If $\sigma > 0$ then $\mathcal{S}_{A+B}(M)$ can only reduce to $\mathcal{S}_{A+B}(M')$ where $M' = x\ M_1 \ldots M_{i-1} M'_i M_{i+1} \ldots M_n$ and $M_i \longrightarrow M'_i$. By induction hypothesis on σ the term $\mathcal{S}_{A+B}(M')$ is reducible and then since Point 4 holds for $A \times B$, then $\mathcal{S}_{A+B}(M)$ is reducible.

We can then conclude that M is reducible.

3. Let M be in $[\![A+B]\!]_{\Phi;\Gamma}$ and M' such that $M \longrightarrow M'$. We have to show that $\mathcal{S}_{A+B}(M')$ is in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$. By definition we have $\mathcal{S}_{A+B}(M)$ in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$. Since Point 3 holds for $A \times B$ and $\mathcal{S}_{A+B}(M) \longrightarrow \mathcal{S}_{A+B}(M')$, the result holds.
4. Let M be a neutral term such that $\Phi;\Gamma \vdash M : A+B$ and all its one-step reducts are in $[\![A+B]\!]_{\Phi;\Gamma}$. We show that the neutral term $\mathcal{S}_{A+B}(M)$ is in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$. Since M is neutral, $\mathcal{S}_{A+B}(M)$ can only reduce to $\mathcal{S}_{A+B}(M')$, where $M \longrightarrow M'$. As M' is reducible by hypothesis then $\mathcal{S}_{A+B}(M')$ is reducible by definition. We can then apply Point 4 on $A \times B$, and conclude that $\mathcal{S}_{A+B}(M)$ is reducible, so that M is also reducible.

Lemma 64 Let M be a term in $[\![A]\!]_{\Phi;\Gamma}$. For all acceptable environment $\Psi;\Delta$ satisfying the conditions of Point 1 of Lemma 41, M is in $[\![A]\!]_{\Phi\Psi;\Gamma\Delta}$

Proof. By induction on the type A.

- $A = \iota$ The result holds immediately by Lemma 41.
- $A = A_1 \rightarrow A_2$ Since by Lemma 41 $\Phi\Psi;\Gamma\Delta \vdash M : A$, all we have to show is that $\forall N \in [\![A_1]\!]_{\Psi\Phi;\Gamma\Gamma'\Delta}, (MN) \in [\![A_2]\!]_{\Psi\Phi;\Gamma\Gamma'\Delta}$. This is true by Lemma 41 and by definition of reducible term.
- $A = A_1 \times A_2$ The result holds by Lemma 41 and induction hypothesis.
- $A = A_1 + A_2$ The result holds by Lemma 41 and the previous point.

We can now deduce from Lemma 63 (Point 2) the following property:

Corollary 3. *All the variables are reducible.*

As for terms, reducible substitutions also enjoy the following expected properties:

Lemma 65

1. If $s \in [\![\Phi;\Gamma]\!]_{\Psi;\Delta}$, then s is strongly normalizing.
2. If $s \in [\![\Phi;\Gamma]\!]_{\Psi;\Delta}$ and $s \longrightarrow s'$ then $s' \in [\![\Phi;\Gamma]\!]_{\Psi;\Delta}$.
3. If s is a neutral substitution such that $\Psi;\Delta \vdash s \triangleright \Phi;\Gamma$ and all its one-step reducts are reducible, then $s \in [\![\Phi;\Gamma]\!]_{\Psi;\Delta}$.

Proof. We prove the properties by cases on Γ.

- Let us suppose $\Gamma = \emptyset$.
  1. Let s be a substitution in $[\![\Phi;\emptyset]\!]_{\Psi;\Delta}$. By Lemma 61, we know that s is a void substitution and then by Remark 12 the result holds.
  2. Let s be a substitution in $[\![\Phi;\emptyset]\!]_{\Psi;\Delta}$. By definition of reducible substitution, we know that $\Psi;\Delta \vdash s \triangleright \Phi;\emptyset$. Let s' be a substitution such that $s \longrightarrow s'$. By subject reduction (Theorem 412), we know that $\Psi;\Delta \vdash s' \triangleright \Phi;\emptyset$ and then s' is reducible by definition.
  3. Let s be a neutral substitution such that $\Psi;\Delta \vdash s \triangleright \Phi;\emptyset$, then s is reducible by definition.

– Let us now suppose $\Gamma \neq \emptyset$.
  1. Let s be in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. Take $(x:A) \in \Gamma$. Then, by definition, the term $x[s]$ is in $[\![A]\!]_{\Psi;\Delta}$. Since the property holds for terms (Lemma 1), then $x[s]$ is strongly normalizing and then s is strongly normalizing.
  2. Let s be in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. Then $\Psi;\Delta \vdash s \rhd \Phi;\Gamma$. Take $(x:A) \in \Gamma$ and let s' such that $s \longrightarrow s'$. By subject reduction (Theorem 412) $\Psi;\Delta \vdash s' \rhd \Phi;\Gamma$ so that by definition we have $s' \in [\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. By definition, $x[s] \in [\![A]\!]_{\Psi;\Delta}$ and since the property holds for terms (Lemma 3), then $x[s'] \in [\![A]\!]_{\Psi;\Delta}$.
  3. Let s be a neutral and well-typed substitution (*i.e.* $\Psi;\Delta \vdash s \rhd \Phi;\Gamma$) and let suppose that all the one-step reducts of s are in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. Take $(x:A)$ in Γ. Since s is neutral, $x[s]$ may only reduce to $x[s']$ with $s \longrightarrow s'$. But $\Psi;\Delta \vdash s' \rhd \Phi;\Gamma$ holds by subject reduction (Theorem 412) and s' is reducible by hypothesis, so that $x[s'] \in [\![A]\!]_{\Psi;\Delta}$ by definition and we are done.

Lemma 66 Let s be a substitution in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. For any acceptable environment $\Phi';\Gamma'$ satisfying the conditions of point 2 of Lemma 41, s is in $[\![\Phi'\Phi;\Gamma'\Gamma]\!]_{\Phi'\Psi;\Gamma'\Delta}$

Proof. By Lemmas 41 and 64.

Since *id* is neutral, well-typed and has no reducts, then we can conclude the following by Lemma 65.

Corollary 4. *Let $\Phi;\Gamma$ be a valid environment. Then $id \in [\![\Phi;\Gamma]\!]_{\Phi;\Gamma}$.*

We are now ready to prove the state statement of this section which allows us to prove that any well-typed expression is reducible.

Lemma 67 Let M be in $[\![A]\!]_{\Psi;\Delta}$ and s be in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. If x is a fresh variable, then $t = (x/M).s \in [\![\Phi;x:A,\Gamma]\!]_{\Psi;\Delta}$.

Proof. First of all we remark that $\Psi;\Delta \vdash t \rhd \Phi;x:A,\Gamma$ holds by definition. So that to prove that t is in $[\![\Phi;x:A,\Gamma]\!]_{\Psi;\Delta}$ it is sufficient to prove that $\forall(y:B) \in (x:A,\Gamma)$ we have $y[t]$ in $[\![B]\!]_{\Psi;\Delta}$.

Take $(y:B) \in (x:A,\Gamma)$. Since $y[t]$ is neutral, if we show that all its one-step reducts are in $[\![B]\!]_{\Psi;\Delta}$ we may conclude that $y[t]$ is in $[\![B]\!]_{\Psi;\Delta}$ by Lemma 63 (Point 4). Since M and s are respectively in $[\![A]\!]_{\Psi;\Delta}$ and $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$, then they are strongly normalizing by Lemma 63 and 65 respectively and thus we may proceed by induction on $\nu(M)+\nu(s)$. The one-step reducts of $y[t]$ are:

– $y[(x/M').s]$ with $M \longrightarrow M'$. Then we conclude by induction hypothesis.
– $y[(x/M).s']$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
– M if $x = y$. Then we conclude by hypothesis.
– $y[s]$ if $x \neq y$. Then we conclude by hypothesis.

Lemma 68 If $s \circ t \in [\![\Phi;\Gamma]\!]_{\Psi;\Delta}$, $M[t] \in [\![A]\!]_{\Psi;\Delta}$, and if x is a fresh variable, then $u = ((x/M).s) \circ t$ is in $[\![\Phi;x:A,\Gamma]\!]_{\Psi;\Delta}$.

Proof. First of all we have to prove that u is well typed in $\Psi;\Delta$. But $\Psi;\Delta \vdash s \circ t \triangleright \Phi;\Gamma$ implies that there exists $\Phi';\Gamma'$ such that $\Psi;\Delta \vdash t \triangleright \Phi';\Gamma'$ and $\Phi';\Gamma' \vdash s \triangleright \Phi;\Gamma$. From $\Psi;\Delta \vdash M[t]:A$ and Lemma 52 we conclude $\Psi;\Delta \vdash t \triangleright \Phi';\Gamma'$ and $\Phi';\Gamma' \vdash M:A$, so that we can build the following typing derivation.

$$\dfrac{\Psi;\Delta \vdash t \triangleright \Phi';\Gamma' \qquad \dfrac{\Phi';\Gamma' \vdash M:A \qquad \Phi';\Gamma' \vdash s \triangleright \Phi;\Gamma}{\Phi';\Gamma' \vdash (x/M).s \triangleright \Phi; x:A,\Gamma}}{\Psi;\Delta \vdash ((x/M).s)\circ t \triangleright \Phi; x:A,\Gamma}$$

Now, it is sufficient to prove that for all $(y:B) \in (x:A,\Gamma)$, $y[u]$ is in $[\![B]\!]_{\Psi;\Delta}$. Let $y : B$ be in $(x : A,\Gamma)$. Since the expressions $M[t]$ and $s \circ t$ are strongly normalizing by Lemma 63 and 65, we can prove the property by induction on $\nu(M)+\nu(t)+\nu(s)$.

Now, the term $y[u]$ is neutral so it is sufficient to prove that all the one-step reducts of $y[u]$ are in $[\![B]\!]_{\Psi;\Delta}$. This reducts are:

- $y[((x/M').s)\circ t]$ with $M \longrightarrow M'$. Then we conclude by induction hypothesis.
- $y[((x/M).s')\circ t]$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
- $y[((x/M).s)\circ t']$ with $t \longrightarrow t'$. Then we conclude by induction hypothesis.
- $y[(x/M[t]).(s\circ t)]$. Since $M[t]$ and $s \circ t$ are respectively in $[\![A]\!]_{\Psi;\Delta}$ and $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$ by hypothesis, then the substitution $(x/M[t]).(s\circ t)$ is in $[\![\Phi;x:A,\Gamma]\!]_{\Psi;\Delta}$ by Lemma 67, and thus $y[(x/M[t]).(s\circ t)]$ is in $[\![B]\!]_{\Psi;\Delta}$ by definition since the following typing judgment can be derived:

$$\dfrac{\Psi;\Delta \vdash (x/M[t]).(s\circ t) \triangleright \Phi;\Gamma,x:A \qquad \Phi;\Gamma,x:A \vdash y:B}{\Psi;\Delta \vdash y[(x/M[t]).(s\circ t)]:B}$$

Lemma 69 Let s be in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. For every fresh sum variable ξ and for $\mathtt{K} \in \{\mathtt{L};\mathtt{R}\}$, we have that $t = (\xi^{P.A}/\mathtt{K}).s$ is in $[\![\xi:\mathtt{K},\Phi;P:A,\Gamma]\!]_{\Psi;\Delta}$.

Proof. First of all we remark that t is well typed (i.e. $\Psi;\Delta \vdash t \triangleright \xi:\mathtt{K},\Phi;P:A,\Gamma$). We have to show that for all $x:A$ in Γ, $x[t]$ is in $[\![A]\!]_{\Psi;\Delta}$. Let us take any $x:A \in \Gamma$. Since $x[t]$ is neutral, it is sufficient to show that all its one step reducts are in $[\![A]\!]_{\Psi;\Delta}$. We proceed by induction on $\nu(s)$. The term $x[t]$ can only reduce to:

- $x[(\xi^{P.A}/\mathtt{K}).s']$ with $s \longrightarrow s'$. Then we conclude by induction hypothesis.
- $x[s]$, which is in $[\![A]\!]_{\Psi;\Delta}$ by definition since s is in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$.

Lemma 610 If M and N are respectively in $[\![A]\!]_{\Phi;\Gamma}$ and $[\![B]\!]_{\Phi,\Gamma}$ then

- $U''_A = (\lambda\_.x)[(x/M).id]N \in [\![A]\!]_{\Phi;\Gamma}$ and $U''_B = ((\lambda x.x)[id])N \in [\![A]\!]_{\Phi;\Gamma}$.
- $U'_A = ((\lambda x.\lambda\_.x)[id]M)N \in [\![A]\!]_{\Phi;\Gamma}$ and $U'_B = ((\lambda\_.\lambda x.x)[id]M)N \in [\![B]\!]_{\Phi;\Gamma}$
- $\langle M,N\rangle$ is in $[\![A\times B]\!]_{\Phi;\Gamma}$

Proof. By Lemma 63 (Point 1), M and N are strongly normalizing so that we can proceed by induction on $\nu(M)+\nu(N)$.

- We only show the first part of the statement, the second one being similar. Since U''_A is neutral, it is sufficient to show that all its one step reducts are in $[\![A]\!]_{\Phi;\Gamma}$. The term U''_A can only reduce to:
  - $(\lambda\_.x)[(x/M').id]\ N$. Then the property holds by induction hypothesis.
  - $(\lambda\_.x)[(x/M).id]\ N'$. Then the property holds by induction hypothesis.
  - $x[(x/M).id]$ which is reducible by definition of reducibility and Lemma 67, using the fact that $M \in [\![A]\!]_{\Phi;\Gamma}$ by hypothesis and $id \in [\![\Phi;\Gamma]\!]_{\Phi;\Gamma}$ by Corollary 4.
- We only show the first part of the statement, the second one being similar. Since U'_A is neutral it is sufficient to show that all its one step reducts are in $[\![A]\!]_{\Phi;\Gamma}$. The term U'_A can only reduce to:
  - $((\lambda x.\lambda\_.x)[id]M')\ N$. Then the property holds by induction hypothesis.
  - $((\lambda x.\lambda\_.x)[id]M)\ N'$. Then the property holds by induction hypothesis.
  - $U''_A = (\lambda\_.x)[(x/M).id]\ N$. This holds by the first point previously shown.
- We have to show that $\mathcal{P}^1_{A\times B}(\langle M,N\rangle)$ is in $[\![A]\!]_{\Phi;\Gamma}$ and $\mathcal{P}^2_{A\times B}(\langle M,N\rangle)$ is in $[\![B]\!]_{\Phi;\Gamma}$. We only show the first part of the statement, the second one being similar. Since $\mathcal{P}^1_{A\times B}(\langle M,N\rangle)$ is neutral it is sufficient to show that all its one step reducts are in $[\![A]\!]_{\Phi;\Gamma}$. The term $\mathcal{P}^1_{A\times B}(\langle M,N\rangle)$ can only reduce to:
  - $\mathcal{P}^1_{A\times B}(\langle M',N\rangle)$ where $M \longrightarrow M'$. Then the property holds by induction hypothesis.
  - $\mathcal{P}^1_{A\times B}(\langle M',N\rangle)$ where $N \longrightarrow N'$. Then the property holds by induction hypothesis.
  - $((\lambda x.\lambda\_.x)[id]\ M)\ N$. The property holds by the second point previously shown.

Lemma 611 If $M[s]$ and $N[s]$ are respectively in $[\![A]\!]_{\Phi;\Gamma}$ and $[\![B]\!]_{\Phi;\Gamma}$, then $\langle M,N\rangle[s]$ is in $[\![A\times B]\!]_{\Phi;\Gamma}$

Proof. Since $U = \langle M,N\rangle[s]$ has type $A\times B$, we have to show that $\mathcal{P}^1_{A\times B}(U)$ and $\mathcal{P}^2_{A\times B}(U)$ are respectively in $[\![A]\!]_{\Phi;\Gamma}$ and $[\![B]\!]_{\Phi;\Gamma}$. We only show the property for $\mathcal{P}^1_{A\times B}(U)$ (for $\mathcal{P}^2_{A\times B}(U)$ is similar). Since $\mathcal{P}^1_{A\times B}(U)$ is neutral it is sufficient to show that all its one step reducts are reducible. By Lemma 63 (Point 1) we know that $M[s]$ and $N[s]$ are strongly normalizing, so that we can proceed by induction on $\nu(M)+\nu(N)+\nu(s)$. The term $\mathcal{P}^1_{A\times B}(U)$ can only reduce to:

- $\mathcal{P}^1_{A\times B}(\langle M',N\rangle[s])$ where $M \longrightarrow M'$. Then the result holds by induction hypothesis.
- $\mathcal{P}^1_{A\times B}(\langle M,N'\rangle[s])$ where $N \longrightarrow N'$. Then, the result holds by induction hypothesis.
- $\mathcal{P}^1_{A\times B}(\langle M,N\rangle[s'])$ where $s \longrightarrow s'$. Then the result holds by induction hypothesis.
- $U' = \mathcal{P}^1_{A\times B}(\langle M[s],N[s]\rangle)$. By hypothesis $M[s]$ and $N[s]$ are reducible so that by Lemma 610 $\langle M[s],N[s]\rangle$ is reducible and thus by definition we conclude that U' is reducible.

Lemma 612

1. Let us suppose that:
   - M is in $[\![A]\!]_{\Phi;P:B,\Gamma}$,
   - N is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$,
   - if ξ is a fresh variable
   - $\Phi;(P\mid_\xi Q):B+C,\Gamma$ is linear

   then $U=[M\mid_\xi N]$ is in $[\![A]\!]_{\Phi;(P\mid_\xi Q):B+C,\Gamma}$.
2. Let us suppose that:
   - M and N are in $[\![A]\!]_{\Phi;\Gamma}$
   - $\xi\in\Phi$

   then $U=[M\mid_\xi N]$ is in $[\![A]\!]_{\Phi;\Gamma}$

Proof. First of all we remark that in both cases U is well typed and is a neutral term. Then, By Lemma 63 (Point 4), to show the reducibility of U it is sufficient to show the reducibility of all its one step reducts. Since M and N are reducible they are strongly normalizing by Lemma 63 and thus we can reason by induction on $\sigma=\nu(M)+\nu(N)$.

- if $\sigma=0$ then U is a well-typed neutral term without any reduct and is then reducible
- if $\sigma\neq 0$ then the possible reduct of U are
  - $U=[M'\mid_\xi N]$ with $M\longrightarrow\ M'$ and then the property hold by induction hypothesis
  - $U=[M\mid_\xi N']$ with $N\longrightarrow\ N'$ and then the property hold by induction hypothesis

Lemma 613 Let Ξ be a strongly normalizing sum term.

1. Let us suppose that:
   - $\Phi;\Gamma\vdash\Xi\rightsquigarrow\xi$
   - $M[s]$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$
   - $N[s]$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$
   - ξ is a fresh variable
   - $\Phi;(P\mid_\xi Q):A+B,\Gamma$ is linear

   then $U=[M\mid^s_\Xi N]$ is in $[\![A]\!]_{\Phi;(P\mid_\xi Q):B+C,\Gamma}$.
2. Let us suppose that:
   - $M[s]$ and $N[s]$ both are in $[\![A]\!]_{\Phi;\Gamma}$.
   - $\Phi;\Gamma\vdash\Xi\rightsquigarrow\mathtt{K}$.

   then $U=[M\mid^s_\Xi N]$ is in $[\![A]\!]_{\Phi;\Gamma}$.

Proof. In both cases, since U is neutral, by Lemma 63 (Point 4) it is sufficient to show that all its one-step reducts are reducible.

Since $M[s]$ and $N[s]$ are reducible, by Lemma 63, we know that M, N and s are strongly normalizing and then we can reason by induction on $\nu(M)+\nu(N)+\nu(s)+\nu(\Xi)$.

1. We reason by case on the possible one-step reduct:

- if $U \longrightarrow [M \mid^{s}_{\Xi'} N]$ with $\Xi \longrightarrow \Xi'$, then $\Phi; \Gamma \vdash \Xi' \rightsquigarrow \xi$ by subject reduction (Theorem 412) and thus the property holds by induction hypothesis.
- if $U \longrightarrow [M' \mid^{s}_{\Xi} N]$ with $M \longrightarrow M'$, then $M'[s]$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$ (point 3 of Lemma 63), and thus the property holds by induction hypothesis.
- if $U \longrightarrow [M \mid^{s}_{\Xi} N']$ with $N \longrightarrow N'$, then $N'[s]$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$ (point 3 of Lemma 63), and thus the property holds by induction hypothesis.
- if $U \longrightarrow [M \mid^{s'}_{\Xi} N]$ with $s \longrightarrow s'$, then $M[s']$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$ and $N[s']$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$ (since by point 3 of Lemma 63), and thus the property holds by induction hypothesis.
- if $\Xi = \xi$ and $U \longrightarrow [M[s] \mid_{\xi} N[s]]$, by Lemma 612 $[M[s] \mid_{\xi} N[s]]$ is reducible, since $M[s]$ and $N[s]$ are reducible by hypothesis.

2. We reason by case on the possible one-step reduct:
   - if $U \longrightarrow [M \mid^{s}_{\Xi'} N]$ with $\Xi \longrightarrow \Xi'$, then $\Phi; \Gamma \vdash \Xi' \rightsquigarrow \mathtt{K}$ by subject reduction (Theorem 412), and thus the property holds by induction hypothesis.
   - if $U \longrightarrow [M' \mid^{s}_{\Xi} N]$ with $M \longrightarrow M'$, $M'[s]$ is in $[\![A]\!]_{\Phi;\Gamma}$ (point 3 of Lemma 63), and thus the property holds by induction hypothesis.
   - if $U \longrightarrow [M \mid^{s}_{\Xi} N']$ with $N \longrightarrow N'$, then $N'[s]$ is in $[\![A]\!]_{\Phi;\Gamma}$ (point 3 of Lemma 63), and thus the property holds by induction hypothesis.
   - if $U \longrightarrow [M \mid^{s'}_{\Xi} N]$ with $s \longrightarrow s'$, then both $M[s']$ and $N[s']$ are in $[\![A]\!]_{\Phi;\Gamma}$ (point 3 of Lemma 63), and thus the property holds by induction hypothesis.
   - if $\Xi = \xi$ and $U \longrightarrow [M[s] \mid_{\xi} N[s]]$, by Lemma 612, $[M[s] \mid_{\xi} N[s]]$ is reducible, since $M[s]$ and $N[s]$ are reducible by hypothesis.
   - If $\Xi = \mathtt{L}$ and $U \longrightarrow M[s]$, $M[s]$ is reducible by hypothesis.
   - If $\Xi = \mathtt{R}$ and $U \longrightarrow N[s]$, $N[s]$ is reducible by hypothesis.

Lemma 614

1. Let us suppose that:
   - $\Phi; (P \mid_{\xi} Q) : B + C, \Gamma \vdash s \triangleright \Phi'\Phi; (P \mid_{\xi} Q) : B + C, \Gamma'\Gamma$
   - $M[s]$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$
   - $N[s]$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$
   - $\Phi'\Phi; \Gamma'\Gamma \vdash \xi \rightsquigarrow \xi$

   then $U = [M \mid_{\xi} N][s]$ is in $[\![A]\!]_{\Phi;(P\mid_{\xi}Q):B+C,\Gamma}$.
2. Let us suppose that:
   - $\Phi; \Gamma \vdash s \triangleright \Phi'\Phi; \Gamma'\Gamma$
   - $M[s]$ is in $[\![A]\!]_{\Phi;\Gamma}$
   - $N[s]$ is in $[\![A]\!]_{\Phi;\Gamma}$
   - $\Phi; \Gamma \vdash \xi \rightsquigarrow \mathtt{K}$

   then $U = [M \mid_{\xi} N][s]$ is in $[\![A]\!]_{\Phi;\Gamma}$.

Proof. In both cases U is neutral. Thus by Lemma 63 (Point 4) it is sufficient to show that any one-step reduct of U is reducible. Since $M[s]$ and $N[s]$ are reducible by hypothesis, then they are strongly normalizing (by Lemma 63) and thus we can reason by induction on $\nu(M)+\nu(M)+\nu(s)$. In both cases we reason as follows:

- If $U \longrightarrow [M' \mid_\xi N][s]$ with $M \longrightarrow M'$ the result holds by induction hypothesis since $M'[s]$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$ by Lemma 63 .
- If $U \longrightarrow [M \mid_\xi N'][s]$ with $N \longrightarrow N'$ the result holds by induction hypothesis since $N'[s]$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$ by Lemma 63.
- If $U \longrightarrow [M \mid_\xi N][s']$ with $s \longrightarrow s'$ the result holds by induction hypothesis since $M[s']$ and $M[s']$ are both reducible by Lemma 63.
- If $U \longrightarrow U' = [M \mid^{s}_{\xi[s]} N]$, the result holds by Lemma 613, since
  - $M[s]$ is reducible by hypothesis.
  - $N[s]$ is reducible by hypothesis.
  - In the first case $\Phi; \Gamma \vdash \xi[s] \rightsquigarrow \xi$ by hypothesis, Remark 6 and Lemma 41.

  it is obvious that U' satisfies the conditions of Lemma 613 and thus that the result holds.

Lemma 615

1. Let us suppose that;
   - $\Phi; (P \mid_\xi Q) : B + C, \Gamma \vdash s \rhd \Phi'\Phi; (P \mid_\xi Q) : B + C, \Gamma'\Gamma$
   - $M[t \circ s]$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$
   - $N[t \circ s]$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$
   - $\Phi'\Phi; \Gamma'\Gamma \vdash \Xi \rightsquigarrow \xi$
   - Ξ is strongly normalizing

   then $U = [M \mid^{t}_{\Xi} N][s]$ is in $[\![A]\!]_{\Phi;(P\mid_\xi Q):B+C,\Gamma}$.
2. Let us suppose that;
   - $\Phi; \Gamma \vdash s \rhd \Phi'\Phi; \Gamma'\Gamma$
   - $M[t \circ s]$ in $[\![A]\!]_{\Phi;\Gamma}$
   - $N[t \circ s]$ in $[\![A]\!]_{\Phi;\Gamma}$
   - $\Phi'\Phi; \Gamma'\Gamma \vdash \Xi \rightsquigarrow \mathtt{K}$
   - Ξ is strongly normalizing

   then $U = [M \mid^{t}_{\Xi} N][s]$ is in $[\![A]\!]_{\Phi;(P\mid_\xi Q):B+C,\Gamma}$.

Proof. In both cases since U is neutral, by Lemma 63 (Point 4), it is sufficient to show that all the one step-reducts of U are reducible. Since $M[s \circ t]$ and $N[s \circ t]$ are reducible then they are strongly normalizing by Lemma 63 and thus we can reason by induction on $\nu(M) + \nu(N) + \nu(s) + \nu(t) + \nu(\Xi)$. We reason by case on the possible one-step reduct of U

- If $U \longrightarrow [M \mid^{t}_{\Xi'} N][s]$ with $\Xi \longrightarrow \Xi'$ the result holds by induction hypothesis and by subject reduction (Lemma 412).
- If $U \longrightarrow [M' \mid^{t}_{\Xi} N][s]$ with $M \longrightarrow M'$ the result holds by induction hypothesis since $M'[t \circ s]$ is in $[\![A]\!]_{\Phi;P:B,\Gamma}$ by Lemma 63.
- If $U \longrightarrow [M \mid^{t}_{\Xi} N'][s]$ with $N \longrightarrow N'$ the result holds by induction hypothesis since $N'[t \circ s]$ is in $[\![A]\!]_{\Phi;Q:C,\Gamma}$ by Lemma 63.
- If $U \longrightarrow [M \mid^{t'}_{\Xi} N][s]$ with $t \longrightarrow t'$ the result holds by induction hypothesis both $M[t' \circ s]$ and $N[t' \circ s]$ are reducible since by Lemma 63.
- If $U \longrightarrow [M \mid^{t}_{\Xi} N][s']$ with $s \longrightarrow s'$ the result holds by induction hypothesis and subject reduction since both $M[t \circ s']$ and $N[t \circ s']$ are reducible by Lemma 63.

- If $\Xi = \texttt{L}$ and $U \longrightarrow M[t][s] \equiv M[t \circ s]$ then the result holds by hypothesis.
- If $\Xi = \texttt{R}$ and $U \longrightarrow N[t][s] \equiv N[t \circ s]$ then the result holds by hypothesis.
- If $\Xi = \xi$ and $U \longrightarrow U' = [M[t] \mid_\xi N[t]][s]$ then we are in the condition of Lemma 614 for U' and then U' is reducible.

Lemma 616 Let M be a term in $[\![A]\!]_{\Phi;\Gamma}$ then

1. $t = (\xi/\texttt{L}^{yB}).id$ is in $[\![\xi:\texttt{L},\Phi;y:B,w_1:A,w_2:B,\Gamma]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$.
2. $s = (x/M).(\xi/\texttt{L}^{yB}).id$ is in $[\![\xi:\texttt{L},\Phi;x:A,y:B,w_1:A,w_2:B,\Gamma]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$.
3. $U'' = [\langle x, w_2\rangle \mid_\xi \langle w_1, y\rangle][(x/M).(\xi^{yB}/\texttt{L}).id]$ is in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$
4. $U' = (\lambda x.[\langle x, w_2\rangle \mid_\xi \langle w_1, y\rangle])[(\xi^{yB}/\texttt{L}).id]\ M$ is in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$
5. $\texttt{inl}_B(M)$ is in $[\![A + B]\!]_{\Phi;\Gamma}$

Proof. 1. By Lemma 69, it is sufficient to show that id is in $[\![\Phi;w_1:A,w_2:B,\Gamma]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$ which is obviously true by Corollary 4.

2. By Lemma 67, it is sufficient to show that M is in $[\![A]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$ which is true by Lemma 64 and that t is in $[\![\xi:\texttt{L},\Phi;y:B,w_1:A,w_2:B,\Gamma]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$ which is true by point 1.
3. To show that U'' is in $[\![A{\times}B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$, it is sufficient o show by Lemma 614 that s is in $[\![\xi:\texttt{L},\Phi;x:A,y:B,w_1:A,w_2:B,\Gamma]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$, $\langle x, w_2\rangle[s]$ is reducible and $\langle w_1, y\rangle[s]$ is reducible.
   The first fact holds by the previous point. For the second one we reason as follows: To show that $\langle x, w_2\rangle[s]$ is reducible, it is sufficient to show by lemma 611 that both $x[s]$ and $w_2[s]$ are reducible. This is obvious by definition of reducible substitution.
4. Since U' is well typed and neutral, we can apply Lemma 63 to reason about the reducts of U' by induction on $\nu(M)$.
   U' can only reduce to
   - $(\lambda x.[\langle x, w_2\rangle \mid_\xi \langle w_1, y\rangle])[(\xi/\texttt{L}^{yB}).id]\ M'$ with $M \longrightarrow M'$. Then the result holds by induction hypothesis since M' is reducible by Lemma 63.
   - U'' which is reducible by point 3.
5. Since $\texttt{inl}(M)$ has type $A{+}B$ in $\Phi;\Gamma$, we have to show that $U = \mathcal{S}_{A+B}(\texttt{inl}_B(M))$ is in $[\![A \times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$. Since U is neutral and well-typed, it suffices by Lemma 63 to show that all its one step reducts are reducible.
   We reason by induction on $\nu(M)$. The possible one-step reducts of U are
   - $\mathcal{S}_{A+B}(\texttt{inl}_B(M'))$ with $M \longrightarrow M'$ and then the result holds by induction hypothesis since M' is reducible by Lemma 63.
   - U' and then the result holds by point 4.

Lemma 617 Let M be a term in $[\![B]\!]_{\Phi;\Gamma}$ then $\texttt{inr}_A(M)$ is in $[\![A + B]\!]_{\Phi;\Gamma}$

Proof. c.f. Lemma 616

Lemma 618 If $M[s]$ and $N[s]$ are respectively in $[\![A]\!]_{\Phi;\Gamma}$ and $[\![B]\!]_{\Phi;\Gamma}$, then

- $\texttt{inl}_B(M)[s]$ is in $[\![A + B]\!]_{\Phi;\Gamma}$
- $\texttt{inr}_A(N)[s]$ is in $[\![A + B]\!]_{\Phi;\Gamma}$

Proof. We only prove the first statement, the second one being similar.

To show that $\texttt{inl}_B(M)[s]$ is in $[\![A+B]\!]_{\Phi;\Gamma}$ we have to show that $U = \mathcal{S}_{A+B}(\texttt{inl}_B(M)[s])$ is in $[\![A\times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$.

Since U is neutral , it is sufficient to show that all its one step reducts are reducible. We then proceed by induction on $\nu(M)+\nu(s)$. The term U can only reduce to:

- $\mathcal{S}_{A+B}(\texttt{inl}_B(M')[s])$ where $M \longrightarrow M'$. Then we conclude by induction hypothesis since $M'[s]$ is reducible by Lemma 63.
- $\mathcal{S}_{A+B}(\texttt{inl}_B(M)[s'])$ where $s \longrightarrow s'$. Then we conclude by induction hypothesis since $M[s']$ is reducible by Lemma 63.
- $U' = \mathcal{S}_{A+B}(\texttt{inl}_B(M[s]))$. By hypothesis $M[s]$ is in $[\![A]\!]_{\Phi;\Gamma}$ and thus by Lemma 616, $\texttt{inl}_B(M[s])$ is in $[\![A+B]\!]_{\Phi;\Gamma}$. By definition of reducibility we can conclude that U' is in $[\![A\times B]\!]_{\Phi;w_1:A,w_2:B,\Gamma}$.

Lemma 619

1. If $\langle N_1, N_2\rangle$ is reducible then so is N_1.
2. If $\langle N_1, N_2\rangle$ is reducible then so is N_2.
3. If $\texttt{inl}(N)$ is reducible then so is N.
4. If $\texttt{inr}(N)$ is reducible then so is N.

Proof.

1. By definition of reducibility we know that $U = \mathcal{P}^1_{A\times B}(\langle N_1, N_2\rangle)$ is reducible. Since U is neutral and N_1 is a reduct of U, the result holds by Lemma 63.
2. *c.f.* previous case.
3. By definition of reducibility we know that $U = \mathcal{S}_{A+B}(\texttt{inl}(N))$ is reducible. We remark that $\langle N, w_2\rangle$ is a reduct of U and thus it is reducible by Lemma 63. N is then reducible by point 1.
4. *c.f.* previous case.

Theorem 620 Let $\Psi;\Delta$ and $\Phi;\Gamma$ be valid environments and s be a substitution in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$.

- For every *substitution* t such that $\Phi;\Gamma \vdash t \triangleright \Phi';\Gamma'$, we have $t \circ s$ is in $[\![\Phi';\Gamma']\!]_{\Psi;\Delta}$.
- For every *term* M such that $\Phi;\Gamma \vdash M : A$, we have $M[s]$ is in $[\![A]\!]_{\Psi;\Delta}$.

Proof. By induction on the structure of the (substitution/term) e.

- If $e = id$, then $id \circ s$ is neutral, so that by Lemma 65 we have just to show by induction (since s is strongly normalizing by Lemma 65 (Point 1)) that all the one-step reducts of $id \circ s$ are in $[\![\Phi;\Gamma]\!]_{\Psi;\Delta}$. These reducts are:
  - $id \circ s'$ with $s \longrightarrow s'$. The property holds by induction hypothesis on $\nu(s)$.
  - s. The property holds by hypothesis and Lemma 65.
- If $e = x$, then Γ is not empty since x must be typed in Γ. Moreover, $x : A$ is in Γ and thus $x[s]$ in $[\![A]\!]_{\Psi;\Delta}$ by hypothesis.

- If $e = (x/N).v$, then by induction hypothesis $N[s]$ and $v \circ s$ are respectively in $[\![A]\!]_{\Psi;\Delta}$ and $[\![\Phi'; \Gamma']\!]_{\Psi;\Delta}$ for a type A and a environment $\Phi'; \Gamma'$. Thus, by Lemma 68 $((x/M).v) \circ s$ is in $[\![\Phi'; x\!:\!A, \Gamma']\!]_{\Psi;\Delta}$.
- If $e = v \circ u$, then there exist two environments $\Phi'; \Gamma'$ and $\Phi''; \Gamma''$ such that $\Phi; \Gamma \vdash u \rhd \Phi''; \Gamma''$ and $\Phi''; \Gamma'' \vdash v \rhd \Phi'; \Gamma'$. By application of the induction hypothesis we obtain consequently $u \circ s$ in $[\![\Phi''; \Gamma'']\!]_{\Psi;\Delta}$ and $v \circ (u \circ s)$ in $[\![\Phi'; \Gamma']\!]_{\Psi;\Delta}$. Since $v \circ (u \circ s) =_{Sub\_ass\_env} (v \circ u) \circ s$ then the property holds.
- If $e = (N_1\ N_2)$, then by induction hypothesis $N_1[s]$ and $N_2[s]$ are respectively in $[\![B \rightarrow A]\!]_{\Psi;\Delta}$ and $[\![B]\!]_{\Psi;\Delta}$ for some type B. By definition of reducibility $(N_1[s]\ N_2[s])$ is in $[\![A]\!]_{\Psi;\Delta}$. But $(N_1[s]\ N_2[s]) =_{Sub\_app} (N_1\ N_2)[s]$, so $(N_1 N_2)[s]$ is in $[\![B]\!]_{\Psi;\Delta}$.
- If $e = \lambda P : A.M'$, we have to show that for all N in $[\![A]\!]_{\Psi;\Delta'\Delta}$ the term $U = (\lambda P : A.M')[s]\ N$ is in $[\![B]\!]_{\Psi;\Delta'\Delta}$. Since U is neutral it is sufficient to show that all its one step reducts are in $[\![B]\!]_{\Psi;\Delta',\Delta}$.
  First of all we remark that from the type derivation of $\Phi; \Gamma \vdash \lambda P : A.M' : A \rightarrow B$ we can obtain a type derivation of $\Phi; \Gamma, P\!:\!A \vdash M' : B$ by Lemma 44. Also, from the derivation of $\Psi; \Delta \vdash s \rhd \Phi; \Gamma$ we get by Lemma 41 a type derivation for $\Psi; \Delta, P : A \vdash s \rhd \Phi; P : A, \Gamma$. Now, we can apply the induction hypothesis and we obtain that $M[s]$ is reducible, so that by Lemma 65 (Point 1) it is strongly normalizing, as well as N. We can now proceed by induction on $(P, \nu(M') + \nu(N) + \nu(s))$ to show that U is reducible. The term U can only reduce on:
  - $(\lambda P\!:\!A.M'')[s]\ N$ where $M' \longrightarrow M''$. Then the result holds by induction hypothesis.
  - $(\lambda P : A.M')[s']\ N$ where $s \longrightarrow s'$. Then the result holds by induction hypothesis.
  - $(\lambda P\!:\!A.M')[s]\ N'$ where $N \longrightarrow N'$. Then the result holds by induction hypothesis.
  - $((\lambda P_1.\lambda P_2.M')[s]\ N_1)\ N_2$ if $P = \langle P_1, P_2\rangle$ and $N = \langle N_1, N_2\rangle$. By Lemma 619 we know that both N_1 and N_2 are reducible. By induction hypothesis on P the term $((\lambda P_1 : A_1.\lambda P_2 : A_2.M')[s]\ N_1)$ is then in $[\![A_2 \rightarrow B]\!]_{\Psi;\Delta'\Delta}$ and by definition of reducibility $((\lambda P_1.\lambda P_2.M')[s]\ N_1)\ N_2$ is in $[\![B]\!]_{\Psi;\Delta'\Delta}$ and we are done.
  - If $P = @(P_1, P_2)$ we proceed as in the previous case.
  - $(\lambda P_1.M')[(\xi^{P_2}/\mathtt{L}).s]\ N_1$ if $P = (P_1 \mid_\xi P_2)$ and $N = \mathtt{inl}_B(N_1)$. By Lemma 619, N_1 is reducible. By Lemma 66, s is in $[\![\Phi; \Delta'\Gamma]\!]_{\Psi;\Delta'\Delta}$. By Lemma 69, $(\xi^{P_2}/\mathtt{L}).s$ is in $[\![\xi\!:\!\mathtt{L}, \Phi; P_2\!:\!B, \Delta'\Gamma]\!]_{\Psi;\Delta'\Delta}$ and by Lemma 410 from a type derivation of $\Phi; \Delta'\Gamma, (P_1 \mid_\xi P_2)\!:\!A\!+\!B \vdash M'\!:\!C$ we can obtain a type derivation of $\xi\!:\!\mathtt{L}, \Phi; \Delta'\Gamma, P_1\!:\!A, P_2\!:\!B \vdash M'\!:\!C$. As a consequence, by induction hypothesis, $(\lambda P_1.M')[(\xi^{P_2}/\mathtt{L}).s]$ is in $[\![A_1 \rightarrow B]\!]_{\Psi;\Delta'\Delta}$ and thus by definition of reducibility $(\lambda P_1.M')[(\xi^{P_2}/\mathtt{L}).s]\ N_1$ is in $[\![B]\!]_{\Psi;\Delta'\Delta}$.
  - If $P = (P_1 \mid_\xi P_2)$ and $N = \mathtt{inr}_A(N_1)$, we proceed as in the previous case.
  - $M'[(x/N).s]$ if $P = x$. By Lemma 66, s is in $[\![\Phi; \Delta'\Gamma]\!]_{\Psi;\Delta'\Delta}$. By Lemma 67, $(x/N).s$ is in $[\![\Phi; x : A, \Delta'\Gamma]\!]_{\Psi;\Delta'\Delta}$ and then by induction hypothesis, $M'[(x/N).s]$ is in $[\![B]\!]_{\Psi;\Delta'\Delta}$.

- $M'[s]$ if $P = \_$. The term $M'[s]$ is in $[\![B]\!]_{\Psi;\Delta'\Delta}$ by induction hypothesis.

– If $e = \langle M_1, M_2 \rangle$, then by induction hypothesis $M_1[s]$ and $M_2[s]$ are respectively in $[\![A_1]\!]_{\Psi;\Delta}$ and $[\![A_2]\!]_{\Psi;\Delta}$ and thus the result holds by Lemma 611.
– If $e = \texttt{inl}_B(M')$ or $e = \texttt{inr}_A(M')$ then by induction hypothesis $M'[s]$ is in $[\![A+B]\!]_{\Psi;\Delta}$. Thus the result holds by Lemma 618.
– If $e = [M_1 \mid_\xi M_2]$, the result holds by induction hypothesis and by Lemma 614.
– If $e = [M_1 \mid^{u}_{\Xi} M_2]$, the result holds by Lemma 615 and induction hypothesis.
– If $e = M_1[u]$, then by induction hypothesis $u \circ s$ is in $[\![\Psi'; \Gamma']\!]_{\Psi;\Delta}$ and thus by induction hypothesis again $M_1[u \circ s]$ is in $[\![A]\!]_{\Psi;\Delta}$. Since $M_1[u \circ s] =_{Sub\_clos} M_1[u][s]$ then we are done.

By Theorem 620 and Corollary 4 we have that the term $M[id]$ and the substitution $t \circ id$ are reducible and thus by Lemmas 63 and 65 $M[id]$ and $t \circ id$ turns out to be strongly normalizing so that the following result holds.

Theorem 621 ($\lambda P_{w/\equiv}$ strong normalization) Any well typed $\lambda P_{w/\equiv}$-expression is $\lambda P_{w/\equiv}$ strongly normalizing.

6.3 Strong normalization for $\boldsymbol{\lambda P_w}$

Theorem 622 (λP_w strong normalization) Any well typed λP_w-expression is λP_w strongly normalizing

Proof. c.f. Theorem 514

7 Conclusion

In this paper we have presented a weak calculus λP_w with explicit operations for pattern matching and substitution. Our formalism is successively inspired by [KPT96] and [CK]. In contrast to [KPT96], which treats substitutions and pattern-matching as meta-level operations, we have incorporated them into the syntax of the language by introducing appropriate reduction and typing rules. In contrast to [CK], we have eliminated the use of α-conversion (as our calculus is weak), and we have considered here a more powerful system of substitutions having composition. However, in our opinion, the major progress w.r.t the calculus TPC_{ES} presented in [CK] is that λP_w incorporates *sum replacement* as an *explicit* operation, where the calculus TPC_{ES} uses *meta-level* substitutions for sum variables. This step was one of the main goals of the formalism we propose in this work.

We have shown that the calculus λP_w enjoys all the classical properties of typed λ-calculi, namely it is confluent on all terms, it has the subject reduction property and it is strongly normalizing on all well-typed terms.

In the future, we would like to extend λP_w with *algebraic data types*, in order to cover more realistic functional programming languages, and with more general syntax for binding structures, as for example Klop's CRS [Klo80], in order to cover not only functional programming, but also other programming paradigms.

We would also like to incorporate to our formalism some ideas of the ρ-calculus [CK98] which deals with explicit *pattern matching* in a *rewriting* formalism. In particular, the λP_w-calculus implements a fix pattern-matching algorithm in contrast to ρ-calculus which can be parametrized with different matching algorithms.

Last, but not least, we are studying different evaluation strategies for λP_w, namely *lazy* and *eager* evaluators, which represent concrete implementations of functional languages via the more theoretical notion of reduction system proposed in this paper. We expect that this future work will allow us to provide a better explanation of the interaction between the operations of pattern matching and substitution.

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