

Effective results on nonlinear ergodic averages in $\text{CAT}(\kappa)$ spaces

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Abstract

In this paper we apply proof mining techniques to compute, in the setting of $\text{CAT}(\kappa)$ spaces (with $\kappa > 0$), effective and highly uniform rates of asymptotic regularity and metastability for a nonlinear generalization of the ergodic averages, known as the Halpern iteration. In this way, we obtain a uniform quantitative version of a nonlinear extension of the classical von Neumann mean ergodic theorem.

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1 Introduction

In this paper we apply methods from mathematical logic to obtain a uniform quantitative version of a generalizations of the classical von Neumann mean ergodic theorem, giving effective rates of metastability for the so-called Halpern iteration, a nonlinear generalization of the ergodic averages. Our results are a contribution to the line of research known as *proof mining*, initiated in the 50's by Kreisel under the name of *unwinding of proofs* and extensively developed by Kohlenbach, beginning with the 90's. The idea of this research direction is to extract new, effective information from mathematical proofs making use of

ineffective principles. Hence, it can be related to Terence Tao’s proposal [32] of *hard analysis*, based on finitary arguments, instead of the infinitary ones from *soft analysis*. We refer to Kohlenbach’s book [12] for an introduction to proof mining.

Let us recall the Hilbert space formulation of the celebrated von Neumann mean ergodic theorem.

Theorem 1.1. *Let H be a Hilbert space and $U : H \rightarrow H$ be a unitary operator. Then for all $x \in H$, the Cesàro mean $x_n = \frac{1}{n} \sum_{i=0}^{n-1} U^i x$ converges strongly to the projection of x on the set of fixed points of U .*

If $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ is a probability measure-preserving system, $H = L^2(\mathcal{X})$ and $U = U_T : L^2(\mathcal{X}) \rightarrow L^2(\mathcal{X})$, $f \mapsto f \circ T$ is the induced operator, the Cesàro mean starting with $f \in L^2(\mathcal{X})$ becomes the ergodic average $A_n f = \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$.

The convergence of the ergodic averages can be arbitrarily slow, as shown by Krengel [22]. Furthermore, one cannot expect, in general, to get effective rates of convergence for the ergodic averages. Avigad, Gerhardy and Towsner [1] applied methods of computable analysis on Hilbert spaces to obtain an example of a computable Lebesgue measure-preserving transformation T on $[0, 1]$ and a computable characteristic function χ_A such that the limit of the sequence $A_n \chi_A$ is not a computable element of $L^2([0, 1])$, which implies that there is no computable bound on the rate of convergence of $(A_n \chi_A)$.

However, one can consider the following equivalent reformulation of the Cauchy property of (x_n) :

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \rightarrow \mathbb{N} \exists N \forall i, j \in [N, N + g(N)] (\|x_i - x_j\| < 2^{-k}). \quad (1)$$

This is known in logic as Kreisel’s [20, 21] no-counterexample interpretation of the Cauchy property and it was popularized in the last years under the name of *metastability* by Tao [32, 33]. In [33], Tao generalized the mean ergodic theorem for multiple commuting measure-preserving transformations, by deducing it from a finitary norm convergence result, expressed in terms of metastability. Recently, Walsh [34] used again metastability to show the L^2 -convergence of multiple polynomial ergodic averages arising from nilpotent groups of measure-preserving transformations.

Logical metatheorems developed by Kohlenbach [14] show that, from wide classes of mathematical proofs one can extract effective bounds on $\exists N$ in (1). Thus, taking $\varepsilon > 0$ instead of 2^{-k} , we define a rate of metastability as a functional $\Phi : (0, \infty) \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ satisfying

$$\forall \varepsilon > 0 \forall g : \mathbb{N} \rightarrow \mathbb{N} \exists N \leq \Phi(\varepsilon, g) \forall i, j \in [N, N + g(N)] (\|x_i - x_j\| < \varepsilon). \quad (2)$$

Avigad, Gerhardy and Towsner [1] computed for the first time explicit and uniform rates of metastability for the ergodic averages, by a logical analysis of Riesz’ proof of the mean ergodic theorem. Their result was generalized, with better bounds, to uniformly convex Banach spaces by Kohlenbach and the first author [15], applying proof mining methods, but this time to a proof of Garrett

Birkhoff [3]. In fact, Avigad and Rute [2] realized that the computations in [16] allow one to compute an effective bound on the number of ε -fluctuations (i.e. pairs (i, j) with $i > j$ and $\|x_i - x_j\| > \varepsilon$). A very nice discussion on the different types of quantitative information (metastability, effective learnability, bounds on the number of oscillations) that can be extracted from convergence proofs is done in a recent paper by Kohlenbach and Safarik [19].

In the important paper [35], Wittmann obtained the following nonlinear generalization of the mean ergodic theorem.

Theorem 1.2. [35] *Let C be a bounded closed convex subset of a Hilbert space X , $T : C \rightarrow C$ a nonexpansive mapping and $(\lambda_n)_{n \geq 1}$ be a sequence in $[0, 1]$. For any $u \in C$, define*

$$x_0 := u, \quad x_{n+1} := \lambda_{n+1}u + (1 - \lambda_{n+1})Tx_n. \quad (3)$$

Assume that (λ_n) satisfies

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty \quad (4)$$

Then for any $x \in C$, the Halpern iteration (x_n) converges to the projection $P_{\text{Fix}(T)}x$ of x on the (nonempty) set of fixed points $\text{Fix}(T)$.

One can easily see that (x_n) coincides with the Cesàro mean when T is linear and $\lambda_n = \frac{1}{n+1}$. The iteration (x_n) is known as the Halpern iteration, as it was introduced by Halpern [10] for the special case $u = 0$. We refer to [17, Section 3] for a discussion on results in the literature on Halpern iterations, obtained by considering different conditions on (λ_n) or more general spaces.

Kohlenbach's logical metatheorem for Hilbert spaces [14] guarantees also in the case of Wittmann's theorem that from its proof one can extract a rate of metastability Φ of (x_n) , uniform in the following sense: it depends only on ε and g , an upper bound on the diameter of C and moduli on (λ_n) , given by the quantitative version of (4). Thus, Φ is independent with respect to the starting point u of the iteration, the nonexpansive mapping T , the Hilbert space X and depends on C only via its diameter. Kohlenbach [13] computed such a uniform rate of metastability, by a logical analysis of Wittmann's proof.

Furthermore, Kohlenbach and the first author [16, 17, 18] extracted rates of metastability from the proofs of two generalizations of Wittmann's theorem given by Shioji and Takahashi [31] for a class of Banach spaces with a uniformly Gâteaux differentiable norm and by Saejung [29] for CAT(0) spaces. Both Saejung's and Shioji-Takahashi's proofs use Banach limits (whose existence requires the axiom of choice), inspired by Lorentz' seminal paper [25], introducing almost convergence. Our quantitative results were obtained by developing in [17] a method to eliminate the use of Banach limits from these proofs and get, in this way, elementary proofs to which general logical metatheorems for CAT(0) spaces [14] and for uniformly smooth Banach spaces [16] can be applied to guarantee

the extractability of effective bounds. We point out that the use of Lorentz' almost convergence (and hence, Banach limits) in nonlinear ergodic theory was introduced by Reich [27], while Bruck and Reich [8] applied Banach limits for the first time to the study of Halpern iterations (see also [9, Sections 12, 14]).

Geodesic spaces provide a suitable setting for extending the notion of sectional curvature from Riemannian manifolds. An important class of geodesic spaces of bounded curvature are $CAT(\kappa)$ spaces, where geodesic triangles are in some sense "thin". Such spaces enjoy nice properties inherited from the comparison with the model spaces and proved to be relevant in various problems and aspects in geometry (see [4]).

In this paper we extract an effective and uniform rate of metastability for the generalization of Wittmann's theorem to $CAT(\kappa)$ spaces with $\kappa > 0$.

To get our main quantitative result (Theorem 3.4) we apply again the general method developed in [17], together with the remark that, in fact, our logical analysis of Saejung's proof for $CAT(0)$ spaces results in the elimination of any contribution of Banach limits, hence even the finitary lemmas proved in [16, Section 8] are not needed anymore (see [18]). Despite of this simplification, the proofs we give in this paper are much more involved, since we work in the setting of $CAT(\kappa)$ spaces. However, in the case $\lambda_n = \frac{1}{n+1}$ we still get a rate of metastability having a very nice form, similar with the one described in [19].

As the first step in the convergence proof is to obtain the asymptotic regularity, our first important result (Proposition 3.2) is the computation of a uniform rate of asymptotic regularity.

For the rest of the paper $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}_+ = \{1, 2, \dots\}$. Furthermore, we consider $CAT(\kappa)$ spaces with $\kappa > 0$.

2 $CAT(\kappa)$ spaces

Let (X, d) be a metric space. A *geodesic path* from x to y is a mapping $c : [0, l] \subseteq \mathbb{R} \rightarrow X$ such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a *geodesic segment* which joins x and y . Note that a geodesic segment from x to y is not necessarily unique. If no confusion arises, we use $[x, y]$ to denote a geodesic segment joining x and y . (X, d) is a (*uniquely*) *geodesic space* if every two points $x, y \in X$ can be joined by a (unique) geodesic path. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$, and we write $z = (1 - t)x + ty$ for simplicity. This, too, may not be unique. A subset C of X is *convex* if C contains any geodesic segment that joins every two points in C . A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ consists of three points x_1, x_2 and x_3 in X (its *vertices*) and three geodesic segments corresponding to each pair of points (its *edges*).

$CAT(\kappa)$ spaces are defined in terms of comparisons with the model spaces M_κ^n . We focus here on $CAT(\kappa)$ spaces with $\kappa > 0$. We give below the precise definition and briefly describe some of their properties that play an essential

role in this work. For a detailed discussion on geodesic metric spaces and on $\text{CAT}(\kappa)$ spaces in particular, the reader may check, for example, [4].

The n -dimensional sphere \mathbb{S}^n is the set $\{x \in \mathbb{R}^{n+1} : (x|x) = 1\}$, where $(\cdot|\cdot)$ stands for the Euclidean scalar product. Consider the mapping $d : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$ by assigning to each $(x, y) \in \mathbb{S}^n \times \mathbb{S}^n$ the unique number $d(x, y) \in [0, \pi]$ such that $\cos d(x, y) = (x|y)$. Then, (\mathbb{S}^n, d) is a metric space called the spherical space. This space is also geodesic and, if $d(x, y) < \pi$, then there exists a unique geodesic joining x and y . Also, balls of radius smaller than $\pi/2$ are convex. The spherical law of cosines states that in a spherical triangle with vertices $x, y, z \in \mathbb{S}^n$ and γ the spherical angle between the geodesic segments $[x, y]$ and $[x, z]$ we have

$$\cos d(y, z) = \cos d(x, y) \cos d(x, z) + \sin d(x, y) \sin d(x, z) \cos \gamma.$$

Let $\kappa > 0$ and $n \in \mathbb{N}$. The classical *model spaces* M_κ^n are obtained from the spherical space \mathbb{S}^n by multiplying the spherical distance with $1/\sqrt{\kappa}$. These spaces inherit the geometrical properties from the spherical space. Thus, there is a unique geodesic segment joining $x, y \in M_\kappa^n$ if and only if $d(x, y) < \pi/\sqrt{\kappa}$. Moreover, closed balls of radius smaller than $\pi/(2\sqrt{\kappa})$ are convex and we have a counterpart of the spherical law of cosines. We denote the *diameter of* M_κ^n by $D_\kappa = \pi/\sqrt{\kappa}$.

For a geodesic triangle $\Delta = \Delta(x_1, x_2, x_3)$, a κ -comparison triangle is a triangle $\bar{\Delta} = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that $d(x_i, x_j) = d_{M_\kappa^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$. For κ fixed, κ -comparison triangles of geodesic triangles (having perimeter less than $2D_\kappa$) always exist and are unique up to isometry.

We say that a geodesic triangle Δ satisfies the *CAT(κ) inequality* if for every κ -comparison triangle $\bar{\Delta}$ of Δ and for every $x, y \in \Delta$ we have

$$d(x, y) \leq d_{M_\kappa^2}(\bar{x}, \bar{y}),$$

where $\bar{x}, \bar{y} \in \bar{\Delta}$ are the comparison points of x and y , i.e., if $x = (1-t)x_i + tx_j$ then $\bar{x} = (1-t)\bar{x}_i + t\bar{x}_j$. A metric space is called a *CAT(κ) space* if every two points at distance less than D_κ can be joined by a geodesic segment and every geodesic triangle having perimeter less than $2D_\kappa$ satisfies the $\text{CAT}(\kappa)$ inequality. $\text{CAT}(0)$ spaces are defined in a similar way considering the model space M_0^2 to be the Euclidean plane with diameter ∞ .

3 Main results

If X is a geodesic space, $C \subseteq X$ a convex subset, $T : C \rightarrow C$ nonexpansive and (λ_n) a sequence $[0, 1]$, one can define the Halpern iteration starting from $u \in C$ by

$$x_0 = u, \quad x_{n+1} := \lambda_{n+1}u + (1 - \lambda_{n+1})Tx_n. \quad (5)$$

The main purpose of our work is to prove a quantitative version of the following generalization of Wittmann's theorem to $\text{CAT}(\kappa)$ spaces, obtained recently by Piątek [26].

Theorem 3.1. *Let X be a complete $CAT(\kappa)$ space, $C \subseteq X$ a bounded closed convex subset with diameter $d_C < \frac{D_\kappa}{2}$ and $T : C \rightarrow C$ a nonexpansive mapping. Assume that (λ_n) satisfies (4). Then for any $u \in C$, the iteration (x_n) starting from u converges to the fixed point of T which is nearest to u .*

A first important result of this paper is the extraction of an effective rate of asymptotic regularity for the Halpern iteration, that is of the convergence of $(d(x_n, Tx_n))$ towards 0. In order to state this result, we need to make the hypotheses (4) on (λ_n) quantitative.

For brevity, we say that (λ_n) , $\alpha : (0, \infty) \rightarrow \mathbb{Z}_+$, $\gamma : (0, \infty) \rightarrow \mathbb{Z}_+$, $\theta : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ satisfy (*) if

- (i) $\lim_{n \rightarrow \infty} \lambda_{n+1} = 0$ with rate of convergence α , i.e. $\lambda_{n+1} \leq \varepsilon$ for all $\varepsilon > 0$ and all $n \geq \alpha(\varepsilon)$;
- (ii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ converges with Cauchy modulus γ , i.e. $\sum_{i=\gamma(\varepsilon)+1}^{\gamma(\varepsilon)+n} |\lambda_{i+1} - \lambda_i| \leq \varepsilon$ for all $\varepsilon > 0$ and all $n \in \mathbb{Z}_+$;
- (iii) $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ with rate of divergence θ , i.e. $\sum_{k=1}^{\theta(n)} \lambda_{k+1} \geq n$ for all $n \in \mathbb{Z}_+$.

Proposition 3.2. *Let X be a $CAT(\kappa)$ space, $C \subseteq X$ a bounded convex subset, $T : C \rightarrow C$ nonexpansive and $M < \frac{D_\kappa}{2}$ an upper bound on the finite diameter d_C of C . Assume furthermore that (λ_n) , α, γ, θ satisfy (*).*

Then $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$ with rate of convergence Φ given by

$$\tilde{\Phi}(\varepsilon, \kappa, M, \gamma, \theta) = \theta \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil \left(\gamma \left(\frac{\varepsilon}{2M} \right) + \max \left\{ \left\lceil \ln \left(\frac{2M}{\varepsilon} \right) \right\rceil, 1 \right\} \right) \right) \quad (6)$$

and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ with rate of convergence Φ given by

$$\Phi(\varepsilon, \kappa, M, \gamma, \theta, \alpha) = \max \left\{ \tilde{\Phi} \left(\frac{\varepsilon}{2}, \kappa, M, \gamma, \theta \right), \alpha \left(\frac{\varepsilon}{2M} \right) \right\}. \quad (7)$$

Proof. See Section 5. □

In the case $\lambda_n = \frac{1}{n+1}$ one can easily obtain rates α, γ, θ :

$$\alpha(\varepsilon) = \gamma(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil, \quad \theta(n) = \exp(n \ln 4). \quad (8)$$

As an immediate consequence we get

Corollary 3.3. *Assume that $\lambda_n = \frac{1}{n+1}$, $n \geq 1$. Then $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ with a common rate of convergence*

$$\Psi(\varepsilon, \kappa, M) = \exp \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil \left(\left\lceil \frac{4M}{\varepsilon} \right\rceil + 2 \right) \ln 4 \right), \quad (9)$$

which is exponential in $1/\varepsilon$.

We point out that exponential rates of asymptotic regularity for the Halpern iteration were obtained by the first author for Banach spaces in [23] and for the so-called W -hyperbolic spaces in [24]. Kohlenbach [13] remarked that the proof in [23] can be simplified and, as a consequence, one gets quadratic rates in Banach spaces. For $\text{CAT}(0)$ spaces, Kohlenbach and the first author provide in [17] a quantitative asymptotic regularity result for general (λ_n) by considering instead of $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ the equivalent condition $\prod_{n=1}^{\infty} (1 - \lambda_{n+1}) = 0$. As a corollary, one obtains again quadratic rates of asymptotic regularity. However, the method used in [17] for $\text{CAT}(0)$ spaces does not hold for $\text{CAT}(\kappa)$ spaces.

The main result of the paper is the following quantitative version of Theorem 3.1, providing an explicit uniform rate of metastability for the Halpern iteration in $\text{CAT}(\kappa)$ spaces. To get such a result we apply again the general method developed by Kohlenbach and the first author in [17] for the Halpern iteration in $\text{CAT}(0)$ spaces and applied again in [16] for uniformly smooth Banach spaces as well as in [30] for a modified Halpern iteration in $\text{CAT}(0)$ spaces. As noticed in [18], in the end we don't need the finitary Lemmas 8.3 and 8.4 from [17], since, as a consequence of the proof mining methods applied to Saejung's proofs, one gets a proof where no contributions of Banach limits can be traced.

Theorem 3.4. *Let X be a complete $\text{CAT}(\kappa)$ space, $C \subseteq X$ a bounded closed convex subset, $T : C \rightarrow C$ nonexpansive and $M < \frac{D_\kappa}{2}$ an upper bound on the finite diameter d_C of C . Assume furthermore that $(\lambda_n), \alpha, \gamma, \theta$ satisfy (*). Then for all $\varepsilon \in (0, 2)$ and $g : \mathbb{N} \rightarrow \mathbb{N}$,*

$$\exists N \leq \Sigma(\varepsilon, g, \kappa, M, \theta, \alpha, \beta) \quad \forall m, n \in [N, N + g(N)] \quad (d(x_n, x_m) \leq \varepsilon),$$

where $N = \Theta_{K_0} \left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \right)$ for some $\left\lceil \frac{1}{\varepsilon_0} \right\rceil \leq K_0 \leq \widetilde{f}^* \left(\left\lceil \frac{M\sqrt{\kappa} \tan(M\sqrt{\kappa})}{1 - \cos(\varepsilon_0)} \right\rceil \right) (0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil$ and $\Sigma := \Sigma(\varepsilon, g, \kappa, M, \theta, \alpha, \beta)$ is defined by

$$\Sigma = \theta^+ \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil \left(\Gamma - 1 + \max \left\{ \left\lceil \ln \left(\frac{3 \sin^2 \frac{M\sqrt{\kappa}}{4}}{\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}} \right) \right\rceil, 1 \right\} \right) \right) + 1,$$

with $\theta^+(n) = \max\{\theta(i) : 1 \leq i \leq n\}$,

$$\Gamma = \max \left\{ \chi_i^* \left(\frac{1}{3} \sin^2 \frac{\varepsilon \sqrt{\kappa}}{4} \right) : \left\lceil \frac{1}{\varepsilon_0} \right\rceil \leq i \leq \widetilde{f}^* \left(\left\lceil \frac{M \sqrt{\kappa} \tan(M \sqrt{\kappa})}{1 - \cos(\varepsilon_0)} \right\rceil \right) (0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil \right\},$$

$$\chi_i^*(\varepsilon) = \chi_i \left(\frac{\varepsilon}{2} \cos(M \sqrt{\kappa}) \right), \quad L_i = \frac{\cos(M \sqrt{\kappa}) \varepsilon}{4M \sqrt{\kappa} (i+1)}, \quad \varepsilon_0 = \frac{\cos(M \sqrt{\kappa})}{36} \sin^2 \frac{\varepsilon \sqrt{\kappa}}{4},$$

$$\chi_i(\varepsilon) = \max \left\{ \theta \left(\left\lceil \frac{1}{\cos(M \sqrt{\kappa})} \right\rceil \left(\gamma(L_i) + \max \left\{ \left\lceil \ln \left(\frac{1}{L_i} \right) \right\rceil, 1 \right\} \right) \right), \alpha(2L_i) \right\},$$

$$\Theta_i(\varepsilon) = \theta \left(\left\lceil \frac{1}{\cos(M \sqrt{\kappa})} \right\rceil \left(\chi_i^* \left(\frac{\varepsilon}{3} \right) - 1 + \max \left\{ \left\lceil \ln \left(\frac{3}{\varepsilon} \sin^2 \frac{M \sqrt{\kappa}}{2} \right) \right\rceil, 1 \right\} \right) \right) + 1,$$

$$g_{\varepsilon, i}(n) = n + g \left(n + \chi_i^* \left(\frac{\varepsilon}{3} \right) \right), \quad \Delta_i^*(\varepsilon, g) = \frac{\varepsilon}{3g_{\varepsilon, i}(\Theta_i(\varepsilon) - \chi_i^*(\frac{\varepsilon}{3}))},$$

$$f(i) = \max \left\{ \left\lceil \frac{M \sqrt{\kappa}}{\Delta_i^*(\sin^2 \frac{\varepsilon \sqrt{\kappa}}{4}, g)} \right\rceil, i \right\} - i, \quad f^*(i) = f \left(i + \left\lceil \frac{1}{\varepsilon_0} \right\rceil \right) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil,$$

$$\widetilde{f}^*(i) = i + f^*(i).$$

Proof. We refer to Section 7 for the proof. We point here only the main steps:

- (i) extract a rate of asymptotic regularity (this is done in Proposition 3.2);
- (ii) obtain a quantitative Browder theorem (see Proposition 6.2);
- (iii) define in an appropriate way an approximate fixed point sequence γ_n^t (see (26));
- (iv) apply Lemma 7.3, a quantitative lemma on sequences of real numbers.

□

As a consequence it follows that

Corollary 3.5. *Assume that $\lambda_n = \frac{1}{n+1}$ for all $n \geq 1$. Then for all $\varepsilon \in (0, 2)$ and $g : \mathbb{N} \rightarrow \mathbb{N}$,*

$$\exists N \leq \Sigma(\varepsilon, g, \kappa, M) \forall m, n \in [N, N + g(N)] (d(x_n, x_m) \leq \varepsilon),$$

where

$$\Sigma(\varepsilon, g, \kappa, M) = A_{\varepsilon, \kappa, M} \left(\widetilde{f}^{*B_{\varepsilon, \kappa, M}}(0) + \left\lceil \frac{1}{\varepsilon_0} \right\rceil \right), \quad (10)$$

$$\begin{aligned}
\text{with } B_{\varepsilon,\kappa,M} &= \left\lceil \frac{M\sqrt{\kappa} \tan(M\sqrt{\kappa})}{1 - \cos(\varepsilon_0)} \right\rceil, \quad A_{\varepsilon,\kappa,M} = A_{\varepsilon,\kappa,M}^1 \circ A_{\varepsilon,\kappa,M}^2 \text{ and} \\
A_{\varepsilon,\kappa,M}^1(n) &= \exp \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil (n - 1 + \max \{ \lceil T_{\varepsilon,\kappa,M} \rceil, 1 \}) \ln 4 \right) + 1, \\
A_{\varepsilon,\kappa,M}^2(n) &= \exp \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil U_{\varepsilon,\kappa,M}(n) \ln 4 \right), \\
T_{\varepsilon,\kappa,M} &= \ln \left(\frac{3 \sin^2 \frac{M\sqrt{\kappa}}{4}}{\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}} \right), \quad S_{\varepsilon,\kappa,M}(n) = \frac{24M\sqrt{\kappa}(n+1)}{\cos^2(M\sqrt{\kappa}) \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}}, \\
U_{\varepsilon,\kappa,M}(n) &= \lceil S_{\varepsilon,\kappa,M}(n) \rceil + \max \{ \lceil \ln(S_{\varepsilon,\kappa,M}(n)) \rceil, 1 \}
\end{aligned}$$

and $\varepsilon_0, \widetilde{f}^*$ are defined as in Theorem 3.4.

Proof. Apply the fact that, in this case, with α, β, θ given by (8), $(\chi_i^*)_i$ is non-decreasing. \square

Thus, as in [16, 17] as well as in other case studies in proof mining, we get a rate of metastability having the form described by Kohlenbach and Safarik [19]: the counterfunction g appears only in the definition of \widetilde{f}^* , the mappings $A_{\varepsilon,\kappa,M}, B_{\varepsilon,\kappa,M}$ do not depend at all on g . We refer to [19] for a logical explanation of this phenomenon.

4 Some technical lemmas

Throughout the paper, we shall use the following well-known facts:

- (i) $x \geq \sin x$ for all $x \geq 0$.
- (ii) $\sin(tx) \geq t \sin x$ for all $x \in [0, \pi]$ and all $t \in [0, 1]$.
- (iii) The function $f : (0, \pi) \rightarrow (0, 1)$, $f(x) = \frac{\sin x}{x}$ is decreasing.
- (iv) Given $t \in [0, 1]$, the mapping $f : (0, \pi) \rightarrow (0, \infty)$, $f(x) = \frac{\sin(tx)}{\sin x}$ is increasing.

The following very useful result is proved in [26] for $k = 1$. The proof for general $k > 0$ is an immediate rescaling.

Lemma 4.1. *Let $\Delta(x, y, z)$ be a triangle in X and $M \leq \frac{D_\kappa}{2}$ be an upper bound on the lengths of the sides of $\Delta(x, y, z)$. Then for all $t \in (0, 1)$,*

$$d((1-t)x + tz, (1-t)y + tz) \leq \frac{\sin((1-t)M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} d(x, y) \leq d(x, y).$$

Let X be a $\text{CAT}(\kappa)$ space. The next result gathers some useful properties which will be needed in the subsequent sections.

Lemma 4.2. *Let $\Delta(x, y, z)$ be a triangle in X with perimeter $< 2D_\kappa$. Let w be a point on the segment joining x and z . Suppose that $\cos(d(y, z)\sqrt{\kappa}) \geq \cos(d(y, w)\sqrt{\kappa}) \cos(d(w, z)\sqrt{\kappa})$. Then $d(x, w) \leq d(x, y)$. Moreover, if $\Delta(\bar{x}, \bar{y}, \bar{z})$ is a κ -comparison triangle for $\Delta(x, y, z)$, then $\angle_{\bar{w}}(\bar{y}, \bar{x}) \geq \frac{\pi}{2}$.*

Proof. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a κ -comparison triangle for $\Delta(x, y, z)$ and $\alpha = \angle_{\bar{w}}(\bar{y}, \bar{z})$. Suppose that $\alpha > \frac{\pi}{2}$. Then

$$\begin{aligned} \cos(d(y, z)\sqrt{\kappa}) &= \cos(d(\bar{y}, \bar{w})\sqrt{\kappa}) \cos(d(\bar{w}, \bar{z})\sqrt{\kappa}) \\ &\quad + \sin(d(\bar{y}, \bar{w})\sqrt{\kappa}) \sin(d(\bar{w}, \bar{z})\sqrt{\kappa}) \cos \alpha \\ &< \cos(d(\bar{y}, \bar{w})\sqrt{\kappa}) \cos(d(\bar{w}, \bar{z})\sqrt{\kappa}) \\ &\leq \cos(d(y, w)\sqrt{\kappa}) \cos(d(w, z)\sqrt{\kappa}), \end{aligned}$$

which contradicts the hypothesis. Thus, $\alpha \leq \frac{\pi}{2}$ and $\beta := \angle_{\bar{w}}(\bar{y}, \bar{x}) \geq \frac{\pi}{2}$. It follows that

$$\begin{aligned} \cos(d(\bar{x}, \bar{y})\sqrt{\kappa}) &= \cos(d(\bar{x}, \bar{w})\sqrt{\kappa}) \cos(d(\bar{w}, \bar{y})\sqrt{\kappa}) \\ &\quad + \sin(d(\bar{x}, \bar{w})\sqrt{\kappa}) \sin(d(\bar{w}, \bar{y})\sqrt{\kappa}) \cos \beta \\ &\leq \cos(d(\bar{x}, \bar{w})\sqrt{\kappa}), \end{aligned}$$

hence $d(\bar{x}, \bar{y}) \geq d(\bar{x}, \bar{w})$. It follows that $d(x, w) \leq d(x, y)$. \square

Assume $C \subseteq X$ is bounded with $M < \frac{D_\kappa}{2}$ an upper bound on its diameter. In the sequel x, y, z are pairwise distinct points of C and $w \in [x, y]$, $v \in [x, z]$.

We shall use the following notation:

$$\begin{aligned} S_1 &= \sin(d(x, w)\sqrt{\kappa}) \sin(d(x, v)\sqrt{\kappa}), & S_2 &= \sin(d(x, y)\sqrt{\kappa}) \sin(d(x, z)\sqrt{\kappa}), \\ S_3 &= \sin(d(x, w)\sqrt{\kappa}) \sin(d(x, z)\sqrt{\kappa}), & S_4 &= \sin(d(y, w)\sqrt{\kappa}) \sin(d(x, z)\sqrt{\kappa}), \\ S_5 &= \sin(d(x, w)\sqrt{\kappa}) \sin(d(z, v)\sqrt{\kappa}), \\ C_1 &= \cos(d(x, w)\sqrt{\kappa}) \cos(d(x, v)\sqrt{\kappa}), & C_2 &= \cos(d(x, y)\sqrt{\kappa}) \cos(d(x, z)\sqrt{\kappa}). \end{aligned}$$

Lemma 4.3.

$$S_2 - S_3 \leq S_4 \cos(d(x, w)\sqrt{\kappa}), \quad (11)$$

$$S_3 - S_1 \leq S_5 \cos(d(x, v)\sqrt{\kappa}), \quad (12)$$

$$S_2 C_1 - S_1 C_2 = S_4 \cos(d(x, v)\sqrt{\kappa}) + S_5 \cos(d(x, y)\sqrt{\kappa}), \quad (13)$$

$$S_2 - S_3 - S_4 \cos(d(x, v)\sqrt{\kappa}) \leq 2S_4 \left(\sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2} \right), \quad (14)$$

$$S_3 - S_1 - S_5 \cos(d(x, y)\sqrt{\kappa}) \leq 2S_5 \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right). \quad (15)$$

Proof.

$$\begin{aligned}
S_2 - S_3 &= (\sin(d(x, y)\sqrt{\kappa}) - \sin(d(x, w)\sqrt{\kappa})) \sin(d(x, z)\sqrt{\kappa}) \\
&= 2 \sin \frac{(d(x, y) - d(x, w))\sqrt{\kappa}}{2} \cos \frac{(d(x, y) + d(x, w))\sqrt{\kappa}}{2} \sin(d(x, z)\sqrt{\kappa}) \\
&= 2 \sin \frac{d(y, w)\sqrt{\kappa}}{2} \cos \left(\left(d(x, w) + \frac{d(y, w)}{2} \right) \sqrt{\kappa} \right) \sin(d(x, z)\sqrt{\kappa}) \\
&\leq 2 \sin \frac{d(w, y)\sqrt{\kappa}}{2} \cos(d(x, w)\sqrt{\kappa}) \cos \frac{d(w, y)\sqrt{\kappa}}{2} \sin(d(x, z)\sqrt{\kappa}) \\
&= \sin(d(w, y)\sqrt{\kappa}) \cos(d(x, w)\sqrt{\kappa}) \sin(d(x, z)\sqrt{\kappa}) = S_4 \cos(d(x, w)\sqrt{\kappa}).
\end{aligned}$$

Similarly, one gets that $S_3 - S_1 = S_5 \cos(d(x, v)\sqrt{\kappa})$.

$$\begin{aligned}
S_2 C_1 - S_1 C_2 &= \sin(d(x, y)\sqrt{\kappa}) \sin(d(x, z)\sqrt{\kappa}) \cos(d(x, w)\sqrt{\kappa}) \cos(d(x, v)\sqrt{\kappa}) \\
&\quad - \sin(d(x, w)\sqrt{\kappa}) \sin(d(x, v)\sqrt{\kappa}) \cos(d(x, y)\sqrt{\kappa}) \cos(d(x, z)\sqrt{\kappa}) \\
&= \sin(d(x, z)\sqrt{\kappa}) \cos(d(x, v)\sqrt{\kappa}) \sin((d(x, y) - d(x, w))\sqrt{\kappa}) \\
&\quad + \sin(d(x, w)\sqrt{\kappa}) \cos(d(x, y)\sqrt{\kappa}) \sin((d(x, z) - d(x, v))\sqrt{\kappa}) \\
&= \sin(d(x, z)\sqrt{\kappa}) \cos(d(x, v)\sqrt{\kappa}) \sin(d(y, w)\sqrt{\kappa}) \\
&\quad + \sin(d(x, w)\sqrt{\kappa}) \cos(d(x, y)\sqrt{\kappa}) \sin(d(z, v)\sqrt{\kappa}) \\
&= S_4 \cos(d(x, v)\sqrt{\kappa}) + S_5 \cos(d(x, y)\sqrt{\kappa}).
\end{aligned}$$

Items (14) and (15) follow easily from (11) and (12), respectively. \square

Proposition 4.4.

$$\sin^2 \frac{d(w, v)\sqrt{\kappa}}{2} \leq \frac{S_1}{S_2} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - C_1) - \frac{S_1}{2S_2}(1 - C_2). \quad (16)$$

Proof. Let $\Delta(\bar{x}, \bar{y}, \bar{z})$ be a κ -comparison triangle for $\Delta(x, y, z)$. Denote $\alpha = \angle_{\bar{x}}(\bar{y}, \bar{z}) = \angle_{\bar{x}}(\bar{w}, \bar{v})$. Using the cosine law we have

$$\begin{aligned}
\cos(d(\bar{w}, \bar{v})\sqrt{\kappa}) &= \cos(d(\bar{x}, \bar{w})\sqrt{\kappa}) \cos(d(\bar{x}, \bar{v})\sqrt{\kappa}) \\
&\quad + \sin(d(\bar{x}, \bar{w})\sqrt{\kappa}) \sin(d(\bar{x}, \bar{v})\sqrt{\kappa}) \cos \alpha
\end{aligned}$$

and

$$\begin{aligned}
\cos(d(\bar{y}, \bar{z})\sqrt{\kappa}) &= \cos(d(\bar{x}, \bar{y})\sqrt{\kappa}) \cos(d(\bar{x}, \bar{z})\sqrt{\kappa}) \\
&\quad + \sin(d(\bar{x}, \bar{y})\sqrt{\kappa}) \sin(d(\bar{x}, \bar{z})\sqrt{\kappa}) \cos \alpha.
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos(d(\bar{w}, \bar{v})\sqrt{\kappa}) &= \cos(d(\bar{x}, \bar{w})\sqrt{\kappa}) \cos(d(\bar{x}, \bar{v})\sqrt{\kappa}) \\
&\quad + \frac{\sin(d(\bar{x}, \bar{w})\sqrt{\kappa}) \sin(d(\bar{x}, \bar{v})\sqrt{\kappa})}{\sin(d(\bar{x}, \bar{y})\sqrt{\kappa}) \sin(d(\bar{x}, \bar{z})\sqrt{\kappa})} \left(\cos(d(\bar{y}, \bar{z})\sqrt{\kappa}) \right. \\
&\quad \left. - \cos(d(\bar{x}, \bar{y})\sqrt{\kappa}) \cos(d(\bar{x}, \bar{z})\sqrt{\kappa}) \right) \\
&= \frac{S_1}{S_2} \cos(d(y, z)\sqrt{\kappa}) + C_1 - \frac{S_1}{S_2} C_2.
\end{aligned}$$

It follows that

$$\frac{1 - \cos(d(w, v)\sqrt{\kappa})}{2} \leq \frac{1}{2} + \frac{S_1}{S_2} \left(\frac{1 - \cos(d(y, z)\sqrt{\kappa})}{2} - \frac{1}{2} \right) - \frac{1}{2}C_1 + \frac{S_1}{2S_2}C_2.$$

Hence,

$$\sin^2 \frac{d(w, v)\sqrt{\kappa}}{2} \leq \frac{S_1}{S_2} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - C_1) - \frac{S_1}{2S_2}(1 - C_2).$$

□

Proposition 4.5. (i)

$$\begin{aligned} \sin^2 \frac{d(w, v)\sqrt{\kappa}}{2} &\leq \frac{\sin(d(x, w)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\ &\quad + \frac{\sin(d(y, w)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \left(\sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2} \right) \\ &\quad + \frac{\sin(d(z, v)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa})} \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right). \end{aligned}$$

(ii) Assume that $v = sx + (1 - s)z$, $s \in [0, 1]$ and $w = rx + (1 - r)y$, $r \in [0, 1]$.
Then,

$$\begin{aligned} \sin^2 \frac{d(w, v)\sqrt{\kappa}}{2} &\leq \frac{\sin((1 - r)M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\ &\quad + \frac{\sin(rM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \max \left\{ \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2}, 0 \right\} \\ &\quad + \frac{\sin(sM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \sin^2 \frac{M\sqrt{\kappa}}{2}. \end{aligned}$$

Proof. (i) We apply Proposition 4.4 to get that

$$\begin{aligned}
\sin^2 \frac{d(w, v)\sqrt{\kappa}}{2} &\leq \frac{S_1}{S_2} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - C_1) - \frac{S_1}{2S_2}(1 - C_2) \\
&= \frac{S_1}{S_2} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\
&\quad + \frac{S_2 - S_1 - S_4 \cos(d(x, v)\sqrt{\kappa}) - S_5 \cos(d(x, y)\sqrt{\kappa})}{2S_2} \\
&\text{by (13)} \\
&\leq \frac{S_1}{S_2} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{S_4}{S_2} \left(\sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2} \right) \\
&\quad + \frac{S_5}{S_2} \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right) \\
&\text{by (14) and (15)} \\
&= \frac{S_1}{S_2} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\
&\quad + \frac{\sin(d(y, w)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \left(\sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2} \right) \\
&\quad + \frac{S_5}{S_2} \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right) \\
&\leq \frac{\sin(d(x, w)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\
&\quad + \frac{\sin(d(y, w)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \left(\sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2} \right) \\
&\quad + \frac{\sin(d(z, v)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa})} \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right).
\end{aligned}$$

(ii) We have that

$$\begin{aligned}
\sin^2 \frac{d(w, v)\sqrt{\kappa}}{2} &\leq \frac{\sin((1-r)d(x, y)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\
&\quad + \frac{\sin(rd(x, y)\sqrt{\kappa})}{\sin(d(x, y)\sqrt{\kappa})} \max \left\{ \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2}, 0 \right\} \\
&\quad + \frac{\sin(sd(x, z)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa})} \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right) \\
&\leq \frac{\sin((1-r)M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\
&\quad + \frac{\sin(rM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \max \left\{ \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, w)\sqrt{\kappa}}{2}, 0 \right\} \\
&\quad + \frac{\sin(sM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \sin^2 \frac{M\sqrt{\kappa}}{2}.
\end{aligned}$$

□

For the rest of the section, we assume that $v = sx + (1 - s)z$, $s \in (0, 1)$. We use the additional notation

$$L_1 = \frac{S_1}{S_3} = \frac{\sin(d(x, v)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa})}, \quad L_2 = \frac{S_5}{S_3} = \frac{\sin(d(v, z)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa})}.$$

Lemma 4.6.

$$0 < 1 - L_1 \leq L_2 \cos(d(x, v)\sqrt{\kappa}), \quad (17)$$

$$\frac{L_1}{1 - L_1} \leq \frac{1}{s \cos(M\sqrt{\kappa})}. \quad (18)$$

Proof.

$$\begin{aligned} (1 - L_1) \sin(d(x, z)\sqrt{\kappa}) &= \sin(d(x, z)\sqrt{\kappa}) - \sin(d(x, v)\sqrt{\kappa}) \\ &= 2 \sin \frac{(d(x, z) - d(x, v))\sqrt{\kappa}}{2} \cos \frac{(d(x, z) + d(x, v))\sqrt{\kappa}}{2} \\ &\leq 2 \sin \frac{d(z, v)\sqrt{\kappa}}{2} \cos \frac{d(z, v)\sqrt{\kappa}}{2} \cos(d(x, v)\sqrt{\kappa}) \\ &= \sin(d(z, v)\sqrt{\kappa}) \cos(d(x, v)\sqrt{\kappa}). \end{aligned}$$

Thus, $1 - L_1 \leq L_2 \cos(d(x, v)\sqrt{\kappa})$.

$$\begin{aligned} \frac{L_1}{1 - L_1} &= \frac{\sin(d(x, v)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa}) - \sin(d(x, v)\sqrt{\kappa})} \\ &\leq \frac{\sin(d(x, z)\sqrt{\kappa})}{\sin(d(x, z)\sqrt{\kappa}) - \sin(d(x, v)\sqrt{\kappa})} \\ &= \frac{\sin(d(x, z)\sqrt{\kappa})}{2 \sin \frac{d(z, v)\sqrt{\kappa}}{2} \cos \left(\left(d(x, z) - \frac{d(z, v)}{2} \right) \sqrt{\kappa} \right)} \\ &\leq \frac{\sin(d(x, z)\sqrt{\kappa})}{2 \sin \frac{sd(x, z)\sqrt{\kappa}}{2} \cos(d(x, z)\sqrt{\kappa})} \leq \frac{1}{s \cos(M\sqrt{\kappa})}. \end{aligned}$$

□

Proposition 4.7. (i)

$$\begin{aligned} \sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} &\leq L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2} - \frac{1}{2}L_1 - \frac{1}{2} \cos(d(x, y)\sqrt{\kappa})L_2 \\ &= L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) + L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2}. \end{aligned}$$

(ii) Let $q \in C$ be such that $d(q, z) \leq d(y, v)$. Assume that

$$\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \leq 0. \quad (19)$$

Then,

$$\begin{aligned} \sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} &\leq \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \\ &\quad + \frac{1}{s \cos(M\sqrt{\kappa})} \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \sin \frac{d(y, q)\sqrt{\kappa}}{2} \right). \end{aligned}$$

Proof. (i) We apply Proposition 4.4 with $w := y$ to get that

$$\begin{aligned} \sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} &\leq L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \\ &\quad + \frac{1}{2}(1 - \cos(d(x, y)\sqrt{\kappa}) \cos(d(x, v)\sqrt{\kappa})) - \\ &\quad - \frac{1}{2}L_1(1 - \cos(d(x, y)\sqrt{\kappa}) \cos(d(x, z)\sqrt{\kappa})) \\ &= L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2} - \frac{1}{2}L_1 \\ &\quad - \frac{\cos(d(x, y)\sqrt{\kappa})}{2 \sin(d(x, z)\sqrt{\kappa})} (\cos(d(x, v)\sqrt{\kappa}) \sin(d(x, z)\sqrt{\kappa}) - \\ &\quad - \sin(d(x, v)\sqrt{\kappa}) \cos(d(x, z)\sqrt{\kappa})) \\ &= L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2} - \frac{1}{2}L_1 \\ &\quad - \frac{\cos(d(x, y)\sqrt{\kappa})}{2 \sin(d(x, z)\sqrt{\kappa})} \sin(d(v, z)\sqrt{\kappa}) \\ &= L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} + \frac{1}{2} - \frac{1}{2}L_1 - \frac{1}{2} \cos(d(x, y)\sqrt{\kappa})L_2. \end{aligned}$$

(ii)

$$\begin{aligned} \sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} &\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) + L_1 \sin^2 \frac{d(y, z)\sqrt{\kappa}}{2} \\ &\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) + L_1 \sin^2 \frac{(d(y, q) + d(q, z))\sqrt{\kappa}}{2} \\ &\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) + L_1 \sin^2 \frac{(d(y, q) + d(y, v))\sqrt{\kappa}}{2} \\ &\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) + \\ &\quad + L_1 \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} + \frac{1}{2} \sin(d(y, q)\sqrt{\kappa}) \right) \\ &\text{since } \sin^2 \frac{a+b}{2} \leq \sin^2 \frac{a}{2} + \sin^2 \frac{b}{2} + \frac{1}{2} \sin a \text{ for } a, b \in [0, \pi]. \end{aligned}$$

It follows that

$$\begin{aligned}
\sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} (1 - L_1) &\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) \\
&\quad + L_1 \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \frac{1}{2} \sin(d(y, q)\sqrt{\kappa}) \right) \\
&\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} + \frac{1}{2}(1 - L_1 - L_2) \\
&\quad + L_1 \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \sin \frac{d(y, q)\sqrt{\kappa}}{2} \right) \\
&\leq L_2 \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \frac{1}{2}L_2(1 - \cos(d(x, v)\sqrt{\kappa})) \\
&\quad + L_1 \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \sin \frac{d(y, q)\sqrt{\kappa}}{2} \right) \\
&\text{by (17).}
\end{aligned}$$

Thus,

$$\begin{aligned}
\sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} &\leq \frac{L_2}{1 - L_1} \left(\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \right) \\
&\quad + \frac{L_1}{1 - L_1} \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \sin \frac{d(y, q)\sqrt{\kappa}}{2} \right).
\end{aligned}$$

By assumption, we have that $\sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \leq 0$. Using the fact that $\frac{L_2}{1 - L_1} \geq 1$ and (18), it follows that

$$\begin{aligned}
\sin^2 \frac{d(y, v)\sqrt{\kappa}}{2} &\leq \sin^2 \frac{d(x, y)\sqrt{\kappa}}{2} - \sin^2 \frac{d(x, v)\sqrt{\kappa}}{2} \\
&\quad + \frac{1}{s \cos(M\sqrt{\kappa})} \left(\sin^2 \frac{d(y, q)\sqrt{\kappa}}{2} + \sin \frac{d(y, q)\sqrt{\kappa}}{2} \right).
\end{aligned}$$

□

5 Effective rates of asymptotic regularity

We assume the hypothesis of Proposition 3.2. As in [23, 16, 17], the main tool in obtaining rates of asymptotic regularity is the following quantitative lemma, which is a slight reformulation of [17, Lemma 1].

Lemma 5.1. *Let $(\alpha_n)_{n \geq 1}$ be a sequence in $[0, 1]$ and $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$ be sequences in \mathbb{R}_+ such that*

$$a_{n+1} \leq (1 - \alpha_{n+1})a_n + b_n \quad \text{for all } n \in \mathbb{Z}_+. \quad (20)$$

Assume that $\sum_{n=1}^{\infty} b_n$ is convergent with Cauchy modulus γ and $\sum_{n=1}^{\infty} \alpha_{n+1}$ diverges with rate of divergence θ .

Then, $\lim_{n \rightarrow \infty} a_n = 0$ with rate of convergence Σ given by

$$\Sigma(\varepsilon, P, \gamma, \theta) = \theta \left(\gamma \left(\frac{\varepsilon}{2} \right) + \max \left\{ \left\lceil \ln \left(\frac{2P}{\varepsilon} \right) \right\rceil, 1 \right\} \right) + 1, \quad (21)$$

where $P > 0$ is an upper bound on (a_n) .

A second useful result, which is also needed in the metastability proof, is the following.

Lemma 5.2. For all $n \geq 1$, let

$$\mu_n = 1 - \frac{\sin((1 - \lambda_n)M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \in (0, 1). \quad (22)$$

Then

(i) $\mu_n \geq \lambda_n \cos(M\sqrt{\kappa})$ for all $n \geq 1$.

(ii) $\sum_{n=1}^{\infty} \lambda_{n+1} = \infty$ with rate of divergence θ yields $\sum_{n=1}^{\infty} \mu_{n+1} = \infty$ with rate of divergence $\tilde{\theta}(n) := \theta \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil n \right)$.

Proof. (i) One has

$$\begin{aligned} \mu_n &= \frac{2 \sin \frac{\lambda_n M \sqrt{\kappa}}{2} \cos \frac{(2 - \lambda_n) M \sqrt{\kappa}}{2}}{\sin(M\sqrt{\kappa})} \geq \frac{2 \sin \frac{\lambda_n M \sqrt{\kappa}}{2} \cos(M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \\ &\geq \lambda_n \cos(M\sqrt{\kappa}). \end{aligned}$$

(ii) Follows immediately from (i). □

Lemma 5.3. For all $n \in \mathbb{Z}_+$

$$d(x_n, x_{n+1}) \leq (1 - \mu_{n+1})d(x_{n-1}, x_n) + M|\lambda_{n+1} - \lambda_n|. \quad (23)$$

Proof. Let us denote for simplicity $u_n = \lambda_{n+1}u + (1 - \lambda_{n+1})Tx_{n-1}$. Then,

$$\begin{aligned} d(x_n, u_n) &= |\lambda_{n+1} - \lambda_n|d(u, Tx_{n-1}) \leq M|\lambda_{n+1} - \lambda_n| \quad \text{and} \\ d(u_n, x_{n+1}) &\leq \frac{\sin((1 - \lambda_{n+1})M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})}d(x_{n-1}, x_n) \quad \text{by Lemma 4.1.} \end{aligned}$$

□

5.1 Proof of Proposition 3.2

Let $\tilde{\Phi}$, Φ be given by (6) and (7). Apply Lemma 5.1 with

$$a_n = d(x_n, x_{n-1}), \quad b_n = M|\lambda_{n+1} - \lambda_n| \quad \text{and} \quad \alpha_n := \mu_n,$$

and use Lemma 5.2(ii) and the fact that $\sum_{n=1}^{\infty} b_n$ is convergent with Cauchy modulus $\tilde{\gamma}(\varepsilon) = \gamma\left(\frac{\varepsilon}{M}\right)$ to conclude that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$ with rate of convergence $\tilde{\Phi}$.

Since $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + M\lambda_{n+1}$ for all $n \geq 1$, it follows easily that Φ is a rate of asymptotic regularity. \square

6 A quantitative Browder theorem

Let X be a complete $\text{CAT}(\kappa)$ space, $C \subseteq X$ a bounded closed convex subset with diameter $d_C < \frac{D_\kappa}{2}$ and $T : C \rightarrow C$ be nonexpansive.

A very important step in the convergence proof for Halpern iterations is the construction of a sequence of approximants converging strongly to a fixed point of T . Given $t \in (0, 1)$ and $u \in C$, Lemma 4.1 yields that the mapping

$$T_t^u : C \rightarrow C, \quad T_t^u(y) = tu + (1-t)Ty. \quad (24)$$

is a contraction, hence it has a unique fixed point $z_t^u \in C$. Thus,

$$z_t^u = tu + (1-t)Tz_t^u. \quad (25)$$

Piątek [26] obtained the following generalization from Hilbert spaces to $\text{CAT}(\kappa)$ spaces of an essential result due to Browder [5, 7].

Theorem 6.1. [26] *In the above hypothesis, $\lim_{t \rightarrow 0^+} z_t^u$ exists and is a fixed point of T .*

In the setting of Hilbert spaces, Browder proved the result using J -monotone operators (where J is a duality mapping) introduced by him in [6]. A new and elementary proof of Browder's result was given by Halpern [10] when C is the closed unit ball and the starting point is $u = 0$. Generalizations of Browder's theorem were obtained by Reich [28] for uniformly smooth Banach spaces, Goebel and Reich [9] for the Hilbert ball and Kirk [11] for $\text{CAT}(0)$ spaces.

Kohlenbach [13] applied proof mining methods to both Browder's original proof and the extension of Halpern's proof to bounded closed convex C and arbitrary $u \in C$ and obtained in this way quantitative versions of Browder's theorem with uniform effective rates of metastability. As pointed out in [13, Remark 1.4], one can not expect in general to get effective rates of convergence.

Since Kirk's proof of the generalization of Browder's theorem to CAT(0) spaces is obtained by a slight change of Halpern's argument, Kohlenbach's quantitative result extends basically unchanged to CAT(0) spaces (see [17, Proposition 9.3]).

In this section we obtain a quantitative version of Theorem 6.1. As a consequence of Halpern's proof, for any nonincreasing sequence (t_n) in $(0, 1)$, one gets that $(z_{t_n}^u)$ converges strongly to some point $z \in C$, which is a fixed point of T if $\lim_{n \rightarrow \infty} t_n = 0$. Our quantitative result gives rates of metastability for such sequences $(z_{t_n}^u)$ and this suffices for the proof of our main Theorem 3.4.

Proposition 6.2. *Let X be a complete CAT(κ) space, $C \subseteq X$ bounded closed convex with diameter $d_C < \frac{D_\kappa}{2}$ and $T : C \rightarrow C$ be nonexpansive. Assume that $(t_n) \subseteq (0, 1)$ is nonincreasing. Then for every $\varepsilon \in (0, 1)$ and $g : \mathbb{N} \rightarrow \mathbb{N}$,*

$$\exists K_0 \leq K(\varepsilon, g, M) \forall i, j \in [K_0, K_0 + g(K_0)] \left(d(z_{t_i}^u, z_{t_j}^u) \leq \frac{\varepsilon}{\sqrt{\kappa}} \right),$$

where

$$K(\varepsilon, g, M) := \tilde{g} \left(\left\lceil \frac{M\sqrt{\kappa} \tan(M\sqrt{\kappa})}{1 - \cos \varepsilon} \right\rceil \right) (0),$$

with $d_C \leq M < \frac{D_\kappa}{2}$ and $\tilde{g}(n) := n + g(n)$.

Proof. Let $\varepsilon \in (0, 1)$ and $g : \mathbb{N} \rightarrow \mathbb{N}$. We assume without loss of generality that $i < j$, hence $t_j \leq t_i$. Denote $u_{i,j} = t_j u + (1 - t_j) T z_{t_i}^u$. Then,

$$d(u, z_{t_i}^u) = (1 - t_i) d(u, T z_{t_i}^u) \leq (1 - t_j) d(u, T z_{t_i}^u) = d(u, u_{i,j}),$$

so $z_{t_i}^u \in [u, u_{i,j}]$. It follows by Lemma 4.1 that $d(z_{t_j}^u, u_{i,j}) \leq d(T z_{t_j}^u, T z_{t_i}^u) \leq d(z_{t_j}^u, z_{t_i}^u)$.

We can apply now Lemma 4.2 with $x := u, y := z_{t_j}^u, z := u_{i,j}, w := z_{t_i}^u$ to get that $d(u, z_{t_i}^u) \leq d(u, z_{t_j}^u)$ and for $\Delta(\bar{u}, \bar{z}_{t_j}^u, \bar{u}_{i,j})$ a κ -comparison triangle of $\Delta(u, z_{t_j}^u, u_{i,j})$, one has $\angle_{\bar{z}_{t_i}^u}(\bar{u}, \bar{z}_{t_j}^u) \geq \pi/2$. Since $(d(u, z_{t_n}^u))_n$ is a nondecreasing sequence in $[0, M]$, there exists $K_0 \leq K(\varepsilon, g, M)$ such that

$$\forall i, j \in [K_0, K_0 + g(K_0)] \left(|d(u, z_{t_j}^u) - d(u, z_{t_i}^u)| \leq \frac{1 - \cos \varepsilon}{\sqrt{\kappa} \tan(M\sqrt{\kappa})} \right).$$

by an application of [13, Lemma 4.1].

Let now $i < j \in [K_0, K_0 + g(K_0)]$. Then,

$$\begin{aligned} \cos(d(u, z_{t_i}^u)\sqrt{\kappa}) - \cos(d(u, z_{t_j}^u)\sqrt{\kappa}) &\leq \sin(M\sqrt{\kappa}) \frac{1 - \cos \varepsilon}{\tan(M\sqrt{\kappa})} \\ &= (1 - \cos \varepsilon) \cos(M\sqrt{\kappa}), \end{aligned}$$

Furthermore, by the cosine law and the fact that $\angle_{\bar{z}_{t_i}^u}(\bar{u}, \bar{z}_{t_j}^u) \geq \frac{\pi}{2}$, we have that

$$\cos(d(\bar{u}, \bar{z}_{t_j}^u)\sqrt{\kappa}) \leq \cos(d(\bar{u}, \bar{z}_{t_i}^u)\sqrt{\kappa}) \cos(d(\bar{z}_{t_i}^u, \bar{z}_{t_j}^u)\sqrt{\kappa}).$$

It follows that

$$\begin{aligned}
& \cos(d(u, z_{t_i}^u)\sqrt{\kappa}) - (1 - \cos(\varepsilon)) \cos(M\sqrt{\kappa}) \\
& \leq \cos(d(u, z_{t_j}^u)\sqrt{\kappa}) = \cos(d(\bar{u}, \bar{z}_{t_j}^u)\sqrt{\kappa}) \\
& \leq \cos(d(\bar{u}, \bar{z}_{t_i}^u)\sqrt{\kappa}) \cos(d(\bar{z}_{t_j}^u, \bar{z}_{t_i}^u)\sqrt{\kappa}) \\
& \leq \cos(d(u, z_{t_i}^u)\sqrt{\kappa}) \cos(d(z_{t_j}^u, z_{t_i}^u)\sqrt{\kappa}).
\end{aligned}$$

Hence

$$\cos(d(z_{t_j}^u, z_{t_i}^u)\sqrt{\kappa}) \geq 1 - (1 - \cos \varepsilon) \frac{\cos(M\sqrt{\kappa})}{\cos(d(u, z_{t_i}^u)\sqrt{\kappa})} \geq \cos \varepsilon.$$

Thus, $d(z_{t_j}^u, z_{t_i}^u)\sqrt{\kappa} \leq \varepsilon$ and the proof is complete. \square

7 Effective rates of metastability

In this section we shall prove the main result of our paper, Theorem 3.4, hence we assume that its hypotheses are satisfied. We give first some technical results that will be needed in the proof.

7.1 Some useful lemmas

As in [16, 17], one of the main ingredients of our proof is a sequence obtained by combining the Halpern iteration (x_n) and the points z_t^u . However, in the setting of $\text{CAT}(\kappa)$ spaces, its definition and the proofs of the necessary properties are based on the much more involved technical lemmas from Section 4.

Let us define

$$\gamma_n^t = \sin^2 \frac{d(u, z_t^u)\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2}. \quad (26)$$

Proposition 7.1. (i) For $n \geq 1$, if $\gamma_n^t \geq 0$, then

$$\gamma_n^t \leq \frac{a_n}{t} - \sin^2 \frac{d(x_{n+1}, z_t^u)\sqrt{\kappa}}{2}, \quad (27)$$

where

$$a_n = \frac{1}{\cos(M\sqrt{\kappa})} \left(\sin^2 \frac{d(x_{n+1}, Tx_{n+1})\sqrt{\kappa}}{2} + \sin \frac{d(x_{n+1}, Tx_{n+1})\sqrt{\kappa}}{2} \right). \quad (28)$$

(ii) $\gamma_n^t \leq \frac{a_n}{t}$ for all $n \geq 1$.

(iii) $\limsup_{n \rightarrow \infty} \gamma_n^t \leq 0$ with effective rate $\Psi(\varepsilon, \kappa, M, t, \gamma, \theta, \alpha)$ given by

$$\Psi := \max \left\{ \theta \left(\left[\frac{1}{\cos(M\sqrt{\kappa})} \right] \left(\gamma(L) + \max \left\{ \left[\ln \left(\frac{1}{L} \right) \right], 1 \right\} \right) \right), \alpha(2L) \right\}, \quad (29)$$

$$\text{where } L := \frac{\cos(M\sqrt{\kappa})t\varepsilon}{4M\sqrt{\kappa}}.$$

(iv) For $n \geq 1$,

$$\begin{aligned} \sin^2 \frac{d(x_{n+1}, z_t^u)\sqrt{\kappa}}{2} &\leq \frac{\sin((1 - \lambda_{n+1})M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \sin^2 \frac{d(x_n, z_t^u)\sqrt{\kappa}}{2} \\ &\quad + \frac{\sin(\lambda_{n+1}M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \max\{\gamma_n^t, 0\} \\ &\quad + \frac{\sin(tM\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \sin^2 \frac{M\sqrt{\kappa}}{2}. \end{aligned}$$

Proof. (i) Apply Proposition 4.7.(ii) with $x := u, y := x_{n+1}, z := Tz_t^u, v := z_t^u, s := t, q := Tx_{n+1}$ and note that

$$\sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, z_t^u)\sqrt{\kappa}}{2} = -\gamma_n^t \leq 0.$$

It follows that $\sin^2 \frac{d(x_{n+1}, z_t^u)\sqrt{\kappa}}{2} \leq -\gamma_n^t + \frac{a_n}{t}$, hence (i).

(ii) Obviously, since $\frac{a_n}{t} \geq 0$.

(iii) Since $a_n \leq \frac{1}{\cos(M\sqrt{\kappa})} d(x_{n+1}, Tx_{n+1})\sqrt{\kappa}$ and, by Proposition 3.2, the sequence $(d(x_n, Tx_n))$ converges to 0 with rate of convergence Φ given by (7), we get that $\limsup_{n \rightarrow \infty} \gamma_n^t \leq 0$ with effective rate

$$\Psi(\varepsilon, \kappa, M, t, \gamma, \theta, \alpha) = \Phi\left(\frac{\cos(M\sqrt{\kappa})t\varepsilon}{\sqrt{\kappa}}, \kappa, M, \gamma, \theta, \alpha\right).$$

(iv) By Proposition 4.5.(ii) with $x := u, y := Tx_n, z := Tz_t^u, w := x_{n+1}, v := z_t^u, r := \lambda_{n+1}$ and $s := t$. □

In fact, it suffices for the proof of the main theorem to consider the case $t_i := \frac{1}{i+1}, i \geq 0$. Then (t_i) converges towards 0 with rate $\left[\frac{1}{\varepsilon}\right]$.

We shall denote $\gamma_n^{t_i}$ with γ_n^i . Furthermore, $z_{t_i}^u$ will be simply denoted by z_i^u . Thus,

$$\gamma_n^i = \sin^2 \frac{d(u, z_i^u)\sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1})\sqrt{\kappa}}{2}. \quad (30)$$

Lemma 7.2. Assume that $i, j \geq 0$ and $\delta > 0$ are such that $d(u, Tz_i^u) - d(u, Tz_j^u) \leq \frac{\delta}{\sqrt{\kappa}}$. Then,

$$\gamma_n^i \leq \gamma_n^j + \sin^2 \frac{M\sqrt{\kappa}}{2(j+1)} + 2 \sin \frac{M\sqrt{\kappa}}{2(j+1)} + \sin^2 \frac{\delta}{2} + 2 \sin \frac{\delta}{2} \sin \frac{M\sqrt{\kappa}}{2}.$$

Proof. We have that

$$\begin{aligned}
\gamma_n^i &= \sin^2 \frac{\frac{i}{i+1} d(u, Tz_i^u) \sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1}) \sqrt{\kappa}}{2} \\
&\leq \sin^2 \frac{d(u, Tz_i^u) \sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1}) \sqrt{\kappa}}{2} \\
&\leq \left(\sin \frac{d(u, Tz_j^u) \sqrt{\kappa}}{2} + \sin \frac{\delta}{2} \right)^2 - \sin^2 \frac{d(u, x_{n+1}) \sqrt{\kappa}}{2} \\
&\leq \sin^2 \frac{d(u, Tz_j^u) \sqrt{\kappa}}{2} - \sin^2 \frac{d(u, x_{n+1}) \sqrt{\kappa}}{2} + \sin^2 \frac{\delta}{2} + 2 \sin \frac{\delta}{2} \sin \frac{M\sqrt{\kappa}}{2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sin^2 \frac{d(u, Tz_j^u) \sqrt{\kappa}}{2} &= \sin^2 \frac{\frac{j}{j+1} d(u, Tz_j^u) \sqrt{\kappa} + \frac{1}{j+1} d(u, Tz_j^u) \sqrt{\kappa}}{2} \\
&\leq \sin^2 \frac{\frac{j}{j+1} d(u, Tz_j^u) \sqrt{\kappa}}{2} + \sin^2 \frac{M\sqrt{\kappa}}{2(j+1)} + 2 \sin \frac{M\sqrt{\kappa}}{2(j+1)}.
\end{aligned}$$

□

Finally, let us recall the following slight reformulation of [17, Lemma 5.2].

Lemma 7.3. *Let $\varepsilon \in (0, 2)$, $g : \mathbb{N} \rightarrow \mathbb{N}$, $L > 0$, $\theta : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ and $\psi : (0, \infty) \rightarrow \mathbb{Z}_+$. Define*

$$\begin{aligned}
\Theta &:= \Theta(\varepsilon, L, \theta, \psi) = \theta \left(\psi \left(\frac{\varepsilon}{3} \right) - 1 + \max \left\{ \left\lceil \ln \left(\frac{3L}{\varepsilon} \right) \right\rceil, 1 \right\} \right) + 1, \\
\Delta &:= \Delta(\varepsilon, g, L, \theta, \psi) = \frac{\varepsilon}{3g_\varepsilon(\Theta - \psi(\varepsilon/3))},
\end{aligned}$$

where $g_\varepsilon(n) = n + g(n + \psi(\varepsilon/3))$.

Assume that

- (i) (α_n) is a sequence in $[0, 1]$ such that the series $\sum_{n=1}^{\infty} \alpha_n$ diverges with rate of divergence θ ;
- (ii) (t_n) is a sequence of real numbers such that $t_n \leq \frac{\varepsilon}{3}$ for all $n \geq \psi \left(\frac{\varepsilon}{3} \right)$.

Let (s_n) be a bounded sequence with upper bound L satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + \Delta \quad \text{for all } n \geq 1. \quad (31)$$

Then $s_n \leq \varepsilon$ for all $n \in [\Theta, \Theta + g(\Theta)]$.

7.2 Proof of Theorem 3.4

Let $\varepsilon \in (0, 2)$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ be fixed. For simplicity, we omit parameters $\kappa, M, \Phi, \theta, \alpha, \beta$ for all functionals in this proof. Let us define $h : (0, 1) \rightarrow \mathbb{R}_+$ by

$$h(\delta) = \sin \frac{\delta}{2} \left(\sin \frac{\delta}{2} + 2 \sin \frac{M\sqrt{\kappa}}{2} \right) + \sin \frac{\delta M\sqrt{\kappa}}{2} \left(\sin \frac{\delta M\sqrt{\kappa}}{2} + 2 \right) \leq 6\delta. \quad (32)$$

Take $\varepsilon_0 := \frac{\cos(M\sqrt{\kappa})}{36} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}$. Then, $h(\varepsilon_0) \leq \frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}$.

Applying Proposition 6.2 for $t_i := \frac{1}{i+1}, \varepsilon_0$ and f^* , we get the existence of

$$K_1 \leq K(\varepsilon_0, f^*) = \widetilde{f}^* \left(\left\lceil \frac{M\sqrt{\kappa} \tan(M\sqrt{\kappa})}{1 - \cos \varepsilon_0} \right\rceil \right)_{(0)}$$

such that $d(z_i^u, z_j^u) \leq \frac{\varepsilon_0}{\sqrt{\kappa}}$ for all $i, j \in [K_1, \widetilde{f}^*(K_1)]$.

Let $K_0 := K_1 + \left\lceil \frac{1}{\varepsilon_0} \right\rceil$ and $J := K_0 + f(K_0) = \widetilde{f}^*(K_1)$. It follows that $d(z_j^u, z_{K_0}^u) \leq \frac{\varepsilon_0}{\sqrt{\kappa}}$, hence

$$\begin{aligned} d(u, Tz_j^u) &\leq d(u, Tz_{K_0}^u) + d(Tz_{K_0}^u, Tz_j^u) \leq d(u, Tz_{K_0}^u) + d(z_{K_0}^u, z_j^u) \\ &\leq d(u, Tz_{K_0}^u) + \frac{\varepsilon_0}{\sqrt{\kappa}}. \end{aligned}$$

An application of Lemma 7.2 with $i := J, j := K_0$ and $\delta := \varepsilon_0$ gives us

$$\begin{aligned} \gamma_n^J &\leq \gamma_n^{K_0} + \sin^2 \frac{M\sqrt{\kappa}}{2(K_0+1)} + 2 \sin \frac{M\sqrt{\kappa}}{2(K_0+1)} + \sin^2 \frac{\varepsilon_0}{2} + 2 \sin \frac{\varepsilon_0}{2} \sin \frac{M\sqrt{\kappa}}{2} \\ &\leq \gamma_n^{K_0} + \sin^2 \frac{\varepsilon_0}{2} + 2 \sin \frac{\varepsilon_0}{2} \sin \frac{M\sqrt{\kappa}}{2} + \sin^2 \frac{M\varepsilon_0\sqrt{\kappa}}{2} + 2 \sin \frac{M\varepsilon_0\sqrt{\kappa}}{2} \\ &= \gamma_n^{K_0} + h(\varepsilon_0) \leq \gamma_n^{K_0} + \frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}. \end{aligned}$$

Applying now Proposition 7.1.(iv) with $t := \frac{1}{J+1}$ and recalling the definition (22) of (μ_n) , it follows that for all $n \geq 1$,

$$\begin{aligned} \sin^2 \frac{d(x_{n+1}, z_j^u)\sqrt{\kappa}}{2} &\leq (1 - \mu_{n+1}) \sin^2 \frac{d(x_n, z_j^u)\sqrt{\kappa}}{2} \\ &\quad + \frac{\sin(\lambda_{n+1}M\sqrt{\kappa})}{\sin(M\sqrt{\kappa})} \max\{\gamma_n^J, 0\} \\ &\quad + \frac{\sin\left(\frac{1}{J+1}M\sqrt{\kappa}\right)}{\sin(M\sqrt{\kappa})} \sin^2 \frac{M\sqrt{\kappa}}{2}. \end{aligned}$$

Since $J = K_0 + f(K_0) \geq \left\lceil \frac{M\sqrt{\kappa}}{\Delta_{K_0}^*(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, g)} \right\rceil$ and

$$\cos(M\sqrt{\kappa}) \sin(\lambda_{n+1}M\sqrt{\kappa}) \leq \sin(M\sqrt{\kappa}) - \sin((1 - \lambda_{n+1})M\sqrt{\kappa}),$$

it follows that

$$\begin{aligned} \sin^2 \frac{d(x_{n+1}, z_J^u)\sqrt{\kappa}}{2} &\leq (1 - \mu_{n+1}) \sin^2 \frac{d(x_n, z_J^u)\sqrt{\kappa}}{2} \\ &\quad + \mu_{n+1} \max \left\{ \frac{\gamma_n^J}{\cos(M\sqrt{\kappa})}, 0 \right\} + \Delta_{K_0}^* \left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, g \right). \end{aligned}$$

Letting $t := \frac{1}{K_0 + 1}$ in Proposition 7.1.(iii), we get that

$$\gamma_n^{K_0} \leq \frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4},$$

for all $n \geq \chi_{K_0}^* \left(\frac{1}{3} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \right) = \chi_{K_0} \left(\frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \right)$. Thus,

$$\gamma_n^J \leq \gamma_n^{K_0} + \frac{\cos(M\sqrt{\kappa})}{6} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \leq \frac{\cos(M\sqrt{\kappa})}{3} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4},$$

and so, $\max \left\{ \frac{\gamma_n^J}{\cos(M\sqrt{\kappa})}, 0 \right\} \leq \frac{1}{3} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}$ for all $n \geq \chi_{K_0}^* \left(\frac{1}{3} \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \right)$.

Furthermore, by Lemma 5.2, we have that $\sum_{n=1}^{\infty} \mu_{n+1} = \infty$ with rate of divergence $\tilde{\theta}(n) := \theta \left(\left\lceil \frac{1}{\cos(M\sqrt{\kappa})} \right\rceil n \right)$. Hence, we can apply Lemma 7.3 with

$$\begin{aligned} s_n &:= \sin^2 \frac{d(x_n, z_J^u)\sqrt{\kappa}}{2}, \quad t_n := \max \left\{ \frac{\gamma_n^J}{\cos(M\sqrt{\kappa})}, 0 \right\}, \quad \alpha_n := \mu_{n+1}, \\ \varepsilon &:= \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, \quad \Delta := \Delta_{K_0}^* \left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, g \right), \quad L := \sin^2 \frac{M\sqrt{\kappa}}{2} \end{aligned}$$

to get that, by letting $N = \Theta_{K_0} \left(\sin^2 \frac{\varepsilon\sqrt{\kappa}}{4} \right)$, we have that for all $n \in [N, N + g(N)]$

$$\sin^2 \frac{d(x_n, z_J^u)\sqrt{\kappa}}{2} \leq \sin^2 \frac{\varepsilon\sqrt{\kappa}}{4}, \quad \text{and so} \quad d(x_n, z_J^u) \leq \frac{\varepsilon}{2}$$

Obviously, $d(x_n, x_m) \leq \varepsilon$ for all $m, n \in [N, N + g(N)]$. One can easily see that $N \leq \Sigma(\varepsilon, g)$. \square

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References

- [1] J. Avigad, P. Gerhardy, H. Towsner, Local stability of ergodic averages, *Trans. Amer. Math. Soc.* 362 (2010), 261-288.
- [2] J. Avigad, J. Rute, Oscillation and the mean ergodic theorem for uniformly convex Banach spaces, arXiv:1203.4124v3 [math.DS].
- [3] G. Birkhoff, The mean ergodic theorem, *Duke Math. J.* 5 (1939), 19-20.
- [4] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.
- [5] F.E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. U.S.A.* 56 (1966), 1080-1086.
- [6] F. E. Browder, Fixed point theorems for nonlinear semicontractive mappings in Banach spaces, *Arch. Rational Mech. Anal.* 21 (1966), 259–269.
- [7] F.E. Browder, Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* 24 (1967), 82-90.
- [8] R.E. Bruck, S. Reich, Accretive operators, Banach limits, and dual ergodic theorems, *Bull. Acad. Polon. Sci.* 29 (1981), 585–589.
- [9] K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, Inc., New York and Basel, 1984.
- [10] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.* 73 (1967), 957-961.
- [11] W.A. Kirk, *Geodesic geometry and fixed point theory*, Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Colecc. Abierta, 64, Univ. Seville Secr. Publ., Seville (2003), 195-225.
- [12] U. Kohlenbach, *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*, Springer, Heidelberg-Berlin, 2008.
- [13] U. Kohlenbach, On quantitative versions of theorems due to F.E. Browder and R. Wittmann, *Adv. Math.* 226 (2011), 2764-2795.
- [14] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *Trans. Amer. Math. Soc.* 357 (2005), 89-128.
- [15] U. Kohlenbach, L. Leuştean, A quantitative mean ergodic theorem for uniformly convex Banach spaces, *Ergodic Theory Dynam. Systems* 29 (2009), 1907-1915. Erratum: *Ergodic Theory Dynam. Systems* 29 (2009), 1995.
- [16] U. Kohlenbach, L. Leuştean, On the computational content of convergence proofs via Banach limits, *Phil. Trans. Royal Society A* 370 (2012), No. 1971, 3449-3463.

- [17] U. Kohlenbach, L. Leuştean, Effective metastability of Halpern iterates in CAT(0) spaces, *Adv. Math.* 231 (2012), 2526-2556.
- [18] U. Kohlenbach, L. Leuştean, Addendum to "Effective metastability of Halpern iterates in CAT(0) spaces", <http://www.mathematik.tu-darmstadt.de/~kohlenbach/Addendum.pdf>, 2013.
- [19] U. Kohlenbach, P. Safarik, Fluctuations, effective learnability and metastability in analysis, Preprint, 2013.
- [20] G. Kreisel, On the interpretation of non-finitist proofs, part I, *J. Symbolic Logic* 16 (1951), 241-267.
- [21] G. Kreisel, On the interpretation of non-finitist proofs, part II, *J. Symbolic Logic* 17 (1952), 43-88.
- [22] U. Krengel, On the speed of convergence in the ergodic theorem, *Monatsh. Math.* 86 (1978/79), 36.
- [23] L. Leuştean, Rates of asymptotic regularity for Halpern iterations of non-expansive mappings, *J. Universal Comp. Sci.* 13 (2007), 1680-1691.
- [24] L. Leuştean, Proof mining in metric fixed point theory and ergodic theory, Habilitation thesis, Technische Universität Darmstadt, 2009.
- [25] G.G. Lorentz, A contribution to the theory of divergent series, *Acta Math.* 80 (1948), 167-190.
- [26] B. Piątek, Halpern iteration in CAT(κ) spaces, *Acta Math. Sin. (Engl. Ser.)* 27 (2011), 635-646.
- [27] S. Reich, Almost convergence and nonlinear ergodic theorems, *J. Approx. Theory* 24 (1978), 269-272.
- [28] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* 75 (1980), 287-292.
- [29] S. Saejung, Halpern iterations in CAT(0) spaces, *Fixed Point Theory Appl.* 2010 (2010), Article ID 471781, 13pp.
- [30] K. Schade, U. Kohlenbach, Effective metastability for modified Halpern iterations in CAT(0) spaces, *Fixed Point Theory Appl.* 2012, 2012:191, 19pp.
- [31] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.* 125 (1997), 3641-3645.
- [32] T. Tao, Soft analysis, hard analysis, and the finite convergence principle, Essay posted May 23, 2007, appeared in: T. Tao, *Structure and Randomness: Pages from Year One of a Mathematical Blog.* AMS, 298pp., 2008.

- [33] T. Tao, Norm convergence of multiple ergodic averages for commuting transformations, *Ergodic Theory Dynam. Systems* 28 (2008), 657-688.
- [34] M. Walsh, Norm convergence of nilpotent ergodic averages, *Ann. Math.* 175 (2012), 1667-1688.
- [35] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math* 58 (1992), 486-491.