

A Generalized Construction of OFDM M -QAM Sequences With Low Peak-to-Average Power Ratio

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Abstract

A construction of 2^{2n} -QAM sequences is given and an upper bound of the peak-to-mean envelope power ratio (PMEPR) is determined. Some former works can be viewed as special cases of this construction.

Keywords. Golay sequences, QAM, multicarrier communications, orthogonal frequency-division multiplexing (OFDM), peak-to-mean envelope power ratio (PMEPR).

1 Introduction

Multicarrier communications have recently attracted much attention in wireless applications. The orthogonal frequency division multiplexing (OFDM) has been employed in several wireless communication standards. Their popularity is mainly due to the robustness to multipath fading channels and the efficient hardware implementation employing fast Fourier transform (FFT) techniques. However, multicarrier communications have the major drawback of the high peak-to-average power ratio (PAPR) of transmitted signals. Please refer to Litsyn [9] for a general source on PAPR control.

A coding method for PAPR control in multicarrier communications is to use Golay complementary sequences [4] [5] for subcarriers such that the sequences provide low peak-to-mean envelope power ratio (PMEPR) of at most 2 for transmitted signals, where the PAPR of the signals is bounded by the PMEPR. An important theoretical research on Golay complementary sequences has been set by Davis

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and Jedwab [2], where they showed the sequences can be constructed as a coset of the first order Reed-Muller codes by using algebraic normal forms. The research on Golay sequences has been flourished in the literature [3], [12]. The reader is referred to Jedwab [6] for a comprehensive survey of Golay sequences.

The approaches above consider phase-shift keying (PSK) signal constellations. However, there are many OFDM systems utilizing quadrature amplitude modulation (QAM) constellations. Some constructions of 16-QAM and 64-QAM complementary sequences were presented sequentially by Chong *et al* [1], Lee and Golomb [7], and Li [8]. In 2001, Rößing and Tarokh [14] gave an upper bound of PMEPR of the set of 16-QAM sequences using 2 quaternary phase-shift keying (QPSK) Golay sequences. In 2003, Tarokh and Sadjadpour [18] generalized the results in [14] from 16-QAM to 2^{2n} -QAM sequences by using n QPSK Golay sequences, and determined an upper bound of the PMEPR of this set. Motivated by these works, we found that the former construction can be generalized in a new way, such that the family size is significantly enlarged while the upper bound of PMEPR changes insignificantly.

The rest of the paper is organized as follows. In Section 2, the mathematical model of the multicarrier communication, the basic concept of Golay sequences, and the main results in [14] and [18] are reviewed. In Section 3, we give a construction of 2^{2n} -QAM sequences set \mathcal{A} , and determine an upper bound of $\text{PMEPR}(\mathcal{A})$. The construction in [14] and [18] can be viewed as a special case of our construction. Section 4 is for discussions and conclusions of this construction.

2 Preliminaries

2.1 Definitions

The transmitted OFDM signal is the real part of the complex signal

$$S_{\mathbf{a}}(t) = \sum_{i=0}^{N-1} a_i e^{2j\pi f_i t},$$

where f_i is the frequency of the i th carrier, $j = \sqrt{-1}$, and $\mathbf{a} = (a_1, a_2, \dots, a_{N-1})$ is a sequence with period N . To ensure orthogonality of different carriers, the i th carrier frequency f_i is set to be $f_0 + i\Delta f$, where f_0 is the smallest carrier frequency and Δf is an integer multiple of the OFDM symbol rate $1/T$, namely, $T\Delta f \in \mathbb{Z}$.

Definition 1 *The instantaneous envelope power of $S_{\mathbf{a}}(t)$ is defined as $P_{\mathbf{a}}(t) = |S_{\mathbf{a}}(t)|^2$. Then*

$$P_{\mathbf{a}}(t) = S_{\mathbf{a}}(t) \cdot \overline{S_{\mathbf{a}}(t)} = \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} a_i \overline{a_k} e^{2j\pi(i-k)\Delta f t}.$$

Thus, the mean power of $S_{\mathbf{a}}(t)$ during the symbol period T is

$$\frac{1}{T} \int_0^T P_{\mathbf{a}}(t) dt = \sum_{i=0}^{N-1} |a_i|^2 = \|\mathbf{a}\|^2.$$

Definition 2 The peak envelope power (PEP) of a codeword \mathbf{a} is defined as $PEP(\mathbf{a}) = \sup_{t \in [0, T]} P_{\mathbf{a}}(t)$.

Definition 3 The peak-to-mean power ratio (PMEPR) of a code \mathcal{C} is defined as

$$PMEPR(\mathcal{C}) = \max_{\mathbf{a} \in \mathcal{C}} PEP(\mathbf{a}) / P_{av}(\mathcal{C}),$$

where $P_{av}(\mathcal{C})$ is the mean envelope power of an OFDM signal averaged over all the OFDM signals in the codebook \mathcal{C} , i.e.,

$$P_{av}(\mathcal{C}) = \frac{1}{T} \sum_{\mathbf{a} \in \mathcal{C}} p(\mathbf{a}) \int_0^T P_{\mathbf{a}}(t) dt = \sum_{\mathbf{a} \in \mathcal{C}} p(\mathbf{a}) \|\mathbf{a}\|^2.$$

2.2 Golay sequences

An H -ary PSK (H -PSK) constellation can be realized as $\{e^{\frac{2\pi j s_i}{H}} | s_i \in \mathbb{Z}_H\}$. Thus any H -PSK sequence $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})$ is associated with the sequence $\mathbf{s} = (s_0, s_1, \dots, s_{N-1})$, where $a_i = e^{\frac{2\pi j s_i}{H}}$. For a given H -PSK code \mathcal{C} , since $P_{av}(\mathcal{C}) = N$, the PMEPR of code \mathcal{C} can be determined as

$$PMEPR(\mathcal{C}) = \max_{\mathbf{a} \in \mathcal{C}} PEP(\mathbf{a}) / N.$$

By the definition, one can get $N \leq PEP(\mathbf{a}) \leq N^2$. Thus $1 \leq PMEPR(\mathcal{C}) \leq N$.

An efficient coding method to reduce the PMEPR to 2 is the Golay sequences which was first introduced by M. J. E. Golay [4] in the context of infrared spectrometry. This approach relegates the main difficulty of reducing the PMEPR from finding the flat polynomials to constructing the sequences with good aperiodic auto-correlation property, i.e., from a continuous problem to a discrete one.

Definition 4 The aperiodic auto-correlation of the sequence $\mathbf{a} = (a_1, a_2, \dots, a_{N-1})$ at shift τ , where $1 \leq \tau \leq N - 1$, is defined as

$$C_{\mathbf{a}}(\tau) = \sum_{i=0}^{N-1-\tau} a_i \overline{a_{i+\tau}}.$$

Thus $P_{\mathbf{a}}(t)$ can be rewritten as the form

$$\begin{aligned}
P_{\mathbf{a}}(t) &= \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} a_i \bar{a}_k e^{2j\pi(i-k)\Delta ft} \\
&= \sum_{i=0}^{N-1} |a_i|^2 + 2 \sum_{\tau=1}^{N-1} \operatorname{Re}(e^{-2j\pi\tau\Delta ft} \sum_{i=0}^{N-1-\tau} a_i \overline{a_{i+\tau}}) \\
&= N + 2 \sum_{\tau=1}^{N-1} \operatorname{Re}(e^{-2j\pi\tau\Delta ft} C_{\mathbf{a}}(\tau)).
\end{aligned}$$

If a pair of sequences \mathbf{a} and \mathbf{b} satisfy

$$C_{\mathbf{a}}(\tau) + C_{\mathbf{b}}(\tau) = 0, \quad \forall \tau \neq 0,$$

then $P_{\mathbf{a}}(t) + P_{\mathbf{b}}(t) = 2N$. This implies that both $\operatorname{PEP}(\mathbf{a})$ and $\operatorname{PEP}(\mathbf{b}) \leq 2N$. Therefore, the PMEPR of the code \mathcal{C} , which is a collection of these sequences, is not larger than 2.

Definition 5 *The pair (\mathbf{a}, \mathbf{b}) satisfying the above condition is called a Golay complementary pair. Each member of a Golay complementary pair is called a Golay complementary sequence, or simply Golay sequence.*

2.3 M -QAM sequences constructed from QPSK Golay sequences

The QPSK constellation can be realized as $\{j^m \mid m \in \mathbb{Z}_4\}$. Therefore the QPSK sequence $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})$ is corresponding to the sequence $\mathbf{s} = (s_0, s_1, \dots, s_{N-1})$, where $a_i = j^{s_i}$ with $s_i \in \mathbb{Z}_4$, and a 2^{2n} -QAM constellation can be realized as

$$2^{2n}\text{-QAM} = \sum_{i=0}^{n-1} 2^{n-1-i} \frac{\sqrt{2}}{2} j^{s_i} e^{\frac{\pi j}{4}} = \frac{\sqrt{2}}{2} e^{\frac{\pi j}{4}} \sum_{i=0}^{n-1} 2^{n-1-i} j^{s_i}.$$

2^4 -QAM constellation can be viewed in both [14] and [1] as a simple example when $n = 2$. In this way, any 2^{2n} -QAM sequence $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})^T$ with period N is associated with a sequence vector or a matrix $\mathbf{s} = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{N-1})$, where $\mathbf{s}_i = (s_{i,0}, s_{i,1}, \dots, s_{i,N-1})^T \in \mathbb{Z}_4^N$ is a quaternary sequence with period N . In particular, the k th element of the 2^{2n} -QAM sequence \mathbf{a} is associated with $(s_{1,k}, s_{2,k}, \dots, s_{n-1,k})$, and can be presented as

$$a_k = \frac{\sqrt{2}}{2} e^{\frac{\pi j}{4}} \sum_{i=0}^{n-1} 2^{n-1-i} j^{s_{i,k}}.$$

Thus the signal $S_{\mathbf{a}}(t)$ can be written as

$$S_{\mathbf{a}}(t) = \frac{\sqrt{2}}{2} \sum_{k=0}^{N-1} \sum_{i=0}^{n-1} 2^{n-1-i} j^{s_{i,k}} e^{2\pi j f_k t + \frac{\pi j}{4}}.$$

Let \mathcal{C} be a collection of the 2^{2n} -QAM sequences \mathbf{a} corresponding to $\mathbf{s} = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{n-1})$, where \mathbf{s}_i is a Golay sequence for any $0 \leq i \leq n-1$. An upper bound of $\text{PMEPR}(\mathcal{C})$ is determined in [14] for 16-QAM and in [18] for the general case, which is shown as follows.

Fact 1

$$\text{PMEPR}(\mathcal{C}) \leq \frac{6(2^n - 1)^2}{2^{2n} - 1}.$$

From Fact 1, it's straightforward to get that $\text{PMEPR}(\mathcal{C}) \leq 3.6$ for 16-QAM, and $\text{PMEPR}(\mathcal{C}) < 6$ for general n .

3 A generalized construction with low PMEPR

For two given numbers x, y with $x > 1$ and $1 \leq y < 2$, let \mathcal{S}_i ($0 \leq i \leq n-1$) be a subset of the QPSK sequences with period n , and satisfy the following conditions:

- (a) $\text{PEP}(\mathbf{s}_i) \leq xy^{2i}N$ for every $\mathbf{s}_i \in \mathcal{S}_i$.
- (b) If $\mathbf{s}_i \in \mathcal{S}_i$, then $j^m \mathbf{s}_i \in \mathcal{S}_i$ for $m \in \mathbb{Z}_4$, where $j^m \mathbf{s}_i = (j^m s_{i,0}, j^m s_{i,1}, \dots, j^m s_{i,n-1})$.

Remark 1 1) It is not required that \mathcal{S}_i contains all the sequences satisfying $\text{PEP}(\mathbf{s}_i) \leq xy^{2i}N$.

2) $\text{PEP}(\mathbf{s}_i) = \text{PEP}(j^m \mathbf{s}_i)$, so it is reasonable to require \mathcal{S}_i satisfy the condition (b).

Theorem 1 Let \mathcal{A} be a collection of the 2^{2n} -QAM sequences \mathbf{a} such that $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})^T = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{n-1})$ and $\mathbf{s}_i \in \mathcal{S}_i$. Then

$$\text{PMEPR}(\mathcal{A}) \leq \frac{3}{4} \cdot \frac{2^{2n}}{2^{2n} - 1} \cdot \left(\frac{1 - (\frac{y}{2})^n}{1 - \frac{y}{2}} \right)^2 \cdot x.$$

For verifying Theorem 1, we first estimate $\text{PEP}(\mathbf{a})$ for every $\mathbf{a} \in \mathcal{A}$ in Lemma 1, then determine $P_{av}(\mathcal{A})$ in Lemma 2.

Lemma 1 Let \mathbf{a} be a 2^{2n} -QAM sequence such that $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})^T = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{n-1})$ and $\mathbf{s}_i \in \mathcal{S}_i$. Then

$$\text{PEP}(\mathbf{a}) \leq 2^{2n-3} \left(\frac{1 - (\frac{y}{2})^n}{1 - \frac{y}{2}} \right)^2 \cdot x \cdot N.$$

Proof: The signal $S_{\mathbf{a}}(t)$ can be written in the form

$$\begin{aligned}
S_{\mathbf{a}}(t) &= \frac{\sqrt{2}}{2} \sum_{k=0}^{N-1} \sum_{i=0}^{n-1} 2^{n-1-i} j^{s_{i,k}} e^{2\pi j f_k t + \frac{\pi j}{4}} \\
&= \frac{\sqrt{2}}{2} e^{\frac{\pi j}{4}} \sum_{i=0}^{n-1} 2^{n-1-i} \sum_{k=0}^{N-1} j^{s_{i,k}} e^{2\pi j f_k t} \\
&= \frac{\sqrt{2}}{2} e^{\frac{\pi j}{4}} \sum_{i=0}^{n-1} 2^{n-1-i} S_{\mathbf{s}_i}(t).
\end{aligned}$$

Thus the instantaneous envelope power of \mathbf{a} is given by

$$P_{\mathbf{a}}(t) = |S_{\mathbf{a}}(t)|^2 = \frac{1}{2} \left| \sum_{i=0}^{n-1} 2^{n-1-i} S_{\mathbf{s}_i}(t) \right|^2.$$

By the triangle inequality, one can get

$$P_{\mathbf{a}}(t) \leq \frac{1}{2} \left(\sum_{i=0}^{n-1} 2^{n-1-i} |S_{\mathbf{s}_i}(t)| \right)^2.$$

From $\mathbf{s}_i \in \mathcal{S}_i$ and $\text{PEP}(\mathbf{s}_i) \leq xy^{2i}N$, we have $|S_{\mathbf{s}_i}(t)| \leq (xy^{2i}N)^{\frac{1}{2}}$. Thus

$$\begin{aligned}
P_{\mathbf{a}}(t) &\leq \frac{1}{2} \left(\sum_{i=0}^{n-1} 2^{n-1-i} (xy^{2i}N)^{\frac{1}{2}} \right)^2 \\
&= \frac{1}{2} xN \left(\sum_{i=0}^{n-1} 2^{n-1-i} y^i \right)^2 \\
&= \frac{1}{2} xN \left(2^{n-1} \sum_{i=0}^{n-1} \left(\frac{y}{2}\right)^i \right)^2 \\
&= 2^{2n-3} \left(\frac{1 - (\frac{y}{2})^n}{1 - \frac{y}{2}} \right)^2 \cdot x \cdot N
\end{aligned}$$

□

Lemma 2 Let \mathbf{a} be a 2^{2n} -QAM sequence such that $\mathbf{a} = (a_0, a_1, \dots, a_{N-1})^T = (\mathbf{s}_0, \mathbf{s}_1, \dots, \mathbf{s}_{n-1})$ and $\mathbf{s}_i \in \mathcal{S}_i$. Then

$$P_{av}(\mathcal{A}) = \frac{1}{2}(2^n - 1) \cdot N.$$

Proof: Regard \mathbf{a} as a discrete random variable such that every \mathbf{s}_i is chosen from \mathcal{S}_i with the same probability, as well as the time t is a continuous random variable uniformly distributed in the interval $[0, T]$. Then P_{av} can be regarded as the expectation of the random function $P_{\mathbf{a}}(t)$. In the following, we also treat the sequence \mathbf{s}_i , and $s_{i,j}$, the j th element of \mathbf{s}_i , as random variables. Therefore

$$\begin{aligned}
P_{av}(\mathcal{A}) &= E(P_{\mathbf{s}}(t)) \\
&= E\left(\frac{1}{2}\left|\sum_{i=0}^{n-1} 2^{n-1-i} S_{\mathbf{s}_i}(t)\right|^2\right) \\
&= \frac{1}{2}E\left(\sum_{i=0}^{n-1} 2^{n-1-i} S_{\mathbf{s}_i}(t) \cdot \sum_{k=0}^{n-1} 2^{n-1-k} \overline{S_{\mathbf{s}_k}(t)}\right) \\
&= \frac{1}{2}\sum_{i=0}^{n-1}\sum_{k=0}^{n-1} 2^{2n-2-i-k} E\left(S_{\mathbf{s}_i}(t)\overline{S_{\mathbf{s}_k}(t)}\right) \\
&= \sum_{i=0}^{n-1} 2^{2n-3-2i} E|S_{\mathbf{s}_i}(t)|^2 + \frac{1}{2}\sum_{i=0}^{n-1}\sum_{k \neq i}^{n-1} 2^{2n-2-i-k} E\left(S_{\mathbf{s}_i}(t)\overline{S_{\mathbf{s}_k}(t)}\right).
\end{aligned}$$

Since \mathbf{s}_i is a random variable with respect to QPSK sequences in \mathcal{S}_i , one can get $E|S_{\mathbf{s}_i}(t)|^2 = N$ immediately. For $k \neq i$,

$$\begin{aligned}
E\left(S_{\mathbf{s}_i}(t)\overline{S_{\mathbf{s}_k}(t)}\right) &= E\left(\sum_{p=0}^{N-1} s_{i,p} e^{2j\pi(f_0+p\Delta f)t} \sum_{q=0}^{N-1} \overline{s_{k,q}} e^{-2j\pi(f_0+q\Delta f)t}\right) \\
&= \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} E\left(s_{i,p} \overline{s_{k,q}} e^{2j\pi(p-q)\Delta ft}\right).
\end{aligned}$$

For given i, k, p, q with $i \neq k$, $s_{i,p}$ and $s_{k,q}$ are random variables with respect to the p th and q th elements of \mathbf{s}_i and \mathbf{s}_k respectively. So $s_{i,p}$ and $s_{k,q}$ are independent. Thus,

$$E\left(s_{i,p} \overline{s_{k,q}} e^{2j\pi(p-q)\Delta ft}\right) = E(s_{i,p}) \overline{E(s_{k,q})} E\left(e^{2j\pi(p-q)\Delta ft}\right).$$

By the definition, if a sequence $\mathbf{s}_i \in \mathcal{S}_i$, then $j^m \mathbf{s}_i \in \mathcal{S}_i$. Therefore $s_{i,p} = j^m$ with the equal probability $1/4$ for any $m \in \mathbb{Z}_4$, which implies $E(a_{i,p}) = 0$. Due to the above, we obtain

$$P_{av}(\mathcal{A}) = N \sum_{i=0}^{n-1} 2^{2n-3-2i} = \frac{N}{6}(2^{2n} - 1).$$

This completes the proof. □

Proof of Theorem 1: By the results of Lemmas 1 and 2, the assertion of Theorem 1 follows immediately from the definition of PMEPR. \square

Corollary 1

$$\text{PMEPR}(\mathcal{A}) < \frac{3}{4} \cdot \frac{x}{\left(1 - \frac{y}{2}\right)^2}.$$

Proof: For $1 \leq y < 2$, it is obvious that

$$\lim_{n \rightarrow +\infty} \frac{2^{2n}}{2^{2n} - 1} \cdot \left(1 - \left(\frac{y}{2}\right)^n\right)^2 = 1.$$

Thus, to prove the result, one needs only to verify $\frac{2^{2n}}{2^{2n} - 1} \cdot \left(1 - \left(\frac{y}{2}\right)^n\right)^2$ is an increasing function with respect to n when $n \geq 1$. From

$$\frac{2^{2n}}{2^{2n} - 1} \cdot \left(1 - \left(\frac{y}{2}\right)^n\right)^2 = \left(1 - \left(\frac{y}{2}\right)^n\right) \cdot \left(1 - \frac{1}{2^n + 1}\right) \cdot \left(1 - \frac{y^n - 1}{2^n - 1}\right),$$

the claim holds since all $1 - \left(\frac{y}{2}\right)^n$, $1 - \frac{1}{2^n + 1}$, and $1 - \frac{y^n - 1}{2^n - 1}$ are positive increasing functions with respect to n when $n \geq 1$ and $1 \leq y < 2$. This completes the proof. \square

Corollary 2 *Let $y = 1 + \epsilon$ ($\epsilon \geq 0$), then*

$$\text{PMEPR}(\mathcal{A}) < 3x(1 + 2\epsilon) + o(\epsilon) \quad \text{and} \quad \text{PMEPR}(\mathcal{A}) < 3xy^2 + o(\epsilon).$$

Proof: Since $y = 1 + \epsilon$, we have

$$\frac{3}{4} \cdot \frac{x}{\left(1 - \frac{y}{2}\right)^2} = \frac{3x}{(1 - \epsilon)^2} = 3x(1 + 2\epsilon) + o(\epsilon)$$

and

$$3xy^2 = 3x(1 + \epsilon)^2 = 3x(1 + 2\epsilon) + o(\epsilon).$$

\square

4 Conclusion

Note that Fact 1 in Section 2.3, the main result in [18] and in [14], can be viewed as a special case of Theorem 1 by setting $x = 2$ and $y = 1$.

In the following, we discuss the case $y > 1$.

First, we consider the QPSK sequences subset \mathcal{S} with $\text{PEP}(\mathbf{s}) \leq \delta$ for all $\mathbf{s} \in \mathcal{S}$. Obviously, there is a trade off between the size $\#\mathcal{S}$ and the upper bound δ of the set \mathcal{S} . Since $\delta = xy^{2i}$ which may

be larger than 2, one can construct \mathcal{S}_i as a larger set than the Golay sequences set. There has been some research on how to enlarge the family size at the cost of increasing the PEP bound. The reader is referred to [13] and [15] for the construction of near-complementary sequences with $\text{PMEPR} < \delta$, and [16], [10], and [17] for the construction of \mathcal{S} with family size 2^{cn} and PEP upper bound $c \log n$.

Since xy^{2i} is an exponential function with respect to i , there exists i_0 such that $xy^{2i} \geq N$ when $i \geq i_0$. This implies that the sequences in the set \mathcal{S}_i can be arbitrary.

If $x = 2$ and $y = 1 + \epsilon$ with a small number ϵ , compared with the set \mathcal{C} presented in Fact 1, $\text{PMEPR}(\mathcal{A})$ changes insignificantly by Corollary 2, while the size of the set \mathcal{A} is significantly enlarged from the above results.

From Corollary 2, $\text{PMEPR}(\mathcal{A})$ is bounded by $3 \cdot \text{PEP}(\mathcal{S}_1)$ if ϵ is small enough. An interesting idea is that if there exist x and y with $xy^2 < 2$ and \mathcal{S}_0 is not an empty set, then one can obtain the bound $\text{PMEPR}(\mathcal{A}) < 6$. Here the size $\#(\mathcal{S}_0)$ and $\#(\mathcal{S}_1)$ may be small, but $\#(\mathcal{S}_i)$ would be very large for large enough i due to the comments above, which ensures that \mathcal{A} is a set with great size.

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