

A KIRCHBERG TYPE TENSOR THEOREM FOR OPERATOR SYSTEMS

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ABSTRACT. We construct operator systems \mathfrak{C}_I that are universal in the sense that all operator systems can be realized as their quotients. They satisfy the operator system lifting property. The Kirchberg type tensor theorem

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$$

is proved independently of Kirchberg's theorem.

1. INTRODUCTION

Every Banach space can be realized as a quotient of $\ell_1(I)$ for a suitable choice of index set I . Moreover, every linear map $\varphi : \ell_1(I) \rightarrow E/F$ lifts to $\tilde{\varphi} : \ell_1(I) \rightarrow E$ with $\|\tilde{\varphi}\| < (1 + \varepsilon)\|\varphi\|$. On noncommutative sides, $\bigoplus_1 T_{n_i}$ (respectively $C^*(\mathbb{F})$) plays such a role in the category of operator spaces (respectively C^* -algebras). The purpose of this paper is to find operator systems that play such a role in the category of operator systems.

We construct operator systems \mathfrak{C}_I that are universal in the sense that all operator systems can be realized as their quotients. The method of construction is motivated by [Bl, Proposition 3.1] and the coproduct of operator systems [F, KL]. The index set I is chosen to be sufficiently large that we can index the set $\mathcal{S}_{\|\cdot\| \leq 1}^+$ of positive contractive elements in an operator system \mathcal{S} . The operator system \mathfrak{C}_I is defined as a certain operator subsystem of the unital free product of $\{M_k(C([0, 1]))\}_{k \in \mathbb{N}}$ admitting copies of $M_k(C([0, 1]))$ up to the cardinality of I . It turns out that \mathfrak{C}_I is unital completely order isomorphic to the infinite coproduct of $\{M_k \oplus M_k\}_{k \in \mathbb{N}}$ admitting copies of $M_k \oplus M_k$ to the same.

We prove that operator systems \mathfrak{C}_I satisfy the operator system lifting property: for any unital C^* -algebra \mathcal{A} and its closed ideal \mathcal{I} , every unital completely positive map $\varphi : \mathfrak{C}_I \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \rightarrow \mathcal{A}$.

$$\begin{array}{ccc} & & \mathcal{A} \\ & \nearrow \tilde{\varphi} & \downarrow \\ \mathfrak{C}_I & \xrightarrow{\varphi} & \mathcal{A}/\mathcal{I} \end{array}$$

For a free group \mathbb{F} and a Hilbert space H , Kirchberg [K, Corollary 1.2] proved that

$$C^*(\mathbb{F}) \hat{\otimes}_{\min} B(H) = C^*(\mathbb{F}) \hat{\otimes}_{\max} B(H).$$

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The proof was later simplified in [P1] and [FP] using operator space theory and operator system theory, respectively. Kirchberg's theorem is striking if we recall that $C^*(\mathbb{F})$ and $B(H)$ are universal objects in the C^* -algebra category: every C^* -algebra is a C^* -quotient of $C^*(\mathbb{F})$ and a C^* -subalgebra of $B(H)$ for suitable choices of \mathbb{F} and H .

Every operator system is a quotient of \mathfrak{C}_I and a subsystem of $B(H)$ [CE] for suitable choices of I and H . We will prove a Kirchberg type tensor theorem

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H).$$

The proof is independent of Kirchberg's theorem. Combining this with Kavruk's idea [K], we give a new operator system theoretic proof of Kirchberg's theorem.

We also prove that the operator system analogue

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$$

of Kirchberg's conjecture

$$C^*(\mathbb{F}) \hat{\otimes}_{\min} C^*(\mathbb{F}) = C^*(\mathbb{F}) \hat{\otimes}_{\max} C^*(\mathbb{F})$$

is equivalent to Kirchberg's conjecture itself.

In the final section, we consider several lifting problems of completely positive maps. It is natural to ask whether the universal operator system \mathfrak{C}_I is a projective object in the category of operator systems. In other words, for any operator system \mathcal{S} and its kernel \mathcal{J} , does every unital completely positive map $\varphi : \mathfrak{C}_I \rightarrow \mathcal{S}/\mathcal{J}$ lift to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \rightarrow \mathcal{S}$? The answer is negative in an extreme manner. An operator system satisfying such a lifting property is necessarily one-dimensional. This is essentially due to Archimedeanization of quotients [PT]. Even though some perturbation is allowed, we will see that there is a certain rigidity. Finally, we present an operator system theoretic approach to the Effros-Haagerup lifting theorem [EH].

2. PRELIMINARIES

Given an operator system \mathcal{S} , we call $\mathcal{J} \subset \mathcal{S}$ the kernel, provided that it is the kernel of a unital completely positive map from \mathcal{S} to another operator system. If we define a family of positive cones $M_n(\mathcal{S}/\mathcal{J})^+$ on $M_n(\mathcal{S}/\mathcal{J})$ as

$$M_n(\mathcal{S}/\mathcal{J})^+ := \{[x_{i,j} + J]_{i,j} : \forall \varepsilon > 0, \exists k_{i,j} \in J, \varepsilon I_n \otimes 1_{\mathcal{S}} + [x_{i,j} + k_{i,j}]_{i,j} \in M_n(\mathcal{S})^+\},$$

then $(\mathcal{S}/\mathcal{J}, \{M_n(\mathcal{S}/\mathcal{J})^+\}_{n=1}^{\infty}, 1_{\mathcal{S}/\mathcal{J}})$ satisfies all the conditions of an operator system [KPTT2, Proposition 3.4]. We call this the quotient operator system. With this definition, the first isomorphism theorem can be proved: If $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive map with $\mathcal{J} \subset \ker \varphi$, then the map $\tilde{\varphi} : \mathcal{S}/\mathcal{J} \rightarrow \mathcal{T}$ given by $\tilde{\varphi}(x + \mathcal{J}) = \varphi(x)$ is a unital completely positive map [KPTT2, Proposition 3.6]. In particular, when

$$M_n(\mathcal{S}/\mathcal{J})^+ = \{[x_{i,j} + J]_{i,j} : \exists k_{i,j} \in J, [x_{i,j} + k_{i,j}]_{i,j} \in M_n(\mathcal{S})^+\}$$

for all $n \in \mathbb{N}$, we call the kernel \mathcal{J} completely order proximal.

Since the kernel \mathcal{J} in an operator system \mathcal{S} is a closed subspace, the operator space structure of \mathcal{S}/\mathcal{J} can be interpreted in two ways: first, as the operator space quotient and second, as the operator space structure induced by the operator system quotient. The two matrix norms can be different. For a specific example, see [KPTT2, Example 4.4].

For a unital completely positive surjection $\varphi : \mathcal{S} \rightarrow \mathcal{T}$, we call $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ a *complete order quotient map* [H, Definition 3.1] if for any Q in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take an element P in $M_n(\mathcal{S})$ so that it satisfies

$$P + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \quad \text{and} \quad \varphi_n(P) = Q,$$

Equivalently, if for any Q in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take a positive element P in $M_n(\mathcal{S})$ satisfying

$$\varphi_n(P) = Q + \varepsilon I_n \otimes 1_{\mathcal{S}}.$$

This definition is compatible with [FKP, Proposition 3.2]: every strictly positive element lifts to a strictly positive element. The map $\varphi : \mathcal{S} \rightarrow \mathcal{T}$ is a complete order quotient map if and only if the induced map $\tilde{\varphi} : \mathcal{S}/\ker \varphi \rightarrow \mathcal{T}$ is a unital complete order isomorphism [H, Proposition 3.2]. In other operator system references, this is termed a complete quotient map. To avoid confusion with complete quotient maps in operator space theory, we use the terminology of a *complete order quotient map* throughout this paper. In this paper, we say that a linear map $\Phi : V \rightarrow W$ for operator spaces V and W is a *complete quotient map* if Φ_n maps the open unit ball of $M_n(V)$ onto the open unit ball of $M_n(W)$. Actually, readers will encounter the two terminologies in the same paragraph.

Let f (respectively g) be a state on $M_k(\mathcal{S})$ (respectively $M_k(\mathcal{T})$). We call (f, g) a compatible pair whenever $f|_{M_k} = g|_{M_k}$. An operator system structure is defined on the amalgamated direct sum $\mathcal{S} \oplus \mathcal{T} / \langle (1_{\mathcal{S}}, -1_{\mathcal{T}}) \rangle$ identifying each order unit. For $s \in M_k(\mathcal{S})$ and $t \in M_k(\mathcal{T})$, we define

- (1) $(s + t)^* = s^* + t^*$,
- (2) $s + t \geq 0$ if and only if $f(s) + g(t) \geq 0$ for all compatible pairs (f, g) .

This is denoted by $\mathcal{S} \oplus_1 \mathcal{T}$ and is called the coproduct of operator systems \mathcal{S} and \mathcal{T} . The canonical inclusion from \mathcal{S} (respectively \mathcal{T}) into $\mathcal{S} \oplus_1 \mathcal{T}$ is a complete order embedding. The coproducts of operator systems satisfy the universal property: for unital completely positive maps $\varphi : \mathcal{S} \rightarrow \mathcal{R}$ and $\psi : \mathcal{T} \rightarrow \mathcal{R}$, there is a unique unital completely positive map $\Phi : \mathcal{S} \oplus_1 \mathcal{T} \rightarrow \mathcal{R}$ that extends both φ and ψ [F, Proposition 3.3].

$$\begin{array}{ccc}
\mathcal{S} & & \\
\downarrow & \searrow \varphi & \\
\mathcal{S} \oplus_1 \mathcal{T} & \xrightarrow{\Phi} & \mathcal{R} \\
\uparrow & \nearrow \psi & \\
\mathcal{T} & &
\end{array}$$

The maximal tensor product and the commuting tensor product are two different means of extending the C^* -maximal tensor product from the category of C^* -algebras to operator systems. For this reason, the weak expectation property of C^* -algebras bifurcates into the weak expectation property and the double commutant expectation property of operator systems. We say that an operator system \mathcal{S} has the *double commutant expectation property* provided that for every completely order isomorphic inclusion $\mathcal{S} \subset B(H)$, there exists a completely positive map $\varphi : B(H) \rightarrow \mathcal{S}''$ that fixes \mathcal{S} . For an operator system \mathcal{S} , the following are equivalent [KPTT2, Theorem 7.6]:

- (i) \mathcal{S} has the double commutant expectation property;
- (ii) \mathcal{S} is (el, c)-nuclear;

(iii) $\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_\infty)$.

We refer to [KPTT1] and [KPTT2] for general information on tensor products and quotients of operator systems.

3. A KIRCHBERG TYPE TENSOR THEOREM FOR OPERATOR SYSTEMS

Suppose that I is an index set and $\{I_k\}_{k \in \mathbb{N}}$ is a sequence of index sets having the same cardinality as I . We consider the unital C^* -algebra free product

$$*_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0, 1])_{\iota_k},$$

where $M_k(C([0, 1])_{\iota_k})$ denotes the copy of $M_k(C([0, 1]))$ for each index $\iota_k \in I_k$. We denote the copy of $1 \in C([0, 1])$ (respectively $t \in C([0, 1])$) in $M_k(C([0, 1])_{\iota_k})$ as 1_{ι_k} (respectively t_{ι_k}). Let \mathfrak{C}_I (respectively \mathfrak{C}_{ι_k}) be an operator subsystem generated by

$$\{e_{ij} \otimes 1_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k, 1 \leq i, j \leq k\} \quad \text{and} \quad \{e_{ij} \otimes t_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k, 1 \leq i, j \leq k\}$$

(respectively $\{e_{ij} \otimes 1_{\iota_k} : \iota_k \in I_k, 1 \leq i, j \leq k\}$ and $\{e_{ij} \otimes t_{\iota_k} : \iota_k \in I_k, 1 \leq i, j \leq k\}$).

in $*_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0, 1])_{\iota_k})$ (respectively in $M_k(C([0, 1])_{\iota_k})$). The operator system \mathfrak{C}_I depends only on the cardinality of the index set I .

Theorem 3.1. *Suppose that \mathcal{S} is an operator system and $\mathcal{S}_{\|\cdot\| \leq 1}^+$ is indexed by a set I . Then, \mathcal{S} is an operator system quotient of \mathfrak{C}_I . Furthermore, the kernel is completely order proximal and every positive element $x \in M_k(\mathcal{S})$ can be lifted to a positive element $\tilde{x} \in M_k(\mathfrak{C}_I)$ with $\|\tilde{x}\| \leq k^2 \|x\|$.*

Proof. Let $\{I_k\}_{k \in \mathbb{N}}$ be a sequence of index sets with the same cardinality as I . Then, each element in $M_k(\mathcal{S})_{\|\cdot\| \leq 1}^+$ can be indexed by I_k . Suppose that $\mathcal{S} \subset B(H)$. For each index $\iota_k \in I_k$, we define a unital completely positive map $\Phi_{\iota_k} : M_k(C([0, 1])_{\iota_k}) \rightarrow B(H)$ as

$$\Phi_{\iota_k}(\alpha \otimes f) = \frac{1}{k} \begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \alpha \otimes f(x_{\iota_k}) \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} = \frac{1}{k} \sum_{i,j} \alpha_{i,j} f(x_{\iota_k})_{i,j}.$$

By [B, Theorem 3.1], their unital free product

$$*_{k \in \mathbb{N}, \iota_k \in I_k} \Phi_{\iota_k} : *_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0, 1])_{\iota_k}) \rightarrow B(H)$$

is still completely positive. It maps \mathfrak{C}_I into \mathcal{S} . Let $\Phi : \mathfrak{C}_I \rightarrow \mathcal{S}$ be its restriction. Since $\mathcal{S}_{\|\cdot\| \leq 1}^+$ is contained in the range of Φ , Φ is surjective.

Choose an element $x_{\iota_k} \in M_k(\mathcal{S})_{\|\cdot\|=1}^+$. From

$$\Phi_k(k[E_{ij} \otimes t_{\iota_k}]_{i,j}) = [k\Phi(E_{ij} \otimes t_{\iota_k})]_{i,j} = [x_{\iota_k}(i, j)]_{i,j} = x_{\iota_k}$$

and

$$k[E_{i,j} \otimes t_{\iota_k}]_{i,j} = k[E_{ij}]_{i,j} \otimes t_{\iota_k} = k \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \begin{pmatrix} e_1^t & \cdots & e_n^t \end{pmatrix} \otimes t_{\iota_k} \in M_{k^2}(C([0, 1])_{\iota_k}^+),$$

we see that $\Phi : \mathfrak{C}_I \rightarrow \mathcal{S}$ is a complete order quotient map whose kernel is completely order proximal. Moreover, we have

$$\begin{aligned} \|k[E_{i,j} \otimes t_{\iota_k}]_{i,j}\|_{M_k(C([0,1]))} &= \|k[E_{i,j}]_{i,j}\| \\ &= k \left\| \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} (e_1^t \ \cdots \ e_k^t) \right\| \\ &= k \left\| (e_1^t \ \cdots \ e_k^t) \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} \right\| \\ &= k^2. \end{aligned}$$

□

Theorem 3.2. *Every finite dimensional operator system is an operator system quotient of a separable operator system $\mathfrak{C}_{\mathbb{N}}$.*

Proof. Let E be a finite dimensional operator system. We index a countable dense subset D_k of $M_k(E)_{\|\cdot\| \leq 1}^+$ by \mathbb{N} . Define a unital completely positive map $\Phi : \mathfrak{C}_{\mathbb{N}} \rightarrow E$ as in Theorem 3.1. Since the range of Φ is a dense subspace of a finite dimensional space E , Φ is surjective.

Choose $\varepsilon > 0$ and an element x in $M_k(E)_{\|\cdot\|=1}^+$. Since E is finite dimensional, the inverse of $\tilde{\Phi} : \mathfrak{C}_{\mathbb{N}}/\text{Ker}\Phi \rightarrow E$ is completely bounded. Let

$$\|\tilde{\Phi}^{-1} : E \rightarrow \mathfrak{C}_{\mathbb{N}}/\text{Ker}\Phi\|_{cb} \leq M.$$

Take $y \in D_k$ so that $\|x - y\| \leq \frac{\varepsilon}{2M}$. Since $\|\tilde{\Phi}_k^{-1}(x - y)\| \leq \frac{\varepsilon}{2}$, we have

$$\tilde{\Phi}_k^{-1}(x - y) + \frac{\varepsilon}{2}I_k \otimes 1_{\mathfrak{C}_{\mathbb{N}}/\text{Ker}\Phi} \in M_k(\mathfrak{C}_{\mathbb{N}}/\text{Ker}\Phi)^+.$$

There exists a positive element z in $M_k(\mathfrak{C}_{\mathbb{N}})$ satisfying

$$z + \text{Ker}\Phi_k = \tilde{\Phi}_k^{-1}(x - y) + \varepsilon I_k \otimes 1_{\mathfrak{C}_{\mathbb{N}}/\text{Ker}\Phi},$$

which implies

$$\Phi_k(z) = \tilde{\Phi}_k(z + \text{Ker}\Phi) = x - y + \varepsilon I_k \otimes 1_E.$$

As in the proof of Theorem 3.1, we can take a positive element \tilde{y} in $M_k(\mathfrak{C}_{\mathbb{N}})$ such that $\Phi_k(\tilde{y}) = y$. It follows that

$$\Phi_k(z + \tilde{y}) = (x - y + \varepsilon I_k \otimes 1_E) + y = x + \varepsilon I_k \otimes 1_E.$$

□

Proposition 3.3. *For $\iota_k \neq \iota'_l$, the canonical inclusion from the coproduct $\mathfrak{C}_{\iota_k} \oplus_1 \mathfrak{C}_{\iota'_l}$ into \mathfrak{C}_I is a complete order embedding.*

Proof. By [F, Proposition 3.4], we have

$$\begin{aligned} &x + y \in M_n(\mathfrak{C}_{\iota_k} \oplus_1 \mathfrak{C}_{\iota'_l})^+ \\ \Leftrightarrow &\exists \alpha \in M_n, x - \alpha \otimes 1 \in M_n(\mathfrak{C}_{\iota_k})^+ \text{ and } y + \alpha \otimes 1 \in M_n(\mathfrak{C}_{\iota'_l})^+ \\ \Leftrightarrow &\exists \alpha \in M_n, x - \alpha \otimes 1 \in M_n(M_k(C([0,1]))_{\iota_k})^+ \text{ and } y + \alpha \otimes 1 \in M_n(M_l(C([0,1]))_{\iota'_l})^+ \\ \Leftrightarrow &x + y \in M_n(M_k(C([0,1]))_{\iota_k} \oplus_1 M_l(C([0,1]))_{\iota'_l})^+ \end{aligned}$$

The operator system coproduct of unital C^* -algebras embeds into their unital C^* -algebra free product [F, Corollary 3.7]. We have the complete order embeddings

$$\begin{aligned} \mathfrak{C}_{\iota_k} \oplus_1 \mathfrak{C}_{\iota'_l} &\subset M_k(C([0, 1]))_{\iota_k} \oplus_1 M_k(C([0, 1]))_{\iota'_l} \\ &\subset M_k(C([0, 1]))_{\iota_k} * M_k(C([0, 1]))_{\iota'_l} \\ &\subset *_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0, 1]))_{\iota_k}. \end{aligned}$$

□

The coproduct of two operator systems studied in [F] can be generalized to the coproduct of a finite family of operator systems in an obvious way. We now define the coproduct of arbitrary family of operator systems to prove the lifting property of \mathfrak{C}_I . Suppose that $\{\mathcal{S}_\iota : \iota \in I\}$ is a family of operator systems. Fix an index $\iota_0 \in I$. We consider their algebraic direct sum $\bigoplus_{\iota \in I} \mathcal{S}_\iota$ consisting of finitely supported elements. Define the elements $n_\iota \in \bigoplus_{\iota \in I} \mathcal{S}_\iota$ ($\iota \neq \iota_0$) as

$$n_\iota(\iota') = \begin{cases} 1_{\mathcal{S}_{\iota_0}}, & \iota' = \iota_0 \\ -1_{\mathcal{S}_\iota}, & \iota' = \iota \\ 0, & \text{otherwise.} \end{cases}$$

The algebraic quotient

$$\mathcal{S} := \bigoplus_{\iota \in I} \mathcal{S}_\iota / \text{span}\{n_\iota : \iota \neq \iota_0\}$$

can be regarded as an amalgamated direct sum of $\{\mathcal{S}_\iota : \iota \in I\}$ identifying each order unit. Hence, a general element $\sum_{\iota \in F} x_\iota$ in \mathcal{S} (F is a finite subset of I) can be regarded as an element in the coproduct $\bigoplus_1 \{\mathcal{S}_\iota : \iota \in F\}$. We define the positive cones $M_n(\mathcal{S})^+$ as

$$\sum_{\iota \in F} x_\iota \in M_n(\mathcal{S})^+ \quad \text{iff} \quad \sum_{\iota \in F} x_\iota \in M_n(\bigoplus_1 \{\mathcal{S}_\iota : \iota \in F\})^+.$$

Since $\bigoplus_1 \{\mathcal{S}_\iota : \iota \in F_1\}$ is an operator subsystem of $\bigoplus_1 \{\mathcal{S}_\iota : \iota \in F_2\}$ if $F_1 \subset F_2$, $M_n(\mathcal{S})^+$ are well defined positive cones. It is routine to check that $(\mathcal{S}, \{M_n(\mathcal{S})^+\}_{n \in \mathbb{N}})$ with the identified order unit is an operator system satisfying the universal property. We denote the operator system as $\bigoplus_1 \{\mathcal{S}_\iota : \iota \in I\}$ and call it the coproduct of $\{\mathcal{S}_\iota : \iota \in I\}$.

$$\begin{array}{ccc} \mathcal{S}_\iota & & \\ \downarrow & \searrow \varphi_\iota & \\ \bigoplus_1 \{\mathcal{S}_\iota : \iota \in I\} & \xrightarrow{\Phi} & \mathcal{R} \end{array}$$

By Proposition 3.3, the operator system \mathfrak{C}_I is unittally completely order isomorphic to the coproduct

$$\bigoplus_1 \{\mathfrak{C}_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

Theorem 3.4. *Suppose that \mathcal{A} is a unital C^* -algebra and \mathcal{I} is a closed ideal in it. Every unital completely positive map $\varphi : \mathfrak{C}_I \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \rightarrow \mathcal{A}$.*

$$\begin{array}{ccc} & & \mathcal{A} \\ & \nearrow \tilde{\varphi} & \downarrow \\ \mathfrak{C}_I & \xrightarrow{\varphi} & \mathcal{A}/\mathcal{I} \end{array}$$

Proof. For $\alpha, \beta \in M_{nk}$, $\alpha \otimes 1_{\iota_k} + \beta \otimes t_{\iota_k}$ is positive in $M_n(\mathfrak{C}_{\iota_k})$ if and only if both α and $\alpha + \beta$ are positive semidefinite matrices, because \mathfrak{C}_{ι_k} is an operator subsystem of $C([0, 1], M_k)$. The mapping

$$\alpha \otimes 1_{\iota_k} + \beta \otimes t_{\iota_k} \in \mathfrak{C}_{\iota_k} \mapsto (\alpha, \alpha + \beta) \in M_k \oplus M_k$$

is a unital complete order isomorphism. Hence, \mathfrak{C}_I is unital completely order isomorphic to the coproduct

$$\oplus_1 \{(M_k \oplus M_k)_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

The Choi matrix z_{ι_k} corresponding to the restriction $\varphi|_{(M_k \oplus M_k)_{\iota_k}}$ belongs to the positive cone of $M_k(\mathcal{A}/I) \oplus M_k(\mathcal{A}/I)$. Let $\tilde{z}_{\iota_k} \in M_k(\mathcal{A}) \oplus M_k(\mathcal{A})$ be a positive lifting z_{ι_k} . Its corresponding mapping

$$\tilde{\varphi}_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \rightarrow \mathcal{A}$$

is a completely positive lifting of $\varphi|_{(M_k \oplus M_k)_{\iota_k}}$. We let

$$\tilde{\varphi}_{\iota_k}(I_{2k}) = 1 + h, \quad h = h^+ - h^- \quad (h \in \mathcal{I}, \quad h^+, h^- \in \mathcal{I}^+)$$

and take a faithful state ω on $(M_k \oplus M_k)_{\iota_k}$. Considering

$$\alpha \in (M_k \oplus M_k)_{\iota_k} \mapsto (1 + h^+)^{-\frac{1}{2}}(\tilde{\varphi}_{\iota_k}(\alpha) + \omega(\alpha)h^-)(1 + h^+)^{-\frac{1}{2}} \in \mathcal{A}$$

as in [KPTT2, Remark 8.3], we may assume that the lifting $\tilde{\varphi}_{\iota_k}$ is unital. By the universal property of the coproduct, there exists a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \rightarrow \mathcal{S}$ that extends all $\tilde{\varphi}_{\iota_k}$. \square

Corollary 3.5. *Let \mathcal{S} be an operator system and $Q : \mathfrak{C}_I \rightarrow \mathcal{S}$ be a complete order quotient map. Let $C_u^*(Q) : C_u^*(\mathfrak{C}_I) \rightarrow C_u^*(\mathcal{S})$ be a unique $*$ -homomorphism that extends Q . The following are equivalent:*

- (i) \mathcal{S} has the operator system lifting property;
- (ii) $C_u^*(Q)$ has a unital $*$ -homomorphic right inverse;
- (iii) $C_u^*(Q)$ has a unital completely positive right inverse.

Proof. (i) \Rightarrow (ii). The inclusion $\iota : \mathcal{S} \subset C_u^*(\mathcal{S})$ lifts to a unital completely positive map $\tilde{\iota} : \mathcal{S} \rightarrow C_u^*(\mathfrak{C}_I)$. Its $*$ -homomorphic extension $\rho : C_u^*(\mathcal{S}) \rightarrow C_u^*(\mathfrak{C}_I)$ is the right inverse of $C_u^*(Q) : C_u^*(\mathfrak{C}_I) \rightarrow C_u^*(\mathcal{S})$.

(ii) \Rightarrow (iii). Trivial

(iii) \Rightarrow (i). Suppose that $\varphi : \mathcal{S} \rightarrow \mathcal{A}/\mathcal{I}$ is a unital completely positive map for a C^* -algebra \mathcal{A} and its closed ideal \mathcal{I} . By Theorem 3.4, $\varphi \circ Q : \mathfrak{C}_I \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\psi : \mathfrak{C}_I \rightarrow \mathcal{A}$. Let $\rho : C_u^*(\mathfrak{C}_I) \rightarrow \mathcal{A}$ (respectively $\sigma : C_u^*(\mathcal{S}) \rightarrow \mathcal{A}/\mathcal{I}$) be a unique $*$ -homomorphic extension of ψ (respectively φ). Suppose that r is a unital completely positive right inverse of $C_u^*(Q)$.

$$\begin{array}{ccc}
 C_u^*(\mathfrak{C}_I) & \xrightarrow{\rho} & \mathcal{A} \\
 \uparrow & \swarrow C_u^*(Q) & \downarrow \pi \\
 & & C_u^*(\mathcal{S}) \\
 \uparrow & \swarrow r & \downarrow \sigma \\
 \mathfrak{C}_I & \xrightarrow{Q} & \mathcal{S} \xrightarrow{\varphi} \mathcal{A}/\mathcal{I} \\
 & & \uparrow \iota
 \end{array}$$

Let us show that

$$\tilde{\varphi} := \rho \circ r \circ \iota : \mathcal{S} \rightarrow \mathcal{A}$$

is a lifting of φ . Since \mathfrak{C}_I generates $C_u^*(\mathfrak{C}_I)$ as a C^* -algebra, $\pi \circ \psi = \varphi \circ Q$ implies that

$$\pi \circ \rho = \sigma \circ C_u^*(Q).$$

For $x \in \mathfrak{C}_I$, we have

$$\pi \circ \tilde{\varphi}(x) = \pi \circ \rho \circ r(x) = \sigma \circ C_u^*(Q) \circ r(x) = \varphi(x).$$

□

For a free group \mathbb{F} and a Hilbert space H , Kirchberg [K, Corollary 1.2] proved that

$$C^*(\mathbb{F}) \hat{\otimes}_{\min} B(H) = C^*(\mathbb{F}) \hat{\otimes}_{\max} B(H).$$

Kirchberg's theorem is striking if we recall that $C^*(\mathbb{F})$ and $B(H)$ are universal objects in the C^* -algebra category: every C^* -algebra is a C^* -quotient of $C^*(\mathbb{F})$ and a C^* -subalgebra of $B(H)$ for suitable choices of \mathbb{F} and H . Every operator system is a quotient of \mathfrak{C}_I and a subsystem of $B(H)$ for suitable choices of I and H . Hence we may say that

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$$

is the Kirchberg type theorem in the category of operator systems.

If \mathcal{S} has the operator system local lifting property, then $\mathcal{S} \otimes_{\min} B(H) = \mathcal{S} \otimes_{\min} B(H)$ [KPTT2, Theorem 8.6]. Hence, Theorem 3.4 immediately yields that $\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$. The proof of [KPTT2, Theorem 8.6] depends on Kirchberg's theorem. We give a direct proof of $\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$ that is independent of Kirchberg's theorem. By combining this with [K], we present a new operator system theoretic proof of Kirchberg's theorem in Corollary 3.9.

Theorem 3.6. *For an index set I and a Hilbert space H , we have*

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H).$$

Proof. Let z be a positive element in $\mathfrak{C}_I \otimes_{\min} B(H)$. We write $z = \sum_{\iota_k \in F} z_{\iota_k}$ for a finite subset F of $\bigcup_{k=1}^{\infty} I_k$ and $z_{\iota_k} \in \mathfrak{C}_{\iota_k} \otimes B(H)$. We regard z as a positive element in $\oplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H)$.

Fix an index $\iota_{k_0} \in F$ and denote the identity matrix in $(M_k \oplus M_k)_{\iota_k}$ by I_{ι_k} . We put $J = \text{span}\{0 \oplus \cdots \oplus I_{\iota_{k_0}} \oplus 0 \oplus \cdots \oplus -I_{\iota'_l} \oplus 0 \oplus \cdots \oplus 0 \in \oplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \iota'_l \in F, \iota'_l \neq \iota_{k_0}\}$.

We can show that the mapping

$$\sum_{\iota_k \in F} z_{\iota_k} \in \oplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \mapsto |F|(z_{\iota_k})_{\iota_k \in F} + J \in \oplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} / J$$

is a unital complete order isomorphism similarly with [K, Proposition 4.7]. Taking their duals, we see that the dual space of $\oplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\}$ is completely order isomorphic to

$$J^\perp = \{(\alpha_{\iota_k}) \in \oplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \text{tr}(\alpha_{\iota_k}) = \text{tr}(\alpha_{\iota'_l}) \text{ for all } \iota_k, \iota'_l \in F\}$$

as a matrix ordered space. Multiplying both sides by $(\sqrt{k}I_{\iota_k})_{\iota_k \in F}$, we see that J^\perp is completely isomorphic to the operator system

$$K = \{(\alpha_{\iota_k}) \in \oplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \frac{\text{tr}(\alpha_{\iota_k})}{k} = \frac{\text{tr}(\alpha_{\iota'_l})}{l} \text{ for all } \iota_k, \iota'_l \in F\}.$$

Let $\varphi : K \rightarrow B(H)$ be a completely positive map corresponding to z [KPTT2, Lemma 8.5]. By the Arveson extension theorem, $\varphi : K \rightarrow B(H)$ extends to a completely positive map $\tilde{\varphi} : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \rightarrow B(H)$. This implies that

$$R \otimes \text{id} : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \otimes_{\min} B(H) \rightarrow \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H)$$

is a complete order quotient map for the restriction

$$R : (\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k})^* \rightarrow K^*.$$

Maximal tensor products of complete order quotient maps are still complete order quotient maps [H, Theorem 3.4]. Hence, we obtain

$$\begin{array}{ccc} \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \otimes_{\min} B(H) & \xlongequal{\quad} & \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \otimes_{\max} B(H) \\ \downarrow & & \downarrow \\ \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H) & & \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\max} B(H). \end{array}$$

The element z is also positive in $\mathfrak{C}_I \otimes_{\max} B(H)$. The same arguments apply to all matricial levels. \square

Theorem 3.7. *An operator system \mathcal{S} has the double commutant expectation property if and only if it satisfies*

$$\mathcal{S} \otimes_{\min} \mathfrak{C}_I = \mathcal{S} \otimes_c \mathfrak{C}_I.$$

Proof. \Rightarrow) Every operator system with the double commutant expectation property is (el, c)-nuclear [KPTT2, Theorem 7.3]. We have

$$\begin{array}{ccc} B(H) \otimes_{\min} \mathfrak{C}_I & \xlongequal{\quad} & B(H) \otimes_{\max} \mathfrak{C}_I \\ \uparrow & & \uparrow \\ \mathcal{S} \otimes_{\min} \mathfrak{C}_I & & \mathcal{S} \otimes_{\text{el}=c} \mathfrak{C}_I. \end{array}$$

\Leftarrow) Let us show that $\ell_\infty^2 \oplus_1 \ell_\infty^3$ is complemented in \mathfrak{C}_I by a unital completely positive map. Fix two indices $\iota'_2 \in I_2$ and $\iota'_3 \in I_3$. We have a completely positive inclusion

$$\begin{aligned} (a_1, a_2) + (b_1, b_2, b_3) &\in \ell_\infty^2 \oplus_1 \ell_\infty^3 \\ \mapsto \text{diag}(a_1, a_2, a_1, a_2) + \text{diag}(b_1, b_2, b_3, b_1, b_2, b_3) &\in (M_2 \oplus M_2)_{\iota'_2} \oplus_1 (M_3 \oplus M_3)_{\iota'_3} \subset \mathfrak{C}_I \end{aligned}$$

For each index $\iota_k \neq \iota'_2, \iota'_3$, we take a state ω_{ι_k} on $(M_k \oplus M_k)_{\iota_k}$ and define a unital completely positive map

$$\varphi_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \rightarrow \ell_\infty^2 \oplus_1 \ell_\infty^3$$

as $\varphi_{\iota_k}(\alpha) = \omega_{\iota_k}(\alpha)1$. For ι'_2 (respectively ι'_3), we also define a unital completely positive map

$$\varphi_{\iota'_2} : (M_2 \oplus M_2)_{\iota'_2} \rightarrow \ell_\infty^2 \oplus_1 \ell_\infty^3 \quad (\text{respectively } \varphi_{\iota'_3} : (M_3 \oplus M_3)_{\iota'_3} \rightarrow \ell_\infty^2 \oplus_1 \ell_\infty^3)$$

as $\varphi_{\iota'_2}(\alpha \oplus \beta) = (\alpha_{11}, \alpha_{22})$ (respectively $\varphi_{\iota'_3}(\alpha \oplus \beta) = (\alpha_{1,1}, \alpha_{22}, \alpha_{33})$). By the universal property of the coproduct, there exists a unital completely positive map $\Phi : \mathfrak{C}_I \rightarrow \ell_\infty^2 \oplus_1 \ell_\infty^3$ that extends all φ_{ι_k} . All elements in $\ell_\infty^2 \oplus_1 \ell_\infty^3$ are fixed by Φ .

Hence, the hypothesis implies that

$$\mathcal{S} \otimes_{\min} (\ell_\infty^2 \oplus_1 \ell_\infty^3) = \mathcal{S} \otimes_{\max} (\ell_\infty^2 \oplus_1 \ell_\infty^3).$$

Every operator system satisfying the above identity has the double commutant expectation property [K, Theorem 5.9]. \square

Since the maximal tensor product and the commuting tensor product are two different means of extending the C^* -maximal tensor product from the category of C^* -algebras to operator systems, we can regard

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I \quad \text{and} \quad \mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$$

as operator system analogues of the Kirchberg's conjecture

$$C^*(\mathbb{F}) \hat{\otimes}_{\min} C^*(\mathbb{F}) = C^*(\mathbb{F}) \hat{\otimes}_{\max} C^*(\mathbb{F}).$$

The former is not true and the latter is equivalent to the Kirchberg's conjecture itself.

Corollary 3.8. (i) $\mathfrak{C}_I \otimes_c \mathfrak{C}_I \neq \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$. In particular, $\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I \neq \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$.
(ii) The Kirchberg's conjecture has an affirmative answer if and only if

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I.$$

Proof. (i) Similarly as in the proof of Theorem 3.7, we can show that $\ell_\infty^2 \oplus_1 \ell_\infty^2$ is complemented in \mathfrak{C}_I by a unital completely positive map. In [FKPT, Theorem 6.11], it is shown that

$$(\ell_\infty^2 \oplus_1 \ell_\infty^2) \otimes_c (\ell_\infty^2 \oplus_1 \ell_\infty^2) \quad \text{and} \quad (\ell_\infty^2 \oplus_1 \ell_\infty^2) \otimes_{\max} (\ell_\infty^2 \oplus_1 \ell_\infty^2)$$

are different.

(ii) We have the equivalences:

The Kirchberg's conjecture has an affirmative answer

$$\Leftrightarrow \ell_\infty^2 \oplus_1 \ell_\infty^3 \text{ has the double commutant expectation property ([K, Theorem 5.14])}$$

$$\Leftrightarrow (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\min} \mathfrak{C}_I = (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_c \mathfrak{C}_I \text{ (Theorem 3.7)}$$

$$\Leftrightarrow \mathfrak{C}_I \text{ has the double commutant expectation property ([K, Theorem 5.9])}$$

$$\Leftrightarrow \mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I \text{ (Theorem 3.7).}$$

□

Corollary 3.9 (Kirchberg). *Let \mathbb{F}_∞ be a free group on a countably infinite number of generators and H be a Hilbert space. We have*

$$C^*(\mathbb{F}_\infty) \hat{\otimes}_{\min} B(H) = C^*(\mathbb{F}_\infty) \hat{\otimes}_{\max} B(H).$$

Proof. Since $\ell_\infty^2 \oplus_1 \ell_\infty^3$ is complemented in \mathfrak{C}_I by a unital completely positive map, Theorem 3.6 immediately implies that

$$(\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\min} B(H) = (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\max} B(H).$$

Alternatively, applying the proof of Theorem 3.6 to commutative algebras instead of matrix algebras, we obtain

$$\begin{array}{ccc} \ell_\infty^5 \otimes_{\min} B(H) & \xlongequal{\quad} & \ell_\infty^5 \otimes_{\max} B(H) \\ \downarrow & & \downarrow \\ (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\min} B(H) & & (\ell_\infty^2 \oplus_1 \ell_\infty^3) \otimes_{\max} B(H). \end{array}$$

For the remaining proof, we follow Kavruk's paper [K]. Since $\ell_\infty^2 \oplus_1 \ell_\infty^3 \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ contains enough unitaries, we have

$$C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\min} B(H) = C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\max} B(H)$$

by [KPTT2, Proposition 9.5]. The free group \mathbb{F}_∞ embeds into the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ [LH]. By [P2, Proposition 8.8], $C^*(\mathbb{F}_\infty)$ is a C^* -subalgebra of $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ complemented by a unital completely positive map. □

The following proof is motivated by [FP, Theorem 2.6].

Theorem 3.10. *The C^* -envelope of \mathfrak{C}_I is $*_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k}$.*

Proof. Suppose that

$$\mathfrak{C}_I \subset B(H) \quad \text{and} \quad *_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k} \subset B(K).$$

Let \mathcal{A} be a C^* -algebra generated by \mathfrak{C}_I in $B(H)$. By the Arveson extension theorem, the canonical inclusion from $\mathfrak{C}_I = \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}$ into $*_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k}$ extends to a unital completely positive map $\rho : \mathcal{A} \rightarrow B(K)$. Let $\rho = V^* \pi(\cdot) V$ be a minimal Stinespring decomposition of ρ for a $*$ -representation $\pi : \mathcal{A} \rightarrow B(\widehat{K})$ and an isometry $V : K \rightarrow \widehat{K}$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi} & B(\widehat{K}) \\ & \searrow \rho & \downarrow V^* \cdot V \\ \mathfrak{C}_I \subset *_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k} & \subset & B(K) \end{array}$$

For a unitary matrix U in $(M_k \oplus M_k)_{\iota_k}$ (U need not be unitary in \mathcal{A}), we can write $\pi(U)$ as the operator matrix form

$$\pi(U) = \begin{pmatrix} U & B \\ C & D \end{pmatrix}.$$

Since U is unitary in $B(K)$ and

$$1 = \|U\| \leq \left\| \begin{pmatrix} U & B \\ C & D \end{pmatrix} \right\| = \|\pi(U)\| \leq 1,$$

we have $B = 0 = C$ by the C^* -axiom. It follows that ρ is multiplicative on

$$\mathcal{U} := \{U \in (U(k) \oplus U(k))_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

By the spectral theorem, every matrix can be written as a linear combination of unitary matrices. It follows that the set \mathcal{U} generates \mathcal{A} as a C^* -algebra. We can regard ρ as a surjective $*$ -homomorphism from \mathcal{A} onto $*_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k}$. Hence, $*_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k}$ is the universal quotient of all C^* -algebras generated by \mathfrak{C}_I . \square

4. LIFTINGS OF COMPLETELY POSITIVE MAPS

It is natural to ask whether the universal operator system \mathfrak{C}_I is a projective object in the category of operator systems. In other words, for any operator system \mathcal{S} and its kernel \mathcal{J} , does every unital completely positive map $\varphi : \mathfrak{C}_I \rightarrow \mathcal{S}/\mathcal{J}$ lift to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \rightarrow \mathcal{S}$? The answer is negative in an extreme manner.

Proposition 4.1. *An operator system \mathcal{S} is one-dimensional if and only if for any operator system \mathcal{T} and its kernel \mathcal{J} , every unital completely positive map $\varphi : \mathcal{S} \rightarrow \mathcal{T}/\mathcal{J}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathcal{S} \rightarrow \mathcal{T}$.*

Proof. Let V^+ be the cone in \mathbb{R}^3 generated by $\{(x, y, 1) : (x - 1)^2 + y^2 \leq 1, y \geq 0\}$ and the origin. The triple $V := (\mathbb{C}^3, V^+, (1, 1, 2))$ is an Archimedean ordered $*$ -vector space. The operator system quotient of $OMAX(V)$ [PTT] by the z -axis is ℓ_∞^2 . Suppose that $\dim \mathcal{S} \geq 2$. Let v be a positive element in \mathcal{S} distinct from the scalar multiple of the identity. Considering $v - \lambda I$ for sufficiently large $\lambda > 0$, we may assume that the spectrum of v contains zero. Let ω_1 (respectively ω_2) be a state on \mathcal{S} that extends the

Dirac measure $\delta_{\{0\}}$ (respectively $\delta_{\{\|v\|\}}$) on the spectrum of v . The unital completely positive map $\varphi : \mathcal{S} \rightarrow \ell_\infty^2$ defined by $\varphi = (\omega_1, \omega_2)$ cannot not be lifted to a unital completely positive map, because the fiber of $\varphi(v) = (0, \|v\|)$ does not intersect V^+ . \square

The absence of unital completely positive liftings in the above proof is essentially due to Archimedeanization of quotients [PT]. In Corollary 4.5, we will see that there is a certain rigidity, even though some perturbation is allowed.

Remark 4.2. Suppose that $T : E \rightarrow F$ is a bounded linear surjection for normed spaces E and F . Let E_0 be a dense subspace of E , and $Q_0 : E_0 \rightarrow Q(E_0)$ be the surjective restriction of Q on E_0 . Then, Q_0 is a quotient map if and only if $\ker Q_0 = \ker Q$ and Q is a quotient map [DF, 7.4]. This is called the quotient lemma.

Thanks to the quotient lemma, the exactness of operator systems can be described via incomplete tensor products. Suppose that \mathcal{S} is an operator system and \mathcal{A} is a C^* -algebra with a closed ideal \mathcal{I} . By the quotient lemma, the sequence

$$0 \rightarrow \mathcal{S} \bar{\otimes}_{\min} \mathcal{I} \rightarrow \mathcal{S} \hat{\otimes}_{\min} \mathcal{A} \rightarrow \mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I} \rightarrow 0$$

is 1-exact if and only if $\text{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete quotient map. This is equivalent to saying that $\text{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map [KPTT2, Theorem 5.1]. Hence an operator system \mathcal{S} is 1-exact if and only if $\text{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map for any unital C^* -algebra \mathcal{A} and its closed ideal \mathcal{I} .

As pointed out in [KPTT2, Section 5], the context along the line of short exact sequences is inappropriate if we replace ideals in C^* -algebras and C^* -quotients by kernels in operator systems and operator system quotients. Even one-dimensional operator system does not satisfy such exactness. The replacement will occur in $\text{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ instead of a short exact sequence.

Theorem 4.3. *Let \mathcal{S} be an operator system. Then, the following are equivalent:*

- (i) \mathcal{S} is nuclear;
- (ii) if $\Phi : \mathcal{T}_1 \rightarrow \mathcal{T}_2$ is a complete order quotient map for operator systems \mathcal{T}_1 and \mathcal{T}_2 , then

$$\text{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 \rightarrow \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

is a complete order quotient map;

- (iii) if $\Phi : \mathfrak{C}_I \rightarrow \mathcal{T}$ is a complete order quotient map for an operator system \mathcal{T} , then

$$\text{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathfrak{C}_I \rightarrow \mathcal{S} \otimes_{\min} \mathcal{T}$$

is a complete order quotient map;

- (iv) if $\Phi : T \rightarrow E$ is a complete order quotient map for an operator system T and a finite dimensional operator system E , then

$$\text{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} T \rightarrow \mathcal{S} \otimes_{\min} E$$

is a complete order quotient map.

Proof. (i) \Rightarrow (ii). Maximal tensor products of complete order quotient maps are still complete order quotient maps [H, Theorem 3.4].

(ii) \Rightarrow (i). The proof is motivated by [ER, Theorem 14.6.1]. Take a finite subset $\{x_1, \dots, x_n\}$ of \mathcal{S} and $\varepsilon > 0$. If \mathcal{S} is a unital exact C^* -algebra, there always exist unital completely positive maps $\varphi : \mathcal{S} \rightarrow E \subset M_n$ and $\psi : E \rightarrow \mathcal{S}$ such that $\|\psi \circ \varphi(x_i) - x_i\| < \varepsilon$

for $1 \leq i \leq n$ [BO, Lemma 3.9.7]. However, the proof of [BO, Lemma 3.9.7] still works, even though \mathcal{S} is merely an exact operator system. Taking \mathcal{T}_1 as a unital C^* -algebra and \mathcal{T}_2 as a C^* -quotient, we see that \mathcal{S} is an exact operator system by Remark 4.2.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\text{id}_{\mathcal{S}}} & \mathcal{S} \\ & \searrow \varphi & \nearrow \psi \\ & E & \subset M_n \\ & & \uparrow \psi' \end{array}$$

By choosing a faithful state ω (respectively $\omega|_E$) on M_n (respectively E), we can regard the dual space M_n^* (respectively E^*) as an operator system. The element z in $E^* \otimes_{\min} \mathcal{S}$ corresponding to $\psi : E \rightarrow \mathcal{S}$ is positive. The restriction $R : M_n^* \rightarrow E^*$ is a complete order quotient map by the Arveson extension theorem. By the hypothesis,

$$R \otimes \text{id}_{\mathcal{S}} : M_n^* \otimes_{\min} \mathcal{S} \rightarrow E^* \otimes_{\min} \mathcal{S}$$

is also a complete order quotient map. There exists a positive lifting $\tilde{z} \in M_n^* \otimes_{\min} \mathcal{S}$ of $z + \varepsilon \omega|_E \otimes 1_{\mathcal{S}}$. The completely positive map $\psi' : M_n \rightarrow \mathcal{S}$ corresponding to \tilde{z} satisfies

$$\|\psi - \psi'|_E\|_{cb} \leq \varepsilon.$$

It follows that

$$\|\psi' \circ \varphi(x_i) - x_i\| \leq \|(\psi' - \psi) \circ \varphi(x_i)\| + \|\psi \circ \varphi(x_i) - x_i\| \leq \varepsilon(\|x_i\| + 1)$$

for each $1 \leq i \leq n$. Hence, we can take nets of unital completely positive maps $\varphi_{\lambda} : \mathcal{S} \rightarrow M_{n_{\lambda}}$ and completely positive maps $\psi'_{\lambda} : M_{n_{\lambda}} \rightarrow \mathcal{S}$ such that $\psi'_{\lambda} \circ \varphi_{\lambda}$ converges to the map $\text{id}_{\mathcal{S}}$ in the point-norm topology.

Since each φ_{λ} is unital, $\psi'_{\lambda}(I_{n_{\lambda}})$ converges to $1_{\mathcal{S}}$. Let us choose a state ω_{λ} on $M_{n_{\lambda}}$ and set

$$\psi_{\lambda}(A) = \frac{1}{\|\psi'_{\lambda}\|} \psi'_{\lambda}(A) + \omega_{\lambda}(A) \left(1_{\mathcal{S}} - \frac{1}{\|\psi'_{\lambda}\|} \psi'_{\lambda}(I_{n_{\lambda}})\right).$$

Then $\psi_{\lambda} : M_{n_{\lambda}} \rightarrow \mathcal{S}$ is a unital completely positive map such that $\psi_{\lambda} \circ \varphi_{\lambda}$ converges to the map $\text{id}_{\mathcal{S}}$ in the point-norm topology.

(ii) \Rightarrow (iii), (ii) \Rightarrow (iv). Trivial

(iii) \Rightarrow (ii). Choose a positive element z in $\mathcal{S} \otimes_{\min} \mathcal{T}_2$ and $\varepsilon > 0$. By Theorem 3.1, we can take a complete order quotient map $\Psi : \mathfrak{C}_I \rightarrow \mathcal{T}_1$. By the assumption, there exists a positive element \tilde{z} in $\mathcal{S} \otimes_{\min} \mathfrak{C}_I$ satisfying $(\text{id}_{\mathcal{S}} \otimes \Phi \circ \Psi)(\tilde{z}) = z + \varepsilon 1$. Thus, $\text{id}_{\mathcal{S}} \otimes \Psi(\tilde{z})$ is a positive lifting of $z + \varepsilon 1$.

(iv) \Rightarrow (ii). Choose a positive element $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{S} \otimes_{\min} \mathcal{T}_2$. Take E as a finite dimensional operator system generated by $\{y_i : 1 \leq i \leq n\}$ and \mathcal{T} as $\Phi^{-1}(E)$. \square

The proof of (ii) \Rightarrow (i) gives an alternative proof of the Choi-Effros-Kirchberg theorem for operator systems [HP, Corollary 3.2].

Remark 4.4. The equivalence of (i) and (ii) was already discovered by Kavruk independently. The proof depends on Kavruk's result that is not yet published.

Corollary 4.5. *Suppose that E is a finite dimensional operator system and ω is a faithful state on E . The following are equivalent:*

- (i) *if $\varepsilon > 0$ and $\varphi : E \rightarrow \mathcal{S}/\mathcal{J}$ is a completely positive map for an operator system \mathcal{S} and its kernel \mathcal{J} , then there exists a self-adjoint lifting $\tilde{\varphi} : E \rightarrow \mathcal{S}$ of φ such that $\tilde{\varphi} + \varepsilon \omega 1$ is completely positive;*
- (ii) *E is unittally completely order isomorphic to the direct sum of matrix algebras.*

Proof. (i) \Rightarrow (ii). Condition (i) can be rephrased to state that

$$\text{id}_{E^*} \otimes \varphi : E^* \otimes_{\min} \mathcal{S} \rightarrow E^* \otimes_{\min} \mathcal{S}/\mathcal{J}$$

is a complete order quotient map for any operator system \mathcal{S} and its kernel \mathcal{J} . Hence, E^* is a finite dimensional nuclear operator system. Every finite dimensional nuclear operator system is unittally completely order isomorphic to the direct sum of matrix algebras [HP, Corollary 3.7]. Suppose that E^* is completely order isomorphic to $\bigoplus_{i=1}^n M_{k_i}$ for some $n, k_i \in \mathbb{N}$. Taking their duals, we see that E is completely order isomorphic to $\bigoplus_{i=1}^n M_{k_i}$. Suppose that the isomorphism maps the order unit of E to a matrix A in $\bigoplus_{i=1}^n M_{k_i}$. Then, A is positive definite. Let

$$A = U^* \text{diag}(\lambda_1, \dots, \lambda_m) U, \quad \lambda_i > 0, m = \sum_{k=1}^n k_i$$

be a diagonalization of A . The mapping

$$\alpha \in \bigoplus_{i=1}^n M_{k_i} \mapsto U^* \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_m}) \alpha \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_m}) U \in \bigoplus_{i=1}^n M_{k_i}$$

is a complete order isomorphism that maps the identity matrix to A .

(ii) \Rightarrow (i) We may assume that $E = \bigoplus_{i=1}^n M_{k_i}$. Let A be a density matrix of ω and $\lambda > 0$ be its smallest eigenvalue. Suppose that $z \in \bigoplus_{i=1}^n M_{k_i}(\mathcal{S}/\mathcal{J})$ is a Choi matrix corresponding to φ . There exists a lifting $\tilde{z} \in \bigoplus_{i=1}^n M_{k_i}(\mathcal{S})$ of z such that $\tilde{z} + \varepsilon \lambda 1$ is positive. Let $\tilde{\varphi} : E \rightarrow \mathcal{S}$ be a self-adjoint map corresponding to \tilde{z} . Then we have

$$\tilde{\varphi} + \varepsilon \omega 1 = \tilde{\varphi} + \varepsilon \text{tr}(\cdot A) 1 \geq_{cp} \tilde{\varphi} + \varepsilon \lambda \text{tr} 1 \geq_{cp} 0.$$

□

When characterizing 1-exact operator systems, in

$$\text{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I},$$

operator systems \mathcal{S} were fixed, and C^* -algebras \mathcal{A} and their closed ideals \mathcal{I} were considered to be variables. In the following, we switch their roles. As a result, we give an operator system theoretic proof of the Effros-Haagerup lifting theorem [EH, Theorem 3.2].

Theorem 4.6. *Suppose that \mathcal{A} is a unital C^* -algebra and \mathcal{I} is its closed ideal. The following are equivalent:*

- (i) $\text{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map for any operator system \mathcal{S} ;
- (ii) $\text{id}_{\mathcal{B}} \otimes \pi : \mathcal{B} \otimes_{\min} \mathcal{A} \rightarrow \mathcal{B} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map for any unital C^* -algebra \mathcal{B} ;
- (iii) $\text{id}_{B(H)} \otimes \pi : B(H) \otimes_{\min} \mathcal{A} \rightarrow B(H) \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map for a separable Hilbert space H ;
- (iv) $\text{id}_E \otimes \pi : E \otimes_{\min} \mathcal{A} \rightarrow E \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map for any finite dimensional operator system E ;
- (v) the sequence

$$0 \rightarrow \mathcal{B} \hat{\otimes}_{\min} \mathcal{I} \rightarrow \mathcal{B} \hat{\otimes}_{\min} \mathcal{A} \rightarrow \mathcal{B} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I} \rightarrow 0$$

is exact for any C^* -algebra \mathcal{B} ;

- (vi) for any finite dimensional operator system E , every completely positive map $\varphi : E \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a completely positive map $\tilde{\varphi} : E \rightarrow \mathcal{A}$;

- (vii) for any finite dimensional operator system E , every unital completely positive map $\varphi : E \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : E \rightarrow \mathcal{A}$;
- (viii) for any index set I , every unital completely positive finite rank map $\varphi : \mathfrak{C}_I \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \rightarrow \mathcal{A}$ with $\text{Ker}\varphi = \text{Ker}\tilde{\varphi}$;
- (ix) every unital completely positive finite rank map $\varphi : \mathfrak{C}_{\mathbb{N}} \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_{\mathbb{N}} \rightarrow \mathcal{A}$ with $\text{Ker}\varphi = \text{Ker}\tilde{\varphi}$.

Proof. (i) \Rightarrow (ii). \Rightarrow (iii), (viii) \Rightarrow (ix) are trivial. (vi) \Rightarrow (vii) follows from [KPTT2, Remark 8.3]. (ii) \Leftrightarrow (v) follows from Remark 4.2. For (iii) \Rightarrow (iv) and (iv) \Rightarrow (i), it is sufficient to consider the first matrix level.

(iii) \Rightarrow (iv). Let $E \subset B(H)$ for a separable Hilbert space H . Take a strictly positive element z in $E \otimes_{\min} \mathcal{A}/\mathcal{I}$ which is an operator subsystem of $B(H) \otimes_{\min} \mathcal{A}/\mathcal{I}$. By the assumption, there exists a positive lifting \tilde{z} in $B(H) \otimes_{\min} \mathcal{A}$. Let $\{x_i : 1 \leq i \leq k\}$ be a self-adjoint basis of E and $\{\hat{x}_i : 1 \leq i \leq k\}$ be its dual basis. Each functional \hat{x}_i on E extends to a continuous self-adjoint functional on $B(H)$ which we still denote by \hat{x}_i . The map $P := \sum_{i=1}^k \hat{x}_i \otimes x_i : B(H) \rightarrow B(H)$ is a self-adjoint projection onto E . Since

$$(\text{id}_{B(H)} - P) \otimes \pi(\tilde{z}) = z - (P \otimes \text{id}_{\mathcal{A}/\mathcal{I}})(z) = 0,$$

we have

$$(\text{id}_{B(H)} - P) \otimes \text{id}_{\mathcal{A}}(\tilde{z}) \in B(H) \otimes \mathcal{I}.$$

We write

$$(\text{id}_{B(H)} - P) \otimes \text{id}_{\mathcal{A}}(\tilde{z}) = \sum_{i=1}^n b_i \otimes h_i, \quad b_i \in B(H)_{sa}, h_i \in \mathcal{I}_{sa}.$$

Each h_i is decomposed into $h_i = h_i^+ - h_i^-$ for $h_i^+, h_i^- \in \mathcal{I}^+$. From

$$\begin{aligned} 0 &\leq \tilde{z} \\ &= (P \otimes \text{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n b_i \otimes h_i^+ - \sum_{i=1}^n b_i \otimes h_i^- \\ &\leq (P \otimes \text{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^- \end{aligned}$$

and

$$(\text{id}_{B(H)} \otimes \pi)((P \otimes \text{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^-) = z,$$

we see that

$$(P \otimes \text{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^- \in E \otimes_{\min} \mathcal{A}$$

is a positive lifting of z .

(iv) \Rightarrow (i). Take a positive element $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$. Let E be a finite dimensional operator system generated by $\{x_i : 1 \leq i \leq n\}$. Since $E \otimes_{\min} \mathcal{A}/\mathcal{I}$ is an operator subsystem of $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$, z is also positive in $E \otimes_{\min} \mathcal{A}/\mathcal{I}$. By the hypothesis, there exists a positive element \tilde{z} in $E \otimes_{\min} \mathcal{A}$ such that $(\text{id}_E \otimes \pi)(\tilde{z}) = z$. This element is also positive in $\mathcal{S} \otimes_{\min} \mathcal{A}$.

(iv) \Leftrightarrow (vi). Suppose that E is a finite dimensional operator system and $\varphi : E \rightarrow \mathcal{A}/\mathcal{I}$ is a completely positive map. The element z in $E^* \otimes_{\min} \mathcal{A}/\mathcal{I}$ corresponding to φ is positive. Since E is finite dimensional, we have $E^* \otimes_{\min} \mathcal{A} = E^* \hat{\otimes}_{\min} \mathcal{A}$. The kernel

$E^* \otimes \mathcal{I}$ of $\text{id}_{E^*} \otimes \pi$ is completely order proximal in $E^* \otimes_{\min} \mathcal{A}$ [KPTT2, Corollary 5.1.5]. By the hypothesis, z lifts to a positive element \tilde{z} in $E^* \otimes_{\min} \mathcal{A}$. The map $\tilde{\varphi} : E \rightarrow \mathcal{A}$ corresponding to \tilde{z} is completely positive. The converse is merely the reverse of the argument.

(vii) \Rightarrow (vi). The inclusion $\iota : \text{ran} \varphi \subset \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\iota} : \text{ran} \varphi \rightarrow \mathcal{A}$. The map $\tilde{\iota} \circ \varphi$ is the completely positive lifting of φ .

(vii) \Rightarrow (viii). Let $Q : \mathfrak{C}_I \rightarrow \mathfrak{C}_I/\text{Ker} \varphi$ be a quotient map. We have a factorization $\varphi = \psi \circ Q$ for $\psi : \mathfrak{C}_I/\text{Ker} \varphi \rightarrow \mathcal{A}/\mathcal{I}$. By the hypothesis, ψ lifts to a unital completely positive map $\tilde{\psi} : \mathfrak{C}_I/\text{Ker} \varphi \rightarrow \mathcal{A}$. Then $\tilde{\psi} \circ Q$ is a unital completely positive lifting of φ and their kernels coincide.

(ix) \Rightarrow (vii). By Theorem 3.2, there exists a complete order quotient map $\Phi : \mathfrak{C}_N \rightarrow E$. The map $\varphi \circ \Phi : \mathfrak{C}_N \rightarrow \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\Psi : \mathfrak{C}_N \rightarrow \mathcal{A}$ such that their kernels coincide. Since $\text{Ker} \Phi \subset \text{Ker} \Psi$, Ψ induces a map $\tilde{\varphi} : E \rightarrow \mathcal{A}/\mathcal{I}$ which is a unital completely positive lifting of φ . \square

The following theorem can be regarded as an operator system version of the quotient lemma discussed in Remark 4.2.

Theorem 4.7. *Suppose that $\Phi : \mathcal{S} \rightarrow \mathcal{T}$ is a unital completely positive surjection for operator systems \mathcal{S} and \mathcal{T} . Let \mathcal{S}_0 be an operator subsystem that is dense in \mathcal{S} , $\mathcal{T}_0 := \Phi(\mathcal{S}_0)$, and $\Phi_0 = \Phi|_{\mathcal{S}_0} : \mathcal{S}_0 \rightarrow \mathcal{T}_0$ be the surjective restriction. Then, the following are equivalent:*

- (i) $\Phi : \mathcal{S} \rightarrow \mathcal{T}$ is a complete order quotient map and for any $\varepsilon > 0$, $k \in \mathbb{N}$ and a self-adjoint element $x \in \text{Ker} \Phi_k$, there exists a self-adjoint element $x_0 \in \text{Ker}(\Phi_0)_k$ such that $x_0 + \varepsilon 1 \geq x$;
- (ii) $\Phi_0 : \mathcal{S}_0 \rightarrow \mathcal{T}_0$ is a complete order quotient map.

Proof. The following arguments apply to all matricial levels.

(i) \Rightarrow (ii). Choose $\varepsilon > 0$ and $\Phi_0(y_0) \in \mathcal{T}_0^+$ for a self-adjoint $y_0 \in \mathcal{S}_0$. By the hypothesis, there exist self-adjoint $x \in \text{Ker} \Phi$ and $x_0 \in \text{Ker} \Phi_0$ such that

$$y_0 + \frac{\varepsilon}{2} 1 + x \in \mathcal{S}^+ \quad \text{and} \quad x \leq x_0 + \frac{\varepsilon}{2} 1.$$

It follows that

$$y_0 + \varepsilon 1 + x_0 \geq y_0 + \frac{\varepsilon}{2} 1 + x \geq 0.$$

(ii) \Rightarrow (i). (a) Take $\varepsilon > 0$ and a self-adjoint element x in $\text{Ker} \Phi$. Since \mathcal{S}_0 is dense in \mathcal{S} , there exists a self-adjoint element y_0 in \mathcal{S}_0 such that

$$x - \frac{\varepsilon}{3} 1 \leq y_0 \leq x + \frac{\varepsilon}{3} 1,$$

which implies that

$$\Phi_0(-y_0 + \frac{\varepsilon}{3} 1) = \Phi(-y_0 + x + \frac{\varepsilon}{3} 1) \in \mathcal{T}^+ \cap \mathcal{T}_0 = \mathcal{T}_0^+.$$

There exists an element x_0 in $\text{Ker} \Phi_0$ such that

$$-y_0 + \frac{2}{3} \varepsilon 1 + x_0 \geq 0.$$

From

$$x - \frac{\varepsilon}{3} 1 \leq y_0 \leq \frac{2}{3} \varepsilon 1 + x_0,$$

it follows that $x \leq \varepsilon 1 + x_0$.

(b) Let $\Phi(y) \in \mathcal{T}^+$ for a self-adjoint $y \in \mathcal{S}$. There exists an element y_0 in \mathcal{S}_0 such that

$$y - \frac{\varepsilon}{3}1 \leq y_0 \leq y + \frac{\varepsilon}{3}1,$$

which implies that

$$\Phi_0(y_0 + \frac{\varepsilon}{3}1) \geq \Phi(y) \geq 0.$$

There exists an element $x_0 \in \text{Ker}\Phi_0$ such that

$$y_0 + x_0 + \frac{2}{3}\varepsilon 1 \geq 0.$$

It follows that

$$y + x_0 + \varepsilon 1 \geq y_0 + x_0 + \frac{2}{3}\varepsilon 1 \geq 0.$$

□

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