Fourth order perturbative expansion for the Casimir energy with a slightly deformed plate

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I- Abstract

We apply a perturbative approach to evaluate the Casimir energy for a massless real scalar field in $3+1$ dimensions, subject to Dirichlet boundary conditions on two surfaces. One of the surfaces is assumed to be flat, while the other corresponds to a small deformation, described by a single function η, of a flat mirror. The perturbative expansion is carried out up to the fourth order in the deformation η, and the results are applied to the calculation of the Casimir energy for corrugated mirrors in front of a plane. We also reconsider the proximity force approximation (PFA) within the context of this expansion.

II- The system

Scalar field φ satisfying Dirichlet boundary conditions on two surfaces (zero width): L and R:

$$\begin{aligned} L)\ x_3 &= 0 \\ R)\ x_3 &= a+\eta(\mathbf{x}_\parallel) \end{aligned}$$

Where η represents a small departure from a flat configuration.

The vacuum energy of the system, $E_{\rm vac}$ is

$$E_{\rm vac} = \lim_{T\to\infty} \Big[\frac{\Gamma(\eta)}{T}\Big]\Big|_{\eta=\eta(\mathbf{x}_\parallel)}$$

where T is the extent of the time dimension, and Γ is given by:

$$\Gamma \equiv -\log\Big(\frac{\mathcal{Z}}{\mathcal{Z}_0}\Big) = \frac{1}{2}{\rm Tr}\big(\log \mathbb{T}\big)$$

with

$$T_{\alpha\beta} = \langle x_\alpha|(-\nabla^2)^{-1}|x_\beta\rangle$$

$\alpha,\beta = L, R$

III- Perturbative expansion

$$\mathbb{T} = \mathbb{T}^{(0)} + \mathbb{T}^{(1)} + \mathbb{T}^{(2)} + \mathbb{T}^{(3)} + \mathbb{T}^{(4)} + \dots$$

we obtain an expansion for Γ of the form

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \Gamma^{(3)} + \Gamma^{(4)} + \dots$$

Thus, the perturbative espansion has the following expression

$$\begin{aligned} E_{\rm vac} &= -\frac{\pi^2 L^2}{1440 a^3} + \frac{1}{T}\sum_{n\geq 1}\frac{1}{a^{3+n}}\int \frac{d^3k_\parallel^{(1)}}{(2\pi)^3}\cdots\frac{d^3k_\parallel^{(n)}}{(2\pi)^3}\delta(k_\parallel^{(1)}+\dots+k_\parallel^{(n)}) \\ &\times\ h^{(n)}(ak_\parallel^{(1)},\dots,ak_\parallel^{(n)})\,\tilde\eta(k_\parallel^{(1)})\dots\tilde\eta(k_\parallel^{(n)}) \end{aligned}$$

We have explicitly calculated the $h^{(n)}$ factors for $n = 1, 2, 3,$ and 4

IV- Applications

Connection with the PFA

Let us assume that the function $\eta(\mathbf{x}_\parallel)$ is slowly varying. We can approximate $h^{(n)}(ak_\parallel^{(1)},\dots,ak_\parallel^{(n)}) \simeq h^{(n)}(0,\dots,0)$. As a consequence:

$$E_{\rm vac} \simeq -\frac{\pi^2 L^2}{1440 a^3} + \sum_{n\geq 1}\frac{1}{a^{3+n}}h^{(n)}(0,.,0)\int d^2\mathbf{x}_\parallel \eta^n$$

In the low momentum approximation the perturbative series can be summed up, the result being

$$E_{\rm vac} \simeq -\frac{\pi^2}{1440}\int \frac{d^2\mathbf{x}_\parallel}{\big(a+\eta(\mathbf{x}_\parallel)\big)^3}$$

which agrees with the PFA.

High momentum expansion

In the limit $|k_\parallel|a \gg 1$, the second order term is given in terms of a non-local operator

$$\frac{\Gamma^{(2)}}{T} \simeq -\frac{\pi^2}{480 a^4}\int d^2\mathbf{x}_\parallel\, \eta(\mathbf{x}_\parallel)(-\nabla^2)^{1/2}\eta(\mathbf{x}_\parallel)$$

The third order has two contributions, a local in configuration space

$$\frac{\Gamma^{\rm Local}}{T} \simeq \frac{\pi^2}{960 a^4}\int d^2\mathbf{x}_\parallel\, \eta^2(\mathbf{x}_\parallel)\nabla^2\eta(\mathbf{x}_\parallel)$$

and a non-local one

$$\begin{aligned} \frac{\Gamma^{\rm Non-local}}{T} &\simeq \frac{\pi^2}{480 a^4}\int d^2\mathbf{x}_\parallel\, \eta(\mathbf{x}_\parallel)(-\nabla^2)^{1/2}\eta(\mathbf{x}_\parallel) \\ &\times\ (-\nabla^2)^{1/2}\eta(\mathbf{x}_\parallel) \end{aligned}$$

The fourth order terms can be treated in a similar way. To leading order, they grow cubically with the momenta.

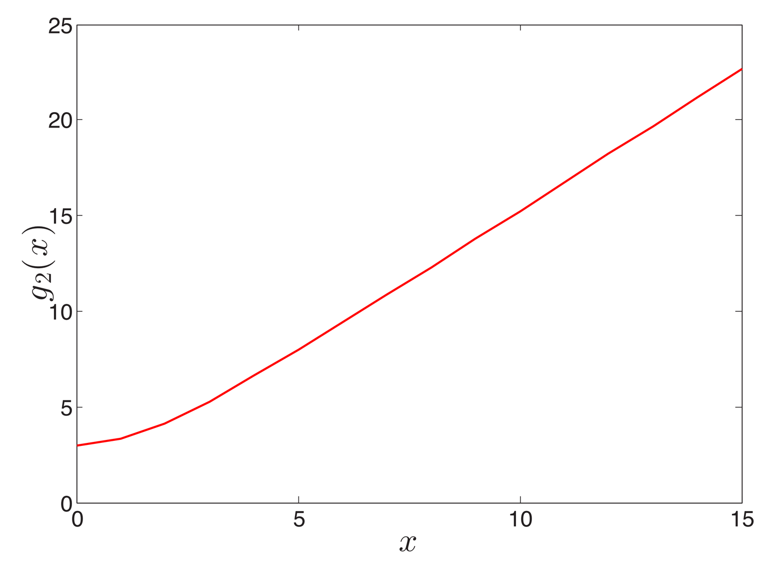
Sinusoidally corrugated surface

In the case $\eta(\mathbf{x}_\parallel) = \epsilon\sin(q_1 x_1)$, with $\epsilon \ll a$

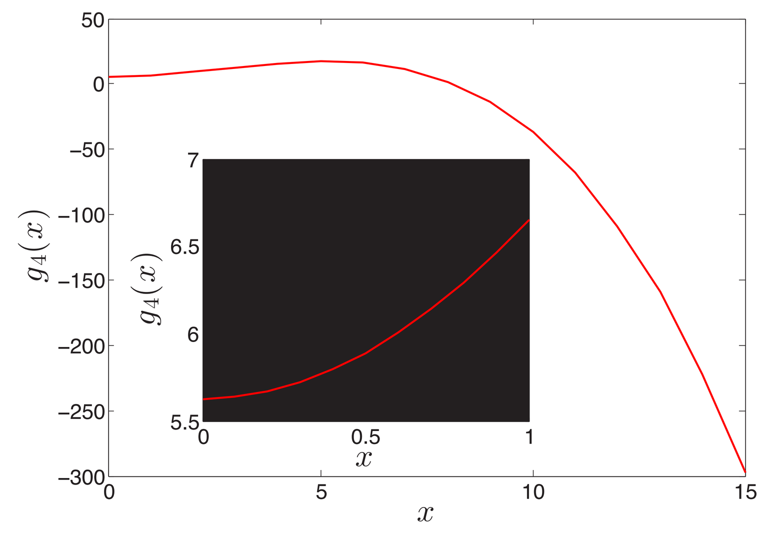
$$\begin{aligned} \mathcal{E}_{\rm vac} &\simeq -\frac{\pi^2}{1440 a^3}\Big[1 + g_2(q_1 a)\Big(\frac{\epsilon}{a}\Big)^2 \\ &+\ g_4(q_1 a)\Big(\frac{\epsilon}{a}\Big)^4 + \dots\Big] \end{aligned}$$

The perturbative expansion breaks down for very high momenta (short wavelength of the corrugations). The ratio of the fourth to second order corrections is proportional to $(\epsilon q_1)^2$. Therefore, the fourth order correction becomes more relevant when approaching the short wavelength limit.

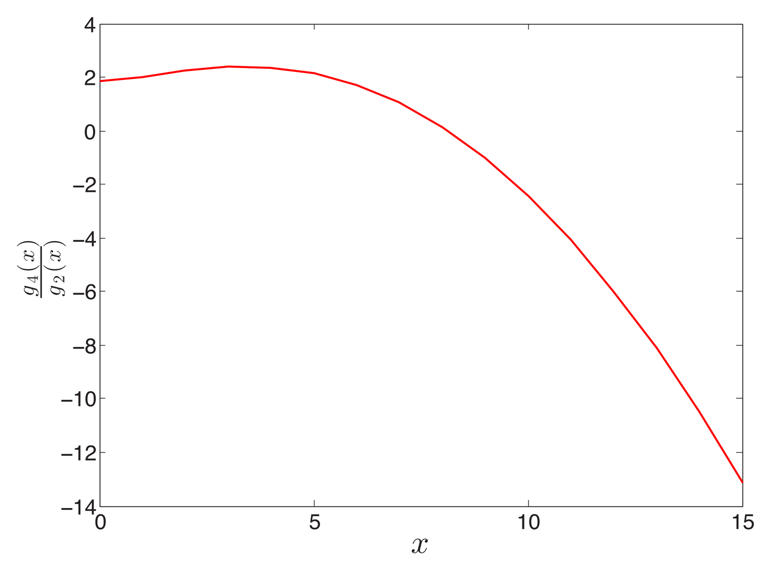
In Fig. c we show a plot of the ratio between the fourth and second order correction coefficients, g_4/g_2, as a function of $x = q_1 a$.



(a) g_2 as a function of $x = q_1 a$. Linear behavior of g_2 for large x is shown



(b) g_4 as a function of the $x = q_1 a$. The inset shows the low momentum behavior of g_4



(c) Fourth order correction becomes more relevant when approaching the short wavelength limit.

V - Main results

- $h^{(n)}$ factors contain all the information about the Casimir interaction energy. We extended previous results for $n=2$ to $n=4$
- For η a slowly varying function, a resummation of the perturbative series is possible, obtaining PFA after approximating the form factors by their zero-momentum values. PFA can be *improved* by expanding the form factors around $k_\parallel = 0$
- The *derivative expansion* for the Casimir energy emerges when the perturbative series only involves even powers of the momentum. The higher order corrections can be written in terms of derivatives of the shape of the surface
- For strongly varying surfaces, the Casimir energy is written in terms of nonlocal operators, suitable for corrugated mirrors
- In the sinosoidally corrugated surface, we have evaluated numerically the energy up to fourth order. The fourth order term is relevant when evaluating the Casimir energy in the limit of short wavelengths, where it becomes dominant

Reference

C.D. Fosco, F.C. Lombardo, and F.D. Mazzitelli, arXiv: 1209.1000[hep-th]