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Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter ☆

Carol L. Smith [a,*], Gregg E.A. Solomon [b], Susan Carey [c]

[a] *Department of Psychology, University of Massachusetts, Boston, MA 02125, USA*
[b] *Department of Research on Learning and Education, National Science Foundation, 4201 Wilson Blvd., Arlington, VA 22230, USA*
[c] *Department of Psychology, Harvard University, Cambridge, MA 02138, USA*

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Abstract

Clinical interviews administered to third- to sixth-graders explored children's conceptualizations of rational number and of certain extensive physical quantities. We found within child consistency in reasoning about diverse aspects of rational number. Children's spontaneous acknowledgement of the existence of numbers between 0 and 1 was strongly related to their induction that numbers are infinitely divisible in the sense that they can be repeatedly divided without ever getting to zero. Their conceptualizing number as infinitely divisible was strongly related to their having a model of fraction notation based on division and to their successful

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* Corresponding author. Fax: +1 617 287 6336.
E-mail address: Carol.smith@umb.edu (C.L. Smith).

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judgment of the relative magnitudes of fractions and decimals. In addition, their understanding number as infinitely divisible was strongly related to their understanding physical quantities as infinitely divisible. These results support a conceptual change account of knowledge acquisition, involving two-way mappings between the domains of number and physical quantity.

Keywords: Conceptual change; Conceptual development; Theory change; Rational number; Matter

1. Introduction

Students' difficulty in acquiring the concept of rational number has been well documented. Part of the problem is notational: What does "1/56" mean? Gelman (1991) showed that many elementary school children cannot explain why a given fraction is written with two numerals. Not only do children fail at explicitly explaining the mathematical role of the numerator and the denominator in representing fractions, their lack of understanding is also revealed in simple ordering tasks, such as determining whether 1/56 is larger than 1/75. Many researchers in different countries have found that this difficulty persists for some children through the high school years (e.g., Behr, Wachsmuth, Post, & Lesh, 1984, for the US; Kerslake, 1986, for England; and Nesher & Peled, 1986, for Israel). Similarly, researchers have found persistent difficulty in ordering two decimals such as 2.09 and 2.9 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Gelman, 1991; Moss & Case, 1999), placing a number like .685 on a number line that goes from 0 to 1 (Rittle-Johnson, Siegler, & Alibali, 2001), and lining up decimals such as 5.1 and .46 so as to add or subtract them (Hiebert & Wearne, 1986).

Such persistent problems with understanding fraction and decimal *notation*, including seemingly simple operations over fractions and decimals, may actually reflect deep conceptual difficulties in understanding *rational number*. Gelman (1991) and Hartnett and Gelman (1998) described the task faced by elementary school children as one of conceptual change within their concept *number*. Throughout the preschool and early elementary school years, children create, and entrench a rich concept of *counting numbers*, positive integers as represented by the integer list and base 10 notation, and as participate in operations of addition and subtraction (Fuson, 1988; Gelman & Gallistel, 1978). The representation of the positive integers is built by the child through mastering the counting algorithm, which in turn implements the successor relation among integers. For each positive integer, there is an answer to the question "Which is the next one?" The answer is: that number plus one, which happens to be the number named by the next word in the count list. In this system of interrelated concepts and operations, addition is conceptualized in terms of counting and multiplication is understood as repeated addition. It is these conceptual relations and operations that constitute the concept *number* for the child in preschool and in the early elementary grades.

Concepts are individuated both by their extensions (i.e., the entities in the world they pick out), and also by their conceptual role (i.e., the network of interrelated concepts and mental operations that define them) and coming to see "1/3" as a specific number on a par with "1" and "3" involves changes with both. Clearly, the extension of *number* is vastly expanded when it comes to include rational numbers, and the conceptual framework in which rational number is embedded differs from positive integers in every respect detailed above. Rational numbers are based on division ("*x*/*y*" means "*x* divided by *y*") and division itself is difficult to understand in terms of integers. Multiplication of integers can be modeled as repeated addition of whole numbers, but division of integers cannot always be modeled as repeated subtraction of whole numbers. To create a representation of rational number, children must develop the following interrelated understandings: that there are numbers between any two successive integers, including between 0 and 1; that the relation between the numerator and denominator in fractions is one of division; and that rational numbers are infinitely divisible and thus there are an infinite number of them between successive integers. For rational numbers, unlike integers, there is no answer to such questions as "What is the next number after 1/2?"

Given an entrenched understanding of natural number, based on the successor relation and the counting algorithm, it is not surprising that all of these interrelated representations of rational number are difficult for children to achieve. Three lines of argument support the claim that coming to represent rational number involves conceptual change. First, the claim rests on the conceptual analysis of the difference between natural number and rational number sketched above, as well as the historical fact that integers were culturally constructed vastly earlier than rational numbers. Second, teaching students about fractions and decimals is notoriously difficult (e.g., Bright, Behr, Post, & Wachsmuth, 1988; Gelman, 1991; Hartnett & Gelman, 1998). There is strong resistance to change; even after instruction, the errors children make in tasks designed to reveal their understanding of fractions are typically whole number intrusions. For example, they say that 1/56 is smaller than 1/75 because 56 is smaller than 75, that 2.9 is smaller than 2.09 because 29 is smaller than 209 (Gelman, Cohen, & Hartnett, 1989; Moss & Case, 1999). Third, various reflections of conceptual understanding of fractions—most notably awareness of the existence of numbers between integers, understanding the relation between numerator and denominator in a fraction as one of division, and being able to judge the relative magnitudes of fractions and decimals—develop in parallel across children (see Gelman, 1991, for a review of such evidence). These findings, conducted across children and in different studies, provide *indirect* evidence for an analysis of conceptual understanding in terms of interconnected principles, definitions, and operations. One goal of the present study is to probe *directly* for such consistency within the individual child, as the conceptual change position presupposes networks of ideas that are differently interrelated on both sides of the divide.

A second goal of the present study is to probe directly for an aspect of rational number understanding that has been less frequently investigated in the literature—children's understanding of the density and infinite divisibility of rational numbers—to see if this aspect of rational number understanding coheres with the others detailed above. This is a critical aspect of rational number understanding that conflicts with children's initial whole number understanding: for rational numbers, unlike whole numbers, there is no next (unique) number in the sequence. Consequently, the finding that this aspect of understanding rational number develops in parallel with others would suggest that children are restructuring the conceptual foundation of number, rather than acquiring more piecemeal knowledge about fractions and decimals.

Prior research has suggested that children's earliest understanding of infinity is as a property of processes (being endless), rather than as an amount (or number-like object) that has an order of magnitude (Fischbein, Tirosh, & Hess, 1979; Monaghan, 2001). There is converging evidence from different in depth clinical interviews that many grade 1 and 2 and the majority of grade 3 and 4 students can be led to induce and articulate the principle that there is no biggest integer and that counting numbers can always be extended by adding one (Evans, 1983; Falk, Gassner, Ben-Zoor, & Ben-Simon, 1986; Hartnett & Gelman, 1998). To date, however, there has only been one study that probes elementary school children's understanding of the infinite divisibility of numbers (Falk et al., 1986). These researchers engaged elementary school students in playing repeated rounds of a two-person game where the winner was the person who picked the smaller positive rational number. Children were asked to choose whether they wanted to go first or second, to explain why, and to judge how long the game could go on or whether it would ever end. On a conceptual change analysis, understanding the infinite divisibility of positive rational numbers would be expected to lag behind students' understanding of the infinite extendability of positive integers, and there was strong evidence that this was the case in Falk's study. Many students who understood the endlessness of the positive integers in the context of an earlier game (in which the winner was the one who chose the largest number) did not understand the endlessness of the positive rationals in the context of the game of choosing the smallest (positive) number. However, by grades 5–7 (ages 10–12), all the children chose to go second in the game, and the majority realized that the game was endless.

In our study we probed elementary school students' understanding of infinite divisibility even more directly: by involving them in a thought experiment about the repeated divisibility of number. In particular, they were asked whether if we started with a (positive) number 2 and kept dividing it in half and in half again, whether we would ever get to 0 and why. Tirosh and Stavy (1996) engaged 10th and 12th grade students with a very similar number thought experiment (given in written test format) and found that almost all these high school students conceptualized number as infinitely divisible. However, they did not give this task to younger students or explore when this understanding develops. We elaborated on their task in several ways, including asking a series of warm-up questions probing children's ideas about the existence and density of fractional numbers and the meaning of fraction and decimal notations. We also asked the questions in clinical interview rather than written test format. An important aspect of using clinical interviews is that it provides the opportunity to rephrase questions that are not understood, to clarify student meanings, and to probe student understanding more fully. Further, by asking students a sustained series of questions about fractional numbers, it may help students induce the infinite divisibility of number in the course of the interview.

If the mastery of rational number requires conceptual change within the concept of number, the question arises where the new concept comes from. As in all cases of conceptual change, if the ancestor concept (counting number) is truly different from its descendent (rational number) then the descendent cannot be represented in terms of the principles and operations that define the ancestor. Learning mechanisms that underlie such increases of representational power in the course of conceptual change have been dubbed "bootstrapping processes" and have been sketched by many historians and philosophers of science, as well as cognitive scientists (e.g., Carey, 2004; Gentner et al., 1997; Hartnett & Gelman, 1998; Kuhn, 1977; Nersessian, 1992). At its heart, bootstrapping makes use of various modeling techniques—creating analogies between different domains, limiting case analyses, thought experiments, and so on. It also makes use of the human symbolic capacity to represent the relations among interrelated concepts directly while only partially interpreting each concept in terms of antecedently understood concepts.

Many authors have suggested that the bootstrapping process through which the new concept of rational number is created involves modeling number in terms of representations of physical quantity, including protoquantitative operations such as splitting, sharing, folding, comparing, and perceiving proportionality, that draw on a qualitative appreciation of some aspects of the inferential role of rational number and ratios (Confrey, 1994; Moss & Case, 1999; Resnick & Singer, 1993). Indeed, Mix, Levine, and Huttenlocher (1999) have shown that even preschool children can manipulate models of physical quantities based on parts and ratios (e.g., know that 1/2 a circle added to 1/4 a circle yields 3/4 a circle). Both Confrey and Moss and Case have implemented curricular interventions based on these ideas. Moss and Case (1999), for example, argued that by the time children are 9 or 10 years of age, they have a global representation of proportions and a numerical structure that supports splitting and doubling. They further argued that coordinating these is part of the bootstrapping process that yields the construction of fractions and decimals. Their innovative fourth grade curriculum began with percents, as a way of numerically representing the qualitative notions of full, nearly full, half full, and nearly empty, as these apply to a beaker of water. Students were led to coordinate intuitive understanding with learned numerical halving strategies. The curriculum then moved to 2-place decimal notation, and finally to fraction notation. Rigorous pretests and posttests demonstrated that students using this curriculum out performed students using standard curricula.

The mapping between number and physical quantity is likely to be particularly important in children's coming to appreciate *the existence of rational numbers*, and that *they are repeatedly divisible*. Although young children may deny that there is a number between 0 and 1, they can see that a line of unit length exists between the origin and the first unit on a number line. Measurement activities support the existence of quantities such as 1 and 1/2 in. and 1/2 a cup. Once children see how, through measurement, natural number maps onto quantities such as length, their representation of the physical quantity as repeatedly divisible could—if the mapping were maintained—support understanding number as repeatedly divisible.

These bootstrapping processes presuppose that young children conceptualize some physical quantities as repeatedly divisible so that their representations of physical quantity can serve as a base domain for modeling rational number. But if the mapping from physical quantity to number is to support the induction of infinite divisibility of number they need something more: namely, they need to conceptualize some physical quantities as *continuous* and hence *infinitely* divisible. Of course, physical quantities are not actually infinitely divisible, both because of technical limitations and because matter is particulate, but there is much evidence that before children understand that matter is particulate, they conceive of matter as continuous, in the sense of being completely solid with no spaces within.[1] For example, prior to learning about atoms, grades 5–7 students draw continuous models of liquids and are puzzled to find that the volume of an alcohol and water mixture was less than the sum of the volume of each liquid (Snir, Smith, & Raz, 2003). Further, when engaged in thought experiments about the repeated halving of matter, many grade 8–12 students claim that the process is endless: there is always some matter left to divide (Smith, Maclin, Grosslight, & Davis, 1997; Stavy & Tirosh, 2000). Similarly when engaged in thought experiments about the serial dilution of sugar or salt water solutions, students often claim that there will always be some sugar or salt left in the solution (Tirosh & Stavy, 1996). Indeed, so strong is their intuition that matter is continuous, one of the major difficulties students have in initially learning about atoms is accepting the basic tenet that there can be empty space between atoms (Lee, Eichinger, Anderson, Berkheimer, & Blakeslee, 1993; Novick & Nussbaum, 1978; Pfundt, 1981).

However, children do not initially consider matter continuous in this sense, nor do they initially conceive of weight or different aspects of spatial extent (e.g., length, area, or volume) as continuous magnitudes. For example, in their pioneering studies of young children's conceptions of space, Piaget and Inhelder (1956) engaged young children in thought experiments about the successive division of a variety of types of geometric figures (e.g., lines, squares, and other shapes) and argued that the ability to abstract from finite sensory experience and to imagine that geometric figures are constituted of an infinite number of points only emerged with the development of formal operational thought in early adolescence (ages 11–12). They found that younger elementary children could only consider a few subdivisions with the final imagined element preserving the same form as the original (i.e., a subdivided line was still a line, a subdivided square a square). Older elementary school children could imagine a larger number of subdivisions, but they still could only consider a finite number. By early adolescence, Piaget and Inhelder claimed that children were able to imagine the infinite divisibility of a figure and to consider the final elements as dimensionless points. Later work has replicated some aspects of Piaget and Inhelder's results with geometric objects, as well as pointed out some inherent limitations and contradictions in children's understanding of infinity that were neither explored in nor the

[1] This continuous conception of matter is a useful "intermediate" construction on the way to constructing an atomic–molecular theory for two reasons. First, it supports students' conceptualizing amount of matter, weight, and volume as true extensive (additive) quantities that are differentiated from density, an intensive quantity (weight per unit volume), and learning about measurement of weight and volume. Second, it allows them to understand the claims of the particulate nature of matter more clearly: students need to be able to imagine that matter might be continuous all the way down to understand the atomic hypothesis as a very different theoretical proposal.

focus of Piaget and Inhelder's work[2] (Fischbein et al., 1979). Later work has also examined children's reasoning about the infinite divisibility of material objects (Fischbein, Tirosh, Stavy, & Oster, 1990; Smith et al., 1997; Stavy & Tirosh, 2000), the amount of space occupied by material objects, and their weight (Carey, 1991; Smith, 2005; Smith et al., 1997). The latter work has confirmed that many students can at best imagine only a limited number of divisions before the matter disappears and the amount of weight or occupied space goes to zero. For example, Carey (1991) found that many 4- to 10-year-old children claim that one can see all the steel in a solid cylinder of steel, and that if one kept dividing a piece of matter in half, one would eventually arrive at a piece that weighed nothing and took up no space, and indeed, contained no matter! Young children's understanding of the continuity of matter and the space it occupies, however, reliably preceded their understanding of the continuity of weight.

Based on these findings as well as other findings about children's differentiation of weight and density, some researchers have argued that coming to conceptualize matter, weight, and volume as continuous physical quantities involves change within these concepts (Carey, 1991; Smith, 2005; Smith et al., 1997). The core of each concept is reanalyzed: moving from unanalyzed features that are directly perceptible (e.g., matter as something that you can see, feel and touch; weight as felt weight) to more abstracted and analyzed features grounded in a network of assumed conceptual interrelationships (e.g., matter as an underlying constituent of objects; weight as a fundamental property of matter). In addition, there are key conceptual differentiations (e.g., length, area, and volume are differentiated as spatial dimensions; occupied space is differentiated from unoccupied space; weight and density are differentiated as extensive and intensive physical magnitudes) and coalescences (e.g., solids, liquids, and gases are coalescenced in a general concept of matter). Difficulties imagining that matter could be continuous and differentiating extensive from intensive physical quantities can persist into the middle school years (grades 6–8). Significantly, an important part of the conceptual change process in the domain of matter may involve cross-domain mappings with number. For example, in curricular interventions with grade 8 students, children's intuitive understanding of number as an extensive variable has been used as part of the bootstrapping process that supports conceptual change in their concepts of matter, weight, and density (Smith, 2005; Smith et al., 1997). Students were involved in cross-domain mappings as they constructed measures of weight and volume. Further, when challenged to explain how they knew a grain of rice must weigh something even though it felt like it weighed nothing at all, some

[2] As Monaghan (2001) points out, Piaget and Inhelder were more interested in children's understanding of a continuum than of infinity per se. Indeed they never explored children's conception of infinity in their work on number. He argues that the study of students' concepts of infinity would need to address a variety of aspects: "infinity as a process and as an object; infinity as a number; infinitesimals; infinite sequences and series; real numbers; the language of infinity; reasoning with the infinite; contexts (numeric/geometric, counting/measuring, and static/dynamic)" (Monaghan, 2001, p. 244). His interviews with 16- to 18-year-olds studying A-level mathematics do just that and reveal the deep difficulties, inconsistencies, and wide gap between even these older students' ideas about infinity and the formal treatments provided in mathematics.

students made mathematical arguments to convince their classmates. For example, they might argue if a single grain of rice weighs nothing at all, 0 g, then one cannot explain the palpable weight of 50 grains of rice, because 0 + 0 is still 0.

We have thus come to a conundrum: Mapping number onto physical quantity supports an infinitely divisible understanding of number only to the extent that the physical quantity is understood as continuous and infinitely divisible. Similarly, mapping a physical quantity onto number supports an infinitely divisible understanding of that physical quantity only to the extent that number is conceptualized as infinitely divisible. Although the logic seems circular, we know that this is how analogy works when part of the bootstrapping processes that support conceptual change in mathematics and science. For example, work in the history of science indicates that exploring the mapping between the mathematical and physical realms often contributes to conceptual change within both (e.g., see Nersessian's, 1992 case study of Maxwell).

Thus, the third goal of our study is to more closely examine the relation between children's understandings of infinite divisibility both within and across different domains, by testing the same children with thought experiments about number, matter, weight, and occupied space. If the conceptual changes in the domains of number and physical quantities are mutually supportive, as has previously been conjectured, then the two developments should largely go hand in hand. At the same time, if it is true that children use insights about the continuity of some physical quantities (e.g., amount of matter) to support their reconceptualization of number as infinitely divisible, then we should find evidence for slightly earlier understanding of the infinite divisibility of these physical quantities than of the infinite divisibility of number.

Previous work has investigated the relation between older (grades 7–12) students' reasoning about the infinite divisibility of a variety of *mathematical entities* (e.g., a number, a line) and their reasoning about the infinite divisibility of *material entities* (e.g., a copper wire) (Fischbein et al., 1990; Tirosh & Stavy, 1996). In general, these researchers have found a great deal of consistency in students' reasoning about both kinds of entities, with students either asserting that both or neither are infinitely divisible, especially at the younger ages (grade 7). Tirosh and Stavy (1996) interpret this consistency as evidence that students are developing the general intuitive rule "Everything can be divided" during this time—a rule that initially applies indiscriminately to mathematical and material entities. Where inconsistencies occur, it is primarily in judging numeric and geometric entities to be infinitely divisible and material objects to reach a stopping point, based on students' increasing awareness of the atomic nature of matter during the high school years. One limitation of the intuitive rules account, however, is that it provides no real developmental story of how these intuitive rules develop and no characterization of the domain to which the intuitive rule applies. Presumably, application of the intuitive rule is constrained by the ways students conceptualize the entity in question. It is unlikely students think *everything* can be divided—for example, that they think a thought or belief can be divided—but only entities that are conceptualized as having some extent. An advantage of conceptual change approach is that it offers detailed analyses of the changing components of the concepts themselves as well as of the cross-domain bootstrapping processes that might lead to their restructuring.

At present there is no systematic data on within child consistency in reasoning about the infinite divisibility of *numeric* and *material entities* during the elementary school period, the time when an understanding of the infinite divisibility of number and matter is presumably constructed. We suspect that during this age period, children's belief in the infinite divisibility of a material object might actually *precede* their understanding of the infinite divisibility of number—a hypothesis that we are directly testing in the present study.

2. Method

2.1. Participants and design

Fifty children (22 third and fourth graders and 28 fifth and sixth graders) from elementary schools in the Boston area took part in the study. The participants were 24 boys and 26 girls of different racial, ethnic, and socio-economic backgrounds, though they were predominantly white. The third and fourth graders ranged in age from 8 to 10, while the fifth and sixth graders ranged in age from 10 to 12. All of the third and fourth graders, as well as many of the fifth and sixth graders, were interviewed during the latter part of the school year.

We selected this age group to study because this is the time when learning about fractions, decimals, and the operation of division is an important focus of their math curriculum, giving children the opportunity to construct an understanding of rational number. Although children have been exposed to commonly used fractions such as 1/2, 1/3, or 1/4 in grades one and two, the primary emphasis in the early grades is on learning about whole numbers and the operations of addition and subtraction. The few activities involving fractions in grades one and two center on dividing a whole region into equal size parts and defining and recognizing fractions as parts of a whole (or parts of a group). By the third grade, however, curriculum goals include not only using a fraction to name a part of a region or a part of a group, but also using visual models to reason about fractions, knowing how to create equivalent fractions, how to compare and order like and unlike unit fractions (i.e., fractions whose numerator is one), and realizing that when the whole is divided into more parts, the size of each fractional part becomes smaller. This is also the grade when the curriculum first introduces students to decimal notation, which is related to their knowledge of fractions. In the fourth grade, students are expected to extend their facility with fractions and decimals to be able to compare and order decimals and non-unit fractions (e.g., 3/4 and 5/6) as well as use visual models and knowledge of equivalent fractions to add and subtract fractions that have unlike denominators. Important topics that are introduced as part of the fifth and sixth grade curriculum include how to find the lowest common denominator, how to convert between percents, fractions and decimals, and how to multiply and divide fractions and decimals.

Students took part in a one-on-one interview that tapped conceptual understanding of matter and number. They were asked to justify all judgments and the interviewer asked follow up questions if their reasoning was not clear. The tasks described below lasted a total of about 10 or 15 min, and were part of a longer interview that took about 30 min, with the Matter-related tasks preceding the Number-related tasks. With the exception of ideas about the infinite divisibility of number and physical quantities (which are not part of the curriculum at any grade), all the key ideas we assessed about fractions—existence of fractional numbers, explanation of the meaning of fraction notation, and ordering of unit fractions—were included in the third grade curriculum and continued to be reviewed and elaborated in the curricula at subsequent grades. The third grade curriculum introduces students to decimal notation, which is related to their understanding of fractions. Explicit ordering of one and two place decimals is only part of the curriculum starting in grade 4.

In the analyses that follow we are not concerned with grade effects per se, but with assessing coherencies in student reasoning about number throughout this period as students are gaining more experience with fractions and decimals. In some analyses we grouped the younger (grades 3 and 4) students together and compared them with the older (grades 5 and 6) students both to show rough grade trends and also to test if coherency was equally strong at both periods. Note that prior to making these groupings, we checked that there were no differences in coherency between the third and fourth graders or between the fifth and sixth graders in reasoning about number.

2.2. Tasks

2.2.1. Matter Tasks[3]

The Matter Tasks explored students' qualitative understanding that all matter must take up space and have weight, and that matter continues to exist even as it is divided into smaller and smaller pieces. Smith and her colleagues have shown these tasks to be successful at diagnosing key elements of children's commonsense theories of matter—namely that amount, spatial extent, and weight of matter are continuous, extensive physical quantities (Carey, 1991; Smith, Carey, & Wiser, 1985; Smith et al., 1997). Students were asked a progressive series of questions that led them from thinking about macroscopic pieces of Styrofoam they held in their hands to thought experiments about pieces that are too small to see or hold. The progression also moved from questions about amount of matter to questions about the weight and amount of space occupied by the Styrofoam.

2.2.1.1. Continuity of matter tasks. Students were handed a piece of Styrofoam about the size of a cracker and were asked whether it was a lot of matter, a tiny bit

[3] These Matter Tasks were preceded by a matter sorting task (described more fully in Smith et al., 1997) that began the interview. In the matter sorting task, children reflected on which entities they thought were made of matter and which were not (e.g., a rock, a grain of sugar, air, water, heat, a shadow, a wish, and a dream) and provided justifications for their classification. This task functioned as a valuable warm-up by helping students to clarify the distinction between entities that were made of some physical stuff and those that were not. However, it is not relevant to the issue of whether students conceptualize physical quantities as continuous, and so will not be discussed further here.

of matter, or none at all. Then they were given another much smaller piece, about the size of a BB pellet, and were asked the same questions. Students were next engaged in a thought experiment about matter. They were asked whether there could be a piece of Styrofoam too small to see with the naked eye. (Most students said yes; if they said no, we told them that there could be, and reminded them of the things we could only see with microscopes.) We then told them to imagine that it was possible, using a laser beam or some other tool, to divide that tiny piece in half. They were asked if we kept dividing that piece in half, would the Styrofoam matter ever disappear completely? That is, would it ever reach a point where there was not any matter left to divide? If they said simply that you would have to stop dividing, students were asked follow-up questions to determine if they believed that there would no longer be anything left to divide, or that the experiment would be technically not feasible. In the latter cases, they were encouraged to consider that such practical considerations could be overcome (e.g., "imagine that there was a machine that could cut anything, no matter how small it was") and then re-asked the question.[4]

2.2.1.2. Continuity of weight and space tasks. We then asked students questions about the properties of matter. We again gave them the two macroscopic pieces of Styrofoam and for each asked whether it weighed a lot, a tiny, tiny bit, or nothing at all. They were next asked whether a piece of Styrofoam too small to see would take up any space and whether it would weigh anything. Students who had answered yes to either of these questions were then engaged in thought experiments about the properties of small pieces of matter. We asked them whether, when we repeatedly divided the piece in half, we would eventually get to a piece that did not take up any space and whether we would get to one that did not weigh anything. Students' understandings of each property of matter were tested and analyzed separately, for Carey (1991) found that children's understanding that matter has weight lags behind their understanding that matter takes up space.

2.2.2. Number tasks

2.2.2.1. Density of numbers between 0 and 1. We began exploring students' understanding that there are an infinite number of numbers between any two integers by asking them whether there are any numbers between 0 and 1. If they said no, they were asked if 1/2 is between 0 and 1. If they said yes, they were asked to give an

[4] Because we were concerned with probing for students' understanding of continuity of matter and physical quantities rather than students' capacity to distinguish mathematical and material entities, we encouraged students to ignore physical limitations in our probing. This fact as well as the fact that we were questioning students in the context of a clinical interview may explain why we get more older elementary school children judging that matter is infinitely divisible than researchers in Israel found with grade 7 children (Fischbein et al., 1990; Tirosh & Stavy, 1996). They presented their task as a written test item and did not ask students to consider that they had tools available to divide the smallest piece. Another reason for the difference in the findings is that some Israeli students, beginning in the seventh grade, denied infinite divisibility because of the existence of atoms. In general, the US students we have worked with in grade 8 seemed less aware of the existence of atoms and did not bring them up in this context.

example. We then asked how many numbers there are between 0 and 1. We use the term "density" rather than "continuity" because the rational numbers are not continuous; there are irrational real numbers between them. Of course, we do not attribute to *children* the distinction between continuity and density, nor any knowledge of irrational numbers, nor any knowledge of different orders of infinity of the rationals and the reals.

After probing students' understanding of fraction and decimal notation (see Fraction and Decimal Quantity Comparisons and Explanation of the Meaning of Fraction Notation tasks described below), students were engaged in a thought experiment about the infinite divisibility of numbers similar to the thought experiment about the divisibility of matter. They were told that if we started with the number 2 and divided it in half, we would get the number 1, and that if we divided that in half, we would get the number 1/2. They were then asked whether we could keep dividing numbers in half forever, or whether we would get to a point where there would be no number left to divide. Finally, they were asked whether, as they were dividing, the numbers were getting bigger or smaller, and whether we would ever get to the number 0.[5]

In prior work, Gelman et al. (1989) found that the majority of kindergarten and second grade children denied the existence of numbers between 0 and 1, as is consistent with their understanding number as counting numbers. However, the few children who said "yes" were not explicitly questioned further to find out whether they thought there were only a limited number of numbers between 0 and 1. In the present study, we are able to assess whether such children conceive of number as infinitely divisible by also asking children "how many" numbers are between 0 and 1 and involving them in the number thought experiment.

2.2.2.2. Fraction and decimal quantity comparisons. Students were shown two cards, one with ".65" written on it, and the other with ".8" written on it, and they were asked to pick which is the larger number and to explain the basis for their judgment. This was repeated with "2.09" vs. "2.9" and again with "1/75" vs. "1/56." This task tapped both students' understanding of the notational conventions as well as their reasoning about the underlying quantities represented. These comparisons have been used widely in the prior mathematics literature (e.g., Gelman, 1991; Moss & Case, 1999) where it has been found that students frequently make whole number errors.

2.2.2.3. Explanation of the meaning of fractional notation. Students were shown a single card with "1/7" written on it and asked: "Why are there two numbers in a fraction? What do the two numbers mean?" Follow-up questions were asked if

[5] Of course, mathematicians would argue that in the limit, one does reach zero through infinite iteration of division by two. We included this question, however, as a further way of probing if elementary students thought the process was truly endless, not as a way of probing if they understood the mathematical idea of a limit. That is, we suspected that students who said that the process is endless would also say that one would never get to zero.

necessary to clarify student meanings. For example, if students simply said that one number is the numerator and the other is the denominator, they were asked what the numerator or denominator referred to. Or if they simply said that it referred to a pie, they were asked what it was about the pie that the two numbers referred to. In coding our data, however, we found that some student responses were ambiguous as to whether they were referring to division or subtraction and had not been fully probed. Hence, this is a task where even more follow-up probing would be desirable.

This task was used in an unpublished study by Cohen, Gelman, and Massey (reported in Gelman, 1991) in which responses were coded as irrelevant/tautological or as showing some insight. We expanded the coding of student responses to make a threefold distinction among incorrect/irrelevant responses, ambiguous responses, and responses that were based on an articulated division model.

3. Results

3.1. Children's concepts of number

Our results were first analyzed to assess multiple aspects of children's understanding of number that must change as children move from an initial concept of number as positive integer to a concept of number that includes fractions and decimals (rational number). This change entails their coming to understand that: (a) there *are* numbers between 0 and 1, or any two integers (fractions and decimals); (b) the two numerals in a fraction are related through a process of division and represent a unique point on the number line; and (c) there are an infinite number of fractions or decimals between any two integers and that these numbers are infinitely divisible. At issue is when children come to develop these understandings and how these understandings relate to one another. If children's numerical concepts cohere in inter-defined networks, then these changes should go hand-in-hand and mutually support each other as children develop an understanding of rational number.

3.1.1. Understanding the infinite divisibility of number

3.1.1.1. Existence of numbers between 0 and 1. Logically, children must first recognize the existence of numbers between 0 and 1 before they can make the inductive leap that numbers are infinitely divisible. Thus, the first question addressed was whether children realize that there are any numbers between 0 and 1, and, if so, how many they think there are.

Thirty-one out of the 50 children (62%) spontaneously agreed that there are numbers between 0 and 1. The remaining children (38%) initially said there were no numbers between 0 and 1, as is consistent with their interpreting number as "whole number." However, when specifically asked about the number "1/2," most of these children then acknowledged there are some numbers between 0 and 1, though six children (12%) continued to deny it.

3.1.1.2. Density of numbers between 0 and 1. Although most children acknowledged the existence of some numbers between 0 and 1, either spontaneously or after being probed, this finding does not mean that they thought there were an infinite number of numbers between 0 and 1. Two further probes were relevant to the issue of their understanding the density and infinite divisibility of number: one an initial open-ended probe in which we asked them how many numbers there were between 0 and 1, and the other the series of directed questions in our Number Thought Experiment.

When asked "How many numbers are there between 0 and 1?," 36% (18 out of 50) thought there were only a few numbers between 0 and 1. That is, these children gave some number less than 10, typically 1, 2, or 3. Altogether almost half (48%) of the children in the sample said that there were either no numbers between 0 and 1 or at most only a few.

In contrast, 15 children (30% of the sample) spontaneously said there were an infinite number of numbers between 0 and 1, in response to our initial query of how many numbers were between 0 and 1. The majority used the word "infinite," one used the word "continuous," and the remainder used semantic equivalents such as "numbers go on forever," "you cannot stop decimals," or there is "an endless amount of numbers."

The remaining 11 children (22% of the sample) said that there were "lots," "hundreds," "millions," or even "trillions" of numbers between 0 and 1, but stopped short of saying there were an infinite number. Indeed, the one child who said "trillions" made clear that he thought "trillions" *was* the highest number, and that you could not go higher than that. For the other children who said "lots" it was ambiguous whether they thought there are a fixed or infinite number of numbers between 0 and 1. Their answers on the Number Thought Experiment (to be discussed next) help resolve this issue.

3.1.1.3. Number thought experiment. The Number Thought Experiment was designed to probe more thoroughly whether students thought there were an infinite number of numbers between 0 and 1 by asking them whether they thought that one could divide forever without ever getting to zero. Children's answers to the Number Thought Experiment were scored in conjunction with their initial answers to the question about whether there were any numbers between 0 and 1 and if so how many. We found two coherent patterns of response (*Get to Zero*; *Never Get to Zero*) consistent with different underlying conceptions of number, and a third pattern labeled "Transitional."

The *Get to Zero* pattern is consistent with an understanding of fractional numbers as occupying finite, separate, points on a number line. Students were coded as having shown this pattern if:

(a) they claimed that there are at most only a limited number of numbers between 0 and 1; and,
(b) they claimed that when one repeatedly divided a positive number in half, one would get to 0, consistent with confusing repeated division, with repeated subtraction. Some even claimed that one would get to 0 and then pass to negative numbers.

The *Never Get to Zero* pattern is consistent with conceptualizing number as an infinitely divisible quantity. Students were coded as having shown this pattern if:

(a) they claimed there to be an infinite number of numbers between 0 and 1, or if they did not use the word "infinite," at least said that there are lots; and
(b) they claimed that you could keep dividing numbers forever, that the numbers would get smaller and smaller, but would never get to 0.

Finally, the *Transitional* pattern was defined to capture the judgments of students who did not have a consistent way of conceptualizing number: some of their judgments were consistent with thinking of number as infinitely divisible while others were not.

Students varied widely in their understanding of the infinite divisibility of number. Overall, 50% of the sample (25/50) had the Get to Zero pattern, consistent with believing there were only a finite number of fractions; 38% of the sample (19/50) had the Never Get to Zero pattern, consistent with having a firm understanding of the infinite divisibility of number; and 12% of the sample (6/50) had Transitional patterns which showed no consistent conceptualization.

Most students with the Get to Zero pattern initially said that there were no numbers between 0 and 1 and then acknowledged grudgingly, when probed directly about "1/2," that there might be a few. The following excerpts are representative of their responses:

S3 (Grade 3):
(*Any numbers between 0 and 1*?) No.
(*How about one half*?) Yes, I think so.
(*About how many numbers are there between 0 and 1*?) A little, just 0 and half, because it is halfway to one.
(*Suppose you divided 2 in half and got 1; and then divided that number in half…. Could you keep dividing forever*?) No because if you just took that half a number, that would be zero and you can't divide zero.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) Yes.

S9 (Grade 3):
(*Any numbers between 0 and 1*?) No.
(*How about one half*?) Yes, because it's not a number.
(*About how many numbers are there between 0 and 1*?) 1/2 and there are four other pieces. Quarters. There are numbers before 0, negative numbers.
(*Suppose you divided 2 in half and got 1; and then divided that number in half…. Could you keep dividing forever*?) Yes, it'll soon be just a black line, just numbers.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) Yes, because if you have 8 parts, then you minus one and minus one until you get a minus 8, then you'll get 0.

S18 (Grade 4):
(*Any numbers between 0 and 1*?) No.
(*How about one-half*?) Yes.

(*About how many numbers are there between 0 and 1*?). One: one-half.
(*Suppose you divided 2 in half and got 1; and then divided that number in half…. Could you keep dividing forever*?) No, I think you might stop. I don't think that there are that much numbers.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) Yes because 0 is, most people know it is the last number and if it is then you eventually get to it.

S39 (Grade 6):
(*Any numbers between 0 and 1*?) No.
(*How about one-half*?) Yes.
(*About how many numbers are there between 0 and 1*?) Wait a minute. There's 1/2, 1/3, 1/1, 1/over all the way up to 10.
(*Suppose you divided 2 in half and got 1; and then divided that number in half…. Could you keep dividing forever*?) No, after 1 is 0. 0 is nothing else. If kept dividing 1/2, then 1/1, then 0/1, and 0/0 and that's it.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) Yes 0 is the last number.

Only three students scored with the Get to Zero pattern acknowledged that there were lots of numbers between 0 and 1, but they still confused repeated division with repeated subtraction and ultimately thought you'd get to 0.

In contrast, students with the Never Get to Zero pattern had an entirely different way of responding to these questions, consistent with conceptualizing number as infinitely divisible. Indeed, all of these students explicitly acknowledged that there were an infinite number of numbers between 0 and 1 (or some semantic equivalent such as an endless amount) at some point during the questioning. The following excerpts give the flavor of their responses:

S12 (Grade 3):
(*Any numbers between 0 and 1*?) Yes.
(Can you give an example?) 0 and a half.
(*About how many numbers are there between 0 and 1*?). 0 and a half, 0 and 1/4, 0 and 3/4, 1/8, 2/8, and on and on and on. (How many?) Lots.
(*Suppose you divided 2 in half and got 1; and then divided that number in half…. Could you keep dividing forever*?) Yes we could just keep on going because if we ran out of numbers we could just make up names for them because numbers go on forever and ever and there's no such thing as counting up to the highest number.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) No.

S20 (Grade 4):
(*Any numbers between 0 and 1*?) Yes.
(Can you give an example?) 1/2, 1/4, 1/3, .5 fractions of a number.
(*About how many numbers are there between 0 and 1*?). It's continuous. I guess, you can keep going and going.

(*Suppose you divided 2 in half and got 1; and then divided that number in half.... Could you keep dividing forever*?) Yes. Numbers just keep on going.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) No because if numbers are getting small, you cannot measure them, and anyway, you just keep doing the same thing.

S35 (Grade 5):
(*Any numbers between 0 and 1*?) Yes.
(Can you give an example?) 1/2 or .5.
(*About how many numbers are there between 0 and 1*?). A lot.
(*Suppose you divided 2 in half and got 1; and then divided that number in half.... Could you keep dividing forever*?) Yes, there always has to be something left when you divide it.
(*Are the numbers getting bigger or smaller*?) Smaller.
(*Would you ever get to zero*?) No because there is an infinite number of numbers below 1 and above 0

S41 (Grade 6):
(*Any numbers between 0 and 1*?) Yes.
(Can you give an example?) .5
(*About how many numbers are there between 0 and 1*?). 99, no more, infinity.
(*Suppose you divided 2 in half and got 1; and then divided that number in half.... Could you keep dividing forever*?) Yes there is an infinite amount of numbers.
(Are the numbers getting bigger or smaller?) Smaller.
(*Would you ever get to zero*?) No, you have zero in your problem; it would get before other numbers but never to the actual number zero.

Only six children had patterns that seemed to reflect inconsistent reasoning about these issues across the different questions and were scored Transitional. They might be beginning to develop an understanding of number as infinitely divisible, but they still were not reasoning consistently from this perspective. Two initially said there were an infinite number of numbers between 0 and 1, but then answered some of the questions in the Number Thought experiment incorrectly. One said as you divide pieces in half, the fractions are getting bigger and hence you never get to zero; the other said as you divide the pieces in half you eventually DO get to zero. Two said that there were lots of numbers between 0 and 1, but then hedged on the questions in the thought experiment (e.g., maybe get to zero, I'm not sure how to divide fractions.) Finally, two said there were only 3 numbers between 0 and 1, but then said you can keep dividing forever and never get to 0.

Table 1 shows the relation between the students' grade level and their understanding of the infinite divisibility of number. Only 9% of third and fourth graders (2 out of 22) thought that numbers were infinitely divisible, whereas 61% of fifth and sixth graders (17 out of 28) did so. This difference is statistically significant ($X^2(1) = 13.94$, $p < .001$, $N = 50$). A major re-organization in children's conceptions of number appears to occur between fourth and fifth grade. This is not to say, of course, that the transition is always accomplished by fifth and sixth grade; 39% of fifth and sixth graders showed either Get to Zero or Transitional patterns.

Table 1
Children's understanding of the infinite divisibility of number as a function of grade

Grade	*n*	Divisibility of number pattern					
		Never Get to Zero		Transitional		Get to Zero	
		n	%	*n*	%	*n*	%
3 and 4	22	2	9	3	14	17	77
5 and 6	28	17	61	3	11	8	28
Overall	50	19	38	6	12	25	50

These data are consistent with those of Falk et al. (1986). Using a very different method (the game in which the player who chooses a smaller fraction wins), they showed that many, but not all, fifth–seventh graders understood that they should want to go second in this game because they could always construct a fraction smaller than any somebody else had chosen. Further, these children sometimes justified their choice by appealing to the infinite divisibility of number.

3.1.2. Existence of fractional numbers and infinite divisibility of number

Although it is logically possible that students would be aware of the existence of fractional numbers long before they induce that numbers are infinitely divisible, our data suggest that the two developments actually go hand in hand. Table 2 shows the strong relation between spontaneously acknowledging the existence of numbers between 0 and 1 and understanding the infinite divisibility of number in the further questions. Whereas all the students with Never Get to Zero and Transitional patterns spontaneously acknowledged there were numbers between 0 and 1, only 24% (6 of 25) of those with Get to Zero patterns did so, $X^2(1) = 25.41$, $p < .001$, $N = 50$. This finding is consistent with their having very different underlying conceptions of numbers. For students with Get to Zero patterns, positive integers are the prototypical numbers. In contrast, for students with Never Get to Zero and Transitional patterns fractions and decimals are numbers on a par with counting numbers.

The above coherence is not simply an effect of differences between third to fourth graders and fifth to sixth graders. *All* students who had given Never Get to Zero or Transitional judgments spontaneously said that there are numbers between 0 and 1, whether they were third and fourth graders or fifth and sixth graders. In contrast, only 29% of the third and fourth graders who showed Get to Zero patterns ($N = 17$)

Table 2
The relation between pattern on number thought experiment and spontaneous judgment of the existence of numbers between 0 and 1

Pattern on number thought experiment	Any numbers between 0 and 1? (spontaneous)	
	Yes	No
Never Get to Zero	19	0
Transitional	6	0
Get to Zero	6	19

Note. $N = 50$.

and only 13% of the fifth and sixth graders who showed Get to Zero patterns ($N = 8$) did so. The third and fourth graders were not significantly different in this regard than were the fifth and sixth graders, as indicated by Fisher exact test.

3.1.3. Conceptual understanding of fractions and fractional notation

When children do acknowledge there are some numbers between 0 and 1, it is common for them to mention only simple fractions, such as 1/2 or 1/4. What is not clear from such a response is the extent to which they have formed an analyzed concept of fractions that is based on division and understand the rationale behind fractional notation. As mentioned in the introduction, previous researchers have repeatedly found that early in learning about fractions, children often misread and misunderstand many aspects of fractional notation in terms of their initial concept of whole number. In this section, we consider children's understanding of fractional notation, using tasks developed by previous researchers.

3.1.3.1. Fraction quantity comparisons. The first way we assessed children's conceptual understanding of fractions and fractional notation was by asking them to make the judgment: which is larger: 1/75 or 1/56? In keeping with past research, we reasoned that if children think of each fraction as two (unrelated) whole numbers rather than defining a new unique number through the process of division, they will judge 1/75 as larger and use whole number justifications (e.g., 1/75 is greater than 1/56 because 75 is greater than 56). In contrast, if they understand that "1/75" means "one divided by 75," they will judge 1/56 as larger and give relevant explanations (e.g., if you divide a whole into 75 pieces, each part is smaller than if you only divide it into 56 pieces.)

We found that 46% of the students (23 of 50) erroneously judged that 1/75 was a larger number than 1/56. In all cases their qualitative justifications indicated that they saw the two numbers in a fraction as two distinct whole numbers rather than related by division.

In contrast, 54% of the students (27 out of 50) made the correct judgment that 1/75 was smaller than 1/56, with all but one of them articulating a relevant justification. Many envisioned what happened when you cut things into pieces; for example, "It's better to have 1/56 of a pizza than 1/75; the larger the bottom number the smaller the fraction." Others simply articulated the general rule—"The smaller the denominator the larger the fraction." Note in the latter explanation, students explicitly acknowledged a distinction between the numeric value of one part of a fraction and the whole fraction, consistent with their understanding that 1/75 defines *one* particular number, not two.

3.1.3.2. Explanation of the meaning of fraction notation. The second way we probed children's understanding of fractional notation was, following Gelman (1991), by asking them to explain why there are two numbers in a fraction such as "1/7." If children understand a fraction in terms of division, then they should be able to give meaning to the two numerals vis-à-vis an explicit division model either by directly stating that a fraction is merely one number divided by another or by explaining that the denominator indicates the number of parts the whole is divided into, and the numerator indicates the number of parts of that size. Students' explanations were coded for whether they indicated that students held *no model* or an *incorrect model* of fractions, held a *division model*, or whether they were *ambiguous*.

Overall, 43% of students (21 out of 49)[6] either had *no model* of a fraction or an *incorrect model*. Those with no model typically said uninformative things, such as "because they are there," "two numbers equal a fraction," "I forget," "don't know—I can't explain" or "top is the numerator, bottom is the denominator." Those with incorrect models all made reference to representations or concrete situations that they had witnessed in teaching, but they provided mistaken interpretations of what the numbers stood for, and thus what concept had informed the teacher's lesson. Fractions are often discussed in terms of cutting pies or other objects and some students thought the "1" referred to the "whole" and the 7 to the slices (e.g., "1 means 1 pie; 7 equal pieces" or "one whole thing, 7 slices"). Others gave incorrect mathematical formulations relating the two numbers in terms of subtraction or multiplication rather than division (e.g., "One is how many you are taking away; 7 is how many you have," "1 out of 7 is 6," or "1 times 7 = 1/7"). Still others incorrectly focused on irrelevant observable features of these situations such as color, shading, or shape rather than division relations (e.g., "If 7 blocks, only color one out of 7"; "Top number is how many pieces are shaded, bottom number is the whole thing"; or "1 is how many you have and what the shape is").

In contrast, 24% of the sample (12 out of 49) were able to articulate a *clear division model* in which they explained that the denominator refers to how many pieces the whole has been divided into and the numerator refers to how many such pieces one has. Note that we required an explicit analysis of the meaning of the two numbers in a fraction, not some vague reference to a part of a whole. For example:

- "The number of parts taken from the whole and the number of parts in the whole"
- "The top number is how many you used; bottom number is how many there are altogether in total; how many pieces to make 1"
- "One is the numerator; 7 is the denominator. The numerator is how many you have and the denominator is how many it takes to make 1"

If students had simply said it refers to "one piece of pizza" or "one piece of pie" we would not have counted it as a clear division model, because they might simply be thinking of a specific object rather than the process of division.

Finally, 31% of the sample (15 out of 49) gave *ambiguous* explanations—one's that could be consistent with either a division or subtraction interpretation. Most typically, these children said "1/7" means "1 out of 7" which could mean either 1 divided by 7 or 1 taken from a set of 7.

3.1.3.3. The relation among the two fraction notation tasks. To the extent that both fraction notation tasks probe whether students understand fractional notation as

[6] One student was inadvertently not asked this question.

expressing division, we should expect children's performance on the two tasks to be related. The results confirm this expectation (see Table 3). Students who gave a Clear Division explanation of fractions were significantly more likely to make the correct judgment about the relative magnitudes of fractions than were those students who failed to give such an explanation, $X^2(1) = 15.26$, $p < .001$, $N = 49$. Strikingly, all 12 children coded as having a Clear Division Model were correct in their comparison judgments. Equally strikingly, 86% of the children (18 of 21) who were coded as having either No Model or an Incorrect Model of fractions made incorrect comparison judgments. Finally, slightly over half of those students (10 of 16) whose explanations of the "1" and "7" in "1/7" had been coded as Ambiguous made correct comparison judgments while the rest (6 of 16) made incorrect judgments. We suspect that those who made correct judgments had probably interpreted "1 out of 7" correctly as division, while those who made incorrect judgments probably made incorrect subtraction interpretations.

A follow-up analysis confirmed that the coherence was not an artifact of an association of each task with age. *All* children (whether third or fourth graders or fifth or sixth graders) who had a clear division model explanation of fractional notation, could also correctly order the two fractions. In contrast, only 17% of third and fourth graders who had incorrect or no explanation of fractional notation ($N = 12$) and only 11% of fifth and sixth graders who had incorrect or no explanation of fractional notation ($N = 9$) were correct in ordering the fractions. This is not a significant difference between ages, as indicated by Fisher Exact Test. The only difference among the two grade levels was the patterning of students with "Ambiguous" answers on the fraction explanation task: 89% of the fifth and sixth graders with Ambiguous answers ($N = 9$) were correct in ordering the fractions as compared with only 28% of the third and fourth graders students with Ambiguous patterns ($N = 7$) ($p < .03$, Fisher Exact Test).

3.1.4. Conceptual understanding of decimal notation

Fractional numbers can, of course, also be expressed in decimal notation. Students' understanding of decimal notation calls for a place value analysis of tenths, hundredths, and so on, and so should be intimately tied to their developing a conceptual understanding of fractions and fractional notation. Just as students initially misinterpret fractional notation in terms of whole numbers, so too should students initially misinterpret decimals, and consequently use the rules for comparing whole numbers when comparing decimals.

Table 3
The relation between the fraction quantity comparison and explanation of fraction tasks

Pattern on fraction quantity comparison	Explanation of meaning of two numbers in fraction		
	Clear Division Model	Ambiguous Model	Incorrect or No Model
Correct judgment with relevant justification	12	10	3
Incorrect or questionably correct judgment	0	6	18

Note. One student was not asked for an explanation of the meaning of two numbers in a fraction; hence the *N* for this table is 49 rather than 50.

Students were asked to make two relative magnitude judgments: .65 vs. .8 and 2.09 vs. 2.9. About half of the students (23 of 50) answered both decimal problems correctly. The other half (27 out of 50) made at least one error with fully 20 of the 27 making erroneous judgments on both problems. There was no difference in the difficulty of the two problems.

Analyses of student justifications revealed that the contrasting judgments reflected different interpretations of decimal notation. In all cases, those students who judged both comparisons correctly also provided justifications that showed that they were making an appropriate interpretation of decimal notation. Many gave an explicit analysis of the place value meaning of decimals in terms of tenths and hundredths. Others implicitly recognized place value by transforming one of the numbers so that both had the same number of decimal places before making the comparison (e.g., adding a 0 so the comparison was between .65 and .80 rather than .8). Some interpreted the decimals as percents (65% vs. 80%) or as money (65 cents vs. 80 cents). Finally, a few determined which decimal was closest to the next whole number (i.e., 2.9 is closer to 3), a process that may have involved visualizing these numbers along a number line.

In contrast, the students who made at least one incorrect judgment, invariably accompanied this judgment with a justification that indicated that they had misinterpreted the decimals as whole numbers. For example, on the .65 vs. .8 comparison, they argued that .65 was larger because 65 is a bigger number than 8. Similarly, on the 2.09 vs. 2.9 comparison, they either argued that 2.09 was larger because 209 is a bigger number than 29, or they said both numbers had the same value because the 0 does not matter. Further, the few correct judgments that were made were accompanied either by comments that they were guessing or by justifications that indicated they got the right answer for the wrong reason. Thus, children with only one incorrect judgment had no more insight than children with both incorrect judgments.

3.1.4.1. Relation between understanding decimal and fraction notation. As was predicted, there was a strong relation between students' conceptual understanding of fractional notation and their understanding of decimal notation (see Table 4). The majority (73%) of those who correctly ordered fractions also correctly ordered both decimals, while the majority (83%) of those who incorrectly ordered fractions also incorrectly ordered at least one of the decimals ($X^2(1) = 15.99$, $N = 49$, $p < .001$). In cases where students understood one without the other, we found that the younger students tended to understand fractions but not decimals (5 of 7) while the older students understood decimals but not fractions (all 4 of 4). This result may reflect the

Table 4
The relation between conceptual understanding of fractions and decimals

Fraction quantity comparison	Decimal quantity comparison patterns	
	Both correct judgments	One or two incorrect judgments
Correct judgment with relevant justification	19	7
Incorrect or questionably correct judgment	4	20

Note. $N = 50$.

fact that fractions are emphasized more early in teaching, although ultimately decimal notation may be conceptually easier (see Moss & Case, 1999, for an argument for teaching fractions *after* percents & decimals).

3.1.5. Coherence analysis: Understanding fractions in terms of division

We are now in a position to ascertain the underlying coherence in children's patterns of responding across all five of the number tasks (Existence of Numbers Between 0 and 1, Number Thought Experiment, Fraction Quantity Comparison, Explanation of Meaning of Fraction Notation, and Decimal Quantity Comparisons). If children are developing a new way of thinking about number that underlies success on all our tasks, then we should see that children are either consistently correct on our number tasks or incorrect on them, with very few children with partial or in between patterns. On the other hand, if children acquire insight about fractions, decimals, and the infinite divisibility of number in more graded, piecemeal fashion, then, assuming no ceiling or floor effects, the "In Between" patterns should be just as common as either of the more extreme patterns. Table 5 shows that In Between patterns were much rarer. In fact, 37% of students were consistently correct on at least 4 out of the 5 tasks, 51% were consistently incorrect on at least 4 out of the 5 tasks, and only 12% were correct on 2 or 3 tasks. What is most striking about the distribution of scores is that those children who showed evidence of some understanding of fractions (i.e., those children who were not consistently incorrect) were three times more likely to be Consistently Correct than to show the In Between pattern of judgments. Moreover, this was as true for the younger children as it was for the older children. Given that more than half of the children were Consistently Incorrect, we can rule out the argument that this simply reflects a ceiling effect.

In addition, if developing a clear model of fractions in terms of division underlies the coherence in student responding, then we should find that students who articulate such a model should have a categorically different way of responding to the other tasks than those who do not. Table 6 shows that they do. There were strong relations between students' articulated model of fractions and their responses on each of the other four tasks. Although one might expect a strong relation between their articulated model of fractions and their ability to order fractions, the fact that there were equally strong relations with the other quite different tasks is striking. A two-way χ^2

Table 5
Within child consistency across number tasks: number of children with consistently correct, in between, and consistently incorrect patterns

Grade	*n*	Number of tasks correct					
		Consistently correct (4 or 5 correct)		In between (2 or 3 correct)		Consistently incorrect (1 or 0 correct)	
		n	%	*n*	%	*n*	%
3 and 4	22	3	14	1	4	18	82
5 and 6	27[a]	15	56	5	18	7	26
Overall	49	18	37	6	12	25	51

[a] One child not included because she had not been given one of the five tasks.

Table 6
The relation between having a Clear Division Model of fractions and one's success on other rational number tasks

Articulated Model of fractions	*n*	Other number tasks (%)				Total other number tasks correct (%)				
		Spontaneously judge numbers between 0 and 1	Correctly order fractions	Correctly order both decimals	Understand divide forever without getting to zero	4	3	2	1	0
Division Model	12	100	100	83	92	75	25	0	0	0
Ambiguous Model	16	69	63	50	38	31	19	6	25	19
Incorrect or No Model	21	33	14	19	10	5	5	5	33	52

Note. $N = 49$ because one child was not asked to explain the meaning of two numbers in a fraction.

analysis crossing whether or not students articulated a clear division model of fractions with whether they performed correctly on all of the other number tasks was significant, $X^2(1) = 14.74$, $N = 49$, $p < .001$. (Seventy-five percent of the children with a clear division model of fractions were correct on all the other tasks, compared to only 16% of students with ambiguous, incorrect, or no model of fraction patterns.) Recall that students who gave Ambiguous explanations typically said that the two numbers in the fraction 1/7 meant "you take one out of 7," an answer for which "out of" could be interpreted either to imply division or subtraction. The fact that some of these students were systematically correct and some systematically incorrect supports our assumptions that this expression can have two quite different meanings for students.

These coherence analyses assuage a worry one might have about these tasks: children were asked out of the blue whether there are any numbers between 0 and 1, and perhaps with more preparation they would have understood what was being asked. But their answers to that question cohered with the much more scaffolded and extensive clinical interviews that followed, and the consistency of responses across such very different kinds of probes suggests that those children who denied the existence of numbers between 0 and 1 had truly different conceptions of number than those who affirmed the existence of numbers between 0 and 1.

In conclusion, the coherence of the pattern of judgments and justifications across the five number tasks used in this study supports the view that a conceptual understanding of rational number is acquired as an inter-related body of representations, including representations of division and density of number. We turn now to children's representations of the divisibility and continuity of matter.

3.2. Children's concepts of matter

Children's judgments and supporting justifications about whether matter continues to exist, take up space, and have weight as it is repeatedly divided into smaller pieces (indeed pieces so small that they are no longer visible to the naked eye) were initially scored separately. At issue was whether students would believe that these physical quantities were continuous in the sense of being infinitely divisible, or whether they would argue that there would be a point at which they would cease to exist and become "nothing at all." Also at issue was whether they would develop these insights at about the same time or in some regular sequence. There is some evidence that student understanding of the continuity of weight lags behind their understandings of the continuity of matter and the space occupied by matter (Carey, 1991).

The thought experiments about the continued existence of matter and its ability to occupy space were of equivalent difficulty. Sixty-four percent of the students (32/50) judged that the macroscopic Styrofoam had some amount of matter and maintained the continued existence of matter (on a microscopic scale) with repeated division. All of these students also acknowledged that the piece would always take up space. The thought experiment about whether the piece would always have weight was more difficult. Only 46% of the students (23/50) acknowledged that the piece would always have weight. Not a single child showed an understanding of the continuity of weight who did not also understand the continuity of matter itself and the space it occupies.

Thus, there were three general patterns of response. The *Matter, Space, and Weight Continuous* pattern characterized the judgments of those children who reasoned that all three physical quantities are infinitely divisible. (By "space" in the names of the patterns we mean, of course, the space occupied by matter. We are probing a physical magnitude, not a geometric one.) The *Only Matter and Space Continuous* pattern characterized the judgments of those children who treated matter and occupied space but not weight as continuous. Finally, the *Matter, Space, and Weight Not Continuous* pattern characterized the judgments of those children who reasoned that with repeated division one gets to pieces that do not take up space or have any weight and that the matter itself has disappeared. We describe and give examples of each pattern below.

Students who showed the *Matter, Space, and Weight Continuous* pattern were confident that pieces of Styrofoam had some amount of matter, took up space, and had weight in all their judgments (both about macroscopic pieces and the smaller and smaller pieces imagined in the thought experiments). Moreover, all of these children were able to provide clear justifications for their answers, showing that they had a principled set of beliefs that supported their pattern of judgments. These beliefs included: all matter takes up space and has weight; things still exist even when you cannot see them; dividing makes parts smaller, but does not destroy the matter itself; and if you put everything together you would still have the same amount. Below are two typical protocols, one from a younger and one from an older student.

S7 (Grade 3):

(*Is there* a *lot, tiny bit, or no amount of matter in this Styrofoam piece*?) A little bit, because even the smallest piece of the Styrofoam, it's all, if you put all those tiny pieces together it makes one huge piece.

(*Can there be a piece of Styrofoam too small to see*?) Yes, because we can't see everything. But you could look through a microscope.

(*Matter Thought Experiment: Imagine it is possible to divide this tiny piece in half and in half again. If we kept dividing the tiny pieces in half and in half, would the Styrofoam matter ever disappear completely*?) No there would always be something there. Because if you had a piece of rock and you keep breaking it and breaking it and breaking it, there would still be something there even if you couldn't see it anymore.

(*Does this piece of Styrofoam weigh a lot, a tiny bit, or nothing at all*?) A little bit because everything weighs something, but sometimes on a scale you can't figure it out, but it still weighs something.

(*Now imagine a tiny piece of Styrofoam, so tiny that you couldn't see it. Would that tiny piece take up any space at all*?) Yes because everything takes up space.

(*Would that tiny piece weigh anything at all?*) Yes.

(*Space Thought Experiment: Now imagine we kept cutting that tiny piece in half, and in half again. If we kept dividing the tiny pieces in half and in half again, would we ever get to a piece that does not take up any space*?) No because you can't just make something disappear, there's always something there.

(*Weight Thought Experiment: Would we ever get to a piece that has no weight*?) Everything weighs something; if you put all those pieces together it would be heavier.

S46 (Grade 6):

(*Is there* a *lot, tiny bit, or no amount of matter in this Styrofoam piece*?) A little bit, it's something, not nothing.

(*Can there be a piece of Styrofoam too small to see*?) Yes, microscopic, human eye can't see because the way we focus our eye.

(*Matter Thought Experiment: Imagine it is possible to divide this tiny piece in half and in half again. If we kept dividing the tiny pieces in half and in half, would the Styrofoam matter ever disappear completely*?) Half of that is still something and half of that is very very tiny but it's still something. There's nothing that half of it is nothing. There's no one object that half of it is nothing.

(*Does this piece of Styrofoam weigh a lot, a tiny bit, or nothing at all*?) A very, very, very, very little bit. Like a trillionth of an ounce. It is something…it is matter.

(*Now imagine a tiny piece of Styrofoam, so tiny that you couldn't see it. Would that tiny piece take up any space at all*?) Yes, it is something taking up a certain amount of space.

(*Would that tiny piece weigh anything at all*?) Probably but I don't think there is a machine or scale that can weigh stuff that small.

(*Space Thought Experiment: Now imagine we kept cutting that tiny piece in half, and in half again. If we kept dividing the tiny pieces in half and in half again, would we ever get to a piece that does not take up any space*?) No, no matter how tiny, as long as it's matter it takes up space because it's there.

(*Weight Thought Experiment: Would we ever get to a piece that has no weight*?) …It would be unmeasurable, but it would have weight. If a tiny person tried to pick it up, it would have weight to him.

Students who showed the *Only Matter and Space Continuous* pattern were confident that as objects become extremely small they still exist and take up space, but denied that they continue to have weight. Indeed, the contrast between their judgments about weight and their judgments about matter and occupied space were striking as they all denied that even a *macroscopic* piece of Styrofoam had weight. Below are protocols typical of students showing this judgment pattern.

S13 (Grade 4):

(*Is there* a *lot, tiny bit, or no amount of matter in this Styrofoam piece*?) A little bit. The table would be bigger.

(*Can there be a piece of Styrofoam too small to see*?) Yes, lots of things are microscopic, but they are still there.

(*Matter Thought Experiment: Imagine it is possible to divide this tiny piece in half and in half again. If we kept dividing the tiny pieces in half and in half, would the Styrofoam matter ever disappear completely*?) No, I think it might be able to

go on forever. I was seeing in my mind and they could get more microscopic and more microscopic, but they'd still be there.
(*Does this piece of Styrofoam [medium piece] weigh a lot, a tiny bit, or nothing at all*?) Nothing at all. (Probe: 0 g?) Yes. A very, very, very, very little bit. Like a trillionth of an ounce. It is something…it is matter. (*Does this piece of Styrofoam [smaller piece] weigh a lot, a tiny bit, or nothing at all*?) Nothing at all, because if I felt this [bigger piece] and it felt like nothing, that wouldn't either.
(*Now imagine a tiny piece of Styrofoam, so tiny that you couldn't see it. Would that tiny piece take up any space at all*?) Yes, because all matter takes up space.
(*Would that tiny piece weigh anything at all*?) No because it is very microscopic.
(*Space Thought Experiment: Now imagine we kept cutting that tiny piece in half, and in half again. If we kept dividing the tiny pieces in half and in half again, would we ever get to a piece that does not take up any space*?) No, because everything takes up space.
(*Weight Thought Experiment: Not asked because student had already denied that a microscopic piece would have any weight.*)

S33 (Grade 5):

(*Is there* a *lot, tiny bit, or no amount of matter in this Styrofoam piece*?) A lot; you can't see everything going on. You don't know where it came from, what it's made from; it could be a rock, like one in your garden.
(*Can there be a piece of Styrofoam too small to see*?) Yes, cut it up into tiny pieces and disintegrate all of it.
(*Matter Thought Experiment: Imagine it is possible to divide this tiny piece in half and in half again. If we kept dividing the tiny pieces in half and in half, would the Styrofoam matter ever disappear completely*?) No, there would be more left every time you tried to divide it. Because you can't make it disappear—that's impossible.
(*Does this piece [medium size] of Styrofoam weigh a lot, a tiny bit, or nothing at all*?) A tiny bit. (*Does this [smaller] piece of Styrofoam weigh a lot, a tiny bit, or nothing at all?*) Nothing at all. Because it is really little. Like a straw wrapper …
(*Now imagine a tiny piece of Styrofoam, so tiny that you couldn't see it. Would that tiny piece take up any space at all*?) Yes, because it still has a little bit of it left so it still takes up space.
(*Would that tiny piece weigh anything at all*?) No because it is a little piece.
(*Space Thought Experiment: Now imagine we kept cutting that tiny piece in half, and in half again. If we kept dividing the tiny pieces in half and in half again, would we ever get to a piece that does not take up any space*?) No, because it still hasn't disappeared.
(*Weight Thought Experiment: Would we ever get to a piece that has no weight*?) Yes, doesn't weigh anything….like the smallest piece here has no weight.

Students showing the *Matter, Space, and Weight Not Continuous* pattern consistently judged that you would get to a point where there was no amount of matter, space, and weight in a piece. The students who showed this pattern typically made judgment errors *prior* to any of the thought experiments, whether for matter, space, or weight. Below are two typical protocols.

S3 (Grade 3):

(*Is there* a *lot, tiny bit, or no amount of matter in this [medium] Styrofoam piece*?) A tiny bit. (*Is there a lot, tiny bit, or no amount of matter in this [smaller] Styrofoam piece*?) Nothing at all because if you have something small and it's a part of it, the small piece would have no amount because the big piece would take up all the matter.

(*Can there be a piece of Styrofoam too small to see*?) No. (*Well, actually some pieces are so small we can't see them with our eyes, which is why we need microscopes and other special instruments.*)

(*Matter Thought Experiment: Imagine it is possible to divide this tiny piece in half and in half again. If we kept dividing the tiny pieces in half and in half, would the Styrofoam matter ever disappear completely*?) Yes you could go forever and ever, but after a year, it would stop, there wouldn't be anything left.

(*Does this piece [medium size] of Styrofoam weigh a lot, a tiny bit, or nothing at all*?) A tiny bit. (*Does this [smaller] piece of Styrofoam weigh a lot, a tiny bit, or nothing at all?*) Nothing at all. [*0 g*?] Yes, because if you took a tiny piece off, it would just feel like your own skin because it doesn't weigh anything.

(*Now imagine a tiny piece of Styrofoam, so tiny that you couldn't see it. Would that tiny piece take up any space at all*?) No because if you have really big things on a table and kept it in the corner, it wouldn't take up any space.

(*Would that tiny piece weigh anything at all*?) No.

(*Weight and Space Thought Experiments: Not asked.*)

S39 (Grade 6):

(*Is there* a *lot, tiny bit, or no amount of matter in this [medium] Styrofoam piece*?) A tiny bit. (*Is there a lot, tiny bit, or no amount of matter in this [smaller] Styrofoam piece*?) None at all, because I can't feel nothing. If I press, I won't feel it. I can feel the bigger piece.

(*Can there be a piece of Styrofoam too small to see*?) No. (*Well, actually some pieces are so small we can't see them with our eyes, which is why we need microscopes and other special instruments.*)

(*Matter Thought Experiment: Imagine it is possible to divide this tiny piece in half and in half again. If we kept dividing the tiny pieces in half and in half, would the Styrofoam matter ever disappear completely*?) If keep cutting and can't see it, it's disappeared, and no more matter left to divide.

(*Does this piece [medium size] of Styrofoam weigh a lot, a tiny bit, or nothing at all*?) A tiny bit. (*Does this [smaller] piece of Styrofoam weigh a lot, a tiny bit, or nothing at all?)* Nothing at all. Like one grain of sugar. You won't feel nothing. If you put (more) you do.

(*Now imagine a tiny piece of Styrofoam, so tiny that you couldn't see it. Would that tiny piece take up any space at all*?) No, because if so tiny how could it take up space?

(*Would that tiny piece weigh anything at all*?) No, because this (small piece) you can't feel it.

(*Weight and Space Thought Experiments: Not asked.*)

These results replicate earlier findings that there are some children who do not appreciate the continuity of matter itself, matter's spatial extent, or its weight, and that understanding the continuity of matter's spatial extent reliably precedes the understanding of the continuity of its weight (Carey, 1991; Smith, Grosslight, Davis, Unger, & Snir, 1994). It goes beyond the earlier studies by including parallel thought experiments about all three quantities and more carefully probing the thought experiment about matter (i.e., making it clearer whether students think the matter itself has disappeared or just cannot be seen) and by confirming that understanding of the continuity of matter goes hand-in-hand with the understanding of the continuity of space. Carey (1991) found a similarly close relation between understanding of continuity of matter and understanding continuity of the space occupied by matter, using a different measure of continuity of matter.

It ought not to be terribly surprising that we find evidence of a developmental trend in the acquisition of increasingly more sophisticated understandings of matter, space, and weight (see Table 7). The modal response for third and fourth graders was the *Matter, Space, and Weight Not Continuous* pattern, shown by 50% (11 out of 22) of these students, whereas for fifth and sixth graders, the modal response was the *Matter, Space, and Weight Continuous* pattern, shown by 64% (18 of 28) of these students. The association between grade and whether or not a student showed the Matter, Space, and Weight Continuous pattern is significant ($X^2(1) = 8.57$, $p < .01$, $N = 50$), though the relation is hardly categorical. About one-quarter of the third and fourth graders already treat matter, occupied space, and weight as continuous, while about one-quarter of fifth and sixth graders still reason that matter, occupied space, and weight are not continuous. What is most striking about these results is not the finding of a developmental trend across grade levels but rather the high degree of coherence within each child for a range of understandings. At both grade levels, the majority of students (73% for grades 3–4 and 89% for grades 5–6) reasoned consistently about all three physical variables: either consistently treating them all as discontinuous or continuous. Further, as will be discussed in the next section, a high degree of coherence was found not only within the domains of matter and number considered separately, but also across the domains of matter and number.

Table 7
Children's understanding of the continuity of physical quantities as a function of grade

Grade	*n*	Pattern on matter, space, and weight thought experiments					
		Matter, space, and weight continuous		Only matter and space continuous		Matter, space, and weight not continuous	
		n	%	*n*	%	*n*	%
3 and 4	22	5	23	6	27	11	50
5 and 6	28	18	64	3	11	7	25
Overall	50	23	46	9	18	18	36

3.3. Interrelations between children's conceptions of matter and number

In a result of theoretical and practical importance, students' patterns on the Number Thought Experiment were found to be strongly related to their patterns on the Matter, Space, and Weight Thought Experiments (see Tables 8 and 9). All 19 students who showed the Never Get to Zero pattern, judging that numbers could be divided ad infinitum, also judged that matter would continue to exist and take up space with repeated divisions. By contrast, only 36% of students who showed the Get to Zero number pattern did so. Indeed, some students who had infinitely divisible number and matter patterns explicitly justified their Number answers by analogy to the Matter questions (which had come earlier in the interview). For example:

- "Same as Styrofoam, could keep going forever." (S45, Grade 6)
- "There's an endless amount of numbers between 1 and zero; like Styrofoam, there's always something there." (S46, Grade 6)
- "It goes back to the matter thing. You could divide a molecule and keep dividing…an infinite number." (S47, Grade 6)

The results also show that student judgment of the infinite divisibility of matter and the space it occupies *reliably preceded* their judgment of the infinite divisibility of number. Thirteen children judged matter itself as infinitely divisible, judging that it would always occupy some space, but judged number not to be infinitely divisible, whereas the reverse pattern never occurred ($X^2(1) = 17.24$, $p < .001$, $N = 50$).

Table 8
The relation between pattern on number thought experiment and matter and space thought experiments

Pattern on number thought experiment	Pattern on matter and space thought experiments	
	Matter always exists and takes up space	Get to point where no matter or space
Never Get to Zero	19	0
Transitional	4	2
Get to Zero	9	16

Note. $N = 50$.

Table 9
The relation between pattern on number thought experiment and pattern on weight thought experiment

Pattern on number thought experiment	Pattern on weight thought experiment	
	Matter always has weight	Get to point where has no weight
Never Get to Zero	18	1
Transitional	3	3
Get to Zero	2	23

Note. $N = 50$.

Finally, the data reveal that, though student judgment of the infinite divisibility of matter itself and the space occupied by matter reliably precedes that of number, their judgment of the infinite divisibility of weight seems to occur at roughly the same time as that of number (see Table 9). Only 2% of students (1 of 50) understood the infinite divisibility of number and not that of weight, and only 4% (2 of 50) understood infinite divisibility of weight while having no insight about infinite divisibility of number (Get to Zero patterns). Further, those students with transitional patterns on number were in between in their understanding of weight: half already understood the infinite divisibility of weight, while the other half did not. These results have implications for what could be a two-way process by which a conceptual change in one domain might reciprocally aid in the change in another.

These data extend to a much younger age the overall consistency Tirosh and Stavy (1996) found between the conceptualization of physical entities and mathematical entities either as both infinitely divisable or both not so. Tirosh and Stavy interpreted this consistency as reflecting children's belief (or lack thereof) in a general rule "everything can be divided." Positing such a rule, however, does not explain the systematic unfolding of the quantities children consider divisible (matter itself, the space occupied by matter first, then number and weight).

4. Discussion

4.1. Developmental changes in children's understanding of number

Not surprisingly, the study reported here replicates, yet again, the oft-replicated findings in the literature concerning young elementary school aged children's misunderstanding of fractional notation. In our study, we found that most third and fourth graders cannot order fractions or decimals, and cannot explain why there are two numbers in a given fraction. By fifth and sixth grade, about one-third of the children still reveal clear misunderstanding of fractional notation by these measures.

In addition, clinical interview questions used in these studies allowed us to explore whether there was also a developmental shift in children's understanding of the density and infinite divisibility of number. Consistent with the claim that children do not understand the density of number, many denied that there are any numbers between 0 and 1, and when reminded about "1/2" claimed that there were only a few. Also, consistent with the claim that children confuse subtraction with division, many children believed that repeated halving would fairly quickly lead to 0. Among these children, those who believed zero was the last number thought one would stop with zero, while those who were aware of other numbers thought one would pass zero and go to the negative numbers. Notice that our thought experiment reflects an inductive leap children made on their own: children were not directly taught that there are an infinite number of fractions between 0 and 1.

4.2. Explaining the developmental shift in children's thinking about number: Conceptual change or knowledge enrichment?

One might think about the developmental shift in children's thinking about number in one of two different ways: as a *conceptual change*, such that the concept *number* before and after the change is incommensurable, or as an *enrichment* of the child's beliefs about *number*, such that new facts about the same entities, numbers, are learned. On the face of it, it might seem that an enrichment position should be preferred because representations of positive integers based on the successor relation continue to play an important role in mathematical thought even after representations of rational number have been constructed. This is so, but coming to see "1/3" as a number on a par with "1" and "3" may nonetheless implicate conceptual change within the concept number, involving a reconceptualization of the integers.

Others have denied the conceptual change position on the grounds that proto-numerical understanding of ratios and division is part of children's mental models for reasoning quantitively about objects and space (e.g., Mix et al., 1999). While this is so, and is presupposed by bootstrapping accounts that draw on these representations in the construction of representations of rational number, these are models of quantities in general and not number in particular. One can partition objects and sets without understanding that there are *numbers* like 1/2 or 1/4. Indeed, Hartnett and Gelman (1998) provide persuasive evidence that many first grade children do not understand that one-half is a number between 0 and 1.

Obviously, deciding whether the construction of rational number requires conceptual change requires *distinguishing* conceptual change from knowledge enrichment, a notoriously difficult but not impossible task (see Carey, 1991; Hartnett & Gelman, 1998; Kitcher, 1988; Kuhn, 1982). Conceptual changes involve differentiations and coalescences, such that the extension of a concept and its relations to other concepts are qualitatively different after the change than before it. The differentiations and coalescences implicated in conceptual change commit the child to concepts that would be incoherent in terms of the conceptual systems on each side of the divide. In contrast, in conceptual enrichment, new properties or subcategories are added without changing the fundamental definition or core of a concept or its network of relations with other concepts. Consequently, learning should be much more difficult in cases of conceptual change.

In previous work, Hartnett and Gelman (1998) found that 5- to 7-year-olds had much more difficulty understanding and ordering fractions (even simple fractions like one-half with which they had the most experience) than they did with understanding the Successor Principle, the principle that every natural number has a successor and that there is no largest number. They argued that inducing the Successor Principle is relatively easy even though children had never been taught this directly in school because it is consistent with young children's conception of number as counting number. In contrast, learning about fractions is difficult despite prior experience because it is inconsistent with their initial conception of number. We agree with their conclusion and turn now to four additional considerations that lead us to favor the hypothesis that genuine conceptual change, rather than knowledge enrichment, is implicated in this transition.

4.2.1. Argument 1: Learning about fractions requires change in children's definition of number

Knowledge enrichment consists in changes of beliefs formulated over the same concepts before and after the change, or in the addition of new concepts that do not implicate the revision or abandonment of antecedent ones. Consider the change that occurs when children learn that there are subtypes of dogs, breeds such as dachshunds or poodles, of which they were initially unaware. This change is a conceptual enrichment rather than a conceptual change because coming to represent these new concepts, *dachshund* and *poodle*, does not challenge the core of the child's initial concept *dog*. Dogs, before and after the change, are four-legged furry mammals of a certain size that bark, make good pets, wag their tails, eat dog food, and so on. Although some aspects of their appearance may be novel (e.g., the long body of the dachshund), once it is pointed out that they are dogs, they are easily recognizable as dogs in good standing—just shorter and longer than the typical dog. Whatever essential characteristics of dogs the child represents (e.g., that they are born from dog parents or that it is something about their insides that makes them dogs; Gelman & Wellman, 1991; Keil, 1989) are true of dogs before and after this change. In other words, children differentiate the concept *dog* into breeds without changing their initial concept, either in its essential features or in its relations to other biological kinds such as plants. Hence, children do not show strong resistance to learning about these new subcategories nor do they show strong misconceptions about their properties.

In contrast, learning that there are new kinds of numbers—such as fractions and decimals—is not so simple. Indeed, acknowledging their existence directly challenges children's initial and entrenched concept of number as counting number. Before the change, "1" and "1/2" are fundamentally different kinds of entities: "1" is a number that occurs in the count list and "1/2" is not. Some children deny that "1/2" is a number, and, although this might merely be a semantic issue to do with the term "number," the fact is that even those children who come to agree that "1/2" is a number still often claim there are only a few numbers between 0 and 1 and that repeated division will get to 0. This implicates differences in their concept of number, not just the meaning of the word. After the change, "3" is reconceptualized as a number of the same status as "1/3"—it is its multiplicative inverse, it can be expressed as "3/1" and, like 1/3, it corresponds to just another point along the number line. This is a classic coalescence in which the coalesced concept (that unites numbers like 1/3 and numbers like 3) makes no sense in the original system. The change also involves differentiating subtraction from division; this is a classic differentiation in which the undifferentiated concept (subtraction/division) is incoherent from the point of view of the attained system. And clearly the extension of the concept number changes radically, as well does as the network of interrelations based on the differentiated operation of division that constitutes the representation of rational number.

Thus, the argument that the core of the concept of *positive integer* remains essentially unchanged before and after the construction of rational number, because it is based on the successor relation, ignores the fact that in developing a concept of rational number, children have developed an entirely different model of number that has transformed their understanding of positive integers. Numbers are no longer solely the counting numbers, and the positive integers are now a subset of all numbers, lying at discrete points along a seemingly continuous number line.

In sum, the first argument that conceptual change is implicated in the child's construction of rational number has two parts: the analysis of the change sketched above, along with the empirical evidence that children progress from an earlier understanding of number embedded in the arithmetic of the counting numbers to a radically different concept that encompasses rational number.

4.2.2. Argument 2: Strong within child consistency in reasoning about number

A second line of argument for the conceptual change position derives from the pattern and extent of coherence found in children's reasoning about number across quite different kinds of tasks. Strong coherence is expected on a conceptual change account, both for younger and older children, because concepts are interrelated differently in the two systems, with understanding of one aspect of the system constraining understanding of others. In contrast, on a knowledge enrichment account, coherence is not an intrinsic part of the change process, as new facts can be added somewhat independently. Further, on the knowledge enrichment view, any coherence that is observed would be seen as resulting from extrinsic factors such as lack of exposure or explicit teaching. On this view, young children may consistently fail on certain tasks when they have not been exposed to relevant information yet; similarly, older children may consistently succeed when they have been explicitly taught all the items in question. But partial patterns of success and failure should also be abundant, especially because one is typically not exposed to all of the information at once.

Three features of the observed patterns of coherence favor a conceptual change interpretation. First, there was exceptionally strong coherence among the diverse number tasks. Being able to articulate a clear division model of fractions was strongly associated with spontaneously acknowledging the existence of numbers between 0 and 1, being able to order fractions and decimals, and understanding the infinite divisibility of number. Indeed, if one considers the five separate number tasks, the vast majority of children were either systematically correct or systematically incorrect with very few children having in between patterns.

Second, coherence was equally striking at both grade groupings. On an exposure account one might have predicted *less* coherence among the third and fourth grade children than among the fifth and sixth grade children because the younger children have had more experience with fractional than decimal notation. Hence, one might have predicted that many third and fourth grade children would have "in between" patterns reflecting only partial mastery of these ideas. Yet this prediction is not borne out. Only one of the third and fourth grade children had an in between pattern; the rest were either consistently correct (3 out of 22) or consistently incorrect (18 out of 22). Indeed the overwhelming failure of the third and fourth grade children on the different fraction problems—*despite* exposure—is quite striking. A detailed examination of their answers revealed that they had heard of fractions and knew something about them, they just did not understand them correctly as numbers. This pattern of systematic misunderstanding of a new idea—by assimilating it to an earlier entrenched understanding—lends support to the conceptual change rather than to the knowledge enrichment account.

Third, one of the tasks—the number thought experiment—tapped an understanding that children had not been explicitly taught. The fact that this task patterned as closely with the other tasks that were more related to direct instruction (such as acknowledging the existence of numbers between 0 and 1, understanding the meaning of the two numbers in a fraction, and correctly ordering fractions and decimals) also lends more support to the conceptual change than the knowledge enrichment position. If coherence is an artifact of direct instruction, then children's understanding of the infinite divisibility of number thought experiment should lag behind because it is not something that has been directly taught. In contrast, if coherence reflects conceptual restructuring, then the internal changes in children's concept of number needed to assimilate the notions of fractions and decimals should be manifest in changed understanding of the number thought experiment as well.

4.2.3. Argument 3: Making sense of the puzzling things that children say

One of the main hallmarks of conceptual change can be dubbed the "huh?" phenomenon. Children say things that make no sense if the terms in their language reflected the same concepts as adults use them to express. The transcripts included in the results sections from the "Get to 0" children contain many examples. We urge you to read these carefully. For example, student S39 said, in response to the Number Thought question about whether one could keep divided by 2 forever, "No, after 1 is 0. 0 is nothing else. If kept dividing 1/2, then 1/1, then 0/1, and 0/0 and that's it." Student S9's response to the question was "Yes, it'll soon be just a black line, just numbers." And in response to the question of whether one would ever get to zero, the student replied"Yes, if you have 8 parts, then you minus one and minus one until you get a minus 8, then you'll get 0." Although these answers seem incoherent from the adult perspective, they make much more sense when one assumes that children are thinking of "division" as "subtraction" and of numbers as discontinuous points in a sequence.

4.2.4. Argument 4: Within-child consistency in reasoning about number and physical quantities

Still a fourth argument in favor of the conceptual change account is the within child consistency we found in children's reasoning about number and physical quantities. Such consistency would be expected on conceptual change accounts that invoke bootstrapping processes across the domains, but not on simple knowledge enrichment accounts. Indeed, on knowledge enrichment accounts, one might even assume that students always are aware of the continuity of matter because these understandings are "perceptually" given, and that the only real challenge is extending these understandings to number. However, our data suggest that the developmental story of how children learn about the infinite divisibility of number is considerably more complicated than that and involves changing their conceptions of physical quantities as well.

4.3. The matter interview

Turning to the results on the matter interview, the present study replicates several phenomena previously observed. Many children of these ages (8–12) do not yet conceptualize matter itself as continuous, let alone properties of matter such as volume or weight, although about one-quarter of the youngest children (8- to 10-year-olds) tested here did. As in previous studies (Carey, 1991), understanding matter as continuous was closely associated with understanding the space occupied by matter as continuous, and understanding weight as a continuous extensive variable lagged somewhat behind. In general, however, there was still considerable coherence among all three understandings. The majority of children either consistently judged all three aspects of matter to be infinitely divisible or none to be. This latter finding is consistent with the considerable coherence in children's conceptions of matter, space, and weight found by Smith (2005) and with her explanation of these and other phenomena as stemming from a conceptual change within the child's intuitive matter theory.

We turn now to the new finding introduced in the present study: namely, the close association between coming to understand matter, weight, and space as continuous, on the one hand, and coming to understand number as continuous, on the other. As shown in Tables 8 and 9, there is a fairly close relation between these two achievements. All children who understood the infinite divisibility of number also had a continuous model of matter. Furthermore, all children who had a discontinuous model of matter also failed to understand number as infinitely divisible. Nonetheless, an understanding of the continuity of matter itself (including its capacity to occupy space) appears to precede the understanding of number as infinitely divisible. As seen in Table 8, about one-fifth of the sample understood the former but not the latter, whereas the reverse was never true. Children understood weight and number as repeatedly divisible quantities at about the same time (see Table 9).

These results are important for three reasons. First, they provide evidence for the assumption in the literature on rational number that protoquantitative conceptions of the physical world may serve as models for fractions and decimals (Confrey, 1994; Moss & Case, 1999; Resnick & Singer, 1993). Clearly, if students are going to use physical understandings to aid in the construction of mathematical understandings, it is necessary that they have the relevant physical insights first. Why is it that a continuous conception of matter might *precede* that of number? We would argue that children's concepts of physical quantities, unlike their initial concept of number, are not inherently discrete. That is, there is no positive impediment to a continuous conception of physical quantity similar to the impediment from the concept of counting number to the concept of rational number. Furthermore, there appears to be perceptual support for a continuous representation of physical extent.

Second, these data are important because they emphasize that at least some of the protoquantitative conceptions relevant to rational number are themselves hard won achievements—a fact that has been largely ignored or underappreciated in the literature on mathematical development and that is often overlooked when researchers take a knowledge enrichment rather than conceptual change perspective. Evidently, the fact that extents of matter may appear continuous, perceptually, does not automatically lead children to a conception of matter as continuous. That is, such a conception does not come for free, but involves a genuine abstraction from perceptual experience. To be sure, it is possible that, by concentrating on matter, we have underestimated the age at which most children come to appreciate the continuity of physical extent. What, for example, if we had looked at length? This question bears further research, but we note that informal piloting of a thought experiment about the infinite divisibility of a line (with most of the younger children at the very end of the interview) yielded the same pattern of data. Success on this task appeared to pattern with the matter and space thought experiments; there was no evidence that understanding the continuity of a line was easier. Children's difficulty in understanding the continuity of a line is consistent with other analyses of the child's concept of *line*. As work in children's understanding of geometry shows, children first understand geometric entities such as lines and circles as concrete physical marks on paper, and deny that a line could connect two points in empty space or could continue off the paper (Piaget, Inhelder, & Szeminska, 1964). The capacity for geometric abstraction to support these latter thought experiments may be important for the construction of a continuous model of physical extent.

Third, in our view, the most important result of the paper is the high level of coherence between children's thinking about the infinite divisibility of weight, on the one hand, and the infinite divisibility of number, on the other. Such mutual dependence is what one would expect if change involved a conceptual bootstrapping process rather than simple knowledge enrichment. At first glance, the mutual dependence may seem inflated by the fact that similar thought experiments probed children's concepts of number, matter, volume, and weight. While that is so, children's responses to the thought experiments predict other indications of their understanding of rational number (e.g., their abilities to order fractions or explain notation) and other indications of their understanding of matter (e.g., their differentiation of weight from density, their sorting of entities as material vs. non-material, their appreciation that solid entities are material throughout, see Carey, 1991; Smith et al., 1997). Thus, we conclude that the thought experiments reflect conceptual changes in each case, and the two conceptual changes are indeed mutually supportive. The association between continuous concepts of number and of weight was nearly perfect (see Table 9).

We return now to the paradox we raised in Section 1. The literature on rational number assumes that proto-quantitative conceptions of physical extent as continuous, together with processes operating over those representations (e.g., computing similarity in proportions, splitting, doubling, and so on), provide models that children can build on in coming to understand rational number. Training studies (e.g., Moss & Case, 1999) support this assumption. We too agree with it, although we note that those prior conceptions are not nearly as robust in young children as has been assumed in that literature. Further, training studies also support the assumption that applying number to physical extent, through operations of measurement, supports conceptual change within concepts of weight, space, and matter (Smith et al., 1994, 1997). In other words there are multiple, two-way interactions between the domains of number and matter, as children construct an understanding of the continuity of matter, space, and weight, and an understanding of rational number. This is how bootstrapping works—exploring the mappings between domains provides explicit representations of some of the conceptual relations among partially understood concepts, and leads to changes within each of the domains under construction. Thus, the present study not only provides strong evidence that the developmental changes in children's understanding of number involve conceptual changes, but also that this change may involve iterative cycles of bootstrapping. A central challenge for future work is to use microgenetic methods in explicit teaching studies in order to get an even more detailed understanding of how such bootstrapping might work.

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