

# THE CHOW RINGS OF THE ALGEBRAIC GROUPS $E_6$ , $E_7$ , AND $E_8$

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ABSTRACT. We determine the Chow rings of the complex algebraic groups of the exceptional type  $E_6$ ,  $E_7$ , and  $E_8$ , giving the explicit generators represented by the pull-back images of Schubert varieties of the corresponding flag varieties. This is a continuation of the work of R. Marlin on the computation of the Chow rings of  $SO_n$ ,  $Spin_n$ ,  $G_2$ , and  $F_4$ . Our method is based on Schubert calculus of the corresponding flag varieties, which has its own interest.

## 1. INTRODUCTION

The problem of computing the Chow rings of complex algebraic groups dates back to the paper by Grothendieck [13], where he showed the Chow rings of  $SL_n$  and  $Sp_{2n}$  are trivial. Later R. Marlin computed the Chow rings of  $SO_n$ ,  $Spin_n$ ,  $G_2$ , and  $F_4$  ([16], [17]). The purpose of the present article is to give the explicit descriptions of the Chow rings for the remaining cases of  $E_6$ ,  $E_7$ , and  $E_8$ <sup>1</sup>.

Let  $G$  be a simply-connected simple algebraic group over the field of complex numbers  $\mathbb{C}$ ,  $B$  a fixed Borel subgroup of  $G$ , and  $H$  a maximal (algebraic) torus contained in  $B$ . The homogeneous space  $G/B$  is known to be a projective variety, called the *flag variety*. The well-known basis theorem [7] tells us that the Chow ring  $A(G/B)$  of  $G/B$  has a distinguished  $\mathbb{Z}$ -module basis which consists of *Schubert varieties*. Our approach is to describe  $A(G)$  via pull-back images of Schubert varieties through the quotient map  $p : G \rightarrow G/B$ . To be precise, the method proceeds as follows: Let

$$(1.1) \quad c_G : S(\hat{H}) \longrightarrow A(G/B)$$

be the characteristic homomorphism for  $G$ , where  $S(\hat{H})$  denotes the symmetric algebra over  $\mathbb{Z}$  of the character group  $\hat{H}$  of  $H$  ([13, (4.1)]). Then Grothendieck showed ([13, p.21, REMARQUES 2°], see also [6, §3] for the proof) that  $p^* : A(G/B) \rightarrow A(G)$  is surjective and its kernel is the ideal generated by  $c_G(\hat{H})$ , in other words, the divisor classes on  $G/B$ . Therefore the determination of  $A(G)$  reduces to that of  $A(G/B)$ . In principle, the ring structure of  $A(G/B)$  could be determined by the classical Chevalley formula [7], but in practice, it could not be applicable to the cases of higher rank groups. On the other hand, the Chow ring  $A(G/B)$  is isomorphic to the integral cohomology ring  $H^*(G/B; \mathbb{Z})$  via the cycle map which assigns to each Schubert variety its fundamental class ([13, §6], [12, Example 19.1.11]). Hence we work with the cohomology instead of the Chow ring, which allows us to access the computational facility of algebraic topology.

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<sup>1</sup>Recently, H. Duan and X. Zhao also determined the Chow rings of  $E_6$ ,  $E_7$ , and  $E_8$  by a different approach from ours ([11]).

To describe the cohomology ring  $H^*(G/B; \mathbb{Z})$ , we have two different ways, namely, the *Schubert presentation* and the *Borel presentation*. In the Schubert presentation, an additive basis for  $H^*(G/B; \mathbb{Z})$  is given by Schubert classes corresponding to the Schubert varieties ([1], [3], [7]). However, the multiplicative structure among Schubert classes is difficult to determine, and hence it does not seem to fit our purpose of computing  $A(G)$ . In fact, Marlin's computation of  $A(G)$  for  $G = \mathrm{SO}_n$ ,  $\mathrm{Spin}_n$ ,  $G_2$ , and  $F_4$  in [16], where he considered the Schubert presentation only, becomes unmanageable when applied to the remaining exceptional groups  $E_6$ ,  $E_7$ , and  $E_8$ . In [2], A. Borel gives another description for  $H^*(G/B; \mathbb{Z})$  in terms of the ring of invariants of the Weyl group action on  $S(\hat{H})$ , which we call the Borel presentation today. This presentation has an advantage that the ring structure of  $H^*(G/B; \mathbb{Z})$  is relatively easy to see. However, the generators in this presentation have little geometric meaning. There is a connection between these two presentations discovered by Bernstein-Gelfand-Gelfand [1] and Demazure [8]. More specifically, they introduced a series of operators called the *divided difference operators* acting on  $S(\hat{H})$ . Using these operators, we can express the ring generators of  $H^*(G/B; \mathbb{Z})$  obtained by the Borel presentation in terms of Schubert classes. Computational difficulties in the cases of the higher rank exceptional groups  $G = E_l$  ( $l = 6, 7, 8$ ) are overcome by an appropriate choice of the generators for  $H^*(E_l/B; \mathbb{Z})$  discovered by Toda [24] and Toda-Watanabe [25] (see §3.1). Once we obtain the correspondence between the Borel and Schubert presentations, the Chow rings  $A(E_l)$  are determined immediately using the result of Grothendieck mentioned earlier. In fact, the second author simplified Marlin's computation by making use of the same method as in this paper ([20]).

Then our main result is stated as follows:

**Theorem 1.1.** *For  $G = E_l$  ( $l = 6, 7, 8$ ), we denote by  $p : G \rightarrow G/B$  the natural projection, by  $p^* : A(G/B) \rightarrow A(G)$  the induced pull-back homomorphism, by  $w_0$  the longest element of the Weyl group of  $G$ , and by  $s_i$  the reflection corresponding to the simple root  $\alpha_i$  ( $1 \leq i \leq l$ ) (for the notation, see §2.1 and §3.1).*

(1) *The Chow ring of  $E_6$  is given by*

$$A(E_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3),$$

where  $X_3$  and  $X_4$  are the images under  $p^*$  of the elements of  $A(E_6/B)$  defined by the Schubert varieties  $X_{w_0 s_5 s_4 s_2}$  and  $X_{w_0 s_6 s_5 s_4 s_2}$  respectively.

(2) *The Chow ring of  $E_7$  is given by*

$$A(E_7) = \mathbb{Z}[X_3, X_4, X_5, X_9]/(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2),$$

where  $X_3, X_4, X_5$ , and  $X_9$  are the images under  $p^*$  of the elements of  $A(E_7/B)$  defined by the Schubert varieties  $X_{w_0 s_5 s_4 s_2}$ ,  $X_{w_0 s_6 s_5 s_4 s_2}$ ,  $X_{w_0 s_7 s_6 s_5 s_4 s_2}$ , and  $X_{w_0 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2}$  respectively.

(3) *The Chow ring of  $E_8$  is given by*

$$A(E_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \\ / (2X_3, 3X_4, 2X_5, 5X_6, 2X_9, 3X_{10}, X_4^3, 2X_{15}, X_9^2, X_5^4, X_3^8, X_6^5, X_{10}^3, X_{15}^2),$$

where  $X_3, X_4, X_5, X_6, X_9, X_{10}$ , and  $X_{15}$  are the images under  $p^*$  of the elements of  $A(E_8/B)$  defined by the Schubert varieties  $X_{w_0 s_5 s_4 s_2}$ ,  $X_{w_0 s_6 s_5 s_4 s_2}$ ,  $X_{w_0 s_7 s_6 s_5 s_4 s_2}$ ,  $X_{w_0 s_1 s_3 s_6 s_5 s_4 s_2}$ ,  $X_{w_0 s_1 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2}$ ,  $X_{w_0 s_1 s_6 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2}$ , and  $X_{w_0 s_1 s_3 s_4 s_2 s_7 s_6 s_5 s_4 s_3 s_8 s_7 s_6 s_5 s_4 s_2}$  respectively.

Combining Theorem 1.1 with the results from [16], [17], we have the following table:

$G$	$A(G)$	generators
$SL_n$	$\mathbb{Z}$	
$Sp_{2n}$	$\mathbb{Z}$	
$SO_{2n}$	$\frac{\mathbb{Z}[X_1, X_3, X_5, \dots, X_{2[\frac{n}{2}-1]}]}{(2X_i, X_i^{p_i})}$ , $p_i = 2^{\lceil \log_2 \frac{n-1}{i} \rceil + 1}$	$[n], [n-2, n], \dots, [1, \dots, n-3, n-2, n]$
$Spin_{2n}$	$\frac{\mathbb{Z}[X_3, X_5, \dots, X_{2[\frac{n}{2}-1]}]}{(2X_i, X_i^{p_i})}$ , $p_i = 2^{\lceil \log_2 \frac{n-1}{i} \rceil + 1}$	$[n-2, n], \dots, [1, \dots, n-3, n-2, n]$
$SO_{2n+1}$	$\frac{\mathbb{Z}[X_1, X_3, X_5, \dots, X_{2[\frac{n+1}{2}-1]}]}{(2X_i, X_i^{p_i})}$ , $p_i = 2^{\lceil \log_2 \frac{n}{i} \rceil + 1}$	$[n], [n-1, n], \dots, [1, \dots, n]$
$Spin_{2n+1}$	$\frac{\mathbb{Z}[X_3, X_5, \dots, X_{2[\frac{n+1}{2}-1]}]}{(2X_i, X_i^{p_i})}$ , $p_i = 2^{\lceil \log_2 \frac{n}{i} \rceil + 1}$	$[n-1, n], \dots, [1, \dots, n]$
$G_2$	$\frac{\mathbb{Z}[X_3]}{(2X_3, X_3^2)}$	[121]
$F_4$	$\frac{\mathbb{Z}[X_3, X_4]}{(2X_3, 3X_4, X_3^2, X_4^3)}$	[123], [1234]
$E_6$	$\frac{\mathbb{Z}[X_3, X_4]}{(2X_3, 3X_4, X_3^2, X_4^3)}$	[542], [6542]
$E_7$	$\frac{\mathbb{Z}[X_3, X_4, X_5, X_9]}{(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2)}$	[542], [6542], [76542], [654376542]
$E_8$	$\frac{\mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}]}{(2X_3, 3X_4, 2X_5, 5X_6, 2X_9, 3X_{10}, X_4^3, 2X_{15}, X_9^2, X_5^4, X_3^8, X_6^5, X_{10}^3, X_{15}^2)}$	[542], [6542], [76542], [136542], [154376542], [1654376542], [134276543876542]

Here a Schubert variety  $X_{w_0 s_{i_1} s_{i_2} \dots}$  is abbreviated to  $[i_1 i_2 \dots]$  in the third column of the table.

**Remark 1.2.** We note that the mod  $p$  Chow rings  $A(G; \mathbb{F}_p) := A(G) \otimes_{\mathbb{Z}} \mathbb{F}_p$  were studied by V. Kac in [15], where he showed that  $A(G; \mathbb{F}_p)$  is isomorphic to  $\mathbb{F}_p$  for a non-torsion prime  $p$  of  $G$ , whereas for a torsion prime  $p$ ,  $A(G; \mathbb{F}_p)$  is the “polynomial part” of  $H^*(G; \mathbb{F}_p)$ .

The organization of this paper is as follows: In §2, we review basic facts on Schubert calculus and fix our notations. In §3, we recall the Borel presentation of the integral cohomology rings of  $H^*(G/B; \mathbb{Z})$  for  $G = E_l$  ( $l = 6, 7, 8$ ) and convert those results to the Schubert presentation using the divided difference operators. Finally in §4, we give descriptions of the Chow rings  $A(G)$  for  $G = E_l$  ( $l = 6, 7, 8$ ).

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be obtained by using the technique of [27]. We also thank Haibao Duan and Xuezi Zhao for discussion about the cohomology of flag varieties. Finally, we thank Mamoru Mimura for giving us various suggestions.

## 2. COHOMOLOGY OF FLAG VARIETIES

In the area of Schubert calculus, we frequently use two ways of describing the integral cohomology of flag varieties; the Schubert presentation and the Borel presentation. The former gives a geometric basis for the cohomology whose product structure is hard to know, while the latter allows us purely algebraic treatment of the cohomology. For our purpose, we need both presentations.

2.1. We begin with a brief review of the Schubert presentation of the integral cohomology rings of flag varieties ([1], [3], [7]).

Let us denote by  $\Delta$  the root system of  $G$  relative to  $H$ , by  $\Pi$  the system of simple roots, by  $N_G(H)$  the normalizer of  $H$  in  $G$ . Then the Weyl group  $W$  of  $G$  is defined by  $N_G(H)/H$ . Denote by  $s_\alpha$  the simple reflection, i.e., the reflection corresponding to the simple root  $\alpha \in \Pi$ , and by  $S = \{s_\alpha \mid \alpha \in \Pi\}$  the set of simple reflections. Then it is known that the Weyl group  $W$  of  $G$  is generated by the simple reflections. Denote by  $l(w)$  the length of an element  $w \in W$  with respect to  $S$ .

As is well known (see for example [4, 14.12]),  $G$  has the Bruhat decomposition

$$G = \bigsqcup_{w \in W} B\dot{w}B,$$

where  $\dot{w}$  denotes any representative of  $w \in W$ . It induces a cell decomposition

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B,$$

where  $X_w^\circ = B\dot{w}B/B \cong \mathbb{C}^{l(w)}$  is called the *Schubert cell*. The *Schubert variety*  $X_w$  is defined to be the closure  $\overline{X_w^\circ}$  of  $X_w^\circ$ . The fundamental class  $[X_w]$  of  $X_w$  lies in  $H_{2l(w)}(G/B; \mathbb{Z})$ . We define a cohomology class  $Z_w \in H^{2l(w)}(G/B; \mathbb{Z})$  as the Poincaré dual of  $[X_{w_0 w}]$ , where  $w_0$  is the longest element of  $W$ . We call  $Z_w$  the *Schubert class* corresponding to  $w \in W$ . The Schubert classes  $\{Z_w\}_{w \in W}$  form an additive basis for the free  $\mathbb{Z}$ -module  $H^*(G/B; \mathbb{Z})$ ; we refer to  $\{Z_w\}_{w \in W}$  as the *Schubert basis*.

The product of two Schubert classes can be expressed as a  $\mathbb{Z}$ -linear combination of the Schubert basis. In order to complete the multiplicative structure of  $H^*(G/B; \mathbb{Z})$ , we have to compute the *structure constants*  $a_{u,v}^w$  for  $u, v, w \in W$ . These integers  $a_{u,v}^w$  are defined by the following equation:

$$Z_u \cdot Z_v = \sum_{\substack{w \in W, \\ l(u) + l(v) = l(w)}} a_{u,v}^w Z_w.$$

Computing the structure constants  $a_{u,v}^w$  becomes a hard task in general. Indeed, one of the main problems of *Schubert calculus* is to give a formula for structure constants (see for example [23]).

**Remark 2.1.** In [9], H. Duan find an effective algorithm for computing the structure constants. Based on it, he and X. Zhao gave descriptions of the integral cohomology rings of flag varieties in terms of Schubert classes ([10]).

To a subset  $I_P$  of simple roots  $\Pi$ , we associate the parabolic subgroup  $P (\supset B)$  of  $G$  whose Weyl group<sup>2</sup>  $W_P$  is the subgroup of  $W$  generated by  $\{s_\alpha \mid \alpha \in I_P\}$ . We can take the explicit minimal coset representatives  $W^P$  of  $W/W_P$  (see [14, §1.10]) as

$$W^P = \{w \in W \mid l(ws_\alpha) = l(w) + 1 \text{ for all } \alpha \in I_P\}.$$

For  $w \in W^P$ , the Schubert class  $Z_w \in H^{2l(w)}(G/P; \mathbb{Z})$  of the partial flag variety  $G/P$  is defined as the class dual to  $[\overline{Bw_0wP/P}]$ . Similarly to the case of  $G/B$ , the Schubert classes  $\{Z_w\}_{w \in W^P}$  form an additive basis for the free  $\mathbb{Z}$ -module  $H^*(G/P; \mathbb{Z})$ . Consider the following fibration:

$$(2.1) \quad P/B \xrightarrow{i} G/B \xrightarrow{\pi} G/P.$$

Since the fibre  $P/B$  and the base  $G/P$  have even dimensional cohomology, the Serre spectral sequence with integer coefficients for the fibration (2.1) collapses at the  $E_2$ -term. Therefore the projection  $\pi$  induces an inclusion  $\pi^* : H^*(G/P; \mathbb{Z}) \hookrightarrow H^*(G/B; \mathbb{Z})$  which is compatible with the inclusion  $W^P \subset W$ , the inclusion  $i$  induces a surjection  $i^* : H^*(G/B; \mathbb{Z}) \rightarrow H^*(P/B; \mathbb{Z})$ , and  $\text{Ker } i^*$  is an ideal of  $H^*(G/B; \mathbb{Z})$  generated by  $\pi^*H^+(G/P; \mathbb{Z})$ , where  $H^+(G/P; \mathbb{Z}) = \bigoplus_{i>0} H^i(G/P; \mathbb{Z})$ . Furthermore, it is known that  $\text{Im } \pi^* = H^*(G/P; \mathbb{Z}) \subset H^*(G/B; \mathbb{Z})$  coincides with the set of  $W_P$ -invariant elements of  $H^*(G/B; \mathbb{Z})$  ([1, Theorem 5.5]). From this, we can describe the ring structure of  $H^*(G/B; \mathbb{Z})$  in terms of  $H^*(G/P; \mathbb{Z})$  and  $H^*(P/B; \mathbb{Z})$  (cf. [25, Lemma 1,1], [16, §3], [10, §2.8]). Certain choice of  $P$  reduces the computation of  $H^*(G/B; \mathbb{Z})$  greatly via the ‘‘splitting’’ above. Especially for our purpose of determining the Chow ring  $A(G) \cong H^*(G/B; \mathbb{Z}) / (H^2(G/B; \mathbb{Z}))$ , we will choose  $P$  so that  $H^*(P/B; \mathbb{Z})$  is generated by degree two elements (see §3.1).

2.2. In this subsection, we review the Borel presentation of the cohomology of flag varieties ([2]).

Let  $K$  be a maximal compact subgroup of  $G$  and  $T = K \cap H$  a compact maximal torus of  $K$ . Then the inclusion  $K \hookrightarrow G$  induces a diffeomorphism  $K/T \cong G/B$ . The inclusion  $T \hookrightarrow K$  induces a fibration

$$K/T \xrightarrow{\iota} BT \xrightarrow{\rho} BK,$$

where  $BT$  (resp.  $BK$ ) denotes the classifying space of  $T$  (resp.  $K$ ). The induced homomorphism in cohomology

$$(2.2) \quad c = \iota^* : H^*(BT; \mathbb{Z}) \longrightarrow H^*(K/T; \mathbb{Z})$$

is called *Borel’s characteristic homomorphism*<sup>3</sup>. The Weyl group  $W$  of  $K$  acts naturally on  $T$ , hence on  $H^2(BT; \mathbb{Z})$ . This action of  $W$  extends to the whole  $H^*(BT; \mathbb{Z})$  and also to  $H^*(BT; \mathbb{F}) = H^*(BT; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$ , where  $\mathbb{F}$  is any field. Then one of Borel’s results can be stated as follows:

**Theorem 2.2** (Borel [2]). *Let  $\mathbb{F}$  be a field of characteristic zero. Then Borel’s characteristic homomorphism induces an isomorphism*

$$\bar{c} : H^*(BT; \mathbb{F}) / (H^+(BT; \mathbb{F})^W) \longrightarrow H^*(K/T; \mathbb{F}),$$

where  $(H^+(BT; \mathbb{F})^W)$  is the ideal of  $H^*(BT; \mathbb{F})$  generated by the  $W$ -invariants of positive degrees.

<sup>2</sup>More precisely,  $W_P$  is the Weyl group of the reductive part of  $P$ .

<sup>3</sup>It is not difficult to see that the symmetric algebra  $S(\hat{H})$  is isomorphic to  $H^*(BT; \mathbb{Z})$ , and Borel’s characteristic homomorphism is identified with the characteristic homomorphism  $c_G : S(\hat{H}) \longrightarrow A(G/B)$ .

In particular, the rational cohomology ring  $H^*(K/T; \mathbb{Q})$  is easy to describe. For the classical types, the rings of invariants of the Weyl groups with rational coefficients are well-known. For the exceptional types, see for example [18]. In order to determine the integral cohomology ring  $H^*(K/T; \mathbb{Z})$ , we need further considerations. In [24], Toda established a method to describe the integral cohomology ring  $H^*(K/T; \mathbb{Z})$  by a minimal system of generators and relations, from the mod  $p$  cohomology rings  $H^*(K; \mathbb{Z}/p\mathbb{Z})$  for all primes  $p$  and the rational cohomology ring  $H^*(K/T; \mathbb{Q})$ . Along the line of Toda's method, the integral cohomology rings of flag varieties for  $G = E_l$  ( $l = 6, 7, 8$ ) have been concretely determined by Toda-Watanabe [25] and the second author [19], [22]. We emphasize that the ring structure is completely determined there. Throughout the rest of the paper, we fix maximal compact subgroups of  $E_l$  ( $l = 6, 7, 8$ ) and denote them by  $E_l$  ( $l = 6, 7, 8$ ) respectively.

### 3. INTEGRAL COHOMOLOGY RING OF $E_l/T$ ( $l = 6, 7, 8$ )

In this section, we focus on the cases of the exceptional groups  $E_l$  ( $l = 6, 7, 8$ ).

3.1. First we introduce a handy set of generators of  $H^2(BT; \mathbb{Z})$  for  $K = E_l$  ( $l = 6, 7, 8$ ). Following [5], we take the simple roots  $\Pi = \{\alpha_i\}_{1 \leq i \leq l}$ . For instance, the Dynkin diagram of  $E_8$  is given as follows:

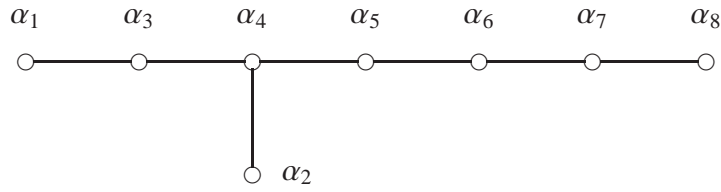


FIGURE 1. Dynkin diagram of  $E_8$

We denote by  $\{\omega_i\}_{1 \leq i \leq l}$  the corresponding fundamental weights. As is customary, we regard roots and weights as elements of  $H^2(BT; \mathbb{Z})$ . Let  $s_i$  ( $1 \leq i \leq l$ ) denote the reflection corresponding to the simple root  $\alpha_i$  ( $1 \leq i \leq l$ ). Then the Weyl group  $W(E_l)$  of  $E_l$  is generated by  $s_i$  ( $1 \leq i \leq l$ ).

Following [25, §4], [26, §1], and [21, §2], we set

$$(3.1) \quad \begin{aligned} t_1 &= -\omega_1 + \omega_2, \quad t_2 = \omega_1 + \omega_2 - \omega_3, \quad t_3 = \omega_2 + \omega_3 - \omega_4, \\ t_i &= \omega_i - \omega_{i+1} \quad (4 \leq i < l), \quad t_l = \omega_l, \quad t = \omega_2, \quad c_i = e_i(t_1, \dots, t_l) \quad (1 \leq i \leq l), \end{aligned}$$

where  $e_i(t_1, \dots, t_l)$  denotes the  $i$ -th elementary symmetric polynomial in the variables  $t_1, \dots, t_l$ . In this setting, we have

$$\begin{aligned} H^*(BT; \mathbb{Z}) &= \mathbb{Z}[\omega_1, \omega_2, \dots, \omega_l] \\ &= \mathbb{Z}[t_1, t_2, \dots, t_l, t]/(c_1 - 3t). \end{aligned}$$

Since we assume the simply-connectedness of the groups, Borel's characteristic homomorphism restricted in degree 2 is an isomorphism  $H^2(BT; \mathbb{Z}) \cong H^2(E_l/T; \mathbb{Z})$ . Under this isomorphism, we denote the images of  $t_i$  ( $1 \leq i \leq l$ ) and  $t$  by the same symbols. Thus  $H^2(E_l/T; \mathbb{Z})$  is a free  $\mathbb{Z}$ -module generated by  $t_i$  ( $1 \leq i \leq l$ ) and  $t$  with a relation  $c_1 = 3t$ .

As for the action of the Weyl group on  $H^2(BT; \mathbb{Z})$  ( $\cong H^2(E_l/T; \mathbb{Z})$ ), one can easily see that

- (i)  $s_i$  ( $i \neq 2$ ) acts on  $\{t_i\}_{1 \leq i \leq l}$  as the transpositions and trivially on  $t$ .

(ii) The action of  $s_2$  on  $\{t_i\}_{1 \leq i \leq l}$  and  $t$  is given by

$$\begin{aligned} s_2(t_i) &= \begin{cases} t - t_1 - t_2 - t_3 + t_i & \text{for } 1 \leq i \leq 3, \\ t_i & \text{for } 4 \leq i \leq l, \end{cases} \\ s_2(t) &= 2t - t_1 - t_2 - t_3. \end{aligned}$$

Let  $P_2$  be the centralizer of a one dimensional torus determined by  $\alpha_i = 0$  ( $i \neq 2$ )<sup>4</sup>. Then the Weyl group  $W_{P_2}$  of  $P_2$  is generated by  $s_i$  ( $i \neq 2$ ). Notice that the elements  $t_i$  ( $1 \leq i \leq l$ ) and  $t$  in  $H^2(E_l/T; \mathbb{Z})$  belong to an orbit under the action of  $W_{P_2}$ . This technical choice of a “maximal parabolic subgroup” of  $E_l$  is made by paying attention to the symmetry seen from the Dynkin diagram of type  $E$  exceptional groups (see Figure 1); Disregarding the root  $\alpha_2$ , the Dynkin diagram of type  $E_l$  is the same as that of type  $A_{l-1}$ . Therefore, by (2.1), we have the following bundle:

$$SU(l)/T' \cong P_2/T \xrightarrow{i} E_l/T \xrightarrow{\pi} E_l/P_2,$$

where  $T'$  denotes the standard maximal torus of  $SU(l)$ . As mentioned in §2.1, the Serre spectral sequence with integer coefficients for the above fibration collapses at the  $E_2$ -term. From this and the fact that  $H^*(SU(l)/T'; \mathbb{Z})$  is generated by degree two classes, it is not hard to see that  $H^*(E_l/T; \mathbb{Z})$  is generated multiplicatively by  $H^2(E_l/T; \mathbb{Z})$  and  $\text{Im } \pi^* = H^*(E_l/P_2; \mathbb{Z})$ . Especially, ring generators of degrees greater than two can be chosen so that they lie in  $H^*(E_l/P_2; \mathbb{Z}) \subset H^*(E_l/T; \mathbb{Z})$  (see Theorems 3.1, 3.3, and 3.5 below). Hence any class in  $H^*(E_l/T; \mathbb{Z}) / (H^2(E_l/T; \mathbb{Z}))$  ( $\cong A(E_l)$ ) can be represented by an element of  $H^*(E_l/P_2; \mathbb{Z}) \subset H^*(E_l/T; \mathbb{Z})$ . This is the crucial point for the computation in the sequel.

3.2. Using the basis for  $H^2(E_l/T; \mathbb{Z})$  described in the previous subsection, we give concrete descriptions of the integral cohomology rings of  $E_l/T$ , following the results of [25], [19], and [22].

**Theorem 3.1** (Toda-Watanabe [25], Theorem B). *The integral cohomology ring of  $E_6/T$  is*

$$H^*(E_6/T; \mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_6, t, \gamma_3, \gamma_4] / (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{12}),$$

where

$$\begin{aligned} \rho_1 &= c_1 - 3t, \quad \rho_2 = c_2 - 4t^2, \quad \rho_3 = c_3 - 2\gamma_3, \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4, \\ \rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3, \quad \rho_6 = \gamma_3^2 + 2c_6 - 3t^2\gamma_4 + t^6, \\ \rho_8 &= 3\gamma_4^2 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \quad \rho_9 = 2c_6\gamma_3 - 3t^3c_6, \\ \rho_{12} &= 3c_6^2 - 2\gamma_4^3 + 6t\gamma_3\gamma_4^2 + 3t^2c_6\gamma_4 + 5t^3c_6\gamma_3 - 15t^4\gamma_4^2 - 10t^6c_6 + 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12}. \end{aligned}$$

**Remark 3.2.** This description is slightly different from the one in [25]. Toda-Watanabe [25] make use of the integral cohomology ring of the Hermitian symmetric space  $EIII = E_6/T^1 \cdot Spin(10)$  to determine the higher relations  $\rho_9$  and  $\rho_{12}$ . For our purpose, it is convenient to have a description directly related to the ring of invariants of the Weyl group  $W(E_6)$ . We will give the outline of the computation in §5.1 for convenience of the reader.

**Theorem 3.3** (Nakagawa [19], Theorem 5.9). *The integral cohomology ring of  $E_7/T$  is*

$$\begin{aligned} H^*(E_7/T; \mathbb{Z}) &= \mathbb{Z}[t_1, \dots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ &\quad / (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18}), \end{aligned}$$

<sup>4</sup>The subgroup  $P_2$  corresponds to the maximal parabolic subgroup of  $E_l$  associated to the subset  $\Pi \setminus \{\alpha_2\}$ .

where

$$\begin{aligned}
\rho_1 &= c_1 - 3t, \quad \rho_2 = c_2 - 4t^2, \quad \rho_3 = c_3 - 2\gamma_3, \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4, \\
\rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5, \quad \rho_6 = \gamma_3^2 + 2c_6 - 2t\gamma_5 - 3t^2\gamma_4 + t^6, \\
\rho_8 &= 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) - 9t^2c_6 + 12t^3\gamma_5 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8, \\
\rho_9 &= 2c_6\gamma_3 + t^2c_7 - 3t^3c_6 - 2\gamma_9, \quad \rho_{10} = \gamma_5^2 - 2c_7\gamma_3 + 3t^3c_7, \\
\rho_{12} &= 3c_6^2 - 2\gamma_4^3 - 2c_7\gamma_5 + 2\gamma_3\gamma_4\gamma_5 + t(4c_7\gamma_4 - 2c_6\gamma_5 + 6\gamma_3\gamma_4^2) + t^2(-3c_7\gamma_3 + 3c_6\gamma_4) \\
&\quad + t^3(-12\gamma_4\gamma_5 + 5c_6\gamma_3) + t^4(-2\gamma_3\gamma_5 - 15\gamma_4^2) - 10t^6c_6 + 12t^7\gamma_5 + 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12}, \\
\rho_{14} &= c_7^2 + 6c_7\gamma_3\gamma_4 - 2c_6\gamma_3\gamma_5 - t^2c_7\gamma_5 + t^3(-9c_7\gamma_4 + 3c_6\gamma_5) - 6t^4c_7\gamma_3 + 9t^7c_7, \\
\rho_{18} &= -\gamma_9^2 + 2c_6c_7\gamma_5 + 6c_7\gamma_3\gamma_4^2 - 2c_7^2\gamma_4 - 2c_6\gamma_3\gamma_4\gamma_5 + 2c_6\gamma_3\gamma_9 + t(-6c_7^2\gamma_3 + 24c_6c_7\gamma_4) \\
&\quad + t^2(-25c_7\gamma_4\gamma_5 + c_7\gamma_9 - 18c_6c_7\gamma_3) + t^3(-45c_7\gamma_4^2 + 20c_7\gamma_3\gamma_5 + 3c_6\gamma_4\gamma_5 - 3c_6\gamma_9) \\
&\quad + t^4(11c_7^2 + 2c_6\gamma_3\gamma_5 + 48c_7\gamma_3\gamma_4) + 51t^5c_6c_7 - 53t^6c_7\gamma_5 + t^7(-69c_7\gamma_4 - 3c_6\gamma_5) \\
&\quad + 16t^8c_7\gamma_3 + 15t^{11}c_7.
\end{aligned}$$

**Remark 3.4.** Similarly to the case of  $E_6$ , this description is slightly different from the one in [19]. The second author [19] make use of the integral cohomology ring of the homogeneous space  $E_7/T^1 \cdot Spin(12)$  to determine the higher relations  $\rho_{12}$ ,  $\rho_{14}$ , and  $\rho_{18}$ . Here we give a description directly related to the ring of invariants of the Weyl group  $W(E_7)$  (see also §5.1).

In a similar fashion, we can give a Borel presentation of  $H^*(E_8/T; \mathbb{Z})$ .

**Theorem 3.5** ([22], Theorem 3.4). *The integral cohomology ring of  $E_8/T$  is*

$$\begin{aligned}
H^*(E_8/T; \mathbb{Z}) &= \mathbb{Z}[t_1, \dots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}] \\
&\quad /(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{15}, \rho_{18}, \rho_{20}, \rho_{24}, \rho_{30}),
\end{aligned}$$

where

$$\begin{aligned}
\rho_1 &= c_1 - 3t, \quad \rho_2 = c_2 - 4t^2, \quad \rho_3 = c_3 - 2\gamma_3, \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4, \\
\rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5, \quad \rho_6 = c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\gamma_6, \\
\rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\gamma_6) + 3t^3\gamma_5 + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8, \\
\rho_9 &= 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6 - 2\gamma_9, \quad \rho_{10} = \gamma_5^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 - 3\gamma_{10}, \\
\rho_{12} &= 15\gamma_6^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2\gamma_3^4 + 10\gamma_3^2\gamma_6 - 3c_8\gamma_4 - 2\gamma_4^3 + t(c_8\gamma_3 - 2\gamma_3^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3\gamma_4^2) \\
&\quad + t^2(3\gamma_{10} - 25\gamma_4\gamma_6 - c_7\gamma_3 - 16\gamma_3^2\gamma_4) + t^3(25\gamma_3\gamma_6 - 3\gamma_4\gamma_5 + 10\gamma_3^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5\gamma_4^2) \\
&\quad + t^5(-3c_7 - 5\gamma_3\gamma_4) + 4t^6\gamma_3^2 - 7t^8\gamma_4 + 4t^9\gamma_3, \\
\rho_{14} &= c_7^2 - 3c_8\gamma_6 + 6\gamma_4\gamma_{10} - 4c_8\gamma_3^2 + 6c_7\gamma_3\gamma_4 - 6\gamma_3^2\gamma_4^2 - 12\gamma_4^2\gamma_6 - 2\gamma_3\gamma_5\gamma_6 \\
&\quad + t(24\gamma_3\gamma_4\gamma_6 - 8c_7\gamma_3^2 - 8c_7\gamma_6 + 4c_8\gamma_5 - 6\gamma_3\gamma_{10} + 12\gamma_3^3\gamma_4) \\
&\quad + t^2(-2\gamma_3\gamma_4\gamma_5 + 6\gamma_4^3 + 2\gamma_3^2\gamma_6 + 20\gamma_6^2 - 4\gamma_3^4 - c_7\gamma_5) + t^3(-12\gamma_3\gamma_4^2 + 8c_8\gamma_3 - 5c_7\gamma_4 + 3\gamma_5\gamma_6) \\
&\quad + t^4(3\gamma_{10} - 26\gamma_4\gamma_6 + 6c_7\gamma_3 - 4\gamma_3^2\gamma_4) + t^5(24\gamma_3\gamma_6 + 3\gamma_4\gamma_5 + 12\gamma_3^3) + t^6(-6c_8 + 2\gamma_4^2) \\
&\quad - 4t^7c_7 + t^8(6\gamma_6 - 6\gamma_3^2) - 6t^{10}\gamma_4 + 12t^{11}\gamma_3 - 2t^{14}, \\
\rho_{15} &= (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2(\gamma_3^2 + c_6)(\gamma_9 - c_6\gamma_3) - 2\gamma_{15},
\end{aligned}$$



$$\begin{aligned}
\rho_{18} = & \gamma_9^2 - 9c_8\gamma_{10} - 6\gamma_4^2\gamma_{10} - 4\gamma_3^3\gamma_9 - 10\gamma_3\gamma_6\gamma_9 + 2\gamma_3\gamma_5\gamma_{10} - 2\gamma_3\gamma_4\gamma_5\gamma_6 - 6c_7\gamma_3\gamma_4^2 + 3c_8\gamma_4\gamma_6 \\
& + c_8\gamma_3^2\gamma_4 + 6\gamma_3^2\gamma_4^3 + 12\gamma_4^3\gamma_6 + 2c_7^2\gamma_4 + 2c_7\gamma_3^2\gamma_5 - 2\gamma_3^3\gamma_4\gamma_5 + 2c_7\gamma_5\gamma_6 + 4\gamma_3^6 - 10\gamma_6^3 + 18\gamma_3^4\gamma_6 \\
& + 15\gamma_3^2\gamma_6^2 - 9c_7c_8\gamma_3 \\
& + t(-2\gamma_3\gamma_5\gamma_9 - 24c_7\gamma_4\gamma_6 + 8c_8\gamma_4\gamma_5 + 4c_7\gamma_3^2\gamma_4 + 4c_7\gamma_{10} - c_8\gamma_9 + 2c_7^2\gamma_3 + 4c_8\gamma_3\gamma_6 + 12\gamma_3\gamma_4\gamma_{10} \\
& - 36\gamma_3\gamma_4^2\gamma_6 + 12\gamma_3^2\gamma_5\gamma_6 + c_8\gamma_3^3 + 6\gamma_3^4\gamma_5 - 18\gamma_3^3\gamma_4^2) \\
& + t^2(24\gamma_3^4\gamma_4 - 2c_8^2 - c_7\gamma_9 - 11\gamma_3^2\gamma_{10} + 2\gamma_3\gamma_4\gamma_9 - 2c_8\gamma_3\gamma_5 + 16c_7\gamma_3\gamma_6 - 3c_7\gamma_4\gamma_5 + 75\gamma_4\gamma_6^2 \\
& - 6\gamma_4^4 - 9c_8\gamma_4^2 + 81\gamma_3^2\gamma_4\gamma_6 - 13\gamma_6\gamma_{10} + 4\gamma_3\gamma_4^2\gamma_5 - c_7\gamma_3^3) \\
& + t^3(-3\gamma_5\gamma_{10} - 150\gamma_3\gamma_6^2 - 135\gamma_3^3\gamma_6 + 6\gamma_3^2\gamma_9 - 2c_7\gamma_3\gamma_5 + 21c_7\gamma_4^2 + 15c_7c_8 + 3\gamma_4\gamma_5\gamma_6 - 3\gamma_3^2\gamma_4\gamma_5 \\
& + 18\gamma_3\gamma_4^3 + 15\gamma_6\gamma_9 + 14c_8\gamma_3\gamma_4 - 30\gamma_3^5) \\
& + t^4(-13c_8\gamma_6 + 2\gamma_4\gamma_{10} - 5c_7^2 - 33\gamma_3^2\gamma_4^2 + 3\gamma_5\gamma_9 - 28\gamma_3\gamma_5\gamma_6 - 45\gamma_4^2\gamma_6 - 41c_7\gamma_3\gamma_4 - 13\gamma_3^3\gamma_5 \\
& - 9c_8\gamma_3^2) \\
& + t^5(3c_7\gamma_6 - 6\gamma_4^2\gamma_5 + 23c_7\gamma_3^2 + 105\gamma_3\gamma_4\gamma_6 - 6c_8\gamma_5 - 3\gamma_4\gamma_9 + 45\gamma_3^3\gamma_4) \\
& + t^6(11\gamma_4^3 - 4\gamma_3\gamma_9 + 4c_7\gamma_5 + 9\gamma_3\gamma_4\gamma_5 + 12\gamma_3^4 + 66\gamma_3^2\gamma_6 + 75\gamma_6^2 + 2c_8\gamma_4) \\
& + t^7(-33\gamma_3\gamma_4^2 + 12\gamma_3^2\gamma_5 + 15\gamma_5\gamma_6) + t^8(-4\gamma_{10} + 21\gamma_3^2\gamma_4 - 5c_7\gamma_3 - 3\gamma_4\gamma_6) \\
& + t^9(6\gamma_9 - 42\gamma_3^3 - 99\gamma_3\gamma_6) + t^{10}(-4c_8 - 6\gamma_4^2 - 13\gamma_3\gamma_5) + t^{11}(3c_7 + 27\gamma_3\gamma_4) + t^{12}(60\gamma_6 + 18\gamma_3^2) \\
& + 6t^{13}\gamma_5 - 9t^{14}\gamma_4 - 12t^{15}\gamma_3 + 10t^{18},
\end{aligned}$$

$$\rho_{20} = 9u^{20} + 45u^{14}v + 12u^{10}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2,$$

$$\rho_{24} = 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2 + 30u^2v^2w + 5v^4,$$

$$\begin{aligned}
\rho_{30} = & -9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w - 35u^6v^4 + 6u^4vw^2 \\
& - 10u^2v^3w - 4v^5 - 2w^3,
\end{aligned}$$

and

$$u = t_8,$$

$$v = 2\gamma_6 + \gamma_3^2 - u\gamma_5 + \gamma_4(-t^2 + u^2) - u^3\gamma_3 + t^6 - t^4u^2 + t^3u^3 + t^2u^4 - tu^5,$$

$$\begin{aligned}
w = & \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3) + \gamma_3^2(2t^2u^2 + 2tu^3 - 2u^4) \\
& + \gamma_6(-5t^2u^2 + 5tu^3) + \gamma_5(t^4u + 3t^3u^2 + t^2u^3) + \gamma_4(6t^4u^2 - 3t^3u^3 - 2t^2u^4 - tu^5 + u^6) \\
& + \gamma_3(-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + u^7) + 4t^7u^3 - 6t^5u^5 + 2t^4u^6 + t^3u^7 - t^2u^8,
\end{aligned}$$

$$\begin{aligned}
x = & \gamma_{15} - 20\gamma_3\gamma_6^2 + 3\gamma_3^2\gamma_9 - 23\gamma_3^3\gamma_6 - 6\gamma_3^5 + 4\gamma_6\gamma_9 + 3u\gamma_4\gamma_{10} - u\gamma_5\gamma_9 - 3u\gamma_3^2\gamma_4^2 + 3uc_7\gamma_3\gamma_4 \\
& - 6u\gamma_4^2\gamma_6 + (-3t + 2u)\gamma_3^3\gamma_5 + (-4t + 4u)\gamma_3\gamma_5\gamma_6 + (-t^2 - u^2)\gamma_4\gamma_9 + (t^2 + tu - u^2)c_7\gamma_3^2 \\
& + (9t^2 + 12tu + 5u^2)\gamma_3\gamma_4\gamma_6 + (5t^2 + 6tu + 2u^2)\gamma_3^3\gamma_4 + (3t^2 + 4tu + u^2)c_7\gamma_6 \\
& + (-6t^3 - 2t^2u - 6tu^2 + 5u^3)\gamma_3^4 - u^3\gamma_3\gamma_9 + (3t^2u + u^3)\gamma_4^3 + (2t^2u + 3tu^2)c_7\gamma_5 \\
& + (-45t^3 + 10t^2u - 40tu^2)\gamma_6^2 + (t^3 - 2t^2u + tu^2 - u^3)\gamma_3\gamma_4\gamma_5 + (-33t^3 + t^2u - 31tu^2 + 13u^3)\gamma_3^2\gamma_6 \\
& + (-2t^4 - 4t^3u - 3tu^3 + 3u^4)c_7\gamma_4 + (-9t^4 - 6t^3u - 18t^2u^2 + 5tu^3 - 3u^4)\gamma_5\gamma_6 \\
& + (-3t^4 - 3t^3u - 7t^2u^2 + 5tu^3 - 4u^4)\gamma_3^2\gamma_5 + (-t^4 - 6t^3u - t^2u^2 - 3tu^3)\gamma_3\gamma_4^2
\end{aligned}$$

$$\begin{aligned}
& + (-3t^4u - 6t^3u^2 + 3t^2u^3 + 15tu^4)\gamma_{10} + (-3t^4u + t^3u^2 + 5t^2u^3 + 10tu^4 - u^5)c_7\gamma_3 \\
& + (15t^5 - 2t^4u + 3t^3u^2 + 14t^2u^3 - 16tu^4 + 3u^5)\gamma_3^2\gamma_4 \\
& + (39t^5 - 13t^4u + 8t^3u^2 + 35t^2u^3 - 31tu^4 - 3u^5)\gamma_4\gamma_6 + (t^6 - t^4u^2 - t^3u^3 - t^2u^4 - tu^5 - u^6)\gamma_9 \\
& + (-13t^6 + 12t^5u + 5t^4u^2 - 56t^3u^3 + 8t^2u^4 + 21tu^5 + 2u^6)\gamma_3\gamma_6 \\
& + (6t^6 + 3t^5u + 2t^4u^2 + 7t^3u^3 + t^2u^4 - 8tu^5 + 3u^6)\gamma_4\gamma_5 \\
& + (-8t^6 + 6t^5u + 2t^4u^2 - 22t^3u^3 + 6t^2u^4 + 8tu^5 - 2u^6)\gamma_3^3 \\
& + (-6t^7 + t^6u - 7t^4u^3 + 5t^3u^4 + 3t^2u^5 + 3tu^6 - 63u^7)\gamma_4^2 \\
& + (-t^7 + 2t^6u + t^5u^2 - 11t^4u^3 + 6t^3u^4 + 5t^2u^5 + 6tu^6 + 39u^7)\gamma_3\gamma_5 \\
& + (2t^8 + 6t^7u + 3t^6u^2 - 4t^5u^3 - 15t^4u^4 + 6t^3u^5 + 3t^2u^6 - 40tu^7 + 59u^8)c_7 \\
& + (3t^8 + t^6u^2 + 11t^5u^3 + 14t^4u^4 - 20t^3u^5 - 4t^2u^6 + 118tu^7 + 3u^8)\gamma_3\gamma_4 \\
& + (-48t^9 + 3t^8u - 41t^7u^2 + 18t^6u^3 + 16t^5u^4 - 13t^4u^5 - 67t^3u^6 + 125t^2u^7 - 15tu^8 - 291u^9)\gamma_6 \\
& + (-18t^9 - 3t^8u - 16t^7u^2 + 10t^6u^3 - 4t^5u^4 - 8t^4u^5 - 16t^3u^6 - 23t^2u^7 - 10tu^8 - 115u^9)\gamma_3^2 \\
& + (-6t^{10} - 3t^9u - 9t^8u^2 + 5t^7u^3 - 5t^6u^4 - 14t^4u^6 - 52t^3u^7 + 6t^2u^8 - 60tu^9 + 117u^{10})\gamma_5 \\
& + (18t^{11} - 3t^{10}u + 5t^9u^2 + 11t^8u^3 - 28t^7u^4 + 8t^6u^5 + 20t^5u^6 - 64t^4u^7 - 15t^3u^8 + 54t^2u^9 \\
& + 178tu^{10} - 177u^{11})\gamma_4 \\
& + (-2t^{12} + 6t^{11}u + 2t^{10}u^2 - 20t^9u^3 + 11t^8u^4 + 22t^7u^5 - 8t^6u^6 + 83t^5u^7 + 15t^4u^8 + 5t^3u^9 \\
& - 116t^2u^{10} + tu^{11} + 117u^{12})\gamma_3 \\
& - 12t^{15} - t^{14}u - 10t^{13}u^2 + 6t^{12}u^3 + 7t^{11}u^4 - 13t^{10}u^5 - 31t^9u^6 + 9t^8u^7 - t^7u^8 - 118t^6u^9 - 18t^5u^{10} \\
& + 131t^4u^{11} - 6t^3u^{12} - 233t^2u^{13} + 175tu^{14} - 58u^{15}.
\end{aligned}$$

**Remark 3.6.** In order to determine the higher relations  $\rho_{20}$ ,  $\rho_{24}$ , and  $\rho_{30}$ , the second author [22] make use of the integral cohomology ring of the homogeneous space  $E_8/T^1 \cdot E_7$  (for the integral cohomology ring of  $E_8/T^1 \cdot E_7$ , see [21]).

Since the elements  $t_i$  ( $1 \leq i \leq l$ ) in  $H^2(E_l/T; \mathbb{Z})$  belong to an orbit of  $W_{P_2}$ , the elementary symmetric polynomials  $c_i$  ( $1 \leq i \leq l$ ) are  $W_{P_2}$ -invariants, and hence they are elements of  $H^*(E_l/P_2; \mathbb{Z})$ . The element  $t$  in  $H^2(E_l/T; \mathbb{Z})$  is also an element of  $H^2(E_l/P_2; \mathbb{Z})$ . Therefore, in any cases of  $E_l$  ( $l = 6, 7, 8$ ), one can see that the higher degree generators  $\gamma_i$  lie in  $H^*(E_l/P_2; \mathbb{Z}) \subset H^*(E_l/T; \mathbb{Z})$ .

3.3. In the previous subsection, we reviewed the Borel presentation of the integral cohomology rings of  $E_l/T$  by generators and relations. In order to express those generators in terms of Schubert classes, we will use the divided difference operators introduced independently by Bernstein-Gelfand-Gelfand [1] and Demazure [8]:

**Definition 3.7** (Bernstein-Gelfand-Gelfand [1], Demazure [8]). (1) For each root  $\alpha \in \Delta$ , the operator of degree  $-2$

$$\Delta_\alpha : H^*(BT; \mathbb{Z}) \longrightarrow H^*(BT; \mathbb{Z})$$

is defined as

$$\Delta_\alpha(u) = \frac{u - s_\alpha(u)}{\alpha} \quad \text{for } u \in H^*(BT; \mathbb{Z}).$$

(2) For  $w \in W$ , the operator  $\Delta_w$  is defined as the composite

$$\Delta_w = \Delta_{\alpha_1} \circ \Delta_{\alpha_2} \circ \cdots \circ \Delta_{\alpha_k},$$

where  $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$  ( $\alpha_i \in \Pi$ ) is any reduced decomposition of  $w$ .

One can show that the definition is well defined, i.e., independent of the choice of the reduced decomposition of  $w$ . The following theorem gives a correspondence between a cycle represented by a polynomial in the Borel presentation and a sum of Schubert classes via the characteristic homomorphism (2.2).

**Theorem 3.8** (Bernstein-Gelfand-Gelfand [1], Demazure [8]). *For a homogeneous polynomial  $f \in H^{2k}(BT; \mathbb{Z})$ , we have*

$$c(f) = \sum_{w \in W, l(w)=k} \Delta_w(f) Z_w.$$

(Note that  $\Delta_w(f) \in H^0(BT; \mathbb{Z}) \cong \mathbb{Z}$ .) In particular, we have  $c(\omega_i) = Z_{s_i}$ .

3.4. Now we combine the result of the previous subsections to have the explicit relation between the ring generators of  $H^*(E_i/T; \mathbb{Z})$  given in §3.2 and the Schubert basis  $\{Z_w\}_{w \in W(E_i)}$ . For simplicity, a reduced decomposition  $w = s_{i_1} s_{i_2} \cdots s_{i_k}$  will be abbreviated to  $[i_1 i_2 \cdots i_k]$ , and we denote by  $Z_{i_1 i_2 \cdots i_k}$  the Schubert class corresponding to  $w$ , although the reduced decomposition of a Weyl group element may not be unique.

First we deal with the case of  $E_6$ . Since  $c(\omega_i) = Z_i$ , it follows immediately from (3.1) that

$$\begin{aligned} t_1 &= -Z_1 + Z_2, & t_2 &= Z_1 + Z_2 - Z_3, & t_3 &= Z_2 + Z_3 - Z_4, \\ t_4 &= Z_4 - Z_5, & t_5 &= Z_5 - Z_6, & t_6 &= Z_6, & t &= Z_2. \end{aligned}$$

For the higher degree generators, we would like to have an expansion of  $\gamma_i$  ( $i = 3, 4$ ) by the sum of Schubert classes. Note that they lie in  $H^*(E_6/P_2; \mathbb{Z}) \subset H^*(E_6/T; \mathbb{Z})$ , which has the Schubert basis indexed by  $W^{P_2}$ . Since the length three and four elements of  $W^{P_2}$  are  $\{[342], [542]\}$ ,  $\{[1342], [3542], [6542]\}$  respectively, the elements  $\gamma_i$  ( $i = 3, 4$ ) should be written as  $\mathbb{Z}$ -linear combinations

$$\begin{aligned} \gamma_3 &= a_{342} Z_{342} + a_{542} Z_{542}, \\ \gamma_4 &= a_{1342} Z_{1342} + a_{3542} Z_{3542} + a_{6542} Z_{6542}. \end{aligned}$$

The coefficients  $a_{342}, a_{542}, a_{1342}, a_{3542}, a_{6542} \in \mathbb{Z}$  can be determined as follows. By Theorem 3.1, we have

$$(3.2) \quad 2\gamma_3 = c_3, \quad 3\gamma_4 = c_4 + 2t^4.$$

Therefore  $2\gamma_3$  and  $3\gamma_4$  are contained in the image of  $c$ . Define the polynomials in  $H^*(BT; \mathbb{Z})$  by

$$\delta_3 = c_3, \quad \delta_4 = c_4 + 2t^4,$$

so that they are mapped under  $c$  to  $2\gamma_3$  and  $3\gamma_4$  in  $H^*(E_6/T; \mathbb{Z})$  respectively. Applying Theorem 3.8 to  $\delta_i$  ( $i = 3, 4$ ), we have

$$(3.3) \quad \begin{aligned} 2\gamma_3 &= 2Z_{342} + 4Z_{542} = 2(Z_{342} + 2Z_{542}), \\ 3\gamma_4 &= 3Z_{1342} + 6Z_{3542} + 6Z_{6542} = 3(Z_{1342} + 2Z_{3542} + 2Z_{6542}). \end{aligned}$$

Since  $H^*(E_6/T; \mathbb{Z})$  is torsion free, we obtain

$$(3.4) \quad \gamma_3 = Z_{342} + 2Z_{542}, \quad \gamma_4 = Z_{1342} + 2Z_{3542} + 2Z_{6542}.$$

Conversely we can express each Schubert class as a polynomial in  $t$ ,  $\gamma_3$  and  $\gamma_4$ ; Consider the following polynomials in  $H^*(BT; \mathbb{Q})$ :

$$\begin{aligned} f_{342} &= -\frac{1}{2}\delta_3 + 2t^3, & f_{542} &= \frac{1}{2}\delta_3 - t^3, \\ f_{1342} &= \frac{1}{3}\delta_4 - t\delta_3 + 2t^4, & f_{3542} &= -\frac{1}{3}\delta_4 + \frac{1}{2}t\delta_3, & f_{6542} &= \frac{1}{3}\delta_4 - t^4. \end{aligned}$$

Applying Theorem 3.8 to the above polynomials, we see that these are indeed the polynomial representatives of the Schubert classes, i.e.,  $c(f_{342}) = Z_{342}$  and so on. Therefore, in the ring  $H^*(E_6/T; \mathbb{Z})$ , we obtain

$$\begin{aligned} Z_{342} &= -\gamma_3 + 2t^3, & Z_{542} &= \gamma_3 - t^3, \\ Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4, & Z_{3542} &= -\gamma_4 + t\gamma_3, & Z_{6542} &= \gamma_4 - t^4. \end{aligned}$$

Thus, for example, we can choose  $Z_{542}, Z_{6542}$  as ring generators in place of  $\gamma_3, \gamma_4$ . Consequently, we obtain the following result.

**Proposition 3.9.** *In  $H^*(E_6/T; \mathbb{Z})$ , the relation between the ring generators  $\{t_1, \dots, t_6, t, \gamma_3, \gamma_4\}$  in Theorem 3.1 and the Schubert classes is given by*

$$\begin{aligned} t_1 &= -Z_1 + Z_2, & t_2 &= Z_1 + Z_2 - Z_3, & t_3 &= Z_2 + Z_3 - Z_4, \\ t_4 &= Z_4 - Z_5, & t_5 &= Z_5 - Z_6, & t_6 &= Z_6, & t &= Z_2, \\ \gamma_3 &= Z_{342} + 2Z_{542}, & \gamma_4 &= Z_{1342} + 2Z_{3542} + 2Z_{6542}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} Z_{342} &= -\gamma_3 + 2t^3, & Z_{542} &= \gamma_3 - t^3, \\ Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4, & Z_{3542} &= -\gamma_4 + t\gamma_3, & Z_{6542} &= \gamma_4 - t^4. \end{aligned}$$

In particular, the set  $\{Z_1, Z_2, \dots, Z_6, Z_{542}, Z_{6542}\}$  is a minimal system of ring generators of  $H^*(E_6/T; \mathbb{Z})$  that consists of Schubert classes.

In a similar manner, we carry out the computation for the case of  $E_7$ . It follows immediately from (3.1) that

$$\begin{aligned} t_1 &= -Z_1 + Z_2, & t_2 &= Z_1 + Z_2 - Z_3, & t_3 &= Z_2 + Z_3 - Z_4, \\ t_4 &= Z_4 - Z_5, & t_5 &= Z_5 - Z_6, & t_6 &= Z_6 - Z_7, & t_7 &= Z_7, & t &= Z_2. \end{aligned}$$

By Theorem 3.3, we have

$$\begin{aligned} (3.5) \quad 2\gamma_3 &= c_3, \\ 3\gamma_4 &= c_4 + 2t^4, \\ 2\gamma_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 = c_5 - tc_4 + t^2c_3 - 2t^5, \\ 2\gamma_9 &= 2c_6\gamma_3 + t^2c_7 - 3t^3c_6 = c_3c_6 + t^2c_7 - 3t^3c_6. \end{aligned}$$

Therefore  $2\gamma_3, 3\gamma_4, 2\gamma_5$ , and  $2\gamma_9$  are contained in the image of  $c$ . Define the polynomials in  $H^*(BT; \mathbb{Z})$  by

$$\delta_3 = c_3, \quad \delta_4 = c_4 + 2t^4, \quad \delta_5 = c_5 - tc_4 + t^2c_3 - 2t^5, \quad \delta_9 = c_3c_6 + t^2c_7 - 3t^3c_6,$$

so that they are mapped under  $c$  to  $2\gamma_3$ ,  $3\gamma_4$ ,  $2\gamma_5$ , and  $2\gamma_9$  in  $H^*(E_7/T; \mathbb{Z})$  respectively. Applying Theorem 3.8 to  $\delta_i$  ( $i = 3, 4, 5, 9$ ), we have

$$(3.6) \quad \begin{aligned} 2\gamma_3 &= 2(Z_{342} + 2Z_{542}), \\ 3\gamma_4 &= 3(Z_{1342} + 2Z_{3542} + 2Z_{6542}), \\ 2\gamma_5 &= 2Z_{76542}, \\ 2\gamma_9 &= 2(2Z_{154376542} + Z_{654376542}), \end{aligned}$$

Since  $H^*(E_7/T; \mathbb{Z})$  is torsion free, we obtain

$$(3.7) \quad \begin{aligned} \gamma_3 &= Z_{342} + 2Z_{542}, \\ \gamma_4 &= Z_{1342} + 2Z_{3542} + 2Z_{6542}, \\ \gamma_5 &= Z_{76542}, \\ \gamma_9 &= 2Z_{154376542} + Z_{654376542}, \end{aligned}$$

Conversely we can express each Schubert class as a polynomial in  $t$ ,  $\gamma_i$  ( $i = 3, 4, 5, 9$ ); Consider the following polynomials in  $H^*(BT; \mathbb{Q})$ :

$$\begin{aligned} f_{342} &= -\frac{1}{2}\delta_3 + 2t^3, & f_{542} &= \frac{1}{2}\delta_3 - t^3, \\ f_{1342} &= \frac{1}{3}\delta_4 - t\delta_3 + 2t^4, & f_{3542} &= -\frac{1}{3}\delta_4 + \frac{1}{2}t\delta_3, & f_{6542} &= \frac{1}{3}\delta_4 - t^4, \\ f_{76542} &= \frac{1}{2}\delta_5, \\ f_{154376542} &= \frac{1}{2}\delta_9 - \frac{1}{6}\delta_4\delta_5 + \frac{1}{2}t^4\delta_5, & f_{654376542} &= -\frac{1}{2}\delta_9 + \frac{1}{3}\delta_4\delta_5 - t^4\delta_5. \end{aligned}$$

Applying Theorem 3.8 again to the above polynomials, we see that these are indeed the polynomial representatives of the Schubert classes, i.e.,  $c(f_{342}) = Z_{342}$  and so on. Therefore, in the ring  $H^*(E_7/T; \mathbb{Z})$ , we obtain

$$\begin{aligned} Z_{342} &= -\gamma_3 + 2t^3, & Z_{542} &= \gamma_3 - t^3, \\ Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4, & Z_{3542} &= -\gamma_4 + t\gamma_3, & Z_{6542} &= \gamma_4 - t^4, \\ Z_{76542} &= \gamma_5, \\ Z_{154376542} &= \gamma_9 - \gamma_4\gamma_5 + t^4\gamma_5, & Z_{654376542} &= -\gamma_9 + 2\gamma_4\gamma_5 - 2t^4\gamma_5. \end{aligned}$$

Consequently, we obtain the following result.

**Proposition 3.10.** *In  $H^*(E_7/T; \mathbb{Z})$ , the relation between the ring generators  $\{t_1, \dots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9\}$  in Theorem 3.3 and the Schubert classes is given by*

$$\begin{aligned} t_1 &= -Z_1 + Z_2, & t_2 &= Z_1 + Z_2 - Z_3, & t_3 &= Z_2 + Z_3 - Z_4, \\ t_4 &= Z_4 - Z_5, & t_5 &= Z_5 - Z_6, & t_6 &= Z_6 - Z_7, & t_7 &= Z_7, & t &= Z_2, \\ \gamma_3 &= Z_{342} + 2Z_{542}, & \gamma_4 &= Z_{1342} + 2Z_{3542} + 2Z_{6542}, \\ \gamma_5 &= Z_{76542}, & \gamma_9 &= 2Z_{154376542} + Z_{654376542}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} Z_{342} &= -\gamma_3 + 2t^3, & Z_{542} &= \gamma_3 - t^3, \\ Z_{1342} &= \gamma_4 - 2t\gamma_3 + 2t^4, & Z_{3542} &= -\gamma_4 + t\gamma_3, & Z_{6542} &= \gamma_4 - t^4, \\ Z_{76542} &= \gamma_5, \\ Z_{154376542} &= \gamma_9 - \gamma_4\gamma_5 + t^4\gamma_5, & Z_{654376542} &= -\gamma_9 + 2\gamma_4\gamma_5 - 2t^4\gamma_5. \end{aligned}$$

In particular, the set  $\{Z_1, Z_2, \dots, Z_7, Z_{542}, Z_{6542}, Z_{76542}, Z_{654376542}\}$  is a minimal system of ring generators of  $H^*(E_7/T; \mathbb{Z})$  that consists of Schubert classes.

Finally, we carry out the computation for the case of  $E_8$ . It follows from (3.1) that

$$\begin{aligned} t_1 &= -Z_1 + Z_2, & t_2 &= Z_1 + Z_2 - Z_3, & t_3 &= Z_2 + Z_3 - Z_4, & t_4 &= Z_4 - Z_5, \\ t_5 &= Z_5 - Z_6, & t_6 &= Z_6 - Z_7, & t_7 &= Z_7 - Z_8, & t_8 &= Z_8, & t &= Z_2. \end{aligned}$$

By Theorem 3.5, we have

$$\begin{aligned} 2\gamma_3 &= c_3, \\ 3\gamma_4 &= c_4 + 2t^4, \\ 2\gamma_5 &= c_5 - tc_4 + t^2c_3 - 2t^5, \\ (3.8) \quad 30\gamma_6 &= 6c_6 - 3c_3^2 - 3tc_5 + 5t^2c_4 - 3t^3c_3 - 2t^6, \\ 2\gamma_9 &= c_3c_6 + tc_8 + t^2c_7 - 3t^3c_6, \\ 12\gamma_{10} &= (c_5 - tc_4 + t^2c_3 - 2t^5)^2 - 4c_7c_3 - 4t^2c_8 + 12t^3c_7, \\ 8\gamma_{15} &= 4(c_8 - t^2c_6 + t^3c_5 + t^5c_3 - t^8)(c_7 - 3tc_6) \\ &\quad - (c_3^2 + 4c_6)(tc_8 + t^2c_7 - 3t^3c_6). \end{aligned}$$

Therefore  $2\gamma_3, 3\gamma_4, 2\gamma_5, 30\gamma_6, 2\gamma_9, 12\gamma_{10}$ , and  $8\gamma_{15}$  are contained in the image of  $c$ . Define the polynomials in  $H^*(BT; \mathbb{Q})$  by

$$\begin{aligned} \delta_3 &= c_3, & \delta_4 &= c_4 + 2t^4, & \delta_5 &= c_5 - tc_4 + t^2c_3 - 2t^5, \\ \delta_6 &= \frac{1}{6}(6c_6 - 3c_3^2 - 3tc_5 + 5t^2c_4 - 3t^3c_3 - 2t^6), \\ \delta_9 &= c_3c_6 + tc_8 + t^2c_7 - 3t^3c_6, \\ \delta_{10} &= \frac{1}{4}\{(c_5 - tc_4 + t^2c_3 - 2t^5)^2 - 4c_7c_3 - 4t^2c_8 + 12t^3c_7\}, \\ \delta_{15} &= (c_8 - t^2c_6 + t^3c_5 + t^5c_3 - t^8)(c_7 - 3tc_6) - \frac{1}{4}(c_3^2 + 4c_6)(tc_8 + t^2c_7 - 3t^3c_6). \end{aligned}$$

By applying Theorem 3.8 to  $\delta_i$  ( $i = 3, 4, 5, 6, 9, 10, 15$ ), we obtain the similar results:

**Proposition 3.11.** *In  $H^*(E_8/T; \mathbb{Z})$ , the relation between the ring generators  $\{t_1, \dots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}\}$  in Theorem 3.5 and the Schubert classes is given by*

$$\begin{aligned}
t_1 &= -Z_1 + Z_2, \quad t_2 = Z_1 + Z_2 - Z_3, \quad t_3 = Z_2 + Z_3 - Z_4, \quad t_4 = Z_4 - Z_5, \\
t_5 &= Z_5 - Z_6, \quad t_6 = Z_6 - Z_7, \quad t_7 = Z_7 - Z_8, \quad t_8 = Z_8, \quad t = Z_2, \\
\gamma_3 &= Z_{342} + 2Z_{542}, \\
\gamma_4 &= Z_{1342} + 2Z_{3542} + 2Z_{6542}, \\
\gamma_5 &= Z_{76542}, \\
\gamma_6 &= -6Z_{136542} - 5Z_{143542} - 2Z_{243542} - 5Z_{376542} - 6Z_{436542} - Z_{876542}, \\
\gamma_9 &= 4Z_{143876542} + 2Z_{154376542} + 4Z_{543876542} + Z_{654376542}, \\
\gamma_{10} &= -Z_{15438765432}, \\
\gamma_{15} &= 58Z_{131426543876542} - 17Z_{134231543876542} + 140Z_{134276543876542} \\
&\quad + 30Z_{135426543876542} + 127Z_{154276543876542} - 22Z_{234231543876542} \\
&\quad + 87Z_{242316543876542} + 271Z_{243176543876542} - 22Z_{245423143876542} \\
&\quad + 52Z_{254316543876542} + 386Z_{314276543876542} + 82Z_{315426543876542} \\
&\quad + 102Z_{342316543876542} - 22Z_{345423143876542} + 30Z_{354276543876542} \\
&\quad + 470Z_{423176543876542} - 17Z_{458765423143542} + 55Z_{465423143876542} \\
&\quad + 139Z_{542316543876542} + 62Z_{543176543876542} + 15Z_{654276543876542} \\
&\quad + 8Z_{658765423143542} + 157Z_{765423143876542}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
Z_{542} &= \gamma_3 - t^3, \\
Z_{6542} &= \gamma_4 - t^4, \\
Z_{76542} &= \gamma_5, \\
Z_{136542} &= \gamma_6 - t\gamma_5 + t^2\gamma_4, \\
Z_{154376542} &= \gamma_9 - 2\gamma_3^3 - 4\gamma_3\gamma_6 - \gamma_4\gamma_5 + t(-6\gamma_4^2 + 5c_8 + 4\gamma_3\gamma_5) + t^2(-4c_7 + 14\gamma_3\gamma_4) \\
&\quad + t^3(-2\gamma_3^2 + 14\gamma_6) - 5t^4\gamma_5 - 10t^5\gamma_4 + 10t^6\gamma_3, \\
Z_{1654376542} &= -\gamma_{10} + \gamma_5^2 - 2\gamma_3^2\gamma_4 - 4\gamma_4\gamma_6 + 2t^2\gamma_4^2 + t^4(2\gamma_3^2 + 4\gamma_6) - 4t^6\gamma_4 + 2t^{10}, \\
Z_{134276543876542} &= \gamma_{15} + 16\gamma_6\gamma_9 + \gamma_5\gamma_{10} + 6\gamma_3\gamma_6^2 + 5\gamma_3^2\gamma_9 - \gamma_3^3\gamma_6 + 4\gamma_3^2\gamma_4\gamma_5 + \gamma_3^5 \\
&\quad - 12\gamma_3\gamma_5c_7 - 29\gamma_3\gamma_4c_8 \\
&\quad + t(-167\gamma_6c_8 - 6\gamma_5\gamma_9 + 165\gamma_4^2\gamma_6 - 96\gamma_3\gamma_5\gamma_6 + 32\gamma_3\gamma_4c_7 - 258\gamma_3^2c_8 \\
&\quad + 276\gamma_3^2\gamma_4^2 - 181\gamma_3^3\gamma_5) \\
&\quad + t^2(107\gamma_6c_7 + 11\gamma_5c_8 + 93\gamma_4\gamma_9 + 48\gamma_3\gamma_{10} - 6\gamma_4^2\gamma_5 - 945\gamma_3\gamma_4\gamma_6 + 190\gamma_3^2c_7 \\
&\quad - 795\gamma_3^3\gamma_4) \\
&\quad + t^3(3\gamma_6^2 - 31\gamma_5c_7 + 134\gamma_3\gamma_9 - 123\gamma_4c_8 - 674\gamma_3^2\gamma_6 - 83\gamma_3\gamma_4\gamma_5) \\
&\quad + t^4(139\gamma_5\gamma_6 + 31\gamma_4c_7 + 26\gamma_3c_8 + 117\gamma_3\gamma_4^2 + 130\gamma_3^2\gamma_5) \\
&\quad + t^5(513\gamma_4\gamma_6 - 194\gamma_3c_7 + 604\gamma_3^2\gamma_4) + t^6(\gamma_9 + 1094\gamma_3\gamma_6 + 133\gamma_4\gamma_5)
\end{aligned}$$

$$\begin{aligned}
& + t^7(3c_8 + 198\gamma_3\gamma_5) + t^8(22c_7 - 685\gamma_3\gamma_4) + t^9(\gamma_6 + 18\gamma_3^2) \\
& + 4t^{10}\gamma_5 + 241t^{11}\gamma_4 + 382t^{12}\gamma_3.
\end{aligned}$$

In particular, the set  $\{Z_1, Z_2, \dots, Z_8, Z_{542}, Z_{6542}, Z_{76542}, Z_{136542}, Z_{154376542}, Z_{1654376542}, Z_{134276543876542}\}$  is a minimal system of ring generators of  $H^*(E_8/T; \mathbb{Z})$  that consists of Schubert classes.

#### 4. CHOW RINGS OF $E_l$ ( $l = 6, 7, 8$ )

In this section, we determine the Chow rings of the exceptional groups  $E_l$  ( $l = 6, 7, 8$ ). As mentioned in the introduction, we have only to compute the quotient ring of  $A(G/B)$  by the ideal generated by  $A^1(G/B)$ . Denote by  $p^* : A(G/B) \rightarrow A(G)$  the projection onto the quotient induced by the natural projection  $p : G \rightarrow G/B$ . It is known that the cycle map  $A(G/B) \rightarrow H^*(G/B; \mathbb{Z})$  which assigns to the Schubert variety  $X_{w_0w}$  the Schubert class  $Z_w$  is an isomorphism of rings. Therefore, the Chow ring  $A(G)$  is isomorphic to the quotient ring of  $H^*(G/B; \mathbb{Z}) = H^*(K/T; \mathbb{Z})$  by the ideal generated by  $H^2(K/T; \mathbb{Z})$ .

By Theorem 3.1, we easily obtain that

$$H^*(E_6/T; \mathbb{Z})/(t_1, \dots, t_6, t) = \mathbb{Z}[\gamma_3, \gamma_4]/(2\gamma_3, 3\gamma_4, \gamma_3^2, \gamma_4^3).$$

Furthermore, by Proposition 3.9, we have

$$Z_{542} \equiv \gamma_3, \quad Z_{6542} \equiv \gamma_4, \quad \text{mod } (t_1, \dots, t_6, t).$$

In a similar manner, by Theorem 3.3, we obtain

$$H^*(E_7/T; \mathbb{Z})/(t_1, \dots, t_7, t) = \mathbb{Z}[\gamma_3, \gamma_4, \gamma_5, \gamma_9]/(2\gamma_3, 3\gamma_4, 2\gamma_5, \gamma_3^2, 2\gamma_9, \gamma_5^2, \gamma_4^3, \gamma_9^2).$$

Furthermore, by Proposition 3.10, we have

$$Z_{542} \equiv \gamma_3, \quad Z_{6542} \equiv \gamma_4, \quad Z_{76542} \equiv \gamma_5, \quad Z_{654376542} \equiv \gamma_9 \quad \text{mod } (t_1, \dots, t_7, t).$$

Finally, by Theorem 3.5, we obtain

$$\begin{aligned}
& H^*(E_8/T; \mathbb{Z})/(t_1, \dots, t_8, t) \\
& = \mathbb{Z}[\gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_9, \gamma_{10}, \gamma_{15}] \\
& \quad / (2\gamma_3, 3\gamma_4, 2\gamma_5, 5\gamma_6, 2\gamma_9, \gamma_5^2 - 3\gamma_{10}, \gamma_4^3, 2\gamma_{15}, \gamma_9^2, 3\gamma_{10}^2, \gamma_3^8, \gamma_{15}^2 + \gamma_{10}^3 + 2\gamma_6^5).
\end{aligned}$$

Furthermore, by Proposition 3.11, we have

$$\begin{aligned}
& Z_{542} \equiv \gamma_3, \quad Z_{6542} \equiv \gamma_4, \quad Z_{76542} \equiv \gamma_5, \quad Z_{136542} \equiv \gamma_6, \quad Z_{154376542} \equiv \gamma_9, \\
& Z_{1654376542} \equiv -\gamma_{10} + \gamma_5^2, \quad Z_{134276543876542} \equiv \gamma_{15} + \gamma_5\gamma_{10} + \gamma_3^2\gamma_9 + \gamma_3^5 \quad \text{mod } (t_1, \dots, t_8, t),
\end{aligned}$$

and the relations are calculated as follows:

$$\begin{aligned}
& \gamma_5^2 - 3\gamma_{10} \equiv Z_{76542}^2 - 3(-Z_{1654376542} + Z_{76542}^2) \equiv 3Z_{1654376542}, \\
& 3\gamma_{10}^2 \equiv Z_{76542}^2(-Z_{1654376542} + Z_{76542}^2) \equiv Z_{76542}^4, \\
& Z_{136542}^5 \equiv \gamma_6^5 \equiv -12(\gamma_{15}^2 + \gamma_{10}^3 + 2\gamma_6^5), \quad Z_{1654376542}^3 \equiv -\gamma_{10}^3 \equiv -10(\gamma_{15}^2 + \gamma_{10}^3 + 2\gamma_6^5), \\
& \gamma_{15} \equiv Z_{134276543876542} + Z_{76542}^3 + Z_{542}^2 Z_{154376542} + Z_{542}^5, \\
& Z_{134276543876542}^2 \equiv \gamma_{15}^2 \equiv 15(\gamma_{15}^2 + \gamma_{10}^3 + 2\gamma_6^5) \quad \text{mod } (t_1, \dots, t_8, t).
\end{aligned}$$

Therefore we obtain our main result of this paper:



**Theorem 4.1.** (1) *The Chow ring of  $E_6$  is given by*

$$A(E_6) = \mathbb{Z}[X_3, X_4]/(2X_3, 3X_4, X_3^2, X_4^3),$$

where  $X_3$  and  $X_4$  are the pull-back images of the elements of  $A(E_6/B)$  defined by the Schubert varieties  $X_{w_0s_5s_4s_2}$  and  $X_{w_0s_6s_5s_4s_2}$  respectively.

(2) *The Chow ring of  $E_7$  is given by*

$$A(E_7) = \mathbb{Z}[X_3, X_4, X_5, X_9]/(2X_3, 3X_4, 2X_5, X_3^2, 2X_9, X_5^2, X_4^3, X_9^2),$$

where  $X_3, X_4, X_5$ , and  $X_9$  are the pull-back images of the elements of  $A(E_7/B)$  defined by the Schubert varieties  $X_{w_0s_5s_4s_2}$ ,  $X_{w_0s_6s_5s_4s_2}$ ,  $X_{w_0s_7s_6s_5s_4s_2}$ , and  $X_{w_0s_6s_5s_4s_3s_7s_6s_5s_4s_2}$  respectively.

(3) *The Chow ring of  $E_8$  is given by*

$$A(E_8) = \mathbb{Z}[X_3, X_4, X_5, X_6, X_9, X_{10}, X_{15}] \\ / (2X_3, 3X_4, 2X_5, 5X_6, 2X_9, 3X_{10}, X_4^3, 2X_{15}, X_9^2, X_5^4, X_3^8, X_6^5, X_{10}^3, X_{15}^2),$$

where  $X_3, X_4, X_5, X_6, X_9, X_{10}$ , and  $X_{15}$  are the pull-back images under  $p^*$  of the elements of  $A(E_8/B)$  defined by the Schubert varieties  $X_{w_0s_5s_4s_2}$ ,  $X_{w_0s_6s_5s_4s_2}$ ,  $X_{w_0s_7s_6s_5s_4s_2}$ ,  $X_{w_0s_1s_3s_6s_5s_4s_2}$ ,  $X_{w_0s_1s_5s_4s_3s_7s_6s_5s_4s_2}$ ,  $X_{w_0s_1s_6s_5s_4s_3s_7s_6s_5s_4s_2}$ , and  $X_{w_0s_1s_3s_4s_2s_7s_6s_5s_4s_3s_8s_7s_6s_5s_4s_2}$  respectively.

**Remark 4.2.** As a Corollary of Theorem 4.1, we can recover the  $p$ -exceptional degrees of  $G = E_l$  ( $l = 6, 7, 8$ ) for each torsion prime  $p$ . In [15], Kac showed that the kernel of the characteristic homomorphism tensored by a prime field  $\mathbb{F}_p$  is an ideal generated by a regular sequence. Hence, there exist homogeneous polynomials  $P_1, \dots, P_l \in S(\hat{H}) \otimes \mathbb{F}_p \cong H^*(BT; \mathbb{F}_p)$  such that

$$\mathcal{I}_p := \ker c \otimes \mathbb{F}_p = (P_1, \dots, P_l).$$

Then he introduced the notion of the  $p$ -exceptional degrees which is a certain sub-sequence of the degrees of the *basic invariants* of  $G$ . The basic invariants of  $G$  are defined to be homogeneous generators of  $\ker c \otimes \mathbb{Q} = ((S^+(\hat{H}) \otimes \mathbb{Q})^W)$ . We only show how to deal with the case of  $(G, p) = (E_8, 2)$  (cf. [15, §7]). The other cases can be obtained in a similar manner. By Theorem 4.1, the mod 2 Chow ring of  $E_8$  is

$$A(E_8; \mathbb{F}_2) = A(E_8) \otimes_{\mathbb{Z}} \mathbb{F}_2 \\ = \mathbb{F}_2[X_3, X_5, X_9, X_{15}]/(X_3^8, X_5^4, X_9^2, X_{15}^2).$$

Therefore, by [15, Theorem 3], the 2-exceptional degrees are  $2^1 \cdot 9 = 18$ ,  $2^2 \cdot 5 = 20$ ,  $2^3 \cdot 3 = 24$ ,  $2^1 \cdot 15 = 30$ . Moreover, by [15, Theorem 4], the degrees of generators of  $\mathcal{I}_2$  can be read off as follows: in the sequence (2, 8, 12, 14, 18, 20, 24, 30) of the degrees of the basic invariants (see the table below), we replace the number 18 by 9, 20 by 5, 24 by 3, and 30 by 15.

The  $p$ -exceptional degrees of  $E_l$  ( $l = 6, 7, 8$ ) are given by the following table (cf. [15, Table 2]).

$(G, p)$	degrees of basic invariants	degrees of generators of $\mathcal{I}_p$	$p$ -exceptional degrees
$(E_6, 2)$	(2, 5, 6, 8, 9, 12)	(2, 3, 5, 8, 9, 12)	(6)
$(E_6, 3)$	↑	(2, 4, 5, 6, 8, 9)	(12)
$(E_7, 2)$	(2, 6, 8, 10, 12, 14, 18)	(2, 3, 5, 8, 9, 12, 14)	(6, 10, 18)
$(E_7, 3)$	↑	(2, 4, 6, 8, 10, 14, 18)	(12)
$(E_8, 2)$	(2, 8, 12, 14, 18, 20, 24, 30)	(2, 3, 5, 8, 9, 12, 14, 15)	(18, 20, 24, 30)
$(E_8, 3)$	↑	(2, 4, 8, 10, 14, 18, 20, 24)	(12, 30)
$(E_8, 5)$	↑	(2, 6, 8, 12, 14, 18, 20, 24)	(30)

## 5. APPENDIX

In this appendix, we first give the Borel presentations of the integral cohomology rings of  $E_l/T$  ( $l = 6, 7, 8$ ) in terms of the rings of invariants of the Weyl groups  $W(E_l)$ . Then we relate these results to the Schubert presentations obtained by Duan-Zhao ([10]).

**5.1. Borel presentation of  $H^*(E_l/T; \mathbb{Z})$ .** As mentioned in §3.2, we derive  $H^*(E_l/T; \mathbb{Z})$  directly from the ring of invariants of the Weyl group  $W(E_l)$ . The notation used here is the same as in §3.2.

First we recall the ring of invariants of the Weyl group  $W(E_l)$  in [25, §5], [26, §2], and [21, §2] (see also [18, 2.1, 2.2, 2.3]). Putting

$$(5.1) \quad \begin{aligned} x_i &= 2t_i - t \quad (1 \leq i \leq 6) \quad (l = 6), \\ x_i &= 2t_i - t \quad (1 \leq i \leq 7) \quad \text{and} \quad x_8 = t \quad (l = 7), \\ x_i &= 2t_i - \frac{2}{3}t \quad (1 \leq i \leq 8) \quad \text{and} \quad x_9 = -\frac{2}{3}t \quad (l = 8), \end{aligned}$$

we see easily that the sets

$$\begin{aligned} S_6 &= \{x_i + x_j \quad (1 \leq i < j \leq 6), \quad t - x_i, -t - x_i \quad (1 \leq i \leq 6)\} \quad (l = 6), \\ S_7 &= \{x_i + x_j, -x_i - x_j \quad (1 \leq i < j \leq 8)\} \quad (l = 7), \\ S_8 &= \{\pm(x_i - x_j) \quad (1 \leq i < j \leq 9), \quad \pm(x_i + x_j + x_k) \quad (1 \leq i < j < k \leq 9)\} \quad (l = 8) \end{aligned}$$

are invariant under the action of  $W(E_l)$  ( $l = 6, 7, 8$ ) respectively. Then we form the  $W(E_l)$ -invariant polynomials

$$I_n = \sum_{y \in S_l} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_l)}.$$

The invariant polynomials  $I_n$  are computed by the formula:

$$(5.2) \quad \begin{aligned} I_n &= \frac{1}{2} \sum_{i=2}^{n-2} \binom{n}{i} s_i s_{n-i} + (6 - 2^{n-1}) s_n + 2(-1)^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} s_{n-2j} t^{2j} \quad (l = 6), \\ I_n &= (16 - 2^n) s_n + \sum_{i=2}^{n-2} \binom{n}{i} s_i s_{n-i} \quad \text{for } n \text{ even} \quad (l = 7), \\ I_n &= \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} s_i s_{n-i} + 2 \cdot 3^{n-1} s_n - \sum_{i=0}^n \binom{n}{i} 2^{n-i} s_i s_{n-i} \\ &\quad + \frac{1}{3} \sum_{i=0}^n \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} s_i s_j s_{n-i-j} \quad \text{for } n \text{ even} \quad (l = 8), \end{aligned}$$

where  $s_n = x_1^n + \cdots + x_6^n$  ( $l = 6$ ),  $s_n = x_1^n + \cdots + x_8^n$  ( $l = 7$ ),  $s_n = x_1^n + \cdots + x_9^n$  ( $l = 8$ ). The power sum symmetric polynomials  $s_n$  are written as polynomials in  $d_i$ 's, where  $d_i = e_i(x_1, \dots, x_6)$  ( $l = 6$ ),  $d_i = e_i(x_1, \dots, x_8)$  ( $l = 7$ ),  $d_i = e_i(x_1, \dots, x_9)$  ( $l = 8$ ) by use of the Newton formula:

$$(5.3) \quad s_n = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-i} d_i + (-1)^{n-1} n d_n.$$

Moreover, using (5.1), we have

$$(5.4) \quad \begin{aligned} d_n &= \sum_{i=0}^n \binom{6-i}{n-i} (-t)^{n-i} 2^i c_i \quad (l=6), \\ d_n &= \sum_{i=0}^n \left\{ \binom{7-i}{n-i} - \sum_{i=0}^n \binom{7-i}{n-1-i} \right\} (-t)^{n-i} 2^i c_i \quad (l=7), \\ d_n &= \sum_{i=0}^n \left\{ \binom{8-i}{n-i} + \binom{8-i}{n-i} \right\} \left( -\frac{2}{3}t \right)^{n-i} 2^i c_i \quad (l=8). \end{aligned}$$

By using (5.2), (5.3), and (5.4),  $I_n$  can be written as polynomials in  $t$  and  $c_2, \dots, c_l$ . Then the next lemma is proved in [25], [26], and [21] (see also [18]).

**Lemma 5.1** ([25], Lemma 5.2, [26], Lemma 2.1, [21], Lemma 2.6). *The rings of invariants of the Weyl group  $W(E_l)$  ( $l = 6, 7, 8$ ) with rational coefficients are respectively given as follows:*

$$\begin{aligned} H^*(BT; \mathbb{Q})^{W(E_6)} &= \mathbb{Q}[I_2, I_5, I_6, I_8, I_9, I_{12}], \\ H^*(BT; \mathbb{Q})^{W(E_7)} &= \mathbb{Q}[I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}], \\ H^*(BT; \mathbb{Q})^{W(E_8)} &= \mathbb{Q}[I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}]. \end{aligned}$$

Hence, by Theorem 2.2, the rational cohomology rings of  $E_l/T$  ( $l = 6, 7, 8$ ) are

$$\begin{aligned} H^*(E_6/T; \mathbb{Q}) &= H^*(BT; \mathbb{Q}) / (H^+(BT; \mathbb{Q})^{W(E_6)}) \\ &= \mathbb{Q}[t_1, t_2, \dots, t_6] / (I_2, I_5, I_6, I_8, I_9, I_{12}), \\ H^*(E_7/T; \mathbb{Q}) &= H^*(BT; \mathbb{Q}) / (H^+(BT; \mathbb{Q})^{W(E_7)}) \\ &= \mathbb{Q}[t_1, t_2, \dots, t_7] / (I_2, I_6, I_8, I_{10}, I_{12}, I_{14}, I_{18}), \\ H^*(E_8/T; \mathbb{Q}) &= H^*(BT; \mathbb{Q}) / (H^+(BT; \mathbb{Q})^{W(E_8)}) \\ &= \mathbb{Q}[t_1, t_2, \dots, t_8] / (I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}), \end{aligned}$$

Then, by [24, Theorem 2.1], the integral relations  $\rho_j$  are determined by the maximal integers  $n_j$  in

$$n_j \cdot \rho_j \equiv I_j \pmod{(\rho_i; i < j)}.$$

These integers are given by

$E_6$	$E_7$	$E_8$
$n_2 = -2^4 \cdot 3$	$n_2 = -2^5 \cdot 3$	$n_2 = -2^5 \cdot 3 \cdot 5$
$n_5 = -2^7 \cdot 3 \cdot 5$	$n_6 = 2^{10} \cdot 3^2$	$n_8 = 2^{15} \cdot 3^2 \cdot 5$
$n_6 = 2^9 \cdot 3^2$	$n_8 = 2^{13} \cdot 3 \cdot 5$	$n_{12} = 2^{18} \cdot 3^4 \cdot 5 \cdot 7$
$n_8 = 2^{12} \cdot 3 \cdot 5$	$n_{10} = 2^{14} \cdot 3^2 \cdot 5 \cdot 7$	$n_{14} = 2^{20} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
$n_9 = 2^{11} \cdot 3^3 \cdot 7$	$n_{12} = -2^{16} \cdot 3^4 \cdot 5$	$n_{18} = 2^{26} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 13$
$n_{12} = -2^{15} \cdot 3^4 \cdot 5$	$n_{14} = 2^{17} \cdot 3 \cdot 7 \cdot 11 \cdot 29$	$n_{20} = 2^{27} \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41$
	$n_{18} = 2^{22} \cdot 3^3 \cdot 1229$	$n_{24} = 2^{32} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 199$
		$n_{30} = 2^{37} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61$

**5.2. Schubert presentation of  $H^*(E_l/T; \mathbb{Z})$ .** For the determination of  $A(E_l)$ , it is enough to give a minimal system of generators of  $H^*(E_l/T; \mathbb{Z})$  that consists of Schubert classes as we did in §3.4. However, it would be desirable to write down the relations explicitly among these Schubert classes, and give the presentation of  $H^*(E_l/T; \mathbb{Z})$  in terms of Schubert classes. Recently, one such presentation has been obtained by Duan-Zhao [10] in a different manner. In this subsection, we compare the generators and the relations in Theorems 3.1, 3.3, and 3.5 with those given in [10].

Following Duan-Zhao [10], we set

$$(5.5) \quad \begin{aligned} y_3 &= Z_{542}, \quad y_4 = Z_{6542} \quad (l = 6, 7, 8), \\ y_5 &= Z_{76542}, \quad y_9 = Z_{154376542} \quad (l = 7, 8). \\ y_6 &= Z_{136542}, \quad y_{10} = Z_{1654376542}, \quad y_{15} = Z_{542316543876542} \quad (l = 8). \end{aligned}$$

In [10], the fundamental weights  $\{\omega_i\}_{1 \leq i \leq l}$  ( $l = 6, 7, 8$ ) are regarded as elements of  $H^2(E_l/T; \mathbb{Z})$  ( $l = 6, 7, 8$ ) via the isomorphism  $c : H^2(BT; \mathbb{Z}) \rightarrow H^2(E_l/T; \mathbb{Z})$ . Therefore  $Z_i = \omega_i$  ( $1 \leq i \leq l$ ) ( $l = 6, 7, 8$ ) and  $t = \omega_2$  in their notation.

Then Duan-Zhao obtained the following description of the integral cohomology ring of  $E_6/T$ :

**Theorem 5.2** (Duan-Zhao [10], Theorem 3).

$$H^*(E_6/T; \mathbb{Z}) = \mathbb{Z}[\omega_1, \dots, \omega_6, y_3, y_4] / (r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{12}),$$

where

$$\begin{aligned} r_2 &= 4t^2 - c_2, \quad r_3 = 2y_3 + 2t^3 - c_3, \quad r_4 = 3y_4 + t^4 - c_4, \\ r_5 &= 2t^2y_3 - tc_4 + c_5, \quad r_6 = y_3^2 - tc_5 + 2c_6, \\ r_8 &= 3y_4^2 - 2c_5y_3 - t^2c_6 + t^3c_5, \quad r_9 = 2y_3c_6 - t^3c_6, \quad r_{12} = y_4^3 - c_6^2. \end{aligned}$$

Then, by Theorems 3.1, 5.2, Proposition 3.9, and (5.5), the correspondence is given as follows:

**Proposition 5.3.**

$$\begin{aligned} y_3 &= \gamma_3 - t^3, \quad y_4 = \gamma_4 - t^4; \\ r_2 &= -\rho_2, \quad r_3 = -\rho_3, \quad r_4 = -\rho_4, \quad r_5 = -\rho_5 + t\rho_4, \quad r_6 = \rho_6 - t\rho_5, \\ r_8 &= \rho_8 - 2y_3\rho_5 + 4t^2\rho_6 + t^3\rho_5, \quad r_9 = -\rho_9, \quad r_{12} = \rho_{12} + y_4\rho_8 + y_3\rho_9 - 2c_6\rho_6. \end{aligned}$$

Likewise, their description of the integral cohomology ring of  $E_7/T$  is given as follows:

**Theorem 5.4** (Duan-Zhao [10], Theorem 4).

$$H^*(E_7/T; \mathbb{Z}) = \mathbb{Z}[\omega_1, \dots, \omega_7, y_3, y_4, y_5, y_9] / (r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{18}),$$

where

$$\begin{aligned} r_2 &= 4t^2 - c_2, \quad r_3 = 2y_3 + 2t^3 - c_3, \quad r_4 = 3y_4 + t^4 - c_4, \\ r_5 &= 2y_5 - 2t^2y_3 + tc_4 - c_5, \quad r_6 = y_3^2 - tc_5 + 2c_6, \\ r_8 &= 3y_4^2 + 2y_3y_5 - 2y_3c_5 + 2tc_7 - t^2c_6 + t^3c_5, \\ r_9 &= 2y_9 + 2y_4y_5 - 2y_3c_6 - t^2c_7 + t^3c_6, \quad r_{10} = y_5^2 - 2y_3c_7 + t^3c_7, \end{aligned}$$

$$\begin{aligned}
r_{12} &= y_4^3 - 4y_5c_7 - c_6^2 - 2y_3y_9 - 2y_3y_4y_5 + 2ty_5c_6 + 3ty_4c_7 + c_5c_7, \\
r_{14} &= c_7^2 - 2y_5y_9 + 2y_3y_4c_7 - t^3y_4c_7, \\
r_{18} &= y_9^2 + 2y_5c_6c_7 - y_4c_7^2 - 2y_4y_5y_9 + 2y_3y_5^3 - 5ty_5^2c_7.
\end{aligned}$$

By Theorems 3.3, 5.4, Proposition 3.10, and (5.5), the correspondence is given as follows:

**Proposition 5.5.**

$$\begin{aligned}
y_3 &= \gamma_3 - t^3, \quad y_4 = \gamma_4 - t^4, \quad y_5 = \gamma_5, \quad y_9 = \gamma_9 - \gamma_4\gamma_5 + t^4\gamma_5 \\
r_2 &= -\rho_2, \quad r_3 = -\rho_3, \quad r_4 = -\rho_4, \quad r_5 = -\rho_5 + t\rho_4, \quad r_6 = \rho_6 - t\rho_5, \\
r_8 &= \rho_8 - 2y_3\rho_5 + 4t^2\rho_6 + t^3\rho_5, \quad r_9 = -\rho_9, \quad r_{10} = \rho_{10}, \\
r_{12} &= \rho_{12} + y_4\rho_8 + y_3\rho_9 - 2c_6\rho_6 + c_7\rho_5, \quad r_{14} = \rho_{14} + y_5\rho_9 + 2y_4\rho_{10}, \\
r_{18} &= \rho_{18} + y_4\rho_{14} + (3y_4^2 + 2y_3y_5 - 5tc_7)\rho_{10} + (y_4y_5 - y_9)\rho_9 + (-2c_7y_3 + t^3c_7)\rho_8 \\
&\quad + (-12tc_7y_4 - 24t^5c_7)\rho_6.
\end{aligned}$$

For the case of  $E_8/T$ , they gave the following presentation:

**Theorem 5.6** (Duan-Zhao [10], Theorem 5).

$$\begin{aligned}
H^*(E_8/T; \mathbb{Z}) &= \mathbb{Z}[\omega_1, \dots, \omega_8, y_3, y_4, y_5, y_6, y_9, y_{10}, y_{15}] \\
&\quad / (r_2, r_3, r_4, r_5, r_6, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}),
\end{aligned}$$

where

$$\begin{aligned}
r_2 &= 4\omega_2^2 - c_2, \quad r_3 = 2y_3 + 2\omega_2^3 - c_3, \quad r_4 = 3y_4 + \omega_2^4 - c_4, \\
r_5 &= 2y_5 - 2\omega_2^2y_3 + \omega_2c_4 - c_5, \quad r_6 = 5y_6 + 2y_3^2 + 10\omega_2y_5 - 2\omega_2c_5 - c_6, \\
r_8 &= 3c_8 - 3y_4^2 - 2y_3y_5 + 2y_3c_5 - 2\omega_2c_7 + \omega_2^2c_6 - \omega_2^3c_5, \\
r_9 &= 2y_9 + 2y_4y_5 - 2y_3y_6 - 4\omega_2y_3y_5 + \omega_2c_8 - \omega_2^2c_7 + \omega_2^3c_6, \\
r_{10} &= 3y_{10} - 2y_5^2 - 2y_3c_7 - 3y_4y_6 + 3y_4c_6 - 6\omega_2y_4y_5 - \omega_2^2c_8 + \omega_2^3c_7, \\
r_{12} \mid_{\omega_2=0} &= y_4^3 - 2c_4c_8 - c_5c_7 + 3c_6y_6 + c_3^2y_6 + c_3y_3^3, \\
r_{14} \mid_{\omega_2=0} &= c_7^2 + c_4y_{10} - c_3^2c_8 - c_4y_4y_6, \\
r_{15} \mid_{\omega_2=0} &= 2y_{15} + c_5y_{10} + 5c_7c_8 - c_8y_3y_4 - 2c_3c_5c_7 + c_3c_6y_6 + 2c_5y_4y_6 - c_3y_3^2y_6 + c_4y_3y_4^2 + c_3y_4^3, \\
r_{18} \mid_{\omega_2=0} &= y_9^2 - 6y_{10}y_8 - 4y_9y_6y_3 + 4y_9y_5y_4 + 5y_8y_4y_3^2 + y_7^2y_4 - 3y_7y_4^2y_3 + 5y_6^3 + 3y_6^2y_3^2 + 10y_6y_5y_4y_3 \\
&\quad + y_6y_4^3 + 6y_5y_4y_3^3, \\
r_{20} \mid_{\omega_2=0} &= (y_{10} + 4y_4y_6 - y_5^2 + 2y_3^2y_4)^2 - y_8(6y_3y_9 + 3y_4y_8 - y_5y_7 + 14y_3^2y_6 + 8y_4^4), \\
r_{24} \mid_{\omega_2=0} &= 5(y_3^2 + 2y_6)^4 - y_8(5y_7y_9 + 5y_8^2 - 4y_3y_5y_8 + 2y_3^3y_7 + 20y_5^2y_6 + 10y_3^2y_4y_6 + 18y_3y_4^2y_5 + 4y_3^4y_4), \\
r_{30} \mid_{\omega_2=0} &= (y_3^2 + 2y_6)^5 + (y_{10} + 4y_6y_4 - y_5^2 + 2y_4y_3^2)^3 + (y_{15} + y_{10}y_5 + y_9y_3^2 + 2y_8y_7 - 4y_7y_5y_3 + 5y_6^2y_3 \\
&\quad + 2y_6y_5y_4 + 2y_5y_4y_3^2 + y_4^3y_3)^2 + y_8(y_{15}y_7 + 8y_{15}y_4y_3 - 9y_{10}y_8y_4 - 10y_{10}y_6y_3^2 - 4y_{10}y_4^3 - 2y_{10}y_3^4 \\
&\quad + y_9y_7y_3^2 + 6y_9y_5y_4^2 + 8y_9y_4y_3^3 - 2y_8^2y_6 - y_8^2y_3^2 + 44y_8y_7^2 + 7y_8y_6y_5y_3 - 49y_8y_5^2y_4 + 7y_8y_5y_3^3 \\
&\quad + 25y_7^2y_5y_3 - 5y_7y_6^2y_3 + 10y_7y_6y_5y_4 - 12y_7y_5y_4y_3^2 - 30y_7y_4^3y_3 - 10y_6^3y_4 + 5y_6^2y_5^2 + 12y_5^3y_4y_3 \\
&\quad + 3y_5y_4^2y_3^3 + y_4^4y_3^3 + 4y_4y_3^6).
\end{aligned}$$

By Theorems 3.5, 5.6, Proposition 3.11, and (5.5), the correspondence is given as follows:

### Proposition 5.7.

$$\begin{aligned}y_3 &= \gamma_3 - t^3, \quad y_4 = \gamma_4 - t^4, \quad y_5 = \gamma_5, \quad y_6 = \gamma_6 - t\gamma_5 + t^2\gamma_4, \\y_9 &= \gamma_9 - 2\gamma_3^3 - 4\gamma_3\gamma_6 - \gamma_4\gamma_5 + t(-6\gamma_4^2 + 5c_8 - 4\gamma_3\gamma_5) + t^2(-4c_7 + 14\gamma_3\gamma_4) + t^3(-2\gamma_3^2 + 14\gamma_6) \\&\quad - 5t^4\gamma_5 - 10t^5\gamma_4 + 10t^6\gamma_3, \\y_{10} &= -\gamma_{10} + \gamma_5^2 - 2\gamma_3^2\gamma_4 - 4\gamma_4\gamma_6 + 2t^2\gamma_4^2 + t^4(2\gamma_3^2 + 4\gamma_6) - 4t^6\gamma_4 + 2t^{10}, \\y_{15} &\equiv \gamma_{15} + 10\gamma_6\gamma_9 - 2\gamma_5\gamma_{10} + 5\gamma_3\gamma_6^2 - \gamma_3^3\gamma_6 + 4\gamma_3^2\gamma_9 + 2\gamma_4\gamma_5\gamma_6 + 3\gamma_3^2\gamma_4\gamma_5 \pmod{(\omega_2)}.\end{aligned}$$

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