L'Hospital-type rules for monotonicity and limits: Discrete case

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1. Introduction

Let $-\infty \le a < b \le \infty$, let f and g be continuously differentiable functions defined on the interval (a, b), and let $r = f/g$ and $\rho = f'/g'$. In [13], general "rules" for monotonicity patterns, resembling the usual l'Hospital rules for limits, were given. For example, according to Proposition 1.9 in [13], one has the following: if ρ is increasing and $gg' > 0$ on (a, b), then $r \searrow\nearrow$, which means that there is some c in $[a, b]$ such that r is decreasing on (a, c) and increasing on (c, b). In particular, if c is either a or b, the result is that r is either increasing or decreasing on the entire interval (a, b). If one also knows whether r is increasing or decreasing in a right neighborhood of a and in a left neighborhood of b, then one can discriminate with certainty between these three patterns. Using such rules, one can generally determine ([13, 16]) the monotonicity pattern of r given that of ρ, however complicated the latter might be.

Clearly, these l'Hospital-type rules for monotonicity patterns are helpful wherever the l'Hospital rules for limits are. Moreover, the monotonicity rules apply even outside such contexts, because they do not require that both f and g (or either of them) tend to 0 or ∞ at any point. In the special case when both f and g vanish at an endpoint of the interval (a, b), l'Hospital-type rules for monotonicity can be found, in different forms and with different proofs, in [1]–[4], [6]–[8], [10], and [12]–[15]. In view of what has been said here, it should not be surprising that a very wide variety of applications of these l'Hospital-type rules for monotonicity patterns were given in those papers; see also [16]. In the present note, discrete analogues of l'Hospital-type rules both for monotonicity and limits are given.

2. Rules

Let $f := (f_n : n \in \overline{a, b})$ and $g := (g_n : n \in \overline{a, b})$ be two real sequences or, equivalently, functions defined on an interval $\overline{a, b} := \{n \in \mathbb{Z} : a \leqslant n \leqslant b\}$ of integers, where a and b are in $\mathbb{Z} \cup \{-\infty, \infty\}$ (so that $\overline{a, b} = \emptyset$ if $a > b$, and no infinite endpoint belongs to $\overline{a, b}$). Let

$$r := \frac{f}{g} \quad \text{and} \quad \rho := \frac{\Delta f}{\Delta g},$$

where $(\Delta f)_n := \Delta f_n := f_n - f_{n-1}$ for $n \in \overline{a + 1, b}$, so that the function Δf is defined on $\overline{a + 1, b}$ (with $\pm\infty + 1 := \pm\infty$). It is assumed throughout that g and Δg do not take on the zero value and do not change their respective signs.

Theorem 1. *Suppose that ρ is either nondecreasing or nonincreasing. Then the dependence of the monotonicity pattern of r on that of ρ (and also on the sign of $g\,\Delta g$) is given by the following table:*

Table 1.

ρ	$g\,\Delta g$	r
$\nearrow$	>0	$\searrow\nearrow$
$\searrow$	>0	$\nearrow\searrow$
$\nearrow$	<0	$\nearrow\searrow$
$\searrow$	<0	$\searrow\nearrow$

Here, for instance, the statement $r \searrow\nearrow$ *can be taken to mean that there is some* k *in* $\overline{a,b}\cup\{a,b\}$ *such that* r *is nonincreasing* ($\searrow$) *on* $\overline{a,k}$ *and nondecreasing* ($\nearrow$) *on* $\overline{k,b}$. *In particular, if* $k=a$ *then* $r\searrow\nearrow$ *will imply* $r\nearrow$ *on the entire interval* $\overline{a,b}$*; similarly, if* $k=b$ *then* $r\searrow\nearrow$ *will imply* $r\searrow$ *on* $\overline{a,b}$.

Remark 1. To discriminate between these three possibilities ($k=a$, $k=b$, and $a<k<b$) in the case when (say) a and b are finite, it suffices to know whether $r_{a+1}\geqslant r_a$ and $r_{b-1}\geqslant r_b$; if (say) $b=\infty$, then one may instead want to know the monotocity pattern of r in a neighborhood of ∞. $\diamondsuit$

Proof of Theorem 1. Without loss of generality, a and b are finite. In view of the "horizontal" and "vertical" reflections $\mathbb{Z}\ni n\leftrightarrow(-n)$ and $f\leftrightarrow(-f)$, it suffices to consider only the first line of Table 1, with $\rho\nearrow$ and $g\,\Delta g>0$. Next, it suffices to show that there is some $k\in\overline{a,b}$ such that $\Delta r\leqslant 0$ on $\overline{a+1,k}$ and $\Delta r>0$ on $\overline{k+1,b}$.

Let now $k:=\sup\{n\in\overline{a+1,b}\colon \Delta r_n\leqslant 0\}$ (here, $\sup\emptyset:=a$). Then $\Delta r>0$ on $\overline{k+1,b}$. If $\Delta r\leqslant 0$ on $\overline{a+1,k}$, the proof is completed. To obtain a contradiction, assume the contrary, that $\Delta r_n>0$ for some $n\in\overline{a+1,k}$ (so that $k\neq a$). Hence, there exists $m:=\max\{n\in\overline{a+1,k}\colon \Delta r_n>0\}$. Then in fact $m\in\overline{a+1,k-1}$; this follows because $k\neq a$ and hence, by the definition of k, one has $\Delta r_k\leqslant 0$, while $\Delta r_m>0$. Now it also follows from the definition of m that $\Delta r_{m+1}\leqslant 0$.

The key observation is that for all $n\in\overline{a+1,b}$

$$g_n g_{n-1}\,\Delta r_n=(\rho_n-r_n)\,g_n\,\Delta g_n=(\rho_n-r_{n-1})\,g_{n-1}\,\Delta g_n. \tag{1}$$

Using these identities (with $n=m$ and $n=m+1$) together with the obtained above inequalities $\Delta r_m>0$ and $\Delta r_{m+1}\leqslant 0$, one concludes that $\rho_m>r_m\geqslant\rho_{m+1}$, which contradicts the condition that ρ is nondecreasing. $\square$

Remark 2. (i) For the case given by the first line of Table 1, when $\rho\nearrow$ and $g\,\Delta g>0$, the above proof shows that there is some $k\in\overline{a,b}$ such that $\Delta r\leqslant 0$ on $\overline{a+1,k}$ and $\Delta r>0$ on $\overline{k+1,b}$. Using the horizontal reflection $\mathbb{Z}\ni n\leftrightarrow(-n)$, one can then see that there also exists some $\ell\in\overline{a,b}$ such that $\Delta r<0$ on $\overline{a+1,\ell}$ and $\Delta r\geqslant 0$ on $\overline{\ell+1,b}$. Hence, the conclusion $r\searrow\nearrow$ for this first-line case in Table 1 can actually be understood in a slightly stronger sense: that there are some k and ℓ in $\overline{a,b}\cup\{a,b\}$ such that r is (strictly) decreasing on $\overline{a,\ell}$, constant on $\overline{\ell,k}$, and increasing on $\overline{k,b}$. Moreover, identities (1) show that r equals a constant C on an interval $\overline{\ell,k}$ of integers only if $\rho=C$ on $\overline{\ell+1,k}$ (indeed, if $r=C$ on $\overline{\ell,k}$, then $\Delta r=0$ on $\overline{\ell+1,k}$, and so, $\rho_n=r_n=C$ for all $n\in\overline{\ell+1,k}$, because of the requirement that neither g nor Δg vanish at any point).

(ii) It follows that, if ρ is (strictly) increasing or decreasing, then the statement $r\searrow\nearrow$ can be taken to mean that there is some k in $\overline{a,b}\cup\{a,b\}$ such that r is decreasing on $\overline{a,k}$ and increasing on $\overline{k+1,b}$. For further details on this point, see the example at the end of this note.

(iii) Similar comments are valid for the other three cases given by the Table 1. $\diamondsuit$

Proposition 1.9 in [13], mentioned earlier, was in fact a corollary of a general result, Proposition 1.2 in [13], which also contains the special case when both f and g vanish at an endpoint of the interval (a,b) (as pointed out in Remark 1.5 in [13]. Here, we shall treat the discrete analogue of that special case separately, as follows.

Proposition 1. *Suppose that either (i) $b=\infty$, $f_\infty := \lim_{n\to\infty} f_n = 0$, and $g_\infty = 0$ or (ii) $a=-\infty$, $f_{-\infty} := \lim_{n\to-\infty} f_n = 0$, and $g_{-\infty}=0$. Suppose also that ρ is nondecreasing or nonincreasing or increasing or decreasing; then r is so, respectively.*

Proof. Consider the case when $b=\infty$, $f_\infty=0$, and $g_\infty=0$. Then the result follows immediately from the identity

$$g_n g_{n-1}\,\Delta r_n = \sum_{j=n}^{\infty} \Delta g_n\,\Delta g_j\,(\rho_j-\rho_n)$$

for all $n\in\overline{a+1,\infty}$. The other case can now be proved by the horizontal reflection; compare with the proof of Theorem 1. □

"Discrete" analogues of l'Hospital's rules for limits are easily stated and proved, and apparently well known. Let us present them here for the sake of completeness.

Proposition 2. *Suppose that $b=\infty$, and either (i) $|g_\infty|=\infty$ or (ii) $f_\infty=g_\infty=0$. Then $r_\infty=\rho_\infty$ provided that the latter limit, ρ_∞, exists.*

Proof. Let us first consider part (i), assuming that $|g_\infty|=\infty$. For $m<n$, one has $r_n = \dfrac{f_m}{g_n} + r_{m,n}\Big(1-\dfrac{g_m}{g_n}\Big)$, where $r_{m,n} := \dfrac{f_n-f_m}{g_n-g_m} = \sum\limits_{j=m+1}^{n}\rho_j\,|\Delta g_j| \Big/ \sum\limits_{j=m+1}^{n}|\Delta g_j|$, since Δg is assumed not to change sign. It follows that $r_{m,n}$ is between the minimum and maximum values of ρ over the interval $\overline{m+1,n}$. To complete the proof of part (i), it remains to let $n\to\infty$ and then $m\to\infty$.

Part (ii) is even simpler to prove, since here $r_m = r_{m,\infty}$ for all m. □

Note that condition $|f_\infty|=\infty$ is not required in part (i) of Proposition 2. Moreover, the above proof can be obviously adapted (using, say, the Mean-Value Theorem for the ratio corresponding to $r_{m,n}$) to l'Hospital's original "differentiable" case, even though the condition $|f_\infty|=\infty$ is traditionally included into the formulation of the corresponding l'Hospital rule for limits.

3. Illustrations

Let $p := (p_j\colon j\in\mathbb{Z})$ be a positive sequence. Then p is called log-convex on $\overline{a,b}$ if $q := \ln p$ is convex on $\overline{a,b}$, in the sense that Δq is nondecreasing on $\overline{a+1,b}$, which is equivalent to the condition that the ratio p_n/p_{n+1} be nonincreasing in $n\in\overline{a,b-1}$. The log-concavity of a sequence and the strict versions of log-convexity and log-concavity are defined similarly.

Corollary 1. *Suppose that p is log-convex or log-concave on $\overline{a,\infty}$, and $f_n := \sum_{j=n}^{\infty} p_j < \infty$ for all $n\in\mathbb{Z}$. Then f is, respectively, log-convex or log-concave on $\overline{a,\infty}$. More generally, the same conclusion holds for any natural k if $f = R^k p$,* *where $R^k p$ is given by the formula*

$$(R^k p)_n := \sum_{j=n}^{\infty} \binom{j-n+k-1}{j-n} p_j \quad \textit{for all } n \in \mathbb{Z}.$$

Proof. Let $g_n := f_{n+1}$ for all $n \in \mathbb{Z}$. Then the first part of Corollary 1 follows immediately from Proposition 1. In turn, this yields the second part of the corollary, because it is easy to see that $(R^k \colon k \in \overline{1,\infty}\,)$ is a semigroup of operators with $(R^1 p)_n = \sum_{j=n}^{\infty} p_j$ for all n; cf. Remark 5 in [11]. □

Corollary 1 is essentially well-known. For the "log-concave" part, see, for example, [11] (where k was allowed to be any positive real number) and, for the continuous counterpart in the case $k = 1$, [5] and [9]. The "log-convex" part can also be obtained from the well-known fact that any linear combination with positive coefficients of log-convex functions is log-convex, having also in mind that the log-convexity is preserved under the shift $n \mapsto n + 1$.

Corollary 2. *Suppose that p is log-concave on $\overline{0,\infty}$, and $f_n := \sum_{j=0}^{n} p_j$ for all $n \in \overline{0,\infty}$. Then f is* strictly *log-concave on $\overline{0,\infty}$. More generally, the same conclusion holds for any natural k if $f = L^k p$, where $L^k p$ is given by the formula*

$$(L^k p)_n := \sum_{j=0}^{n} \binom{n-j+k-1}{n-j} p_j \quad \textit{for all } n \in \overline{0,\infty}.$$

Proof. Let again $g_n := f_{n+1}$ for all $n \in \overline{0,\infty}$. Then the first part of Corollary 2 follows immediately from part (i) of Remark 2 (since $g_0 g_1 \,\Delta r_1 > p_1^2 - p_0\, p_2 \geqslant 0$, and so, $r_1 > r_0$). In turn, this yields the second part of the corollary, since $(L^k \colon k \in \overline{1,\infty}\,)$ is a semigroup with $(L^1 p)_n = \sum_{j=0}^{n} p_j$ for all n. (One can note that $(L^k p)_n = (T^{-1} R^k T p)_n$ for all $n \in \overline{0,\infty}$, all natural k, and all p such that $p = 0$ on $\overline{-\infty,-1}$, where $(Tp)_n := p_{-n}$ for all n.) □

However, the "log-convex" analogue of Corollary 2 does not hold. Indeed, if a sequence p is both log-convex and log-concave (that is, geometric) then, by Corollary 2, f is strictly log-concave and hence not log-convex.

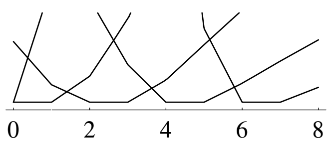
Let us conclude this paper with another illustration of presented results and methods; note the use of Remarks 1 and 2 and identities (1).

Example. Let sequences $f := f^{(\alpha)}$ and g on $\overline{0,\infty}$ be given by the formulas $f_n^{(\alpha)} := \alpha + \sum_{j=0}^{n} p_j$ and $g_n := \sum_{j=0}^{n} q_j$ for any $\alpha \geqslant 0$ and all $n \in \overline{0,\infty}$, where p and q are positive sequences such that $\rho = p/q$ is increasing on $\overline{1,\infty}$ and $r_0^{(0)} < r_1^{(0)}$, where in turn $r^{(\alpha)} := f^{(\alpha)}/g$. (For instance, one can take $p_j = j!$ and $q_j = (j/e)^j$ for all $j \in \overline{0,\infty}$, assuming that $0^0 = 1$.)

Then, by Theorem 1 (or, rather, by part (ii) of Remark 2; cf. Remark 1), the sequence $r^{(0)}$ is increasing on the entire interval $\overline{0,\infty}$. Moreover, for every $\alpha > 0$ there is some k_α in $\overline{0,\infty} \cup \{\infty\}$ such that $r^{(\alpha)}$ is decreasing on $\overline{0,k_\alpha}$ and increasing on $\overline{k_\alpha + 1,\infty}$.

The observed condition that $r^{(0)}$ is increasing on $\overline{0,\infty}$ implies $\Delta r^{(0)} > 0$ and hence, by (1), $\rho > r^{(0)}$ on $\overline{1,\infty}$. On the other hand, $r^{(\alpha)} = r^{(0)} + \alpha/g$. Therefore, for each $k \in \overline{1,\infty}$, the number $\alpha_k := (\rho_k - r_k^{(0)})\, g_k$ is positive and satisfies the equation $\rho_k = r_k^{(\alpha_k)}$, so that, by (1), $(\Delta r^{(\alpha_k)})_k = 0$; that is, $r_k^{(\alpha_k)} = r_{k-1}^{(\alpha_k)}$. Using again part (ii) of Remark 2, one sees that $r^{(\alpha_k)}$ is decreasing on $\overline{0, k-1}$, constant on $\{k-1, k\}$, and increasing on $\overline{k, \infty}$. It follows, in particular, that one cannot replace $\overline{k+1, b}$ in part (ii) of Remark 2 by $\overline{k, b}$ (keeping the rest unchanged).

In view of (1), observe also that $r_k^{(\alpha)} \geqslant r_{k-1}^{(\alpha)} \iff \rho_k \geqslant r_k^{(\alpha)} \iff \alpha \leqslant \alpha_k$, for all $\alpha \geqslant 0$, and these equivalences hold if all three non-strict inequalities here replaced by the corresponding strict ones. In particular, one has $\alpha \leqslant \alpha_k \implies r_k^{(\alpha)} \geqslant r_{k-1}^{(\alpha)} \implies r_{k+1}^{(\alpha)} > r_k^{(\alpha)} \implies \alpha < \alpha_{k+1}$; the second implication here follows again by part (ii) of Remark 2. Hence, α_k is increasing in k. Moreover, if $\alpha \in (\alpha_k, \alpha_{k+1})$ then $r_{k-1}^{(\alpha)} > r_k^{(\alpha)} < r_{k+1}^{(\alpha)}$, so that $r_n^{(\alpha)}$ is decreasing in $n \in \overline{1, k}$ and increasing in $n \in \overline{k, \infty}$; recall that here k was taken to be any natural number.



To illustrate, here are parts of the graphs of the linear interpolations of the sequences $n \mapsto r_n^{(\alpha_k)}/r_k^{(\alpha_k)} - 0.97$ for $k = 0, 1, 3, 5, 7$ where $\alpha_0 := 0$ and, as above, $p_j = j!$ and $q_j = (j/e)^j$ with $0^0 := 1$.

To visualize the idea of this example, one can imagine a tank with the solution of a liquid in water. Initially, at time $n = 0$, the amounts of the liquid and water are $f_0^{(\alpha)} = \alpha + p_0$ and $g_0 = q_0$, respectively, so that the initial relative concentration of the liquid (with respect to water) is $r_0^{(\alpha)} = (\alpha + p_0)/q_0$. Suppose that, at each of the time moments $n = 1, 2, \dots$, the liquid and water are added to the tank in the amounts of $\Delta f_n = p_n$ and $\Delta g_n = q_n$, respectively, so that the relative concentration of the liquid in the nth addition is $\rho_n = p_n/q_n$ and that in the tank at time n is $r_n^{(\alpha)}$. If α is large enough, then initially the relative concentration ρ of the liquid in what is added is less than the relative concentration $r^{(\alpha)}$ of the liquid in the tank, so that $r^{(\alpha)}$ will be decreasing in time. However, at least in the case when ρ is increasing to ∞, ρ will eventually overtake $r^{(\alpha)}$, and the latter will then be forever increasing. ♢

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