Relevance and Conjunction

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June 1, 2008

Abstract

This paper gives an interpretation and justification of extensional and intensional conjunction in the relevant logic R. The interpretive frameworks are Anderson and Belnap's natural deduction system and the theory of situated inference from Mares (2004).

1 Introduction

This paper develops the theory of Mares (2004). On that theory, the implication of the relevant logic **R** is interpreted in terms of *situated inference*. Although I am happy with the basic theory of that book, the treatment of some of the other connectives needs to be improved. In this paper I analyze conjunction in terms of situated inference. I think that when presented with relevant logic many philosophers find its treatment of conjunction counterintuitive. It is the aim of this paper to justify the way in which relevant logic treats conjunction and also to argue that this treatment is integral to the framework of situated inference.

The plan of the paper is as follows. First, I introduce the natural deduction system for **R** and show why there is a need to abandon the standard conjunction introduction rule. Then we look at the relevant logician's distinction between extension and intensional conjunction. It is the central aim of this paper to provide interpretations of these two forms of conjunction and to use those interpretations to justify the natural deduction rules for them. I then introduce the theory of situated inference, which provides the semantic framework for these

*I would like to thank Bob Meyer, Rob Goldblatt, Nuel Belnap, Greg Restall, Mike Dunn, Wen-fen Wong, and Lloyd Humberstone for discussions over the past few years about conjunction and related matters. The research for this paper is supported by grant 05-VUW-079 from the Marsden Fund of the Royal Society of New Zealand.

interpretations. I then develop a justification for the seemingly weak introduction rule for extensional conjunction using Belnap's criterion that a system of deduction should be a conservative extension of its pure structural fragment. After that, I use a holistic view of information to argue that this rule, although it may appear to be so, is also not too strong. I then turn to intensional conjunction. I present a interpretation of it that is based on its elimination rule. I argue that a close connection between the meaning of intensional conjunction and its elimination rule is appropriate, given the function of intensional conjunction to bind premises in an inference. Thus, I provide a justification for the treatment of both extensional and intensional conjunction in the framework of the theory of situated inference.

2 Natural Deduction

The motivation for relevant logic is to avoid the so-called paradoxes of material and strict implication. One such paradox is $p \to (q \to p)$ (which is called "positive paradox"). The following is a derivation of positive paradox in a classical natural deduction system:

1.	$\quad p$	*hyp*
2.	$\quad\quad q$	*hyp*
3.	$\quad\quad p$	$1,\ reit$
4.	$\quad q \to p$	$2-3,\ \to I$
5.	$p \to (q \to p)$	$1-4,\ \to I$

Anderson and Belnap's relevant natural deduction system avoids this problem by adding an extra device to the system – steps of the proof are indexed – and using this device new constraints are placed on proofs. When we add an hypothesis to a proof, we subscript the hypothesis with a number in set brackets; the number of a hypothesis is one that has not been used before. We constrain the rules of proof such that the hypotheses have really to be used when deriving a conclusion of a subproof. Only if a hypothesis is really used can it be discharged. For example, the following is a proof of $p \to ((p \to q) \to q)$:

1.	$\quad p_{\{1\}}$	*hyp*
2.	$\quad\quad p \to q_{\{2\}}$	*hyp*
3.	$\quad\quad p_{\{1\}}$	$1, reit$
4.	$\quad\quad q_{\{1,2\}}$	$2, 3, \to E$
5.	$\quad (p \to q) \to q_{\{1\}}$	$2-4, \to I$
6.	$p \to ((p \to q) \to q)_{\emptyset}$	$1-5, \to I$

We say that there is a proof from $\{A^1_{\{1\}}, ..., A^n_{\{n\}}\}$ to B_α in this system if there is some derivation in which B_α is the conclusion and the undischarged hypotheses are just $\{A^1_{\{1\}}, ..., A^n_{\{n\}}\}$. If there is a proof from $\{A^1_{\{1\}}, ..., A^n_{\{n\}}\}$ to B_α, we write '$\{A^1_{\{1\}}, ..., A^n_{\{n\}}\} \vdash B_\alpha$' We say that there is a *strictly relevant proof* from

$\{A^1_{\{1\}}, ..., A^n_{\{n\}}\}$ to B_α if $\{A^1_{\{1\}}, ..., A^n_{\{n\}}\} \vdash B_\alpha$ and $\alpha = \{1, ..., n\}$.[1,2]

When we apply the method of subscripts to our classical proof of positive paradox, we see that it does not go through in relevant logic:

1.	$p_{\{1\}}$	*hyp*
2.	$q_{\{2\}}$	*hyp*
3.	$p_{\{1\}}$	1, *reit*
4.	$q \rightarrow p_{???}$	$2-3, \rightarrow I$
5.	$p \rightarrow (q \rightarrow p)_{???}$	$1-4, \rightarrow I$

Step 4 cannot be derived using implication introduction. We cannot discharge the second hypothesis, since '2' does not appear in the subscript to line 3. Thus it would seem that the method of subscripts together with the constraint that hypotheses must really be used in order to be discharged saves our system from at least some of the paradoxes.

If only things were this easy. The addition of conjunction to our language appears to allow us to construct a proof for positive paradox:

1.	$p_{\{1\}}$	*hyp*
2.	$q_{\{2\}}$	*hyp*
3.	$p_{\{1\}}$	1, *reit*
4.	$p \wedge q_{\{1,2\}}$	$2, 3, \wedge I$
5.	$p_{\{1,2\}}$	$4, \wedge E$
6.	$q \rightarrow p_{\{2\}}$	$2-5, \rightarrow I$
7.	$p \rightarrow (q \rightarrow p)_{\emptyset}$	$1-6, \rightarrow I$

So, the attempt to avoid the paradoxes has driven us to consider conjunction. We need to restrict the conjunction rules or we risk a commitment to positive paradox.

3 Two Conjunctions

Relevant logicians have dealt with this problem by distinguishing between *extensional conjunction* and *intensional conjunction*. The reason why they are called "intensional" and "extensional" will become clear when we discuss their

[1] Readers might be surprised that I take the premises of an argument to constitute a set rather than a multiset, but in a natural deduction system, when subscripts are included, premises do form a set. Readers may also object to the use of consecutive subscripts in the premise set. But given the way in which proofs are constructed in relevant logic, we can rearrange hypotheses at will. So we can put all the undischarged hypotheses at the start of the proof.

[2] François Rivenc says the following: "I would like to begin by clarifying the analysis of the "partial failure" which Avron sees in the approach of Anderson and Belnap. Brutally put the question is as follows: Is A deducible from the premises A, B? The authors do not furnish a clear answer to this question" (Rivenc (2005) 191). On the present version of the natural deduction framework, the answer to the question is 'yes' if we are talking about the weaker notion of relevant proof but 'not always' if we are talking about *strictly* relevant proofs.

semantics. Extensional conjunction has the standard elimination rules (with subscripts added), that is,

$$(\wedge E) \text{ From } A \wedge B_\alpha \text{ to infer } A_\alpha$$

and

$$(\wedge E) \text{ From } A \wedge B_\alpha \text{ to infer } B_\alpha.$$

Intensional conjunction, or "fusion", has the standard introduction rule with subscripts added, that is,

$$(\circ I) \text{ From } A_\alpha \text{ and } B_\beta \text{ to infer } A \circ B_{\alpha \cup \beta}.$$

The introduction rule for extensional conjunction is a slightly restricted version of the standard rule, viz.,

$$(\wedge I) \text{ From } A_\alpha \text{ and } B_\alpha \text{ to infer } A \wedge B_\alpha.$$

The restriction is that the two formulas to be conjoined are required to have the same subscript. The elimination rule for fusion is quite different than the standard conjunction rules:

$$(\circ E) \text{ From } A \circ B_\alpha \text{ and } A \rightarrow (B \rightarrow C)_\beta \text{ to infer } C_{\alpha \cup \beta}.$$

The use of these two versions of conjunction raises some important (and interrelated) questions:

1. What, apart from the need to avoid the paradoxes, is the justification for these rules?
2. What are the meanings of these two sorts of conjunction?
3. What is the connection between our understanding of these connectives and our theory of relevance?

The purpose of this paper is to answer these questions.

4 Situated Inference

The philosophical framework of this paper is the theory of situated inference that is set out in Mares (2004). The theory of situated inference is based on the situation semantics of Barwise and Perry (Barwise and Perry (1983)). Instead of taking possible worlds as the basic indices of our semantics, we use situations. A concrete situation is a part of the world. It need not be a spatio-temporally continuous part, but it needs to be an informational part of the world. A concrete situation contains some of the information that is contained in the world. The semantics for relevant logic (at least as I interpret it) uses abstract situations. An abstract situation is an abstract representation. An abstract situation is a perhaps partial, perhaps inconsistent, representation of a part of a world.

A possible situation accurately represents a part of a possible world. When a situation accurately represents a part of a world, we say that the situation is in that world. Although we use abstract situations in the semantics, we want our semantics to reflect (and idealize) the inferential practices that people actually have in concrete situations. Thus, our theory should not stray too far from describing concrete situations.

In Mares (2004), I talk of a statement being true or false in a situation. I now think that it is more correct to talk of information carried by a situation rather than what is true in it (Mares (forthcoming)). I think of the objective information in a concrete situation as potential data in that situation. An information condition for a statement is an indealization of the way in which we extract the information that the statement is true from concrete situations. Like Dummett, I think that there are *canonical* means for extracting information from contexts. The canonical means are those that I deem to be the information conditions for a statement. For example, there may be various ways of determining in a situation that a negation holds. But the canonical means is to find some information that conflicts with the negated sentence. Suppose that I want to determine that the desk I upon which my computer sits is not green. Then I can do this by determining that it has some incompatible colour, in this case brown. Thus the information condition associated with $\neg A$ is that there is some information in the situation which conflicts with A.

The theory of situated inference is a means of interpreting the natural deduction system. In a natural deduction derivation, when we make our first hypothesis, say $A_{\{1\}}$, we are saying, let us postulate a situation s_1 in a world w, such that A holds in s_1. When we make further hypotheses, say, $B_{\{2\}}$, we are assuming that there is a situation s_2 also in w such that B holds in s_2. And when we derive a line, say, $C_{\{1,2\}}$, we are saying that *really using* the information in s_1 and s_2, we derive that there is also some situation in w which satisfies C. Note that the notion of real use is not explained by the theory of situated inference, but rather is employed by it. This will be crucial for the argument of this paper.

Mares (2004) uses the theory of situated inference to give an interpretation to relevant implication. An implication $A \to B$ is true in a world w if and only if there is some situation s in that world in which we can derive B from A in the following sense. We need to be able to show that, from the perspective of A, given the assumption that there is some situation in w which satisfies A, using a situated inference we can derive that there is a situation in w which satisfies B. In this derivation, we can appeal to the information in s. This information may include particular matters of fact and *informational links.* An informational link is a piece of information, like a law of nature, a law-like exceptionless regularity, a linguistic convention, and so on, that connects other pieces of information together. For example, if in s I have as information sufficient laws governing the behaviour of our eyes, then from the assumption that there is a situation in which no artificial light in my room is on and the information that my desk is clearly visible, then we can derive that the information that there is light coming from outside my room at that time is also accurate of this world.

Instead of taking as primary conditions under which statements are true at worlds, I take conditions under which the information that a statement is true is carried by a situation. Thus, a situation s carries the information that $A \rightarrow B$ if and only if there is sufficient information in s to derive from the assumption of a situation that satisfies A to there being (in the same world) a situation that satisfies B. In what follows, we adopt the notation '$s \models A$' to mean 's satisfies A'.

5 Extensional Conjunction

The information condition for extensional conjunction is a straightforward adaptation of its truth condition:

$$s \models A \wedge B \text{ iff } s \models A \text{ and } s \models B.$$

This just says that a situation carries the information that a conjunction obtains if and only if it contains the information that each of the conjuncts obtain. This seems to me the only plausible interpretation of extensional conjunction.

This information condition supports the relevant introduction and elimination rules for extensional conjunction. In addition, it does not support the problematic introduction rule, from A_α and B_β to infer $A \wedge B_{\alpha \cup \beta}$. This is good, but the problem of justifying the rules does not end here. The question arises as to why we cannot in a situated inference infer from there being a situation which satisfies A and one which satisfies B to there being a situation which satisfies both A and B. Regardless of the introduction and elimination rules, why is this additional inference not a logical inference?

To answer this question, let us consider the nature of information a little more carefully. In the theory of situated inference, we can look at information in two ways. First, there is a perspective in which all that matters is what information we have concerning the world as a whole. From this perspective, information is encoded by formulas in the object language. We will call this "o-information" for "object-level information". Second, there is a perspective in which what matters is both what o-information exists in the world and in which situations it holds. Let us call this "m-information" for "meta-information". In our natural deduction system, m-information is encoded a steps in a proof, by the combination of an object language formula and the subscript on it.

Now, with this distinction between sorts of information in mind, let us consider again the standard version of the extensional conjunction rule. If we allow the rule, which we will call '!', we will allow the following inference:

1.	$A_{\{1\}}$	*hyp*
2.	$\quad B_{\{2\}}$	*hyp*
3.	$\quad A_{\{1\}}$	1, *reit*
4.	$\quad A \wedge B_{\{1,2\}}$	2, 3, !
5.	$\quad A_{\{1,2\}}$	4, $\wedge E$

Generalizing, the addition of the standard introduction rule forces us to accept a consequence ($\vdash$) relation with the *subscript extension* property:

$$\{A^1_{\{1\}}, ..., A^n_{\{n\}}\} \vdash A^k_\alpha,$$

where $A^k \in \{A^1, ..., A^n\}$ and α is any subset of $\{1, ..., n\}$ that includes k. From an m-informational point of view, subscript extension condones ampliative inferences. Where α contains numbers other than k, the premise list $A^1_{\{1\}}, ..., A^n_{\{n\}}$ does not tell us that A^k_α. Since situated inference is a form of deductive inference, it would seem that we would want to bar rather than justify ampliative inferences.

I think we can push this point further. The addition of the standard conjunction introduction rule leads, from the m-informational perspective, to a violation of the Belnap's conservative extension condition. To understand this claim let us look briefly at the argument of Belnap (1960). Belnap's paper is a response to Prior's famous tonk argument (Prior (1960)). The tonk argument is supposed to show that we cannot merely stipulate the meaning of a logical connective by associating with it an introduction and elimination rule. Suppose that we were to introduce a connective $*$ ("tonk") such that it had a disjunction-like introduction rule, viz.,

From A to infer $A * B$

and a conjunction-like elimination rule, such as

From $A * B$ to infer B.

Then the following inference would seem to be valid in a natural deduction system for classical logic:

1.	A	*hyp*
2.	$A * B$	$*I$
3.	B	$*E$

Thus we have derived a rule that allows the proof of any formula from any hypothesis.

Prior's argument threatens the proof theoretic approach to logic. On that approach, the meaning of a connective is given by the role that it plays in a system of inference. It would seem that on the proof-theoretic approach we can stipulate any meaning we want for a connective by associating with it whatever rules we wish. Belnap argues in reply that we can still maintain the proof theoretic approach to logic, despite Prior's argument. But Belnap disagrees with the characterization of the proof-theoretic theory of meaning; he thinks that a proof theory has to satisfy a coherence condition. Belnap claims that what is wrong with adding tonk as a connective to our system of inference is that it does not fit in with that system. Belnap uses a Gentzen system for classical logic (LK) as an example. The structural rules of that system are supposed completely to define the basic notion of a deduction. Belnap says

> In accordance with the opinions of experts (or even perhaps on more substantial grounds) we may take this little system [the pure structural rules of LK] as expressing all and only universally valid statements and rules expressible in the given notation [the connective-free fragment of the logic]: it completely determines the context. ((1960) 134)

The idea is that the rules for the fragment of the logic without connectives – i.e. the structural rules – determine completely the valid deductions for that fragment. So, when we add rules for connectives, the deductions that can be phrased in the connective-free vocabulary (i.e. that can be stated using only propostional variables) that can be proven in the new system have to be exactly the same as those that can be proven using only the structural rules. In other words, the new system has to be a *conservative extension* of the old system.[3] The addition of tonk violates the conservative extension requirement, and hence is illegitimate.

I suggest that the basic notion of a relevant proof is captured by the structural rules of Anderson and Belnap's natural deduction system. The reader may not be used to thinking of a Fitch-style natural deduction system as having structural rules, but it does. Just think of the connective-free system – the system of those rules that do not need to have any connective to represent them. This system allows the free reiteration of hypotheses into subproofs. It enables the use of the permutation; we can take the hypotheses of a proof and rearrange them and produce a proof of the same formula. It also enables us to use contraction; we can use hypotheses in derivations as often as we wish.[4]

The system also allows us to associate hypotheses in whichever manner we wish.

This connective-free system, I maintain, captures our basic notion of a relevant deduction. In particular, it captures the notion of the real use of hypotheses. Since the notion of the real use of information is incorporated in an essential way into the notion of a relevant deduction, any system of relevant logic must be a conservative extension of the connective free system.

In the appendix below, I prove that the full natural deduction system for **R** is a conservative extension of the connective free system. There I show that for a

[3] Dummett generalizes this idea in his notion of *harmony*. See Dummett (1991) Chapters 9 and 10. Dummett's view of harmony is bound up with his view that the meaning of the connectives is determined by their introduction rules (which also determine in a certain sense the corresponding elimination rules). This view cannot be maintained for Anderson and Belnap's natural deduction systems. These systems include rules that are rather disconnected from either introduction or elimination rules, such as the rule that allows the distribution of conjunction over disjunction. For a natural deduction system that is produced explicitly to satify harmony, see Leslie and Mares (2004). The logic of our paper does not include distribution. For a system that allows the derivation of distribution without a distribution rule, see Brady (2006). Brady's system still includes negation rules that are not strictly speaking introduction or elimination rules, and so does not satisfy Dummett's harmony constraint.

[4] Just as in Gentzen's sequent systems, our system the the free association of hypotheses built into its structure. In Gentzen's systems one does not bracket together some of the formulas on the left hand side of a sequent, nor do we bracket together hypotheses in a derivation.

set of hypotheses, $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash p^k_\alpha$ in the connective-free system if and only if $p^k_\alpha \in \{p^1_{\{1\}}, ..., p^n_{\{n\}}\}$, and so the only strictly relevant proofs that there prove something of the form $p_{\{n\}} \vdash p_{\{n\}}$. This holds also for the connective-free fragment of the full system. Adding the standard rule of conjunction introduction for extensional conjunction is non-conservative since, as we have seen, adding it yields a consequence relation with the subscript extension property.

6 Is $\wedge I$ too Strong?

We have been examining the complaint that the introduction rule for extensional conjunction is too weak. But one might also maintain that it is too strong. For consider the following derivation:

1.	$(A \to B) \wedge (A \to C)_{\{1\}}$	*hyp*
2.	$A \to B_{\{1\}}$	$1,\ \wedge E$
3.	$A \to C_{\{1\}}$	$1,\ \wedge E$
4.	$\quad A_{\{2\}}$	*hyp*
5.	$\quad A \to B_{\{1\}}$	2, *reit*
6.	$\quad B_{\{1,2\}}$	$4, 5,\ \to E$
7.	$\quad A \to C_{\{1\}}$	3, *reit*
8.	$\quad C_{\{1,2\}}$	$4, 7,\ \to E$
9.	$\quad B \wedge C_{\{1,2\}}$	$6, 8,\ \wedge I$
10.	$A \to (B \wedge C)_{\{1\}}$	$4-9,\ \to I$

At step nine we conjoin two formulas with the subscript $\{1, 2\}$. It might seem that this is not legitimate. For step six says that we can derive from the information in s_1 and s_2 that there is a situation in which B holds. Step eight says that we can derive from the information in s_1 and s_2 that there is a situation in which C holds. But step to get step nine we need to assume that the situations in which B and C hold are the same situation. Thus, it looks as though our interpretation of the proof theory does not justify the $\wedge I$ rule in its full form.[5]

In Mares (2004), I argue for the introduction rule on the basis of the intuitiveness of the inference derived in the above argument, that is, from $s \models A \to B$ and $s \models A \to C$ to $s \models A \to (B \wedge C)$. Although I think that this still does provide us with evidence of the acceptability of $\wedge I$ I also think that we are in the position to give a stronger justification for the rule.

When we confront information in a concrete situation (as opposed to an abstract representation of a part of a world), we confront it as a whole. In our minds we separate out the various pieces of information – the data available in the situation. In our modellings of situations, we have tended to represent situations as collections of atoms – states of affairs, "infons", and so on. This may be the correct metaphysics of abstract situations, but the phenomenology of situations is that we meet information as a conglomerate and then carve it

[5] We do of course have a justification for the form in which the subscript has at most one number in it.

up into data. This idea is not new. In a book published posthumously in 1938, Edmund Husserl used situations to deal with the problem of converse relations.[6] The statements 'the earth is larger than the moon' and 'the moon is smaller than the earth' are logically equivalent. In Husserl's earlier work, like contemporary logical atomists and situation theorists, Husserl claimed that these two sentences express different states of affairs. But if these are different states of affairs, the question arises as to why we cannot have one without the other. In terms of situations, why can there not be a situation in which the moon is smaller than the earth and the earth fails to be larger than the moon? Husserl's answer, at least as I read him, is that *we* extract these distinct states of affairs from a single situation (Husserl (1973) §59). Husserl claims that although they appear different to us, the judgments that the moon is smaller than the earth and the earth is larger than the moon refer to the same unified situation (ibid. 239).

I think we should generalize Husserl's view to apply not just to logically equivalent judgments. When we consider our real interaction with situations, say in perception, we do not meet a lot of distinct states of affairs. When we make judgments about the situations, we do distinguish features of the situation. But we confront situations in perception as complete wholes. I suggest that our semantics should reflect the holistic nature of the primary way in which we interact with information.

Let us apply this more holistic view of information to the issue of conjunction introduction. Suppose that $s \models A \rightarrow B$ and $s \models A \rightarrow C$. If we think of these facts as revealing just two aspects of the same informational whole – the situation s – then it makes sense to say that the information in s tells us that if there is a situation s' in the same world as s which satisfies A, then there is a situation also in that world that satisfies both B and C. And this is the feature of $\wedge I$ that we wished to justify.

The holistic view of information also strengthens the interpretation in Mares (2004) of the ternary accessibility relation of the Routley-Meyer semantics. In Mares (2004), I postulate a relation I (for "implication") between pairs of situations and pieces of information (or propositions). Thus, if Is_1s_2X, then really using the information in s_1 and s_2 from the assumption that they both are in the same world, we can derive that there is a situation also that world which carries the information that X.[7] Given the holistic theory of information, we can say that given all the information in s_1 and s_2 we can derive that there are situations s_3 such that s_3 carries all the information X for which Is_1s_2X. Instead of a relation between pairs of situations and pieces of information, then, we can use a ternary relation on situations such that $Rs_1s_2s_3$ if and only if s_3 satisfies all the information that can be derived from the information in s_1 and s_2.

[6] I take the name "the problem of converse relations" from Charles Cross (2002).

[7] Pieces of information here are propositons, which are just sets of situations.

7 Intensional Conjunction

Let us now turn to the issue of fusion. The most pressing problem about fusion is to give a philosophically satisfying interpretation of it. We cannot just ignore fusion; as we shall see it has a central place in the proof theory of relevant logic.

The information condition for fusion is, by itself, rather uninformative. Where R is the ternary relation of the Routley-Meyer semantics (see §6 above), $s \models A \circ B$ if and only if, for some s', s'' such that $Rs's''s$, $s' \models A$ and $s'' \models B$. In what follows, we interpret this condition.

We begin with the way Alonzo Church introduces fusion in his (1951). There Church gives the following definition:

$$A \circ B \ =_{df} \ \forall p((A \rightarrow (B \rightarrow p)) \rightarrow p)$$

As Church notes, this definition echoes Russell's definition of a "logical product" (which, in classical logic, is just conjunction) in the 1903 *Principles of Mathematics*:

> ... it is desirable to define the joint assertion of two propositions, or what is called their logical product. The definition is highly artificial, and illustrates the great distinction between mathematical and philosophical definitions. It is as follows: If p implies p, then, if q implies q, pq (the logical product of p and q) means that if p implies that q implies that r, then r is true. In other words, if p and q are propositions, their joint assertion is equivalent to saying that every proposition is true which is such that the first implies that the second implies it. (Russell (1937) §18, page 16)

The preamble about p and q implying themselves is Russell's way in the *Principles* of saying that p and q are both propositions. After that, Russell defines logical product in the same way as Church defines fusion.

Why define fusion in this way? The reason why fusion is included in the language of relevant logic is that it acts as a *premise binder*. In relevant logics we have a strictly relevant proof $\{A^1_{\{1\}}, ..., A^n_{\{n\}}\} \vdash B_{\{1,...,n\}}$ if and only if $(A^1 \circ ... \circ A^n) \rightarrow B$ is a theorem. In order to represent this "deduction equivalence" in relevant logic we need fusion (see Read (1988) §3.2).[8] The Russell-Church definition is closely related to the fusion-elimination rule of our natural deduction system:

$(\circ E)$ From $A \circ B_\alpha$ and $A \rightarrow (B \rightarrow C)_\beta$ to infer $C_{\alpha \cup \beta}$.

This elimination rule is a generalized form of modus ponens (i.e. of $\rightarrow E$). It allows us to detach a consequent from a string of implications and a list (i.e. a fusion) of the antecedents. It enables inferences from $A^1 \circ ... \circ A^n$ and $A^1 \rightarrow (...(A^n \rightarrow C)...)$ to C.

[8] In a Gentzen-style sequent system, then, $A^1; ...; A^n \vdash B$ if and only if $(A^1 \circ ... \circ A^n) \vdash B$. For sequent calculi for relevant logics, see Restall (2000).

I think that we can use this elimination rule to interpret the information condition for fusion. Simply put, a situation s carries the information that $A \circ B$ if and only if s carries the information that all the consequences of there being in the world a situation in the same world which carries the information that A and there being a situation in the world which carries the information that B. In terms of the formal semantics, $s \models A \circ B$ if and only if there are situations s' and s'' such that $s' \models A$, $s'' \models B$, and $Rs's''s$. And this is just the information condition given above. Note that $s \models A \circ B$ does not say that there are situations in the same world in which A and B hold. It only says that it carries all the information that are the consequences of there being such situations.

This is a minimal interpretation of fusion. The main role of fusion, as we have said, is to bind premises in inferences. What we have seen about its logical behaviour and its semantics shows that it may well be the minimal particle that can do this. On our interpretation, $A \circ B$, tells us that we have all the consequences of there being a situation in which A holds and one in which B holds. In other words, we have the information that $A \circ B$ if and only if we have all the consequences that can be drawn from deductive inferences the only premises of which are A and B. This seems to be the minimum that we want from the interpretation of fusion. I suggest that it is also all we want in an interpretation.

There is something else interesting in this interpretation of fusion. Dummett says that justificationism holds that the meaning of the connectives is determined by their introduction rules and that pragmatism takes their meanings to be determined by their elimination rules. The informational interpretation of logic that I give here fits into neither category. The meaning of fusion is connected more closely with its elimination rule than with its introduction rule. But the meaning of implication given in section 4 is more closely connected with its introduction rule. Let us call those connectives the meanings of which are more closely connected with their introduction rules, *I-connectives*, and those whose meanings are more closely connected with their elimination rules, *E-connectives*.[9]

As I said in section 1 information conditions are idealizations of the ways in which we extract information from concrete situations. It would seem, then, that extensional disjunction should really be treated as an E-connective. Consider the following circumstance. I am lying in bed in the morning. Neither of my two dogs is in sight. I hear a crash. I think 'either Zermela or Lola has knocked something over'. I have made an abductive inference from the sound of the crash to a disjunction about the activity of my dogs. The disjunction elimination rule tells us that if a formula C is a consequence (perhaps given other information) of both A and B, then we can infer from $A \vee B$ to C. The inference I made lying in bed reverses the disjunction elimination rule.[10]

[9]Extensional conjunction fits into both categories. Its rules are so symmetrical that one can extract its meaning from either.

[10]In the Routley-Meyer semantics, extensional disjunction is given a standard information condition which is more closely linked with the introduction rule. But I think that this should

The interpretation of fusion would seem to work the same way. If we have some of the consequences of there being a real situation in which A holds and one in which B holds, we can infer that $A \circ B$ holds in our current situation. Moreover, a very standard way of determining that we have a situation in which A holds and one which satisfies B is to find that both A and B hold in our current situation. Thus, a usual means for extracting that a fusion obtains is to determine that the corresponding extensional conjunction holds. This, I think, is a good explanation of why logicians have traditionally conflated intensional and extensional conjunction. The reason why discovering an extensional conjunction is not the canonical means for determining that an intensional conjunction holds is that it is too strong. Once we have realized that there is a good logical reason to distinguish between intensional and extensional conjunction, we need to postulate a weaker information condition for the former.

Note that although I claim that the meanings of the connectives are closely related to their introduction or elimination rules, I do not claim that their meanings are completely determined by those rules. The derivation rules point to information conditions, and these information conditions are the meanings of the connectives. Moreover, the information conditions may themselves indicate the need for further rules that are needed in the system. These further rules may be required to prove the system complete over the semantics.

Before we end, I should say a few words about the fusion introduction rule. Given the elimination rule, and its closely connected information condition, we can see that the introduction rule is correct. Under what (canonical) conditions do we have the information that a situation s contains every consequence of A together with B? The simple answer seems to be when we can derive that s is in the world from the information in a situation which satisfies A and one which satisfies B. Again, we can see that a limiting case of this is when we already are given s and that s satisfies both A and B. Thus, we have justified both the introduction and elimination rules for fusion.

8 Summary

In this paper I set out to answer three questions:

1. What, apart from the need to avoid the paradoxes, is the justification for the introduction and elimination rules for extensional and intensional conjunction?

2. What are the meanings of these two sorts of conjunction?

3. What is the connection between our understanding of these connectives and our theory of relevance?

be altered. There are other semantics for relevant logic (such as that of Fine (1974)) which give an alternative treatment of disjunction. I think that a new semantics developed by Rob Goldblatt, in which disjunction is treated in terms of sets of situations called "covers", may well be more philosophically satisfying.

Answer to question 1: With regard to extensional conjunction, the introduction and elimination rules closely correspond to the information condition for the connective. Moreover, the addition of the traditional introduction rule leads to a violation of the Belnap conservative extension requirement. An acceptable natural deduction system for relevant logic must be a conservative extension of the connective-free fragment of the Anderson-Belnap system. This connective-free fragment captures the core notion of relevant proof, including the concept of the real use of hypotheses in an inference. With regard to fusion, the key inference rule is its elimination rule. The function of fusion is to bind premises. In order to do this, it must admit of the complex form of modus ponens that is set out in the elimination rule. The information condition, thus, reflects very closely the elimination rule. The introduction rule can be taken to be a consequence of the acceptance of this information condition.

Answer to question 2: The meanings of connectives, on my view, are given by their information conditions. Thus, in setting out these information conditions, we have given the meanings of the two sorts of conjunction.

Answer to question 3: The need to restrict the introduction rule for extensional conjunction is bound up directly with a central aim of relevant logic – to provide a system of proof in which premises are really used in derivations. As we have said, the real use criteria is captured by the connective-free fragment of the system (its structural rules). In order to maintain that the full proof system satisfy the real use condition, we need to restrict implication introduction and disallow any rule that enables the derivation of the standard conjunction introduction rule. The function of fusion in relevant logic is as a premise binder (and allows us to state a version of the deduction equivalence). Its introduction and elimination are the minimal rules that allow it to perform this function.

9 Appendix: The Conservative Extension Argument

In this appendix, I prove that the complete natural deduction system is a conservative extension of its pure structural (or connective-free) fragment. My argument for this is model theoretic. Thus, I begin by setting out the model theory for relevant logic and then I will use it to prove the result.

A Routley-Meyer frame for **R** is a quadruple $\mathcal{F} =< K, 0, R, * >$ where K is a non-empty set (of situations), 0 is a non-empty subset of K (of situations at which valid formulas are verified), R is a ternary relation on K, and $*$ is an operator on K such that the following definitions and postulates hold (see (Anderson, Belnap, and Dunn (1992) §48 and Routley, et. al. (1982) chapter 4):

$$R^2stuv \text{ iff } (R \times R)stuv \text{ iff } \exists x(Rstx \ \& \ Rxuv)$$
$$s \leq t \text{ iff } \exists x(x \in 0 \ \& \ Rxst)$$

- $\leq$ is transitive and reflexive;

- $Rsss$;
- if $Rstu$, then $Rtsu$;
- if R^2sutv, then R^2stuv;
- if $Rstu$, then Rsu^*t^*;
- $s^{**} = s$.

We use other products of the ternary relation with itself. Instead of writing R^2, R^3, and so on, we just write R, since the number of situation names or variables following disambiguates between them.

In order to turn a frame into a model, we must add a value assignment. A value assignment, v, is a function from propositional variables to up-sets of situations. A value assignment determines a satisfaction relation, $\models$, which is defined as follows:

- $s \models p$ if and only if $s \in v(p)$;
- $s \models A \wedge B$ if and only if $s \models A$ and $s \models B$;
- $s \models A \vee B$ if and only if $s \models A$ or $s \models B$;
- $s \models A \rightarrow B$ if and only if $\forall x \forall y((Rsxy \ \& \ x \models A) \implies y \models B)$.

For our proof, we also add an interpretation function, I, from finite sets of natural numbers into K. We write $I(n)$ instead of $I(\{n\})$, when we are dealing with singletons of numbers. Not any such function is an interpretation function, only those that satisfy the following constraint:

$$RI(k)...I(n)I(\{k,...,n\}),$$

where $|\{k,...,n\}| \geq 2$. A sequent $\{A^1_{\{1\}},...,A^n_{\{n\}}\} \Rightarrow B_\alpha$ is valid on a model relative to I if and only if, if $I(\{i\}) \models A^i$ for all i, $1 \leq i \leq n$, then $I(\alpha) \models B$.

In my proofs I appeal to the following soundness and completeness theorems:

> **Soundness:** If $\{A^1_{\{1\}},...,A^n_{\{n\}}\} \vdash B_\alpha$, then $\{A^1_{\{1\}},...,A^n_{\{n\}}\} \Rightarrow B_\alpha$ is valid on the class of models and the class of interpretation functions on models.
>
> **Completeness**: If $\{A^1_{\{1\}},...,A^n_{\{n\}}\} \Rightarrow B_\alpha$ is valid on the class of models and the class of interpretation functions on models, then $\{A^1_{\{1\}},...,A^n_{\{n\}}\} \vdash B_\alpha$.

I also appeal to the canonical model construction from Routley and Meyer's completeness proof (see Routely, et al. (1983)). A canonical model is a structure $< K, 0, R, * >$, where

- K is the set of prime R-theories (a theory is a set of formulas closed under conjunction and provable implication and a prime theory is a theory such that if $A \vee B$ is in it, then either A or B is in it);

- 0 is the set of prime theories that contain all the theorems of **R**;
- R is a ternary relation on K such that $Rstu$ if and only if for any formulas A and B, if $A \rightarrow B \in s$ and $A \in t$, then $B \in u$;
- $*$ is an operator on K such that $s^* = \{A : \neg A \notin s\}$.

Now we can begin the proof of the conservative extension theorem.

Lemma 1 *If* $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash q_\alpha$*, then* $q_\alpha \in \{p^1_{\{1\}}, ..., p^n_{\{n\}}\}$.

Proof. Suppose that $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash q_\alpha$. Then, by the soundness theorem, in any model for any interpretation I, if $\mathcal{M}, I(i) \models p^i$ for all i $(1 \leq i \leq n)$, then $\mathcal{M}, I(\alpha) \models q$.

First I show that α must be a subset of $\{1, ..., n\}$. Suppose that there is some number m in α that is not among $1, ..., n$. Now consider the canonical model. We set $I(m) = \emptyset$. In the canonical model, $R\emptyset ss'$ for all situations s' and, in particular, $R\emptyset s\emptyset$. Generalizing, for all $s_1, ..., s_r$, $R\emptyset s_1 ... s_r \emptyset$. Thus, we can set $I(\alpha) = \emptyset$. So now we have a model $\mathcal{C}$ such that $\mathcal{C}, I(\alpha) \not\models q$ since the canonical model does not satisfy any formulas at the empty set. Thus, if α is not a subset of $\{1, ..., n\}$ we have a countermodel for $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash q_\alpha$, but by the soundness theorem this is impossible.

Second, I show that q must be one of the p^is. This is obvious, for if we take an arbitrary frame and a valuation v such that $v(r) = \emptyset$ for any propositional variable r that is not a p^i we can define a model in which q (if it is not a p^i) is satisfied nowhere.

Third, I show that if q_α is not just $p^i_{\{i\}}$ then α must contain at least two numbers. Clearly α cannot be empty, for q is not a theorem. If $\alpha = \{j\}$ then we find a frame with at least two situations s and t such that neither is great than or equal to the other. Then we can construct a model such that $v(q)$ is the upset of s and we can construct an interpretation function I such that $I(j) = t$. This would give us a countermodel for $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \Rightarrow q_\alpha$ but that is impossible.

Next, I show that $\alpha = \{i\}$ for some i between 1 and n. Suppose not. Then, α is some set that contains more than one number all of which are between 1 and n. Let us represent this set as $\{m, ..., r\}$. By the third point above, we know that this set contains at least two numbers.

Case 1. At least one of $p^m, ..., p^r$ is distinct from q. We choose one of these propositional variables distinct from q. Let's call it p^j. Take the canonical frame and set $v(p^j)$ to be the entire frame and for all the other propositional variables p^i set $v(p^i) = \{m\}$, where m is the set of all formulas. We then set $I(j) = \emptyset = I(\alpha)$. Then we have a countermodel for $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \Rightarrow q_\alpha$, which once again is impossible.

Case 2. Every $p^m, ..., p^r$ is just q. Then we have as a purely relevant deduction, $[q, ..., q] \vdash q$, with the requisite number of repetitions of q in the multiset. Thus, we have as a theorem of R,

$$\underbrace{(q \circ ... \circ q)}_{l-times} \rightarrow q.$$

And so we also have

$$\underbrace{(q \circ ... \circ q)}_{(l-1)-times} \rightarrow (q \rightarrow q).$$

In R, we have as a theorem $q \rightarrow (q \circ ... \circ q)$, for any number of fusions of q. Thus, if we were to have $(q \circ ... \circ q) \rightarrow (q \rightarrow q)$, we would also be able to derive $q \rightarrow (q \rightarrow q)$, which is the mingle axiom. And the mingle axiom is known to be invalid in **R** (see Anderson and Belnap (1975)).

Putting this altogether, I have shown that if $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash q_\alpha$, then q_α is just $p^i_{\{i\}}$ for some i between 1 and n. ■

Theorem 2 (Conservative Extension) *If* $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash q_\alpha$, *then* $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \Longrightarrow q_\alpha$ *is provable in the connective-free fragment of the natural deduction system.*

Proof. Suppose that $\{p^1_{\{1\}}, ..., p^n_{\{n\}}\} \vdash q_\alpha$. Then, by lemma 1, q_α is just one of the $p^i_{\{i\}}$s. And the following deduction is valid in the (impure) natural deduction system:

$p^1_{\{1\}}$		*hyp*
...	$p^n_{\{n\}}$	*hyp*
	$p^i_{\{i\}}$	*i, reit*

Thus, we can derive q_α from $p^1_{\{1\}}, ..., p^n_{\{n\}}$ in the connective free fragment of the natural deduction system. ■

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