

Beta-functions in Yang-Mills Theory and Wilson-Loop

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Abstract

The renormalization group equations of the Yang-Mills theory are examined in the non-critical string theory according to the framework of the holography. Under a simple ansatz for the tachyon, we could find several interesting solutions which are classified by the value of the tachyon potential at the vacuum. We show two typical, asymptotic-free solutions which are different in their infrared behaviors. For both types of solutions, we could obtain quark-confinement potential from the Wilson-loop. The stability of the tachyon and the ZigZag symmetry are also discussed for these solutions.

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1 Introduction

The idea of string description of the super-symmetric gauge theory has been proposed firstly by Maldacena [1] and developed to the case of non-supersymmetric case by several ways [2, 3, 4]. One step in this direction has been proposed by [5, 2] in terms of the non-critical string theory, which is based on the super-symmetric Liouville theory, and developed [6, 7, 8, 9, 10]. This proposal has studied also in the critical type 0B [11, 12, 15, 13, 14, 16] and also in the type IIB [17, 18, 19, 20, 21] string theories.

In the approaches based on the non-supersymmetric global theory, several asymptotic solutions have been discussed both in the ultraviolet and infrared regions. And many authors have tried to connect these fixed points by one renormalization group flow. In order to clear this point, we should find such solutions, which connect those fixed points, by solving the renormalization group equations. Our purpose here is to show such a solution explicitly in the framework of the non-critical string theory for the pure Yang-Mills theory, where ZigZag invariance would be expected [5, 9].

The solutions obtained here are classified into the asymptotically free and non-free types. The asymptotic-free solutions are further separated into two types by their infrared behaviors. One type of solutions has a infrared fixed point with the anti-de Sitter (AdS) background at a finite coupling constant, where the ZigZag invariance is realized. While the β -function of the other type decreases monotonically with the increasing coupling-constant. Estimating the Wilson loops for these asymptotic free solutions, we could obtain the quark-confining potential.

In section two we give the gravitational equations to be solved as the renormalization group equations of the Yang-Mills theory. And the conformal invariant solution with AdS background is shown. The equation to be solved is given as the one of β -function in section three, and the running solutions with and without this AdS fixed point are obtained in the next two sections by assuming that the tachyon is independent on the energy-scale. In section five, the stability of the tachyon is discussed, and the concluding remarks are given in the final section.

2 Gravitational equations and conformal fixed point

The effective action of the non-critical string theory, which is dual to the Yang-Mills theory, could be represented by including the Ramond-Ramond (RR) $p+1$ -form field A_{p+1} .² And it is expected that N D_{p+1} -branes are stacked on the boundary to make the $U(N)$ gauge theory there. Then we start from the following action,

$$S_D = \frac{1}{2\kappa^2} \int dx^D \sqrt{|g|} \left\{ e^{-2\Phi} \left(R - 4(\nabla\Phi)^2 + (\nabla T)^2 + V(T) + c \right) + \frac{1}{2(p+2)!} f(T) F_{p+2}^2 \right\}, \quad (1)$$

² For the sake of the simplicity, we consider only one type of R-R field.

where $c = -(10 - D)/2\alpha'$, and $F_{p+2} = dA_{p+1}$ is the field strength of A_{p+1} . Hereafter we take as $\alpha' = 1$. The total dimension D includes the Liouville direction, which is denoted by r . The tachyon potential is represented by $V(T)$, and $f(T)$ denotes the couplings between the tachyon and the RR field investigated in [11]. Although they might be important, we don't know their precise form. So we search for solutions in the case of $f(T) = 1$ and $T = T_0 = \text{constant}$, for the sake of the simplicity. And the value of $V(T_0)$ is taken as a parameter, which plays an important role to determine the behaviour of the renormalization group flows of the solutions.

The equations of motion are written as

$$R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\Phi = -\nabla_\mu T\nabla_\nu T + e^{2\Phi}T_{\mu\nu}^A \quad (2)$$

$$4\nabla_\mu\Phi\nabla^\mu\Phi - 2\nabla^2\Phi = \frac{D - 2d - 2}{4(p+2)!}e^{2\Phi}F_{p+2}^2 + V_c(T) \quad (3)$$

$$\nabla^2T - 2\nabla_\mu\Phi\nabla^\mu T = \frac{1}{2}V'_c(T) \quad (4)$$

$$\partial_\mu(\sqrt{|g|}F^{\mu\nu_1\cdots\nu_{p+1}}) = 0, \quad (5)$$

where $V_c(T) = c + V(T)$ and

$$T_{\mu\nu}^A = -\frac{1}{2(p+1)!} \left(F_{\mu\nu_1\cdots\nu_{p+1}}F_{\nu_1\cdots\nu_{p+1}} - \frac{g_{\mu\nu}}{2(p+2)}F_{\nu_1\cdots\nu_{p+2}}F^{\nu_1\cdots\nu_{p+2}} \right). \quad (6)$$

For the simplicity, the dimension D is set as $D = p + 2$ and $p = 3$ to consider the case dual to the pure 4-dimensional Yang-Mills theory.

We solve the above equations according to the following ansatz;

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}dr^2 \quad (7)$$

$$\Phi \equiv \Phi(r), \quad T \equiv T(r) \quad \text{and} \quad A_{01\cdots p} = -e^{c(r)}, \quad (8)$$

where $x^\mu, \mu = 0 \sim p$, denote the space-time coordinates. And r represents the Liouville coordinate, which plays the role of the energy-scale in the $p+1$ -dimensional gauge theory [5]. The equation (5) is solved as

$$\partial_r e^{c(r)} = N e^{dA+B} \quad (9)$$

where $d = p + 1$ and N denotes the number of the p -brane. Then the remaining equations (2) and (3) are rewritten as,

$$-\dot{A}\dot{B} + \ddot{A} + d\dot{A}^2 - 2\dot{A}\dot{\Phi} = \frac{N^2}{4}e^{2B+2\Phi}, \quad (10)$$

$$d(\ddot{A} + \dot{A}^2 - \dot{A}\dot{B}) - 2(\ddot{\Phi} - \dot{B}\dot{\Phi}) = \frac{N^2}{4}e^{2B+2\Phi} - \dot{T}^2, \quad (11)$$

$$2(\ddot{\Phi} + \dot{\Phi}(d\dot{A} - \dot{B})) - 4\dot{\Phi}^2 = \frac{(d+1)}{4}N^2e^{2B+2\Phi} + e^{2B}V_c(T) \quad (12)$$

$$\ddot{T} + (d\dot{A} - \dot{B})\dot{T} - 2\dot{\Phi}\dot{T} = \frac{1}{2}e^{2B}V_c'(T) \quad (13)$$

where the dot denotes the derivative with respect to r .

We notice that the above equations have the AdS as a conformal invariant fixed point. We review it briefly. The solution is found by assuming that $\Phi(= \Phi_0)$ and $T(= T_0)$ are independent on r and

$$A = \gamma\rho, \quad B = -\rho, \quad (14)$$

where $\rho = \ln(r/r_0)$ and r_0 denotes an appropriate scale parameter which is taken as unit hereafter for the simplicity. Then we get

$$\lambda_0 \equiv Ne^{\Phi_0} = \sqrt{-\frac{4}{5}V_c(T_0)}, \quad \gamma = \pm \frac{\lambda_0}{4}, \quad V'(T_0) = 0. \quad (15)$$

where λ_0 denotes the 't Hooft coupling constant at this fixed point.

3 Equations for β -Function

Usual way to find renormalization group flows is to see the deviations from the above AdS solution by taking the following functional forms,

$$A(\rho) = \gamma\rho + a(\rho), \quad B(\rho) = -\rho + b(\rho), \quad (16)$$

$$\Phi(\rho) = \Phi_0 + \phi(\rho), \quad T(\rho) = T_0 + t(\rho), \quad (17)$$

where γ, Φ_0 and T_0 are given in (15). This setting is useful for finding the fixed point with the AdS background, where the deviations, $\{a(\rho), b(\rho), \phi(\rho), t(\rho)\}$, vanish at $\rho = \pm\infty$.

While we want to search for more general solutions which are not necessarily asymptotic AdS background. Then we use $A(\rho)$ instead of $a(\rho)$ here. The equation for T , (13), is solved as $t(\rho) = 0$ since we are considering the case of $T = T_0$. Then the remaining equations (10) \sim (12) are rewritten by using $\{A(\rho), b(\rho), \phi(\rho)\}$ as follows:

$$\ddot{A} + \dot{A}(d\dot{A} - \dot{b} - 2\dot{\phi}) = \frac{\lambda_0^2}{4}e^{2(b+\phi)}, \quad (18)$$

$$d[\ddot{A} + \dot{A}(\dot{A} - \dot{b})] - 2(\ddot{\phi} - \dot{b}\dot{\phi}) = \frac{\lambda_0^2}{4}e^{2(b+\phi)}, \quad (19)$$

$$\ddot{\phi} + \dot{\phi}(d\dot{A} - \dot{b} - 2\dot{\phi}) = \frac{5\lambda_0^2}{8}e^{2(b+\phi)} + \frac{V_c(T)}{2}e^{2b}, \quad (20)$$

where the dot denotes the derivative with respect to ρ .

By introducing the following new notations

$$\dot{A} = Q, \quad \dot{\phi} = S, \quad (21)$$

these three equations can be rewritten into the two first order differential equations and one constraint as follows

$$\dot{Q} - Q\dot{b} = b_1, \quad (22)$$

$$\dot{S} - S\dot{b} = b_3, \quad (23)$$

$$b_2 - db_1 + 2b_3 = 0, \quad (24)$$

where

$$b_1 = \frac{\lambda_0^2}{4}e^{2(b+\phi)} - Q(dQ - 2S) \quad (25)$$

$$b_2 = \frac{\lambda_0^2}{4}e^{2(b+\phi)} - dQ^2 \quad (26)$$

$$b_3 = \frac{5\lambda_0^2}{8}e^{2(b+\phi)} - S(dQ - 2S) + \frac{1}{2}V_c(T_0)e^{2b}. \quad (27)$$

We can see that the equation (24) represents the "Hamiltonian" constraint which comes from the reparametrization invariance of the gravitational system with respect to the coordinate r or ρ . In fact, the action (1) can be written as the kinetic term (K) minus the potential term U by using the ansatz (7) \sim (9) as

$$K = \sqrt{g} \left(4(3\dot{A}^2 - 4\dot{A}\dot{\Phi} + \dot{\Phi}^2)e^{-2(b+\Phi)} + N^2/2 \right), \quad (28)$$

$$U = \sqrt{g}V_c(T_0)e^{-2\Phi}, \quad (29)$$

where we used $T = T_0 = \text{const.}$. Then the zero-energy constraint, $T + U = 0$, is written as

$$4(3\dot{A}^2 - 4\dot{A}\dot{\Phi} + \dot{\Phi}^2) + \left(\frac{\lambda^2}{2} + V_c(T_0) \right) e^{2b} = 0, \quad (30)$$

where $\lambda = Ne^\Phi$, which is the running 't Hooft coupling constant. This equation is equivalent to (24). Usually the gauge $b = 0$ is considered, and (24) is used as the boundary condition to solve the equations of A and Φ .

Here, b is not however fixed for a while. And the equation (30) (or (24)) is used to eliminate b in the equations (22) and (23). After that, we find that (22) and (23) are not independent, and we obtain a trivial identity ($0 = 0$) and the following equation

$$(1 - Sb_{1S})\dot{S} - Sb_{1Q}\dot{Q} = b_3 + Sb_{11}, \quad (31)$$

where

$$b_{1Q} = \frac{3Q - 2S}{3Q^2 - 4QS + S^2} \quad (32)$$

$$b_{1S} = \frac{-2Q + S}{3Q^2 - 4QS + S^2} \quad (33)$$

$$b_{11} = -\frac{S\lambda^2}{2V_c + \lambda^2}. \quad (34)$$

This implies the following facts.

(i) One of the two functions, Q or S , can be given as an arbitrary function and the remaining one should be obtained by solving the equation (31). This arbitrariness would come from the fact that b is not yet fixed at this stage.

(ii) As far as the condition (30) is satisfied, it is enough to solve the equation (31) for obtaining the solutions of the original equations.

Our strategy is to rewrite Eq.(31) as the equation of β -function by fixing b and determining Q in an appropriate way as shown below. Before fixing b , we define the β -function of the Yang-Mills theory as follows

$$\beta(\lambda) \equiv \dot{\lambda}, \quad . \quad (35)$$

The functions to be solved (Q and S) are originally introduced as functions of ρ , and the 't Hooft coupling is also the function of ρ as $\lambda = \lambda(\rho)$. So it would be possible to consider Q and S as functions of λ if λ could be regarded as a single valued function of ρ . Then the variable ρ and the function S in the equation (31) can be replaced to λ and β by using the relations, $d/d\rho = \beta(\lambda)d/d\lambda$ and $S = \dot{\Phi} = \beta/\lambda$.

Next we determine Q as a function of β and λ by fixing b as follows. Consider a new coordinate u defined as

$$e^{2b}d\rho^2 = du^2. \quad (36)$$

For this coordinate, we can define new β -function as $\tilde{\beta} = d\lambda/du$. We restrict the analysis to the case where both functions β and $\tilde{\beta}$ represent essentially the same β -function. Then we assume $\beta/\tilde{\beta} = 1$ simply. This assumption leads to the gauge,

$$\left(\frac{du}{d\rho}\right)^2 = e^{2b} = 1. \quad (37)$$

From this we obtain,

$$Q = \frac{1}{3\lambda} \left(2\beta \pm \sqrt{\beta^2 - \frac{3}{8}\lambda^2(\lambda^2 + 2V_c)} \right). \quad (38)$$

Using this, Eqs.(31) is rewritten as

$$\beta' = \frac{\beta}{3\lambda} + \frac{4V_c + 5\lambda^2}{8\beta}\lambda \mp \frac{4}{3\lambda}\sqrt{\beta^2 - \frac{3}{8}\lambda^2(\lambda^2 + 2V_c)}, \quad (39)$$

where prime denotes the derivative with respect to λ , for example $\beta' = d\beta/d\lambda$. $V_c = V_c(T_0)$ is a constant. The sign \mp of the third term on the right hand side implies that if β would be a solution of the equation of either sign then $-\beta$ is the solution of the equation of the opposite sign. In this sense, it is enough to consider the equation of either sign.

In deriving equation (39), the overall factor $(3Q - 2S)$ has been divided out since we can see that $3Q = 2S$ is not the solution of the original equations.

4 Analytic solutions: $V_c(T_0) = 0$

First, we consider the analytic solution obtained for $V_c = 0$. It is easy to find the following solution,

$$\beta = \frac{5}{2}\lambda Q = -\sqrt{\frac{25}{24}}\lambda^2, \quad (40)$$

where we solved the equation (39) with the minus sign of the third term. This is the exact solution, but it contains only one-loop term. In this sense, this corresponds to the β -function of the super-symmetric Yang-Mills theory, but this should be considered as the approximate form at small λ in our model as mentioned below.

For this solution, we obtain $e^A = \lambda^{2/5}$. Then the background in the asymptotic free limit of this solution is not the AdS_5 as found in [14, 12], and it is considered as the "ZigZag" horizon [5, 9] since $e^A \rightarrow 0$ in the limit of $\lambda \rightarrow 0$. This horizon is situated at the ultraviolet limit, $\rho \rightarrow \infty$, contrary to the expectation of [9]. As for the five dimensional scalar curvature $R^{(5)}$, it is obtained as $R^{(5)} = R_0\lambda^2$ with $R_0 = 20/3$. Then $R^{(5)}$ grows with increasing λ , so the higher-order terms of the curvature would become important in the effective action at large λ . And there is no reason to prevent those terms in our model. As a result this solution would be modified in the infrared region

This point is understood from the viewpoint of the availability of the holographic method considered here. For $D = 10$ and $p = 3$, the conditions of the reliability of correspondence between the gravitational equations and the renormalization group equations of the Yang-Mills theory are given by $N, \lambda \gg 1$. While they are changed to $N \gg \lambda^{5/2}$ and $1 \gg \lambda$ if we take the setting, $D = p+2$ and $p = 3$, seriously. The latter condition, $1 \gg \lambda$, is obtained from the requirement that the higher order terms of α' in the effective action are negligible in the D-brane system. It can be expressed as $\lambda^{1/(D-p-3)} \gg 1$ for general D and p . Then the higher curvature terms would be necessary to apply the holographic picture at large λ region contrary to the case of the critical string theory.

Although the solution would be unreliable at large λ , we can estimate the Wilson loop since the solution is exact. In [22], the Wilson-loop is expressed by the integral with respect to the coordinate transverse. Here, we rewrite this expression by replacing the integral variable from the coordinate to the coupling constant λ as

$$\int_{r_0}^{r_M} \frac{dr}{r} = - \int_{\lambda_M}^{\lambda_{IR}} \frac{d\lambda}{\beta(\lambda)}, \quad (41)$$

where λ_{IR} (λ_M) represents the upper (lower) bound of λ at the infrared (ultraviolet) side. Then the distance L between $q\bar{q}$ and the extremized string action S are given as

$$L = -2Er_0 \int_{\lambda_M}^{\lambda_1} \frac{d\lambda}{\beta} \frac{e^{-A+b}}{\sqrt{e^{4A} - E^2}}, \quad (42)$$

$$S = -\frac{\tau r_0}{2\pi} \int_{\lambda_M}^{\lambda_1} \frac{d\lambda}{\beta} \frac{e^{3A+b}}{\sqrt{e^{4A} - E^2}}, \quad (43)$$

where E is an arbitrary constant introduced in minimizing the string action [22], and τ denotes the time-interval of the Wilson-loop. Here the lower bound is given as $\lambda_M = E^{5/4}$ since $e^A = \lambda^{2/5}$, and the upper bound is infinite. Then we obtain the following result,

$$S = S_3 L^{3/7}, \quad S_3 = \tau \frac{\sqrt{3/2r_0} B(3/8, 1/2)}{4\pi [\sqrt{3/2r_0} B(11/8, 1/2)]^{1/5}}. \quad (44)$$

This result shows the confinement potential, but S increases more slowly with L than the linearly rising potential which is expected from the well known QCD. The second unexpected point is that the potential does not show the Coulomb behaviour near $L = 0$. This point could be understood as follows. We obtain $L = L_0 E^{-7/4}$, where L_0 is a positive constant, then the potential at small L is determined by the information at large E . While the lower bound of λ is given by $\lambda_M = E^{5/4}$, so the behaviour of the potential at small L is given by the large coupling dynamics. As a result the Coulomb behaviour could not be seen at small L .

Where is the AdS fixed point in this solution? This solution is obtained for $V_c(T_0) = 0$, then the AdS fixed point at $\lambda_0 (= \sqrt{-4V_c/5})$ is pushed to $\lambda = 0$. And the scalar curvature becomes zero since the radius of the AdS space becomes infinite. In this sense, the usual AdS solution could not be seen. As shown in the next section, it is necessary to consider the case of $V_c < 0$ in order to find a solution including the AdS fixed point at finite λ .

5 Numerical solutions: $V_c(T_0) < 0$

In the case of $V_c(T_0) < 0$, we could show solutions which contain the AdS fixed point by solving Eq.(39) numerically.

Before performing the numerical analysis, we notice some points implied by Eq.(39) to consider an appropriate boundary conditions to solve the equation. (i) Firstly, it can be read that two fixed points are possible at $\lambda = 0$ and $\lambda_0 (= \sqrt{-4V_c/5})$. If β would be zero at some other point, then the third term of the right hand side of (39) diverges. So the solutions for β could not have zero points except for the above two points. (ii) Secondly, Eq.(39) represents two equations discriminated by the sign of the third term. As mentioned above, this freedom can be reduced to the sign of β . Then it is enough to solve the equation in either sign since the other solution can be obtained by reflecting its sign. (iii) Thirdly, β could be obtained near $\lambda = 0$ as

$$\beta = \beta_0^\pm \lambda + \dots, \quad \beta_0^\pm = \pm \frac{1}{2} \sqrt{-V_c}. \quad (45)$$

The coefficient should be taken as $\beta_0^- = -\frac{1}{2} \sqrt{-V_c}$ for the asymptotic-free solution. But we notice that this result is a little undesirable since the leading term is not quadratic with respect to λ as expected from the perturbative Yang-Mills theory.

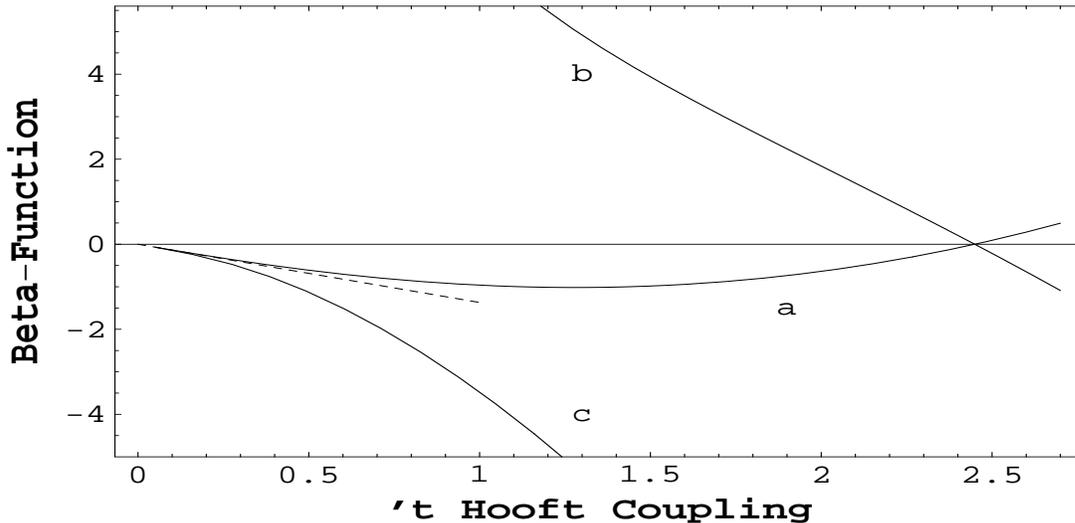


Figure 1: The numerical solution of $\beta(\lambda)$ obtained by (39) with the minus sign of the third term and $V_c = -15/2$. The three solid curve, (a), (b) and (c), are obtained by taking the boundary condition given in the text. The dotted line is the asymptotic solution approximated near $\lambda = 0$, $\beta = \beta_0^- \lambda$.

This linear form has also been found in the type 0 string model [14, 12]. (iii) Near the AdS fixed point, $\lambda = \lambda_0$, the following expansion form can be found,

$$\beta = \beta'_\pm(\lambda_0)(\lambda - \lambda_0) + \dots, \quad \beta'_\pm(\lambda_0) = \frac{-1 \pm \sqrt{6}}{2} \lambda_0. \quad (46)$$

This result indicate that the AdS fixed point is either the ultraviolet or the infrared fixed point depending on the sign of $\beta'(\lambda_0)$. In principle, both solutions are possible.

We solve Eq.(39) with the minus sign of the third term of its right hand side since the asymptotic free solutions can be obtained by this choice. The value of the potential is taken as $V_c(T_0) = -15/2$ for simplicity. Although there are many solutions depending on the boundary condition, they are separated into two groups, the asymptotic free and non-free solutions. The typical solutions are shown in the Fig.1. We should notice that the functions given by the reflection, $\beta \rightarrow -\beta$, of the solutions shown in the Fig.1 are also the solutions which are not shown in the figure.

The solution (b) has been obtained by the boundary condition, $\beta(\lambda_0 + \epsilon) = -\beta'_+(\lambda_0)\epsilon$, where ϵ is a very small number. This solution has the ultraviolet fixed point at λ_0 and diverges to ∞ ($-\infty$) at $\lambda = 0$ ($\lambda = \infty$).

While the solution (a) is obtained by the boundary condition, $\beta(\lambda_0 + \epsilon) = \beta'_+(\lambda_0)\epsilon$. This condition is the reflection on the λ -axis of the above boundary condition for the solution (b), but it does not leads to the solution given by changing the sign of the solution (b). The reflected solution of (b) is obtained from Eq.(39) of the plus sign of the third term with the same boundary condition for the solution (b).

The solution (a) crosses the λ -axis at $\lambda = \lambda_0$ as the infrared fixed point and approaches to zero as $\lambda \rightarrow 0$ according to the asymptotic solution $\beta = \beta_0^- \lambda = -\frac{1}{2}\sqrt{-V_c}\lambda$ which is shown by the dotted line in the Fig.1. Then the asymptotic-free and the AdS infrared fixed points are smoothly connected by this solution. We can further see $e^{A(\lambda)} \rightarrow 1$ as $\lambda \rightarrow 0$, then the background at the asymptotic-free fixed point is not the AdS space but the flat space-time. So this solution denotes the renormalization group flow between two different background configurations, flat and AdS spaces. It should be noticed for this solution that $e^{2A} \rightarrow 0$ as $\lambda \rightarrow \lambda_0$, near the AdS fixed point. This is seen from the approximate form of e^{2A} obtained near $\lambda \sim \lambda_0$,

$$e^{2A} = e^{\frac{1}{2}\lambda_0\rho} = (\lambda_0 - \lambda)^{\frac{\sqrt{6}+1}{5}}. \quad (47)$$

Here we notice the following relation, $\rho = \ln(\lambda - \lambda_0)/\beta'_+(\lambda_0)$, in this region. Then this fixed point can be considered as the ZigZag horizon, which is situated at the infrared boundary $\rho = -\infty$ as expected in [9]. The five dimensional scalar curvature $R^{(5)}$ is finite since this point corresponds to the AdS space, and it is given by $R^{(5)} = \frac{5}{4}\lambda_0^2 = |V_c|$ at $\lambda = \lambda_0$.

The third solution (c) in the Fig.1 is obtained by the condition, $\beta(\epsilon) = \beta_0^- \epsilon$, but it can not arrive at the AdS fixed point and diverges to $-\infty$ at large λ . This solution has the similar form to the one given in the previous section for $V_c = 0$, but the asymptotic form near $\lambda = 0$ is different. As in the solution (a), $(Q =) \dot{A}$ approaches to zero as $\lambda \rightarrow 0$ and we obtain $e^{2A} = 1$ at $\lambda = 0$, while e^{2A} increases monotonically with increasing λ . Then the background of the asymptotic-free fixed point of this solution is also the flat five dimensional space-time, and there is no ZigZag horizon in this solution. These points are also different from the solution of $V_c = 0$.

Next task is to evaluate the Wilson loop for these solutions. For solutions (b) and (c), we need the information of the infinite range of λ for the analysis in the asymptotic-free phase. But it seems to be difficult to do it since we do not know the analytic form of the solutions contrary to the case of the previous section. While, the asymptotic free phase for the solution (a) is restricted to the finite region of λ , $0 < \lambda < \lambda_0$. Then it is possible to estimate numerically the Wilson loop for this solution according to the formula given above. We notice some remarks in performing this numerical analysis.

(i) The form of e^A obtained here is shown in the Fig.2. It implies that the analysis should be performed in the region $0 < E < 1$, and the upper limit in the integration of λ should be introduced for each E .

(ii) e^A approaches one at $\lambda = 0$ as stated above, so the integration has the logarithmic divergence near $\lambda = 0$ since $\beta \rightarrow \beta_0^- \lambda$ for $\lambda \rightarrow 0$. Then we should evaluate the integrals by subtracting this divergence, and the following subtracted form for S and L are used,

$$L = -2Er_0 \int_{\lambda_M}^{\lambda_1} \frac{d\lambda}{\beta} \left(1 - \frac{\beta}{\beta_0\lambda}\right) \frac{e^{-A+b}}{\sqrt{e^{4A} - E^2}}, \quad (48)$$

$$S = -\frac{\tau r_0}{2\pi} \int_{\lambda_M}^{\lambda_1} \frac{d\lambda}{\beta} \left(1 - \frac{\beta}{\beta_0\lambda}\right) \frac{e^{3A+b}}{\sqrt{e^{4A} - E^2}}. \quad (49)$$

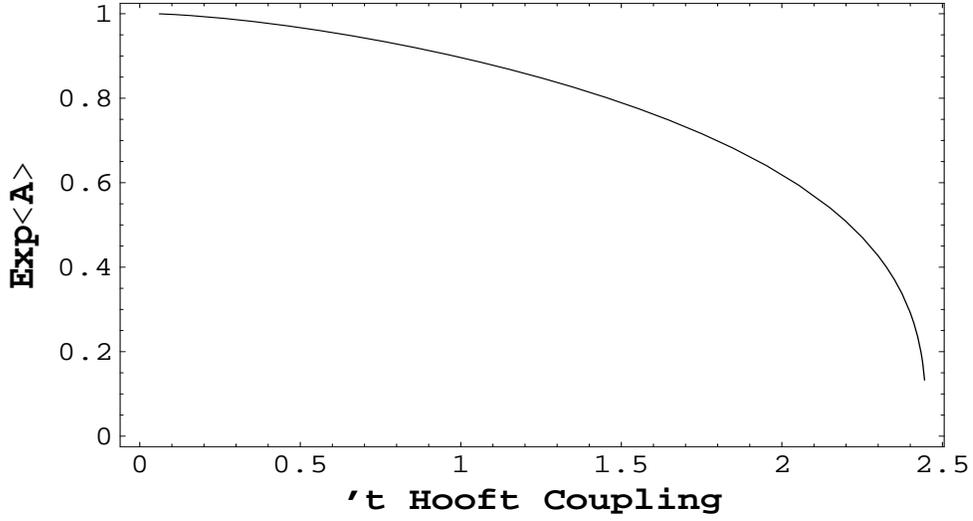


Figure 2: The form of $e^{A(\lambda)}$ for the solution (a) is shown by the curve. The horizontal line is the λ -axis.

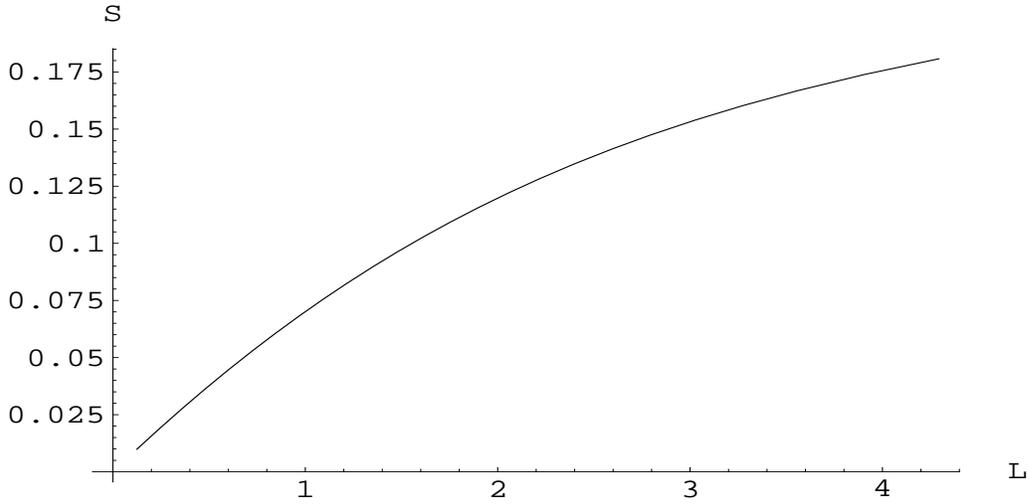


Figure 3: The numerical result of $Q\bar{Q}$ potential extracted from the Wilson-loop is shown for the solution (a). The potential near $L = 0$ is estimated analytically in the text.

The result is shown in the Fig.3. The estimation in the region of small L is suppressed in this figure since a special care is needed in this region where both β and e^A approach to zero. So we estimate S and L analytically in this region as shown below. For $\lambda \sim \lambda_0$, we obtain

$$\beta = -\beta'_+(\lambda_0)x, \quad e^A = x^\alpha, \quad (50)$$

where $x = \lambda_0 - \lambda$, $\alpha = \lambda_0/(4\beta'_+(\lambda_0))$ and the higher order terms of x are suppressed. By using these approximate formula, we obtain the following result

$$S = -\frac{B^2(3/4, 1/2)}{16\pi} \frac{1}{L}. \quad (51)$$

where $B(a, b)$ denotes the beta function. Then the attractive Coulomb potential can be seen for this solution at small L , but this Coulomb behaviour does not reflect the dynamics of the asymptotic free region. It should be the reflection of the dynamics near the infrared fixed point.

6 Stability of Tachyon

Finally, we give a brief comment on the stability of the tachyon fluctuation around our solutions. Here the tachyon was not "running" since the classical equations are solved for the case of $T = T_0$, which is independent on the energy scale ρ . But we must pay attention for its fluctuation, which is denoted by t , around T_0 from the viewpoint of the stability of our classical solutions in the five dimensional theory. This is examined by solving the linearized equation for t , which is obtained from (13) and (15) as

$$\ddot{t} + (d\dot{A} - 2\dot{\phi})\dot{t} = \frac{1}{2}V''(T_0)t. \quad (52)$$

Here dot denotes the derivative with respect to the new variable u , but it is equivalent to ρ since we set as $du/d\rho = 1$. The condition for the stability is the existence of the solution of Eq.(52) in the form $t = e^{\alpha u}$ with real α . Since the coefficient of \dot{t} is dependent on u , then we examine this equation near the fixed points where the coefficient can be approximated by a constant.

First, we consider near the asymptotic free region, $\lambda \sim 0$. (i) For the solution of $V_c = 0$, the coefficient disappears at $\lambda = 0$, namely $d\dot{A} - 2\dot{\phi} = 0$. Then t will be stable if

$$V''(T_0) > 0, \quad (53)$$

which means the positive mass-squared of the tachyon around $T = T_0$. (ii) For the solutions of $V_c < 0$, solutions (a) and (c), we have $d\dot{A} - 2\dot{\phi} = -2\beta_0^-$. So the stability condition is obtained as

$$\frac{1}{2}V''(T_0) - \frac{V_c(T_0)}{4} > 0. \quad (54)$$

In this case, the condition is satisfied even if $V''(T_0)$ is negative for its small absolute value. This situation is similar to the case of the AdS background.

In the case of solution (a), the coefficient can be estimated at the AdS fixed point $\lambda = \lambda_0$, where we get $d\dot{A} - 2\dot{\phi} = 3\lambda_0$. Then the stability condition at this point is given by $\frac{1}{2}V''(T_0) - \frac{9V_c(T_0)}{5} > 0$. Since $V_c < 0$, this condition is satisfied if (54) is fulfilled. Then we can say that the tachyon would be stable if the condition (54) is fulfilled for the solution (a).

7 Conclusions and Discussion

We have examined the equations of non-critical string theory as the renormalization group equations of the Yang-Mills theory on the boundary. The analysis is restricted to the five dimensions to see the properties of the pure Yang-Mills theory. The tachyon is assumed to be a constant to simplify the model, and the equations are rewritten as the differential equations with respect to the 't Hooft coupling constant λ .

In terms of this simple model, we could find several interesting solutions. The β -function of the corresponding Yang-Mills theory could have two zero (or fixed) points at $\lambda = 0$ and λ_0 . The latter point (λ_0) is corresponding to the well-known fixed point with the AdS background. While, the background at the asymptotic-free fixed point ($\lambda = 0$) is the flat space or non-AdS space with zero curvature. Among the various solutions, two types (called as type (a) and (c)) of asymptotic-free solutions are found. They are classified by their infrared behaviour. For the solution of type (c), the β -function decreases monotonically with increasing λ . While the type (a) solution connects two fixed points at $\lambda = 0$ and λ_0 with different background configurations smoothly, and the asymptotic-free phase is restricted to the finite region of λ .

For the type (a) and the analytic type (c) solutions, the ZigZag horizons have been found at the infrared and ultraviolet fixed point respectively. The Wilson loops have been estimated for these solutions and we found the quark-confinement potentials for both solutions. But we should notice several quantitative points which are undesirable from the standpoint of the QCD. (i) The potential obtained grows with the distance between quarks more slowly than the linear rising one. (ii) For the solutions with negative tachyon-potential, the β -functions is proportional to λ near $\lambda = 0$ contrary to the expected form of $\beta \propto \lambda^2$. While this quadratic behaviour has been obtained as an analytic solution in the case of zero tachyon-potential. (iii) But the Coulomb like potential at small distance could not be seen for the latter case.

The availability of our analysis based on the holographic idea would be restricted to the small λ region as mentioned in section four. In this sense, the solution of type (a) would be reliable even in the infrared region if we consider in the asymptotic free phase and V_c is taken to be small.

We could obtain the stability condition for the tachyon-fluctuation around our solutions at the ultraviolet and infrared fixed points, and we found that condition would be satisfied even if tachyon has negative mass-squared as in the AdS background case.

References

- [1] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231, [hep-th/9711200](#).
- [2] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov Phys. Lett. **B 428** (1998) 105, [hep-th/9802109](#).
- [3] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253, [hep-th/9802150](#).
- [4] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 505, [hep-th/9803131](#).
- [5] A.M. Polyakov, Int. J. Mod. Phys. **A14** 645, [hep-th/9809057](#).
- [6] G. Ferretti and D. Martelli, Adv. Theor. Math. Phys. **3**(1999) 119 ([hep-th/9811208](#)).
- [7] G. Ferretti and D. Martelli, Nucl. Phys. **B555** (1999)135 ([hep-th/9904013](#)).
- [8] A. Armoni, E. Fuchs and J. Sonnenschein, J. High Energy Phys. JHEP **06**(1999) 027 ([hep-th/9903090](#)).
- [9] E. Alvarez and C. Gomez, Nucl. Phys. **B550** (1999) 169 ([hep-th/9902012](#)).
- [10] K. Ghoroku, J. Phys. G: Nucl. Phys.**26** (2000) 233.
- [11] I.R. Klebanov and A.A. Tseytlin, Nucl. Phys. **B546** (1999) 155 ([hep-th/9811035](#)).
- [12] I.R. Klebanov and A.A. Tseytlin, Nucl. Phys. **B547** (1999) 143 ([hep-th/9812089](#)).
- [13] J.A. Minahan, J. High Energy Phys. JHEP **01** (1999) 020 ([hep-th/9811156](#)).
- [14] J.A. Minahan, J. High Energy Phys. JHEP **04** (1999) 007 ([hep-th/9902074](#)).
- [15] I.R. Klebanov and A.A. Tseytlin, J. High Energy Phys. JHEP **03** (1999) 015 ([hep-th/9901101](#)).
- [16] R. Grena, S. Letti, M. Maggiore and A. Rissone, [hep-th/0005213](#).
- [17] A. Kehagias and K. Sfetsos, Phys. Lett. **B454** (1999) 270 ([hep-th/9902125](#)); Phys. Lett. B **456** (1999) 22 ([hep-th/9903109](#)).
- [18] S. Nojiri and S.D. Odintsov, [hep-th/9905200](#); [hep-th/9906216](#); Phys. Lett. B **449** (1999) 39.
- [19] S.S. Gubser, [hep-th/9902155](#).
- [20] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, J. High Energy Phys. JHEP **05** (1999) 026 ([hep-th/9903026](#)).

- [21] N.R. Constable and R.C. Myers, J. High Energy Phys. JHEP **11** (1999) 020
([hep-th/9905081](#)).
- [22] J. Maldacena, Phys. Rev. Lett. **80** (1998) 4859,