Chernoff's Inequality - A very elementary proof

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Abstract

We give a very simple proof of a strengthened version of Chernoff's Inequality. We derive the same conclusion from much weaker assumptions.

The theorem

Theorem. *Let $X_1, \ldots, X_n$ be indicator random variables, $0 < \beta < 1$ and $0 < k < \beta n$. Then*

$$\Pr\left(\sum_{i=1}^{n} X_i \geq \beta n\right) \leq \frac{1}{\binom{\beta n}{k}} \sum_{|S|=k} \Pr\left(\wedge_{i \in S}(X_i = 1)\right). \tag{1}$$

In particular, if

$$\Pr\left(\wedge_{i \in S}(X_i = 1)\right) \leq \alpha^k$$

for every S of size $k = \left(\frac{\beta-\alpha}{1-\alpha}\right) n$, where $0 < \alpha < \beta$ then

$$\Pr\left(\sum_{i=1}^{n} X_i \geq \beta n\right) \leq e^{-D(\beta || \alpha) n}.$$

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If one makes the stronger assumption that $\Pr(\wedge_{i\in S}(X_i = 1)) \le \alpha^{|S|}$ for *every* $S \subseteq \{1,\ldots,n\}$, this reduces to Theorem 1.1 from [1]. Under the even more restrictive assumption that the X_i are i.i.d., this reduces to the usual statement of Chernoff's Inequality.

Proof. Let $w_S = \Pr(X_i = 1 \Leftrightarrow i \in S)$. Then

$$\Pr(\wedge_{i\in S}(X_i = 1)) = \sum_{T:S\subseteq T} w_T$$

and

$$\Pr\left(\sum_{i=1}^{n} X_i \ge \beta n\right) = \sum_{T:|T|\ge\beta n} w_T.$$

Since $\binom{x}{k}$ is an increasing function of x and since $w_S \ge 0$, we have

$$\sum_{|T|\ge\beta n} w_T \le \frac{1}{\binom{\beta n}{k}} \sum_T \binom{|T|}{k} w_T = \frac{1}{\binom{\beta n}{k}} \sum_{|S|=k} \sum_{T:S\subseteq T} w_T = \frac{1}{\binom{\beta n}{k}} \sum_{|S|=k} \Pr(\wedge_{i\in S}(X_i = 1)),$$

as claimed. For the second part of the theorem note that if we assume that $\Pr(\wedge_{i\in S}(X_i = 1)) \le \alpha^k$ for every S of size $k = \left(\frac{\beta-\alpha}{1-\alpha}\right) n$ then the above inequality becomes

$$\Pr\left(\sum_{i=1}^{n} X_i \ge \beta n\right) \le \frac{\binom{n}{k}}{\binom{\beta n}{k}} \alpha^k \le e^{-D(\beta||\alpha)n}$$

where the last inequality follows from standard entropy estimates of binomials. □

It is easy to see that inequality 1 is tight. It holds with equality iff $\Pr(\sum X_i = \beta n \text{ or } \sum X_i < k) = 1$.

References

[1] Impagliazzo, R. and Kabanets, V., *Constructive proofs of concentration bounds*, Proceedings of APPROX-RANDOM, pages 617 - 631, 2010.