K-STABILITY OF CONSTANT SCALAR CURVATURE POLARIZATION

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ABSTRACT. In this paper, we shall show that a polarized algebraic manifold is K-stable if the polarization class admits a Kähler metric of constant scalar curvature. This generalizes the results of Chen-Tian [1], Donaldson [4] and Stoppa [20].

1. Introduction

Yau's conjecture often referred to as the Hitchin-Kobayashi correspondence for manifolds suggests a strong correlation between stability of polarized algebraic manifolds and the existence of extremal metrics in the polarization class. Especially, for Kähler metrics of constant scalar curvature, the following is known as an interesting conjecture:

Conjecture (Tian [21], Donaldson [2]). *A polarized algebraic manifold (M, L) is K-stable if and only if the polarization class $c_1(L)_{\mathbb{R}}$ admits a Kähler metric of constant scalar curvature.*

For "if" part of this conjecture, Chen-Tian [1] and Donaldson [4] showed that a polarized algebraic manifold (M, L) is K-semistable when the class $c_1(L)_{\mathbb{R}}$ admits a Kähler metric of constant scalar curvature. Very recently, by an effective use of moduli spaces, Stoppa [20] proved a stronger result showing that a polarized algebraic manifold (M, L) with a Kähler metric of constant scalar curvature in $c_1(L)_{\mathbb{R}}$ is K-stable if the group $\mathrm{Aut}(M, L)$ of holomorphic automorphisms of (M, L) is discrete. The purpose of this paper is to extend Stoppa's result to the following general case without assuming such discreteness:

Main Theorem. *A polarized algebraic manifold (M, L) is K-stable if the class $c_1(L)_{\mathbb{R}}$ admits a Kähler metric of constant scalar curvature.*

It should be emphasized that one of the main ingredients of this paper is the energy-theoretic approach to K-stability as in Tian [21] (see also [1]), where in our actual proof of Main Theorem, the Chow norm is used in place of the K-energy.

This paper is organized as follows: In Section 2, we fix notation by defining K-stability. In Section 3, by [10], we describe the asymptotic behavior of the k-th weighted balanced metric ω_k, as $k \to \infty$. In Section 4, we study the relationship between the Futaki invariant of a test configuration and the asymptotic behavior of the Chow norm of fibers (cf. Lemma 4.8). Finally in Section 5, based on the preceding sections, we give a proof for Main Theorem by developing the Chow norm method in [8] and [9], where the results of Phong and Sturm [17] are used to estimate the second derivative of the Chow norm.

2. K-stability

In this paper, by a *polarized algebraic manifold* (M, L), we mean a pair of a smooth projective algebraic variety M, defined over $\mathbb{C}$, and a very ample line bundle L over M. Let H be the maximal connected linear algebraic subgroup of the identity component $\mathrm{Aut}^0(M)$ of the group of all holomorphic automorphisms of M, so that $\mathrm{Aut}^0(M)/H$ is an Abelian variety (cf. [5]). Replacing L by its suitable positive integral multiple if necessary, we can choose an H-linearization of L (cf. [15]). Fix the natural action of the group $T := \mathbb{C}^*$ on the complex affine line $\mathbb{A}^1 := \{\, z\,;\, z \in \mathbb{C}\,\}$ by multiplication of complex numbers,

$$T \times \mathbb{A}^1 \to \mathbb{A}^1, \qquad (t, z) \mapsto tz.$$

Let $\pi : \mathcal{M} \to \mathbb{A}^1$ be a T-equivariant projective morphism between complex varieties with an invertible sheaf $\mathcal{L}$ on $\mathcal{M}$, relatively very ample over $\mathbb{A}^1$, where the algebraic group T acts on $\mathcal{L}$, linearly on fibers, lifting the T-action on $\mathcal{M}$. For each $z \in \mathbb{A}^1$, we put

$$\mathcal{L}_z := \mathcal{L}_{|\mathcal{M}_z},$$

where $\mathcal{M}_z := \pi^{-1}(z)$ denotes the scheme-theoretic fiber of π over z. Then the following notion of a test configuration is defined by Donaldson [2] in his reformulation of K-stability by Tian [21]. Actually, the pair $(\mathcal{M}, \mathcal{L})$ with a flat family

$$\pi : \mathcal{M} \to \mathbb{A}^1$$

is called a *test configuration* for (M, L), if for some positive integer ℓ, there exist the following isomorphisms of polarized algebraic manifolds

$$(\mathcal{M}_z, \mathcal{L}_z) \cong (M, \mathcal{O}_M(L^\ell)), \qquad 0 \neq z \in \mathbb{A}^1. \tag{2.1}$$

In the special case when $\mathcal{M} = M \times \mathbb{A}^1$, a test configuration is called a *product configuration*, where for such a configuration, T does not necessarily act on the first factor M trivially.

Given a test configuration $\pi : \mathcal{M} \to \mathbb{A}^1$ for (M, L), we consider the vector bundles E_m over $\mathbb{A}^1$ by

$$\mathcal{O}_{\mathbb{A}^1}(E_m) \;=\; \pi_* \mathcal{L}^m, \qquad m = 1, 2, \dots,$$

associated to the direct image sheaves $\pi_* \mathcal{L}^m$. Then E_m admits a natural T-action $\rho_m : T \times E_m \to E_m$ induced by the T-action on $\mathcal{L}$. Consider the fibers $(E_m)_z$, $z \in \mathbb{A}^1$, of the bundle E_m over z. Since the fiber

$$(E_m)_0 \;=\; (\pi_* \mathcal{L}^m)_0 \otimes \mathbb{C}$$

over the origin is preserved by the T-action ρ_m, we can talk about the weight w_m of the T-action on $\det (E_m)_0$. Put $n := \dim_{\mathbb{C}} M$, and we consider the degree d_m of the image of the Kodaira embedding

$$\Phi_{|L^{\ell m}|} : M \hookrightarrow \mathbb{P}^*(V_m), \tag{2.2}$$

where $\mathbb{P}^*(V_m)$ is the set of all hyperplanes in $V_m := H^0(M, \mathcal{O}_M(L^{\ell m}))$ through the origin. Put $N_m := \dim (E_m)_0 = \dim V_m$. Then for $m \gg 1$,

$$\begin{cases} N_m &= a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0, \\ w_m &= b_{n+1} m^{n+1} + b_n m^n + \ldots + b_1 m + b_0, \end{cases} \tag{2.3}$$

for some rational numbers a_i, $b_j \in \mathbb{Q}$ independent of the choice of m. Note here that $a_n = \ell^n c_1(L)^n[M]/n! > 0$. If $m \gg 1$, then we have

$$\frac{w_m}{m\, N_m} \;=\; F_0 + F_1 m^{-1} + F_2 m^{-2} + \ldots, \tag{2.4}$$

with coeficients $F_i = F_i(\mathcal{M}, \mathcal{L}) \in \mathbb{Q}$ independent of the choice of m. In particular $F_1 = F_1(\mathcal{M}, \mathcal{L})$ is called the *Futaki invariant* for the test configuration. The following concept originally introduced by Tian [21] is given in the present form by Donaldson [2]:

Definition 2.5. (i) (M, L) is said to be *K-semistable*, if the inequality $F_1(\mathcal{M}, \mathcal{L}) \leq 0$ holds for all test configurations $(\mathcal{M}, \mathcal{L})$ for (M, L).

(ii) Let (M, L) be K-semistable. Then (M, L) is said to be *K-stable*, if for every test configuration $(\mathcal{M}, \mathcal{L})$ for (M, L), it reduces to a product configuration if and only if $F_1(\mathcal{M}, \mathcal{L})$ vanishes.

In this paper, we fix once for all a test configuration $(\mathcal{M}, \mathcal{L})$ of a polarized algebraic manifold (M, L) which admits a Kähler metric ω_∞ in $c_1(L)_{\mathbb{R}}$ of constant scalar curvature. Obviously $F_i(\mathcal{M}, \mathcal{L}^j)$ coincides with $F_i(\mathcal{M}, \mathcal{L})$ for all positive integers i and j, and hence to discuss K-stability of (M, L), replacing $\mathcal{L}$ by its suitable positive multiple if necessary, we may assume that $\dim H^0(\mathcal{M}_z, \mathcal{L}_z^m) = \dim H^0(\mathcal{M}_0, \mathcal{L}_0^m)$ and that the natural homomorphisms

$$\otimes^m H^0(\mathcal{M}_z, \mathcal{L}_z) \to H^0(\mathcal{M}_z, \mathcal{L}_z^m), \qquad m = 1, 2, \dots,$$

are surjective for all $z \in \mathbb{A}^1$. We can see this easily by the fact that, if $\mathcal{L}$ is replaced by its very high multiple while L is fixed, then ℓ becomes large so that the assumptions above are automatically satisfied (cf. [13]; see also [11], Remark 4.6).

Finally, as remarked in [4], Lemma 2, we have the following theorem of equivariant trivialization for E_m:

Fact 2.6. *Let H_1 be a Hermitian metric on the vector space $(E_m)_z$ at $z = 1$. Then there is a T-equivariant trivialization*

$$E_m \cong \mathbb{A}^1 \times (E_m)_0 \tag{2.7}$$

taking H_1 to a Hermitian metric, denoted by H_0, on the central fiber $(E_m)_0$ which is preserved by the action of $S^1 \subset \mathbb{C}^* (= T)$ on $(E_m)_0$.

3. Asymptotic behavior of weighted balanced metrics

Now choose a Hermitian metric h_∞ for L such that $\omega_\infty = c_1(L, h_\infty)$. Let ℓ be as in (2.1). Following [10], Section 2, we here study the asymptotic behavior of the weighted balanced metrics for polarized algebraic manifolds $(M, L^{m\ell})$ as $m \to \infty$. For the linear algebraic group H in the previous section, choose the maximal compact subgroup K of H such that ω_∞ is K-invariant (cf. [6]). Then for the identity component Z of the center of K, take its complexfication $Z^{\mathbb{C}}$ in H. For the H-linearization of L in the previous section, there exist mutually distinct characters $\chi_{m;1}, \chi_{m;2}, \dots, \chi_{m;\nu_m} \in \mathrm{Hom}(Z^{\mathbb{C}}, \mathbb{C}^*)$ such that the vector space V_m is written as a direct sum

$$V_m \;=\; \bigoplus_{i=1}^{\nu_m} V(\chi_{m;i}),$$

where $V(\chi) := \{\,\sigma \in V_m\,;\, g\cdot\sigma = \chi(g)\sigma \text{ for all } g \in Z^{\mathbb{C}}\}$ for all $\chi \in \mathrm{Hom}(Z^{\mathbb{C}}, \mathbb{C}^*)$. Let $\mathfrak{z}$ be the real Lie subalgebra of $H^0(M, \mathcal{O}(T^{1,0}M))$ corresponding to the real Lie subgroup Z of $\mathrm{Aut}(M)$. Put $\hat{\mathfrak{z}} := \sqrt{-1}\,\mathfrak{z}$. Let h be a Hermitian for L such that $\omega = c_1(L;h)$ is a K-invariant Kähler form. Define a K-invariant Hermitian pairing $\langle\ ,\ \rangle_h$ for V_m by

$$\langle \sigma, \sigma'\rangle_h \;:=\; \int_M (\sigma,\sigma')_h\,\omega^n, \qquad \sigma, \sigma' \in V_m, \tag{3.1}$$

where $(\sigma,\sigma')_h$ denotes the pointwise Hermitian inner product of σ, σ' by the m-multiple of h. Then by this Hermitian pairing $\langle\ ,\ \rangle_h$, we have

$$V(\chi_{m;i}) \;\perp\; V(\chi_{m;j}), \qquad i \neq j.$$

Put $n_{m;i} := \dim_{\mathbb{C}} V(\chi_{m;i})$. Let P_m be the set of all pairs (i,α) of integers such that $1 \leq i \leq \nu_m$ and $1 \leq \alpha \leq n_{m;i}$. For the Hermitian pairing in (3.1), we say that an orthonormal basis $\{\,\sigma_{i,\alpha}\,;\,(i,\alpha)\in P_m\,\}$ for V_m is *admissible* if $\sigma_{i,\alpha} \in V(\chi_{m;i})$ for all $(i,\alpha) \in P_m$. Fixing an admissible orthonormal basis $\{\,\sigma_{i,\alpha}\,;\,(i,\alpha)\in P_m\,\}$ of V_m with $\langle\ ,\ \rangle_h$, we now define $Z_m(\omega, \mathcal{Y}, x)$ to be

$$(n!/m^n)\,\Sigma_{i=1}^{\nu_m}\Sigma_{\alpha=1}^{n_{m;i}} \exp\{-(\chi_{m;i})_*(\mathcal{Y}) + 2x_i\}\,|\sigma_{i,\alpha}|_h^2, \tag{3.2}$$

for each $\mathcal{Y} \in \hat{\mathfrak{z}}$ and $x = (x_1, x_2, \dots, x_{\nu_m}) \in \mathbb{R}^{\nu_m}$, where we put $|\sigma|_h^2 := (\sigma,\sigma)_h$ for all $\sigma \in V_m$, and $(\chi_{m;i})_* : \hat{\mathfrak{z}} \to \mathbb{R}$, $i = 1, 2, \dots$, denote the differentials at $g = 1$ of the restriction to $\hat{\mathfrak{z}}$ of the characters $\chi_{m;i} : Z^{\mathbb{C}} \to \mathbb{C}^*$. Put $r_0 := n\{2c_1(L)^n[M]\}^{-1}\{c_1(L)^{n-1}c_1(M)[M]\}$, and consider

$$B_m := \{\, x = (x_1, x_2, \dots, x_{\nu_m}) \in \mathbb{R}^{\nu_m}\,;\, \|x\| \leq q^2\,\},$$

where $q := m^{-1}$ and $\|x\| := (\Sigma_{i=1}^{\nu_m} n_{m;i}\, x_i^2)^{1/2}$. Then fixing a sufficiently large positive integer k, we see from [7], Theorem B, that there exist vector fields $\mathcal{Y}_j \in \hat{\mathfrak{z}}$, real numbers $r_j \in \mathbb{R}$, $j = 1, 2, \dots, k$, and a K-invariant Hermitian metric u_m for L such that

$$Z_m(v_m, \mathcal{Y}, 0) \; = \; (1 + \Sigma_{j=0}^k r_j q^{j+1}) \; + \; O(q^{k+2}), \tag{3.3}$$

$$u_m \to h_\infty \text{ in } C^\infty, \quad \text{as } m \to \infty, \tag{3.4}$$

where $v_m := c_1(L; u_m)_{\mathbb{R}}$ and $\mathcal{Y} := \Sigma_{j=1}^k q^{j+2} \mathcal{Y}_j$. In view of the definition of δ_0 in [9], Step 5, the proof of Lemma 3.4 in [8] allows us to make a perturbation of h_m via the action of $\exp(\mathfrak{p}''_m)$ (see [9] for the definition of $\mathfrak{p}''_m$) to obtain a critical point for the Chow norm. Then by (3.3) and (3.4), we obtain from [7], pp.574–576, a K-invariant Hermitian metric h_m for L such that, for some $b_m = (b_{m;1}, b_{m;2}, \dots, b_{m;\nu_m}) \in B_m$,

$$Z_m(\omega_m, \mathcal{Y}, b_m) \; = \; 1 + \Sigma_{j=0}^k r_j q^{j+1}, \qquad k \gg 1, \tag{3.5}$$

$$h_m \to h_\infty \text{ in } C^\infty, \quad \text{as } m \to \infty, \tag{3.6}$$

where we set $\omega_m := c_1(L; h_m)$. For an admissible orthonormal basis $\{ \sigma_{i,\alpha} \, ; \, (i, \alpha) \in P_m \}$ for V_m with the pairing $\langle \; , \; \rangle_{h_m}$, by setting

$$\beta_{m;i} \; := \; \exp\{ (\chi_{m;i})_* (\mathcal{Y}) + 2 b_{m;i} \} \; - 1, \tag{3.7}$$

we see from (3.5) and (3.7) the following:

$$(n!/m^n) \, \Sigma_{i=1}^{\nu_m} \Sigma_{\alpha=1}^{n_{m;i}} \; (1 + \beta_{m;i}) \, |\sigma_{i,\alpha}|^2_{h_m} \; = \; 1 + \Sigma_{j=0}^k r_j q^{j+1}, \tag{3.8}$$

where, in view of [9], Lemma 2.6, there exists a positive constant C_1 independent of the choice of $m \gg 1$ and i such that

$$|\beta_{m;i}| \; \leq \; C_1 \, q^2 \qquad \text{for all } m \gg 1 \text{ and } i. \tag{3.9}$$

Then by (3.8) and (3.9), we obtain

$$\frac{\sqrt{-1}}{2\pi} \, \partial \bar{\partial} (\Sigma_{i=1}^{\nu_m} \Sigma_{\alpha=1}^{n_{m,i}} \, |\sigma_{i,\alpha}|^2_{h_m}) \; - \; m \, \omega_m \; = \; O(q^2). \tag{3.10}$$

4. The Chow norm and the Futaki invariant

In this section, we fix a Hermitian metric H_1 on V_m, where $(E_m)_s$ at $s = 1$, denoted by $(E_m)_1$, is identified with V_m. By the trivialization (2.7), H_1 induces a Hermitian metric H_0 on $(E_m)_0$. Then

$$W_m := \{ \mathrm{Sym}^{d_m} ((E_m)_0) \}^{\otimes n+1} \tag{4.1}$$

admits the Chow norm (cf. Zhang [22], 1.5; see also §4 in [7])

$$W_m^* \ni w \;\mapsto\; \|w\|_{\mathrm{CH}(H_0)} \in \mathbb{R}_{\geq 0}.$$

Choose an element $\hat{M}_m$ of W_m^* such that the corresponding point $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point for the reduced effective algebraic cycle

$$\gamma_1 \;:=\; \Phi_{|L^{\ell m}|}(M)$$

on $\mathbb{P}^*((E_m)_0)$. Here each $(E_m)_s$, $s \neq 0$, is identified with $(E_m)_0$ via the trivialization (2.7), and by letting $s = 1$, we regard $\Phi_{|L^{\ell m}|}(M)$ on $\mathbb{P}^*(V_m)$ as the algebraic cycle γ_1 on $\mathbb{P}^*((E_m)_0)$. Since the T-action on E_m preserves $(E_m)_0$, we have a representation

$$\psi_m : T \to \mathrm{GL}((E_m)_0) \tag{4.2}$$

induced by the T-action on E_m. Note that this T-action on $(E_m)_0$ naturally induces a T-action on $\mathbb{P}^*((E_m)_0)$. By the complete linear systems $|\mathcal{L}_s^m|$, $s \in \mathbb{A}^1$, we have the relative Kodaira embedding

$$\mathcal{M} \;\hookrightarrow\; \mathbb{P}^*(E_m),$$

over $\mathbb{A}^1$, where by (2.6) the projective bundle $\mathbb{P}^*(E_m)$ over $\mathbb{A}^1$ is viewed as product bundle $\mathbb{A}^1 \times \mathbb{P}^*((E_m)_0)$. Then each fiber $\mathbb{P}^*(E_m)_s$ of $\mathbb{P}^*(E_m)$ over $s \in \mathbb{A}^1$ is naturally identified with $\mathbb{P}^*((E_m)_0)$, so that all $\mathcal{M}_z$, $z \in \mathbb{A}^1$, are regarded as subschemes of $\mathbb{P}^*((E_m)_0)$. Then

$$\mathcal{M}_z \;=\; \psi_m(z) \cdot \mathcal{M}_1, \qquad z \in \mathbb{C}^*, \tag{4.3}$$

where on the right-hand side, the element $\psi_m(s)$ in $\mathrm{GL}((E_m)_0)$ acts naturally on $\mathbb{P}^*((E_m)_0)$ as a projective linear transformation. Note that $\mathcal{M}_1$ is nothing but γ_1 as an algebraic cycle, and that $\mathcal{M}_0$ is preserved by the T-action on $\mathbb{P}^*((E_m)_0)$.

Let us now consider the N_m-fold covering $\hat{T} := \{\, \hat{t} \in \mathbb{C}^* \,\}$ of the algebraic torus $T := \{\, t \in \mathbb{C}^* \,\}$ by setting

$$t \;=\; \hat{t}^{N_m}$$

for t and $\hat{t}$. Then the mapping $\psi_m^{\mathrm{SL}} : \hat{T} \to \mathrm{SL}((E_m)_0)$ defined by

$$\psi_m^{\mathrm{SL}}(\hat{t}) := \frac{\psi_m(\hat{t}^{N_m})}{\det(\psi_m(\hat{t}))} = \frac{\psi_m(t)}{\det(\psi_m(\hat{t}))}, \qquad \hat{t} \in \hat{T}, \tag{4.4}$$

is also an algebraic group homomorphism. Consider the quotient group $G_m := \mathrm{SL}((E_m)_0)/\Pi_m$, where $\Pi_m := \{\, \zeta^\alpha\, \mathrm{id}\, ;\, \alpha = 1, 2, \dots, N_m \,\}$ for a primitive N_m-th root ζ of unity. We then define an algebraic group homomorphism $\hat{\psi}_m : T \to G_m$ by sending each $t \in T$ to

$$\hat{\psi}_m(t) : \text{ natural image of } \psi_m^{\mathrm{SL}}(\hat{t}) \text{ in } G_m.$$

Consider $\psi_m(t)$, $\hat{\psi}_m(t)$, $\psi_m^{\mathrm{SL}}(\hat{t})$ above. Then these all induce exactly the same projective linear transformation on $\mathbb{P}^*((E_m)_0)$. Let γ_t be the algebraic cycle on $\mathbb{P}^*((E_m)_0)$ obtained as the image of γ_1 by this projective linear transformation. Then as $t \to 0$, we have a limit algebraic cycle

$$\gamma_0 \;:=\; \lim_{t \to 0} \gamma_t \tag{4.5}$$

on $\mathbb{P}^*((E_m)_0)$. To have another understanding of γ_z, $z \in \mathbb{A}^1$, recall that we can regard each $\mathcal{M}_z$ as a subscheme

$$\mathcal{M}_z \;\hookrightarrow\; \mathbb{P}^*((E_m)_0), \qquad z \in \mathbb{A}^1.$$

Then by (4.3), the algebraic cycle γ_z is nothing but $\mathcal{M}_z$ viewed just as an algebraic cycle on $\mathbb{P}^*((E_m)_0)$ counted with multiplicities. In particular, γ_0 is the T-invariant algebraic cycle on $\mathbb{P}^*((E_m)_0)$ associated to the subscheme $\mathcal{M}_0$ counted with multiplicities.

By $\hat{M}_m^{(0)} \in W_m^*$, we denote the element in W_m^* such that the associated element $[\hat{M}_m^{(0)}] \in \mathbb{P}^*(W_m)$ is the Chow point for the cycle γ_0 on $\mathbb{P}^*((E_m)_0)$. Then (4.5) is interpreted as

$$\lim_{\hat{t} \to 0} [\, \psi_m^{\mathrm{SL}}(\hat{t}) \cdot \hat{M}_m \,] \;=\; [\hat{M}_m^{(0)}] \tag{4.6}$$

in $\mathbb{P}^*(W_m)$. Here by (4.1), the group $\mathrm{GL}((E_m)_0)$ acts naturally on W_m^*, and hence acts also on $\mathbb{P}^*(W_m)$. We now consider the function

$$f_m(s) \;:=\; \log \|\hat{\psi}_m(\exp(s)) \cdot \hat{M}_m\|_{\mathrm{CH}(H_0)}, \qquad s \in \mathbb{R}. \tag{4.7}$$

Put $\dot{f}_m(s) := (df_m/ds)(s)$. The purpose of this section is to show the following (see Phong and Sturm [18], eqn 7.29, for the leading term; see also [4], p.464–467):

Lemma 4.8. *Let a_n be as in (2.3). Then the function $\dot{f}_m(s)$ has a limit, as $s \to -\infty$, written in the following form for $m \gg 1$:*

$$(4.9)\qquad \lim_{s\to-\infty} \dot{f}_m(s) \;=\; (n+1)!\, a_n(F_1 m^n + F_2 m^{n-1} + F_3 m^{n-2} + \dots) \\ \;=\; (n+1)!\, a_n \left(\frac{w_m}{mN_m} - F_0\right) m^{n+1}.$$

Proof: Since γ_0 is preserved by the $\hat{T}$-action on $(E_m)_0$, the Chow point $[\hat{M}^{(0)}]$ for γ_0 is fixed by the $\hat{T}$-action on $\mathbb{P}^*(W_m)$, i.e., for some $q_m \in \mathbb{Z}$,

$$\psi_m^{\mathrm{SL}}(\hat{t}) \cdot \hat{M}_m^{(0)} \;=\; \hat{t}^{q_m} \hat{M}_m^{(0)}, \qquad t \in \mathbb{C}^*,$$

where the left-hand side is $\hat{\psi}_m(t) \cdot \hat{M}^{(0)}$ modulo the action of Π_m. Since the $\hat{T}$-action on W_m^* is diagonalizable, we can write $\hat{M}_m$ in the form

$$(4.10)\qquad \hat{M}_m \;=\; \Sigma_{\nu=1}^N\, w_\nu,$$

where $0 \neq w_\nu \in W_m^*$, $\nu = 1, 2, \dots, N$, are such that, for an increasing sequence of integers $e_1 < e_2 < \dots < e_N$, the equality

$$(4.11)\qquad \psi_m^{\mathrm{SL}}(\hat{t}) \cdot w_\nu = \hat{t}^{e_\nu} w_\nu$$

holds for all $\nu \in \{1, 2, \dots, N\}$ and $\hat{t} \in \hat{T}$. In particular, in view of (4.6), we can find a complex number $c \neq 0$ such that

$$\hat{M}_m^{(0)} \;=\; c\, w_1,$$

and hence q_m coincides with e_1. Then by (4.10) and (4.11), it is easy to check that

$$(4.12)\qquad \lim_{s\to-\infty} \dot{f}_m(s) \left(= \frac{e_1}{N_m}\right) \;=\; \frac{q_m}{N_m}.$$

Hence it suffices to show that q_m/N_m admits the asymptotic expansion as in the right-hand side of (4.9) above. Consider the graded algebra

$$\bigoplus_{k=0}^{\infty} (E_{km})_0,$$

where via ψ_m^{SL}, the group $\hat{T}$ acts on $(E_m)_0$ and hence on $(E_{km})_0$. Then by [14], Proposition 2.11, the weight p_k for the $\hat{T}$-action on $\det(E_{km})_0$ satisfies the following:

$$(4.13)\qquad p_k + \frac{q_m}{(n+1)!}\, k^{n+1} \;=\; O(k^n), \qquad k \gg 1,$$

i.e., there exists a constant $C > 0$ independent of k, possibly depending on m, such that the left-hand side of (4.13) has absolute value bounded by Ck^n for positive integers k. Recall the definition of w_{km} and w_m in Section 2. Then by the expression of ψ_m^{SL} in (4.4), the weight p_k for $\det(E_{km})_0$ induced by the $\hat{T}$-action on $(E_m)_0$ via ψ_m^{SL} is expressible as

$$p_k \;=\; N_m w_{km} \;-\; k\, w_m N_{km}. \tag{4.14}$$

Here the term $N_m w_{mk}$ in the right-hand side of (4.14) is the weight in $\hat{t}$ for $\det(E_{km})_0$ induced from the action on $(E_m)_0$ by the numerator $\psi_m(t)$ in (4.4), since it is nothing but the weight in $\hat{t}$ for the action of $\psi_{mk}(t)$ on $\det(E_{km})_0$, while in view of the natural surjective homomorphism

$$S^k((E_m)_0) \to (E_{km})_0,$$

the term $k\, w_m N_{km}$ is just the weight in $\hat{t}$ induced from the scalar action on $(E_m)_0$ by the denominator of (4.4). Then for $k \gg 1$, by (4.14) and (2.4), we obtain

$$\begin{aligned} p_k \;&=\; (km)\, N_m N_{km} \left\{ \frac{w_{km}}{(km)N_{km}} \;-\; \frac{w_m}{mN_m} \right\} \\ &=\; -\,(km)\, N_m N_{km} \{\, (F_1 m^{-1} + F_2 m^{-2} + F_3 m^{-3} + \dots\,) \,+\, O(k^{-1})\,\} \\ &=\; -\,k^{n+1} a_n N_m \{\, (F_1 m^{n} + F_2 m^{n-1} + F_3 m^{n-2} + \dots\,) \,+\, O(k^{-1})\,\}, \end{aligned}$$

where the last equality ifollows from (2.3) applied to km. Then by comparing this with (4.13), and then by (2.4), we obtain

$$\begin{aligned} \frac{q_m}{N_m} \;&=\; (n+1)!\, a_n (F_1 m^n + F_2 m^{n-1} + F_3 m^{n-2} + \dots\,) \\ &=\; (n+1)!\, a_n \left(\frac{w_m}{mN_m} - F_0 \right) m^{n+1}. \end{aligned}$$

as required. □

5. Proof of Main Theorem

In this section, by using the notation in (3.1), we choose $\langle\ ,\ \rangle_{h_m}$ as the Hermitian metric H_1 for V_m in Section 4, where h_m is as in (3.6). For the corresponding Chow norm

$$W_m^* \ni w \;\mapsto\; \|w\|_{\mathrm{CH}(H_0)} \in \mathbb{R}_{\geq 0},$$

we consider the real-valued function f_m on $\mathbb{R}$ as in (4.7). For the one-parameter group $\hat{\psi}_m : \hat{T} \to G_m$, the vector space $(E_m)_0$ admits an orthonormal basis $\mathcal{T} := \{\tau_1, \tau_2, \dots, \tau_{N_m}\}$ such that, for some $e_i \in \mathbb{Q}$ with $\Sigma_{\alpha=1}^{N_m} e_\alpha = 0$, we have

$$\hat{\psi}_m(t)\,\tau_\alpha \;\equiv\; t^{e_\alpha}\tau_\alpha, \qquad t \in T, \tag{5.1}$$

modulo the action of Π_m. By the associated Kodaira embedding $\mathcal{M}_0 \hookrightarrow \mathbb{P}^*((E_m)_0)$, we regard $\mathcal{M}_0$ as a subscheme

$$\mathcal{M}_0 \;\hookrightarrow\; \mathbb{P}^{N_m-1}(\mathbb{C}), \qquad p \mapsto (\tau_1(p) : \tau_2(p) : \dots : \tau_{N_m}(p)),$$

where we identify $\mathbb{P}^*((E_m)_0)$ with

$$\mathbb{P}^{N_m-1}(\mathbb{C}) := \{(z_1 : z_2 : \dots : z_{N_m})\}$$

by the basis $\mathcal{T}$. Since we regard $(E_m)_1$ just as V_m, the identification (2.7) allows us to obtain a basis $\mathcal{T}' := \{\tau'_1, \tau'_2, \dots, \tau'_{N_m}\}$ for V_m corresponding to the basis $\mathcal{T}$ for $(E_m)_0$. Note that this basis $\mathcal{T}'$ is orthonormal with respect to the Hermitian metric $H_1 = \langle\ ,\ \rangle_{h_m}$. Then the Kodaira embedding $\Phi_{|L^{\ell m}|} : M\,(=\mathcal{M}_1) \hookrightarrow \mathbb{P}^*(V_m)$ is given by

$$M \;\hookrightarrow\; \mathbb{P}^{N_m-1}(\mathbb{C}), \qquad p \mapsto (\tau'_1(p) : \tau'_2(p) : \dots : \tau'_{N_m}(p)), \tag{5.2}$$

where we identify $\mathbb{P}^*(V_m) = \mathbb{P}^{N_m-1}(\mathbb{C}) = \mathbb{P}^*((E_m)_0)$ by the bases $\mathcal{T}'$ and $\mathcal{T}$. Then the Fubini-Study form ω_{FS} on $\mathbb{P}^{N_m-1}(\mathbb{C})\,(=\mathbb{P}^*(V_m))$ is

$$\omega_{\mathrm{FS}} \;:=\; \frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log(\Sigma_{\alpha=1}^{N_m}|z_\alpha|^2).$$

By (3.10), we here observe that

$$\omega_{\mathrm{FS}} - m\omega_m = O(q^2), \tag{5.3}$$

on M. For the function $f_m(s)$ in (4.7), we first give an estimate of the fist derivative $\dot{f}_m(0)$. In view of [22] (see also [7]),

$$\dot{f}_m(0) \;=\; (n+1)\int_M \frac{\Sigma_{\alpha=1}^{N_m} e_\alpha\,|\tau'_\alpha|^2_{h_m}}{\Sigma_{\alpha=1}^{N_m}|\tau'_\alpha|^2_{h_m}}\,\omega_{\mathrm{FS}}^n, \tag{5.4}$$

where by (3.8) and (3.9), we observe that

$$\begin{aligned}(5.5)\qquad \Sigma_{\alpha=1}^{N_m} |\tau'_\alpha|^2_{h_m} \;&=\; \Sigma_{i=1}^{\nu_m}\Sigma_{\alpha=1}^{n_{m,i}} |\sigma_{i,\alpha}|^2_{h_m}\\ &=\; (m^n/n!)\,(1+\Sigma_{j=0}^k r_j q^{j+1}) \;-\; \Sigma_{i=1}^{\nu_m}\Sigma_{\alpha=1}^{n_{m,i}} \beta_{m;i}\,|\sigma_{i,\alpha}|^2_{h_m}\\ &=\; (m^n/n!)\,(1+\Sigma_{j=0}^k r_j q^{j+1})\,\{1+O(q^2)\}.\end{aligned}$$

Now, we can rewrite (5.4) in the form

$$\begin{aligned}(5.6)\qquad \dot f_m(0) \;&=\; (n+1)!\int_M \frac{\Sigma_{\alpha=1}^{N_m} e_\alpha\,|\tau'_\alpha|^2_{h_m}}{1+\Sigma_{j=0}^k r_j q^{j+1}}\,\{1+O(q^2)\}\,\omega_m^n\\ &=\; \int_M O(q^2)(\Sigma_{\alpha=1}^{N_m} e_\alpha\,|\tau'_\alpha|^2_{h_m})\,\omega_m^n,\end{aligned}$$

where the last equality follows from

$$\int_M \Sigma_{\alpha=1}^{N_m} e_\alpha|\tau'_\alpha|^2_{h_m}\omega_m^n \;=\; \Sigma_{\alpha=1}^{N_m} e_\alpha \;=\; 0.$$

All weights e_α for $\hat\psi_m$ have absolute value bounded by $C_2 m$ for some constant $C_2 > 0$ independent of both α and m, i.e.,

$$(5.7)\qquad |e_\alpha| \;\le\; C_2 m, \qquad \alpha = 1, 2, \dots, N_m.$$

In view of (5.5), $\Sigma_{\alpha=1}^{N_m} |\tau'_\alpha|^2_{h_m} = O(m^n)$. Then by (5.6) and (5.7),

$$(5.8)\qquad \dot f_m(0) \;=\; O(m^{n-1}).$$

By [22] (see also [7], 4.5), $\ddot f_m(s) \ge 0$ for all $s \in \mathbb{R}$. In (4.9), let $m \to \infty$. Then by (5.8), we obtain $F_1 \le 0$, i.e., K-semistability of (M, L) follows.

To show K-stability, we now assume that $F_1 = 0$ for the test configuration above. Then by Lemma 4.8,

$$(5.9)\qquad \lim_{s\to-\infty} \dot f_m(s) = O(m^{n-1}).$$

Here we consider the second derivative $\ddot f_m(s)$. From now on, by setting $\delta := C_3(\log m)q$, we require the real number s to satisfy

$$(5.10)\qquad |s| \;\le\; \delta,$$

where C_3 is a positive real number independent of the choice of m. For local one-parameter group

$$\mu_{m,s} := \hat\psi_m(\exp(s)) \in G_m, \qquad -\delta \le s \le \delta,$$

we regard each $\mu_{m,s}$ as a linear isomorphism of $\mathbb{C}^{N_m-1}\,(= V_m)$, modulo the action by Π_m, via the identification of $(E_m)_0$ with $(E_m)_1\,(= V_m)$. Note also that G_m acts on $\mathbb{P}^*((E_m)_0)\,(= \mathbb{P}^*(V_m))$ via the projection

$$\pi_m : G_m\,(= \mathrm{SL}(N_m;\mathbb{C})/\Pi_m) \;\to\; \mathrm{PGL}((E_m)_0)\,(= \mathrm{PGL}(N_m;\mathbb{C})).$$

Now by Appendix, the family of Kähler manifolds

$$(5.11) \qquad (M,\, q\,(\mu_{m,s}^*\omega_{\mathrm{FS}})_{|M}), \qquad -\delta \le s \le \delta,\; m = 1, 2, \dots,$$

has bounded geometry. Let us now consider the holomorphic vector field $\mathcal{V}^m := (\pi_m \circ \hat{\psi}_m)_*(\partial/\partial s)$ on $\mathbb{P}^{N_m-1}(\mathbb{C})$ which generates the local one-parameter group $\pi_m(\mu_{m,s})$, $-\delta \le s \le \delta$. For each s, consider the holomorphic tangent bundle TM_s of $M_s := \mu_{m,s}(M)$, where M is viewed as a subvariety of $\mathbb{P}^{N_m-1}(\mathbb{C})$ by (5.2). Metrically, for the orthogonal complement $TM_s^{\perp}$ of TM_s in $T\mathbb{P}^{N_m-1}(\mathbb{C})_{|M_s}$ by the metric ω_{FS}, we can regard the normal bundle of M_s in $\mathbb{P}^{N_m-1}(\mathbb{C})$ as the subbundle $TM_s^{\perp}$ of $T\mathbb{P}^{N_m-1}(\mathbb{C})_{|M_s}$. Hence $T\mathbb{P}^{N_m-1}(\mathbb{C})_{|M_s}$ is differentiably a direct sum $TM_s \oplus TM_s^{\perp}$, and we can uniquely write

$$(5.12) \qquad {\mathcal{V}^m}_{|M_s} \;=\; \mathcal{V}^m_{TM_s} + \mathcal{V}^m_{TM_s^{\perp}},$$

where $\mathcal{V}_{TM}$ and $\mathcal{V}_{TM^{\perp}}$ are smooth sections of TM_s and $TM_s^{\perp}$, respectively. Consider the exact sequence of holomorphic vector bundles

$$0 \to TM_s \to T\mathbb{P}^{N_m-1}(\mathbb{C})_{|M_s} \to TM_s^{\perp} \to 0$$

over M_s. Then the pointwise estimate (cf. [17], (5.16)) for the second fundamental form for this exact sequence is valid also in our case (cf. [8], Step 2), and as in [17], (5.15), we obtain the inequality

$$(5.13) \qquad \int_{M_s} |\mathcal{V}^m_{TM_s^{\perp}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n \;\ge\; C_4 \int_{M_s} |\bar{\partial}\mathcal{V}^m_{TM_s^{\perp}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n,$$

where C_4 is a positive real constant independent of the choice of m. The second derivative $\ddot{f}_m(s)$ is (see for instance [7], [17]) given by

$$(5.14) \qquad \ddot{f}_m(s) \;=\; \int_{M_s} |\mathcal{V}^m_{TM_s^{\perp}}|^2_{\omega_{\mathrm{FS}}}\,\omega_{\mathrm{FS}}^n \;\ge\; 0.$$

Put $\varphi_m := (\Sigma_{\alpha=1}^{N_m} e_\alpha |z_\alpha|^2)/(m\Sigma_{\alpha=1}^{N_m}|z_\alpha|^2)$ on $\mathbb{P}^{N_m-1}(\mathbb{C})$. Then by (5.7) φ_m, $m = 1, 2, \dots$, are uniformly bounded satisfying (cf. [8], (4.5))

$$i_{\mathcal{V}^m}(q\,\omega_{\rm FS}) \;=\; \frac{\sqrt{-1}}{2\pi}\,\bar\partial\varphi_m. \tag{5.15}$$

For M as a submanifold of $\mathbb{P}^{N_m-1}(\mathbb{C})$ in (5.2), we consider the sheaves $\mathcal{A}^{0,j}(TM)$, $j = 0, 1, \dots$, of germs of smooth $(0, j)$-forms on M with values in the holomorphic tangent bundle TM of M, and endow M with the Kähler metric $q\mu_{m,s}^*\omega_{{\rm FS}|M}$ for s as in (5.10). Now on $\mathcal{A}^{(0,0)}(TM)$, we consider the operator

$$\Delta_{TM,s} \;:=\; -\,\bar\partial^{\#}\bar\partial,$$

where $\bar\partial^{\#}$ is the formal adjoint of $\bar\partial : \mathcal{A}^{(0,0)}(TM) \to \mathcal{A}^{(0,1)}(TM)$. For $\Gamma := H^0(M, \mathcal{A}^{(0,0)}(TM))$, we consider the Hermitian L^2-pairing

$$\langle V_1, V_2\rangle_s \;:=\; \int_M (V_1, V_2)_{q\mu_{m,s}^*\omega_{\rm FS}}\,(q\mu_{m,s}^*\omega_{\rm FS})^n, \qquad V_1, V_2 \in \Gamma,$$

where $(V_1, V_2)_{q\mu_{m,s}^*\omega_{\rm FS}}$ is the pointwise Hermitian pairing of V_1 and V_2 by the Kähler metric $q\mu_{m,s}^*\omega_{\rm FS}$. For the subspace $\mathfrak{g} := H^0(M, \mathcal{O}(TM))$ of Γ, we consider its orthogonal complement $\mathfrak{g}_s^\perp$ in Γ by the pairing $\langle\ ,\ \rangle_s$. Then $\mathcal{V}_{TM_s}^m$ in (5.12) is expressible as

$$\mathcal{V}_{TM_s}^m \;=\; \mathcal{V}_{m,s}^\circ + \mathcal{V}_{m,s}^\bullet,$$

where $\mathcal{V}_{m,s}^\circ$ and $\mathcal{V}_{m,s}^\bullet$ belong to $(\mu_{m,s})_*\mathfrak{g}$ and $(\mu_{m,s})_*\mathfrak{g}_s^\perp$, respectively. Since the left-hand side of (5.12) is holomorphic,

$$\bar\partial\mathcal{V}_{TM_s^\perp}^m \;=\; -\,\bar\partial\mathcal{V}_{TM_s}^m \;=\; -\,\bar\partial\mathcal{V}_{m,s}^\bullet. \tag{5.16}$$

Since the family (5.11) has bounded geometry, the first positive eigenvalue λ_1 of the operator Δ_{TM} on $\mathcal{A}^{(0,0)}(TM)$ is bounded from below by some positive constant C_5 independent of the choice of m. Hence

$$\int_{M_s} |\bar\partial\mathcal{V}_{m,s}^\bullet|^2_{q\omega_{\rm FS}}(q\omega_{\rm FS})^n \;\geq\; C_5\int_{M_s}|\mathcal{V}_{m,s}^\bullet|^2_{q\omega_{\rm FS}}(q\omega_{\rm FS})^n. \tag{5.17}$$

From (5.13), (5.16) and (5.17), it now follows that

$$\int_{M_s} |\mathcal{V}_{TM_s^\perp}^m|^2_{\omega_{\rm FS}}\,\omega_{\rm FS}^n \;\geq\; C_4C_5\,q\int_{M_s}|\mathcal{V}_{m,s}^\bullet|^2_{\omega_{\rm FS}}\,\omega_{\rm FS}^n. \tag{5.18}$$

In view of (5.14), $\dot{f}_m(0) - \dot{f}_m(-\delta) = \int_{-\delta}^{0} \ddot{f}_m(s)\, ds \geq 0$, and it follows from (5.8) and (5.9) that

$$O(m^{n-1}) = \dot{f}_m(0) - \lim_{s\to-\infty} \dot{f}_m(s) \geq \dot{f}_m(0) - \dot{f}_m(-\delta)$$
$$= \int_{-\delta}^{0} \ddot{f}_m(s)\, ds \geq \ddot{f}_m(s_m)\,\delta$$

where s_m, $m = 1, 2, \dots$, are real numbers at which the functions $\ddot{f}_m(s)$, $-\delta \leq s \leq 0$, attain their minima, i.e.,

$$\ddot{f}_m(s_m) = \min_{-\delta\leq s\leq 0} \ddot{f}_m(s).$$

Therefore, in view of (5.14) and $\delta = O(q \log m)$, we obtain

$$\int_{M_{s_m}} |\mathcal{V}^m_{TM^{\perp}_{s_m}}|^2_{q\omega_{\rm FS}}\, (q\omega_{\rm FS})^n = O\left(\frac{q}{\log m}\right), \tag{5.19}$$

since the left-hand side is $\ddot{f}_m(s_m)/m^{n+1}$. Then by (5.18),

$$\int_{M_{s_m}} |\mathcal{V}^{\bullet}_{m,s_m}|^2_{q\omega_{\rm FS}}\, (q\omega_{\rm FS})^n = O\left(\frac{1}{\log m}\right). \tag{5.20}$$

Infinitesimally, (5.1) is written as $\mathcal{V}^m \cdot \tau'_\alpha = e_\alpha \tau'_\alpha$, $\alpha = 1, 2, \dots, N_m$, and for φ_m in (5.15), by setting

$$\bar{\varphi}_m := \left\{\int_{M_{s_m}} (q\omega_{\rm FS})^n\right\}^{-1} \int_{M_{s_m}} \varphi_m (q\omega_{\rm FS})^n,$$

we can write the pullback $\mu^*_{m,s_m}(\varphi_m - \bar{\varphi}_m)$ as

$$\eta_m := (\mu^*_{m,s_m}\varphi_m)_{|M} - \bar{\varphi}_m = \frac{\Sigma_{\alpha=1}^{N_m} e_\alpha\, \tau'_\alpha \bar{\tau}'_\alpha \exp(2 s_m e_\alpha)}{m \Sigma_{\alpha=1}^{N_m} \tau'_\alpha \bar{\tau}'_\alpha \exp(2 s_m e_\alpha)} - \bar{\varphi}_m \tag{5.21}$$

when restricted to $M \subset \mathbb{P}^*(V_m)$. Note that $\bar{\varphi}_m$, $m = 1, 2, \dots$, is a bounded sequence of real numbers. Then we can write the uniformly bounded real-valued functions η_m on M as

$$\eta_m := \frac{\Sigma_{\alpha=1}^{N_m} e_{\alpha,m}\, \tau'_\alpha \bar{\tau}'_\alpha \exp(2 s_m e_{\alpha,m})}{m \Sigma_{\alpha=1}^{N_m} \tau'_\alpha \bar{\tau}'_\alpha \exp(2 s_m e_{\alpha,m})}, \qquad m = 1, 2, \dots,$$

where $e_{\alpha,m} := e_\alpha - m\bar{\varphi}_m$. Put $\omega_m := q\mu^*_{s_m}\omega_{\rm FS}{}_{|M}$. Hereafter, replace the sequence s_m, $m \gg 1$, by its suitable subsequence s_{m_j}, $j = 1, 2, \dots$, if necessary. We write m_j, m_j^{-1}, N_{m_j}, s_{m_j}, ω_{m_j}, η_{m_j}, e_{α,m_j} as $m(j)$, $q(j)$,

$N(j)$, $s(j)$, $\omega(j)$, $\eta(j)$, $e_\alpha(j)$, respectively. Then by Appendix, we may assume that $\omega(j)$ converges to ω_∞ in C^∞ as $j \to \infty$. Moreover, we set

$$\begin{cases} \mathcal{V}(j) := \mathcal{V}^{m(j)} = (\mu_j^{-1})_* \mathcal{V}^{m(j)}, \;\; \mathcal{V}_{TM}(j) := (\mu_j^{-1})_* \mathcal{V}^{m(j)}_{TM_{s(j)}}, \\ \mathcal{V}^\circ(j) := (\mu_j^{-1})_* \mathcal{V}^\circ_{m(j),s(j)}, \; \mathcal{V}^\bullet(j) := (\mu_j^{-1})_* \mathcal{V}^\bullet_{m(j),s(j)}, \end{cases}$$

where $\mu_j := \mu_{m(j),s(j)}$. Then the following cases 1 and 2 are possible:

Case 1: $I_j^\circ := \int_M |\mathcal{V}^\circ(j)|^2_{\omega(j)}\, \omega(j)^n$, $j = 1, 2, \dots$, are bounded. In this case, by $|\mathcal{V}^2_{TM}(j)|^2_{\omega(j)} = |\mathcal{V}^\circ(j)|^2_{\omega(j)} + |\mathcal{V}^\bullet(j)|^2_{\omega(j)}$, this boundedness together with (5.20) implies that

$$\int_M |\mathcal{V}_{TM}(j)|^2_{\omega(j)}\, \omega(j)^n, \; j = 1, 2, \dots, \text{ are bounded.} \tag{5.22}$$

Note that $\omega(j) \to \omega_\infty$ in C^∞, as $j \to \infty$. Hence in view of (5.22), since $|\mathcal{V}_{TM}(j)|^2_{\omega(j)} = |\bar\partial \eta(j)|^2_{\omega(j)}$ by (5.15), the sequence of integrals $\int_M |\bar\partial\eta(j)|^2_{\omega_\infty} \omega^n_\infty$, $j = 1, 2, \dots$, is bounded, so that $\eta(j)$, $j = 1, 2, \dots$, is a bounded sequence in the Sobolev space $L^{1,2}(M, \omega_\infty)$. Now by [19], we may assume that $n \geq 2$. Then replacing $\eta(j)$, $j = 1, 2, \dots$, by its subsequence if necessary, we may further assume the convergence

$$\eta(j) \to \eta_\infty \text{ in } L^2(M, \omega^n_\infty), \qquad \text{as } j \to \infty, \tag{5.23}$$

where η_∞ is a real-valued function in $L^2(M, \omega_\infty)$. Recall that the Lichnerowich operator $\Lambda_j : C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}}$ for the Kähler manifold $(M, \omega(j))$ is an elliptic operator, of order 4, with kernel consisting of all Hamiltonian functions for the holomorphic Hamiltonian vector fields on M. Now, to each smooth function $f \in C^\infty(M)_{\mathbb{C}}$, we associate a complex vector field $X_{f,j}$ of type $(1, 0)$ on M such that

$$i(X_{f,j})\, \omega(j) \;=\; \frac{\sqrt{-1}}{2\pi}\, \bar\partial f.$$

Note that $X_{\eta(j),j} = \mathcal{V}_{TM}(j)$ by (5.15) and (5.21). Then for the formal adjoint $\Lambda_j^\# : C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}}$ of the operator Λ_j, we have

$$\int_M \eta(j)\, \{\Lambda_j^\# f\}\, \omega(j)^n \;=\; \int_M \{\Lambda_j \eta(j)\}\, f\, \omega(j)^n \;=\; \langle\, \bar\partial X_{\eta(j),j}, \bar\partial X_{f,j} \rangle_{s(j)}$$
$$=\; \langle\, \bar\partial\{\mathcal{V}_{TM}(j)\}, \bar\partial X_{f,j} \rangle_{s(j)} \;=\; \langle\, \bar\partial\{\mathcal{V}^\bullet(j)\}, \bar\partial X_{f,j} \rangle_{s(j)},$$

for all $f \in C^\infty(M)_{\mathbb{C}}$. Here the last equality follows from the identities $\mathcal{V}_{TM}(j) = \mathcal{V}^\circ(j) + \mathcal{V}^\bullet(j)$ and $\bar\partial \mathcal{V}^\circ(j) = 0$. Hence, for each fixed f in $C^\infty(M)_{\mathbb{C}}$, we obtain

$$(5.24)\qquad \begin{cases} \left|\int_M \eta(j)\,\{\Lambda_j^{\#} f\}\,\omega(j)^n\right| &= \;|\langle\,\mathcal{V}^\bullet(j), \Delta_j X_{f,j}\rangle_{s(j)}| \\ \le\; \left\{\int_M |\Delta_j X_{f,j}|^2_{\omega(j)}\,\omega(j)^n\right\}^{1/2} I_j^\bullet, \end{cases}$$

where $I_j^\bullet := \{\int_M |\mathcal{V}^\bullet(j)|^2_{\omega(j)}\,\omega(j)^n\}^{1/2}$ and $\Delta_j := \Delta_{TM,s(j)}$. Let $j \to \infty$ in (5.24). Since $I_j^\bullet \to 0$ by (5.20), and since $\omega(j) \to \omega_\infty$ in C^∞, by passing to the limit, we see from (5.23) and (5.24) that

$$\int_M \eta_\infty\,\{\Lambda_\infty^{\#} f\}\,\omega_\infty^n \;=\; 0 \qquad \text{for all } f \in C^\infty(M)_{\mathbb{C}},$$

where $\Lambda_\infty : C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}}$ is the Lichnerowich operator for the Kähler manifold (M, ω_∞), and $\Lambda_\infty^{\#}$ is its formal adjoint. Since Λ_∞ is elliptic, any weak solution $\eta = \eta_\infty$ for the equation

$$\Lambda_\infty \eta = 0$$

is always a strong solution. In particular η_∞ is a real-valed smooth function on M such that the complex vector field W of type $(1,0)$ on M defined by $i(W)\omega_\infty = \bar\partial\eta_\infty$ is holomorphic. Then by [12], the test configuration $\pi : \mathcal{M} \to \mathbb{A}^1$ is a product configuration.

Case 2: $I_j^\circ \to +\infty$ as $j \to \infty$. In this case, we put $\hat{\mathcal{V}}_{TM}(j) := \mathcal{V}_{TM}(j)/\sqrt{I_j^\circ}$, $\hat{\mathcal{V}}^\circ(j) := \mathcal{V}^\circ(j)/\sqrt{I_j^\circ}$, and $\hat{\mathcal{V}}^\bullet(j) := \mathcal{V}^\bullet(j)/\sqrt{I_j^\circ}$. Then

$$(5.25)\qquad \int_M |\hat{\mathcal{V}}^\circ(j)|^2_{\omega(j)}\,\omega(j)^n \;=\; 1, \qquad j = 1, 2, \dots,$$

where by setting $\hat\eta(j) := \eta(j)/\sqrt{I_j^\circ}$, we see from $X_{\eta(j),j} = \mathcal{V}_{TM}(j)$ that the complex vector field $\hat{\mathcal{V}}_{TM}(j)$ of type $(1,0)$ on M satisfies

$$(5.26)\qquad i(\hat{\mathcal{V}}_{TM}(j))\,\omega(j) \;=\; \sqrt{-1}\,\bar\partial(\hat\eta(j)).$$

Since the functions η_m, $m \gg 1$, are uniformly bounded on M, and since $\omega(j)$ converges to ω_∞ as $j \to \infty$, we obtain the convergence

$$(5.27)\qquad \hat\eta(j) \to 0 \text{ in } C^0(M), \qquad \text{as } j \to \infty.$$

By (5.25), replacing $\hat{\mathcal{V}}^\circ(j)$, $j = 1, 2, \dots$, by its subsequence if necessary, we may assume that

$$\hat{\mathcal{V}}^\circ(j) \to \hat{\mathcal{V}}^\circ_\infty \text{ in } \mathfrak{g}, \qquad \text{as } j \to \infty, \tag{5.28}$$

for some $0 \neq \hat{\mathcal{V}}^\circ_\infty \in \mathfrak{g}$. Let $\hat{\eta}^\circ(j)$ and $\hat{\eta}^\bullet(j)$ be the Hamiltonian functions associated to the vector filelds $\hat{\mathcal{V}}^\circ(j)$ and $\hat{\mathcal{V}}^\bullet(j)$ by

$$\begin{cases} i(\hat{\mathcal{V}}^\circ(j))\,\omega(j) & = \sqrt{-1}\,\bar{\partial}(\hat{\eta}^\circ(j)), \\ i(\hat{\mathcal{V}}^\bullet(j))\,\omega(j) & = \sqrt{-1}\,\bar{\partial}(\hat{\eta}^\bullet(j)), \end{cases}$$

where the functions $\hat{\eta}^\circ(j)$ and $\hat{\eta}^\bullet(j)$ are normalized by the vanishing of the integrals $\int_M \hat{\eta}^\circ(j)\omega(j)^n$ and $\int_M \hat{\eta}^\bullet(j)\omega(j)^n$. Then

$$\hat{\eta}(j) \;=\; \hat{\eta}^\circ(j) + \hat{\eta}^\bullet(j). \tag{5.29}$$

In view of (5.28), there exists a non-constant real-valued C^∞ function ρ on M such that $i(\hat{\mathcal{V}}^\circ_\infty)\,\omega_\infty \;=\; \sqrt{-1}\,\bar{\partial}\rho$ and that

$$\hat{\eta}^\circ(j) \to \rho \text{ in } C^\infty(M), \text{ as } j \to \infty.$$

Hence by (5.29), it follows from (5.27) that

$$\hat{\eta}^\bullet(j) \to -\rho \text{ in } C^0(M), \text{ as } j \to \infty. \tag{5.30}$$

On the other hand, by (5.20), $\int_M |\bar{\partial}\hat{\eta}^\bullet(j)|^2_{\omega(j)}\,\omega(j)^n \to 0$ as $j \to \infty$, and hence for each fixed smooth $(0,1)$-form θ on M, we have

$$\begin{aligned} \left|\,(\hat{\eta}^\bullet(j), \bar{\partial}(j)^*\theta)_{L^2(M,\,\omega(j)^n)}\,\right| \;&=\; \left|\int_M (\bar{\partial}\hat{\eta}^\bullet(j), \theta)_{\omega(j)}\,\omega(j)^n\right| \\ \le\; \left\{\int_M |\bar{\partial}\hat{\eta}^\bullet(j)|^2_{\omega(j)}\,\omega(j)^n\right\}^{1/2} &\left\{\int_M |\theta|^2_{\omega(j)}\,\omega(j)^n\right\}^{1/2} \to 0, \end{aligned}$$

as $j \to \infty$, where $\bar{\partial}(j)^*$ and $\bar{\partial}^*_\infty$ are the formal adjoints of the operator $\bar{\partial}$ on functions for the Kähler manifolds $(M, \omega(j))$ and (M, ω_∞), respectively. Then by letting to $j \to \infty$, we obtain the vanishing for the Hermitian L^2-inner product of functions ρ and $\bar{\partial}^*\theta$,

$$(\rho, \bar{\partial}^*\theta)_{L^2(M,\,\omega_\infty^n)} \;=\; 0,$$

for every smooth $(0,1)$-form θ on M, i.e., $\bar{\partial}\rho = 0$ in a weak sense, and therefore in a strong sense. Hence we conclude that ρ is constant on M in contradiction to $\hat{\mathcal{V}}^\circ_\infty \neq 0$.

6. Appendix

In this appendix, we shall show that the family of Kähler manifolds

$$(M,\, q\,(\mu_{m,s}^*\omega_{\mathrm{FS}})_{|M}), \qquad -\delta \le s \le \delta,\ m = 1, 2, \dots, \tag{6.1}$$

has bounded geometry. By Fact 2.6 applied to $m = 1$, we identify $\mathbb{P}^*(E_1)$ with $\mathbb{A}^1 \times \mathbb{P}^*((E_1)_0)$, and let $\mathrm{pr}_2 : \mathbb{P}^*(E_1) \to \mathbb{P}^*((E_1)_0)$ be the projection to the second factor. As in Section 4, we have

$$\mathcal{M} \;\hookrightarrow\; \mathbb{P}^*(E_1), \tag{6.2}$$

where the pullback $\mathcal{H} := \mathrm{pr}_2^*\,\mathcal{O}_{\mathbb{P}^*((E_1)_0)}(1)$ to $\mathbb{P}^*(E_1)$ of the the hyperplane bundle $\mathcal{O}_{\mathbb{P}^*((E_1)_0)}(1)$ on $\mathbb{P}^*((E_1)_0)$ is written as

$$\mathcal{H}_{|\mathcal{M}} \;=\; \mathcal{L}\ (=\ L^{\ell}). \tag{6.3}$$

Recall that the action of $T = \mathbb{C}^*$ on $\mathcal{M}$ lifts to an action of T on $\mathcal{L}$, and hence T acts on $E_1 = \mathbb{A}^1 \times (E_1)_0$ by

$$T \times (\mathbb{A}^1 \times (E_1)_0) \to \mathbb{A}^1 \times (E_1)_0, \qquad (t, (z, e)) \mapsto (tz, \psi_1(t)\cdot e).$$

This induces a T-action on $\mathbb{P}^*(E_1)\,(= \mathbb{A}^1 \times \mathbb{P}^*((E_1)_0))$, and for (6.2), $\mathcal{M}$ is preserved by the T-action. Note that the T-action on $\mathcal{L}$ lifts the T-action on $\mathcal{M}$. By

$$T \times \mathcal{M} \to \mathcal{M}, \qquad (t, p) \mapsto g_{\mathcal{M}}(t)\cdot p,$$

we mean the T-action on $\mathcal{M}$, and the corresponding T-action on $\mathcal{L}\otimes\bar{\mathcal{L}}$ upstairs will be denoted by

$$T\,\times\,(\mathcal{L}\otimes\bar{\mathcal{L}}) \to \mathcal{L}\otimes\bar{\mathcal{L}}, \qquad (t, h) \mapsto g_{|\mathcal{L}|^2}(t)\cdot h.$$

Since $\mathrm{GL}((E_m)_0)$ acts on $\mathbb{P}^*((E_m)_0)$ via the projection of $\mathrm{GL}((E_m)_0)$ onto $\mathrm{PGL}((E_m)_0)$, by setting $\tilde{\mu}_{m,s} := \psi_m(\exp(s))$, we have

$$q\,\mu_{m,s}^*\omega_{\mathrm{FS}} \;=\; q\,\tilde{\mu}_{m,s}^*\omega_{\mathrm{FS}}. \tag{6.4}$$

In view of $\delta = C_3(\log m)q$, $m \gg 1$, we estimate $\exp(s)$ in the form

$$1 - \epsilon \le e^{-C_3(\log m)/m} \le \exp(s) \le e^{C_3(\log m)/m} \le 1 + \epsilon \tag{6.5}$$

for some $0 < \epsilon \ll 1$. As in Section 5, by the bases $\{\,\tau_1, \tau_2, \dots, \tau_{N_m}\}$ and $\{\,\tau_1', \tau_2', \dots, \tau_{N_m}'\}$ for $(E_m)_0$ and $(E_m)_1\,(= V_m)$, respectively, the spaces

$\mathbb{P}^*((E_m)_0)$ and $\mathbb{P}^*((E_m)_1)\,(=\mathbb{P}^*(V_m))$ are identified with

$$\mathbb{P}^{N_m-1}(\mathbb{C}) = \{\,(z_1 : z_2 : \cdots : z_{N_m})\,\}.$$

Note that $q\omega_{\mathrm{FS}} = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log\Omega_{\mathrm{FS},m}$, where $\Omega_{\mathrm{FS},m}$ denotes the positive real smooth section $\{(n!/m^n)\Sigma_{\alpha=1}^{N_m}|z_\alpha|^2\}^q$ of $\mathcal{H}\otimes\bar{\mathcal{H}}$. In view of (6.3), identifying M with $\mathcal{M}_1$, we easily see that $q\,\tilde{\mu}^*_{m,s}\omega_{\mathrm{FS}}$ is

$$(6.6)\qquad (\sqrt{-1}/2\pi)\,g_{\mathcal{M}}(\exp(s))^*\partial\bar{\partial}\log\{g_{|\mathcal{L}|^2}(\exp(s))\cdot\Omega_{\mathrm{FS},m}\},$$

when restricted to M. By (3.6) and (5.3), we now conclude from (6.4), (6.5) and (6.6) that the family of Kähler manifolds in (6.1) has bounded geometry, as required.

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