Asymptotic properties of regularized differential-algebraic equations

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1 Introduction

In the present paper we consider asymptotic properties of a regularization method for linear fully implicit differential-algebraic equations (dae's)

$$A(t)x' + B(t)x(t) = 0, \quad t \in [t_0, \infty). \tag{1.1}$$

We will assume that $A(t)$ is singular for all $t \in [t_0, \infty)$. Such problems arise naturally in a number of applications, e.g. electrical networks, constraint mechanical systems of rigid bodies, chemical reaction kinetics at least as linearizations of nonlinear problems. This is why great interest has been devoted to the analysis, geometry and numerical treatment of dae's in recent years.

Nowadays, it is well-known that dae's (1.1) incorporate a great deal of new features compared with explicit ordinary differential equations (i.e. equations solved for x'). A rough criterion for distinguishing between different classes of dae's is given by the notion of an index of a dae. Although there are a number of different notions (with varying aims and applicability) in common use, their common aim is to classify how far a given dae differs from an explicit ordinary differential equation. Ordinary dae's (i.e. $A(t)$ is nonsingular for all $t \in [0, \infty)$) are characterized by the index 0. The index 1 describes the "most simple" class of dae's. Higher index equations are those with an index greater than 1. Analytically, the algebraic relations contained in (1.1) cause the higher index equations to include differentiation problems such that finite difference methods become unstable. This instability may become very dangerous if the nullspace of $A(t)$ varies with t ([9]). On the other hand, a number of numerical methods is available for the solution of index 1 dae's. So a possible way to solve higher index problems is the index reduction.

In the present paper we consider a regularization method. Eq. (1.1) is perturbed by a small parameter ε such that the resulting system has index 1 and, for $\varepsilon \to 0$, the solution of the regularized systems tends to that of (1.1). We consider the approach of [7] whose convergence properties are analyzed in [4, 5]. Note that this approach is closely related to other parametrizations, among them Baumgarte stabilization (cf. [1]).

We are interested in stability properties of the regularization approach. Namely, we will show that if (1.1) is asymptotically stable (in a sense given below), then the regularized systems are so, too. Similar results for *autonomous* quasilinear systems are given in [10] relating on stability criteria proved in [2] and [8]. We emphasize that our results are valid if the nullspace of $A(t)$ depends on t.

The paper is organized as follows. In Section 2 we introduce the solution representation of linear index 1 and index 2 dae's. Motivated by the notion of exponential asymptotic stability [3] we generalize this notion to the dae case. Section 3 deals with the regularization of linear index 2 dae's and some of its properties. Finally, in Section 4, we present our main result concernig the exponential asymptotic stability of the regularization.

2 Exponential asymptotic stability for linear dae's

In a first step consider the explicit ordinary differential equation

$$x' = B(t)x, \quad t \in [t_0, \infty), \tag{2.1}$$

with continuous coefficients $B \in C([t_0, \infty), L(\mathbb{R}^m))$, $x(t) \in \mathbb{R}^m$.

Definition 1 (cf. [3, p. 84])
The trivial solution of (2.1) is called exponentially asymptotically stable (eas) if there are constants $\alpha, K > 0$ such that, for all $\bar{t} \geq t_0$, $x^0 \in \mathbb{R}^m$, the solution of the initial value problem

$$\begin{aligned} x' + B(t)x &= 0, \quad t \in [\bar{t}, \infty) \\ x(\bar{t}) &= x^0 \end{aligned}$$

fulfils the estimate

$$|x(t)| \leq K|x^0|e^{-\alpha(t-\bar{t})}, \quad t_0 \leq \bar{t} \leq t < \infty.$$

Remark:

(i) If the trivial solution of (2.1) is eas, then it is asymptotically stable (in Lyapunov's sense).

(ii) If $B(t) \equiv B$ is a constant matrix function, then it is eas if and only if the real parts of the eigenvalues of B are strictly negative.

It is convenient to generalize Definition 1 slightly.

Definition 2 For every $t \in [t_0, \infty)$, let $V(t)$ denote a subspace of $\mathbb{R}^m$. The trivial solution of (2.1) is called exponentially asymptotically stable with respect to V if there are constants $\alpha, K > 0$ such that, for all $\bar{t} \geq t_0$, $x^0 \in V(\bar{t})$, the solution of the initial value problem

$$\begin{aligned} x' + B(t)x &= 0, \quad t \in [\bar{t}, \infty) \\ x(\bar{t}) &= x^0 \end{aligned}$$

fulfils the estimate

$$|x(t)| \leq K|x^0|e^{-\alpha(t-\bar{t})}, \quad t_0 \leq \bar{t} \leq t < \infty.$$

In contrast to Definition 1 the set of admissible initial values is restricted. Definition 2 seems to be formaly. However it is useful if the subspaces $V(t)$ are invariant solution spaces, i.e., if $x(t)$ is a solution of (2.1) on $[\bar{t}, \infty)$ with $x(\bar{t}) \in V(\bar{t})$, then $x(t) \in V(t)$ for all $t \in [\bar{t}, \infty)$.

Consider now the linear equation

$$A(t)x' + B(t)x = 0, \quad t \in [t_0, \infty), \tag{2.2}$$

with continuous coefficients. Assume that the nullspace $N(t)$ of $A(t)$ is smooth, i.e. there exists a continuously differentiable matrix function $Q \in C^1([t_0,\infty), L(\mathbb{R}^m))$ such that $Q(t)$ is a projection onto $N(t)$. Note that the rank of $A(t)$ is constant then. The trivial case $N(t) \equiv \{0\}$ is equivalent to (2.1) and shall be excluded. Furtheron, let $P = I - Q$. For simplicity we will drop the argument t if no confusion can arise.
The nullspace N of the leading coefficient matrix $A(t)$ determines which kind of functions we should accept for solutions of (1.1). Namely, $AQ \equiv 0$ implies $Ax' = APx' = A((Px)' - P'x)$. Thus, (2.2) may be rewritten

$$A(Px)' + (B - AP')x = 0, \quad t \in [t_0, \infty). \tag{2.3}$$

Therefore, we are looking for solutions of (2.2) in the function space

$$C^1_N[t_0,\infty) := \{y \in C[t_0,\infty) \mid Py \in C^1[t_0,\infty)\}.$$

More precisely, a function $x : [t_0,\infty) \to \mathbb{R}^m$ is called a solution of (2.2) if it belongs to $C^1_N[t_0,\infty)$ and fulfils (2.1). Let

$$B_0 := B - AP', \quad A_1 := A + B_0Q.$$

If x is a solution of (2.2), then, for all $t \in [t_0,\infty)$, $x(t)$ belongs to

$$S(t) := \{z \in \mathbb{R}^m \mid B_0(t)z \in R(A(t))\},$$

where $R(A(t))$ denotes the range of $A(t)$. (2.2) is called transferable (or index 1 dae) if, for all $t \in [t_0,\infty)$,

$$N(t) \oplus S(t) = \mathbb{R}^m. \tag{2.4}$$

(2.4) is equivalent to the condition that $A_1(t)$ is nonsingular (cf. [2, Theorem A.13]). Obviously, (2.3) is equivalent to

$$A_1\{P(Px)' + Qx\} + B_0Px = 0. \tag{2.5}$$

If (2.2) has index 1, we obtain by multiplication of (2.5) by QA_1^{-1} and PA_1^{-1}, respectively, the equivalent system

$$\begin{aligned} (Px)' - P'Px + PA_1^{-1}B_0Px &= 0, \\ Qx + QA_1^{-1}B_0Px &= 0. \end{aligned}$$

Lemma 1 *If (2.2) has index 1, then it is equivalent to the system*

$$\begin{aligned} u' + (PA_1^{-1}B_0 - P')u &= 0, \\ v + QA_1^{-1}B_0u &= 0, \end{aligned} \qquad t \in [t_0,\infty), \tag{2.6}$$

where $u = Px$ and $v = Qx$. Moreover, if $u^0 \in R(P(\bar{t}))$ for some $\bar{t} \in [t_0,\infty)$, then the solution u of the initial value problem $u' + (PA_1^{-1}B_0 - P')u = 0$, $u(\bar{t}) = u^0$ fulfils $u(t) \in R(P(t))$, $t \in [t_0,\infty)$.

The proof of Lemma 1 is obvious. In order to show the second assertion, simply multiply the differential equation by Q leading to $(Qu)' - Q'Qu = 0$.

Especially, (2.6) implies $x = Px + Qx = u - QA_1^{-1}B_0 u = (I - QA_1^{-1}B_0)Pu$. Note that $Q_s(t) := QA_1^{-1}B_0(t)$ is the projection of $\mathbb{R}^m$ onto $N(t)$ along $S(t)$. Once the Px-component of a solution is found, the nullspace components are given by a simple assignment. Therefore, initial conditions can only be given for $Px(\bar{t})$. This suggests the following definition.

Definition 3 Let (2.2) be an index 1 dae. The trivial solution of (2.2) is called exponentially asymptotically stable if there are constants $\alpha, K > 0$ such that, for all $\bar{t} \geq t_0$, $x^0 \in \mathbb{R}^m$, the solution of the initial value problem

$$\begin{aligned} A(t)x' + B(t)x = 0, & \qquad t \in [\bar{t}, \infty), \\ P(\bar{t})(x(\bar{t}) - x^0) = 0, & \end{aligned}$$

fulfils the estimate

$$|x(t)| \leq K|P(\bar{t})x^0|e^{-\alpha(t-\bar{t})}, \quad t_0 \leq \bar{t} \leq t < \infty.$$

Remark: In [2, p. 74] asymptotic stability in the sense of Lyapunov is defined for general nonlinear transferable dae's. If the trivial solution of (2.2) is eas, then it is asymptotically stable in the sense of the latter definition.

Theorem 1 *Let $Q_s = QA_1^{-1}B_0$ be bounded on $[t_0, \infty)$. Then the trivial solution of* (2.2) *is eas if and only if the trivial solution of $u' + (PA_1^{-1}B_0 - P')u = 0$, $t \in [t_0, \infty)$, is eas with respect to $R(P(t))$.*

The proof follows simply from (2.6) and the representation $x = (I - QA_1^{-1}B_0)Pu$.

Remark: In [2, p. 78] a notion of contractivity for nonlinear transferable dae's is defined. Lemma 1.2.44 of that monograph shows that contractivity implies exponential asymptotic stability.

In contrast to (2.4), higher index dae's are characterized by nontrivial intersections $N(t) \cap S(t)$. Equivalently, A_1 is singular. Let $Q_1(t)$ denote a projection onto the kernel of $A_1(t)$,

$$B_1 = B_0 P, \quad A_2 = A_1 + B_1 Q_1.$$

The dae (2.2) is called tractable with index 2 if, for all $t \in [t_0, \infty)$, $A_1(t)$ is singular but $A_2(t)$ is nonsingular. In this case (2.5) is equivalent to

$$A_2[P_1\{P(Px)' + Qx\} + Q_1 x] + B_1 P_1 x = 0. \tag{2.7}$$

Due to Lemma A.13 of [2] $Q_1 A_2^{-1} B_1(t)$ is a projection onto $N(A_1(t))$ again. Assume that we have chosen this special projection from the beginning, i.e. $Q_1 \equiv Q_1 A_2^{-1} B_1$. Especially, $Q_1 Q = 0$ holds such that Q, PQ_1, PP_1 are projections again and the pairwise product of two of them vanishes. Then, x can be decomposed like $x = Qx + PQ_1 x + PP_1 x$. Multiplying now (2.7) by $PP_1 A_2^{-1}$, $PQ_1 A_2^{-1}$ and $QP_1 A_2^{-1}$, respectively, we obtain

$$\begin{aligned} (PP_1 x)' + (PP_1 A_2^{-1} B_1 - (PP_1)')PP_1 x &= 0, \\ PQ_1 x &= 0, \\ Qx + ((QQ_1)' + QP_1 A_2^{-1} B_1)PP_1 x &= 0, \end{aligned}$$

provided that Q_1 is continuously differentiable. The counterpart of Lemma 1 is now

Lemma 2 *Let (2.2) be tractable with index 2, where* $Q_1 \equiv Q_1 A_2^{-1} B_1$ *is continuously differentiable. Then (2.2) is equivalent to the system*

$$
\begin{aligned}
z' + (PP_1 A_2^{-1} B_1 - (PP_1)')z &= 0, \\
y &= 0, \\
v &= -((QQ_1)' + QP_1 A_2^{-1} B_1)z,
\end{aligned} \tag{2.8}
$$

where $z = PP_1 x$, $y = PQ_1 x$ *and* $v = Qx$. *Moreover, if* $z^0 \in R(PP_1(\bar{t}))$ *for some* $\bar{t} \in [t_0, \infty)$, *then the solution* z *of the initial value problem* $z' + (PP_1 A_2^{-1} B_1 - (PP_1)')z = 0$, $z(\bar{t}) = z^0$ *fulfils* $z(t) \in R(PP_1(t))$, $t \in [t_0, \infty)$.

Especially we have $x = z + y + v = (I - (QQ_1)' - QP_1 A_2^{-1} B_1)PP_1 z$, where z solves an explicit ode with initial conditions $z(\bar{t}) \in R(PP_1(\bar{t}))$. Note again that $\pi(t) = (I - (QQ_1)'(t) - QP_1 A_2^{-1} B_1(t))PP_1(t)$ is also a projector with $N(\pi(t)) = N(PP_1(t))$. Similarly to the index 1 case, initial conditions can only be given for $PP_1(\bar{t})$. We are led to the following definition.

Definition 4 Let (2.2) be tractable with index 2, where $Q_1 = Q_1 A_2^{-1} B_1$ is continuously differentiable. The trivial solution of (2.2) is called exponentially asymptotically stable if there are constants $\alpha, K > 0$ such that, for all $\bar{t} \geq t_0$, $x^0 \in I\!R^m$, the solution of the initial value problem

$$
\begin{aligned}
A(t)x' + B(t)x = 0, &\qquad t \in [\bar{t}, \infty), \\
PP_1(\bar{t})(x(\bar{t}) - x^0) = 0, &
\end{aligned}
$$

fulfils the estimate

$$|x(t)| \leq K |PP_1(\bar{t}) x^0| e^{-\alpha(t-\bar{t})}, \quad t_0 \leq \bar{t} \leq t < \infty.$$

Lemma 2 and Definition 4 yield immediately the following theorem.

Theorem 2 *Let the assumptions of Lemma 2 be fulfilled. Moreover, let* $(QQ_1)'$ *and* $QP_1 A_2^{-1} B_1$ *be bounded. Then the trivial solution of (2.2) is eas if and only if the trivial solution of*

$$z' + (PP_1 A_2^{-1} B_1 - (PP_1)')z = 0, \quad t \in [t_0, \infty),$$

is eas with respect to $R(PP_1(t))$.

Remark:

(i) Again, if the trivial solution of (2.2) is eas, it is stable in the sense of Lyapunov.

(ii) Statements concerning Lyapunov stability of autonomous quasilinear index 2 and 3 systems can be found in [8].

(iii) Under the assumptions of Theorem 2, contractivity as defined in [6] is a sufficient condition for eas of the trivial solution.

3 Regularization of index 2 dae's

A well-known method for approximating higher index dae's is the index reduction method. We will consider a method based on regularization. More precisely, the index 2 dae (2.3) is replaced by the perturbed dae [4]

$$(A + \varepsilon B_0 P)(Px)' + B_0 x = 0. \tag{3.1}$$

It is easy to see that for sufficiently small $\varepsilon \neq 0$, (3.1) is an index 1 dae if (2.2) is tractable with index 2. For that denote $A_\varepsilon = A + \varepsilon B_0 P$. For the nullspace we have $N(A(t)) = N(A_\varepsilon(t))$ for all $t \in [t_0, \infty)$. Obviously, $N(A(t)) \subseteq N(A_\varepsilon(t))$. If, for some $t \in [t_0, \infty)$, $z \in N(A_\varepsilon(t))$, then $0 = A_\varepsilon z = (A_2 + \varepsilon B_0 P - B_0 Q - B_0 P Q_1)z$. Multiplying this equation by $Q_1 A_2^{-1}$ yields $\varepsilon Q_1 z = 0$. (Note that $Q_1 = Q_1 A_2^{-1} B_0 P$, $A_2^{-1} B_0 Q = Q$.) On the other hand, by multiplication by $PP_1 A_2^{-1}$ one obtains $PP_1 z + \varepsilon PP_1 A_2^{-1} B_0 PP_1 z = 0$. If $PP_1 A_2^{-1} B_0$ is uniformly bounded on $[t_0, \infty)$, $PP_1 z = 0$ for all sufficiently small ε. But this is equivalent to $z \in N(A(t))$ because of $PQ_1 z = 0$. The relevant matrix for proving the transferability of (3.1) is now

$$A_{1,\varepsilon} = A_\varepsilon + B_0 Q = A + \varepsilon B_0 P + B_0 Q. \tag{3.2}$$

The nonsingularity of $A_{1,\varepsilon}$ can be shown similarly. Assume $A_{1,\varepsilon} z = y$ to hold, for fixed $t \in [t_0, \infty)$. Since $A_{1,\varepsilon} = A_2 - B_0 P Q_1 + \varepsilon B_0 P$, we obtain by multiplication by $Q_1 A_2^{-1}$, $PP_1 A_2^{-1}$ and QA_2^{-1}, respectively,

$$\begin{aligned} \varepsilon Q_1 z &= Q_1 A_2^{-1} y, \\ (I + \varepsilon PP_1 A_2^{-1} B_0) PP_1 z + PP_1 B_0 P Q_1 z &= PP_1 A_2^{-1} y, \\ Qz - QQ_1 z + \varepsilon Q A_2^{-1} B_0 P (P_1 + Q_1) z &= Q A_2^{-1} y. \end{aligned}$$

For given $y \in \mathbb{R}^m$, the first equation gives $Q_1 z$, the second can be solved for $PP_1 z$ provided that ε is sufficiently small and $PP_1 A_2^{-1} B_0$ is uniformly bounded. The nullspace component Qz is given by the third equation. This proves the desired nonsingularity. Note that $\|A_{1,\varepsilon}(t)^{-1}\| = O(\varepsilon^{-1})$ since $Q_1 z = \frac{1}{\varepsilon} Q_1 A_2^{-1} y$.

The convergence behaviour of solutions of initial value problems for (3.1) towards that of (2.2) can be characterized by asymptotic expansions *on compact intervals*. To be more precise, consider (2.2) on a compact interval $[t_0, T]$ together with the initial conditions

$$PP_1(t_0)(x(t_0) - x^0) = 0. \tag{3.3}$$

Since (3.1) has index 1, we need additional initial conditions for $PQ_1 x(t_0)$ in order to specify a unique solution. In view of (2.8) it is reasonable to choose $PQ_1 x(t_0) = 0$. For sufficiently small $\varepsilon > 0$, let x_ε denote the solution of (3.1) subject to the initial condition (3.3) and

$$PQ_1 x(t_0) = 0 \tag{3.4}$$

on $[t_0, T]$. In [4] it is shown that, under suitable smoothness assumptions, an asymptotic expansion of the form

$$x_\varepsilon(t) = \sum_{j=0}^{N} (x_j(t) + \bar{x}_j(\tau))\varepsilon^j + O(\varepsilon^{N+1}) \tag{3.5}$$

with $\tau = t/\varepsilon$, $|\bar{x}_j(\tau)| \leq Ce^{-\tau}$ holds true. x_0 is the solution of (2.2) subject to (3.3). If $PQ_1x(t_0)$ is not chosen consistently (i.e. (3.4)), an additional term $\bar{x}_{-1}(\tau)\varepsilon^{-1}$ with $P\bar{x}_{-1} \equiv 0$ appears. The form of (3.5) leads to the conjecture that, if the index 2 problem (2.2) is asymptotically stable in some sense, then this should be true for the regularized system, too. Unfortunately in general such a statement does not hold. In the next section we will give sufficient conditions for preserving exponential asymptotic stability. Since a number of regularizations introduced by other authors are closely related to (3.1), ([1]), for them similar results should be expected.

It is convenient to decompose (3.1) as (2.8) (cf. [4]).

Lemma 3 *Let the assumptions of Lemma 2 be fulfilled. Moreover, let $C_1 := PP_1A_2^{-1}B_1$ be continuously differentiable and $C_2 := QP_1A_2^{-1}B_1$. Then (3.1) is equivalent to the system*

$$\begin{aligned}(I+\varepsilon C_1)z' + (C_1 - (PP_1)')z - ((PP_1)' + \varepsilon C_1')PQ_1y = 0,\\ \varepsilon y' - \varepsilon(PQ_1)'z + (I - \varepsilon(PQ_1)')y = 0,\\ v = -C_2z - \varepsilon C_2(z'+y') + QQ_1(z'+y'),\end{aligned} \tag{3.6}$$

where $z = PP_1x$, $y = PQ_1x$, $v = Qx$. Moreover, if $z^0 \in R(PP_1(\bar{t}))$ for some $\bar{t} \in [t_0,\infty)$, then the solution of the initial value problem $(I+\varepsilon C_1)z' + (C_1 - (PP_1)')z - ((PP_1)' + \varepsilon C_1')PQ_1y = 0$, $z(\bar{t}) = z^0$ fulfils $z(t) \in R(PP_1(t))$ for $t \in [t_0,\infty)$ and any continuous function y. A similar assertion holds for y as a solution of the second equation.

4 Exponential asymptotic stability of regularized linear dae's

The aim of this section is to show that exponential asymptotic stability of (2.2) carries over to (3.1) under some smoothness and boundedness assumptions.

Lemma 4 *Let $M(t) \in C^1[t_0,\infty)$ such that M and M' are bounded. Then, for sufficiently small ε, it holds:*

(i) *$X_\varepsilon(t) := I + \varepsilon M(t)$ is nonsingular.*

(ii) *$X_\varepsilon^{-1}(t)$ is bounded on $[t_0,\infty)$ and the bound does not depend on ε.*

(iii) *$\|\frac{\mathrm{d}}{\mathrm{d}t}X_\varepsilon^{-1}(t)\| \leq \varepsilon K$ uniformly on $[t_0,\infty)$.*

Proof: (i) and (ii) are simple consequences of Banach's theorem. (iii) follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}X_\varepsilon^{-1}(t) = -X_\varepsilon^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}X_\varepsilon(t)X_\varepsilon^{-1}(t) = -\varepsilon X_\varepsilon(t)^{-1}M'(t)X_\varepsilon^{-1}(t).$$

Lemma 5 *Consider the system*

$$x' + A(t,\varepsilon)x = 0, \quad t \in [t_0,\infty), \tag{4.1}$$

for $0 < \varepsilon \leq \varepsilon_0$ and the perturbed system

$$x' + A(t,\varepsilon)x = F(t,\varepsilon)x, \quad t \in [t_0,\infty). \tag{4.2}$$

Let $A, F : [t_0,\infty) \times [0,\varepsilon_0] \to L(\mathbb{R}^m)$ be continuous matrix functions. Suppose that

(i) (4.1) *is eas with respect to the set of subspaces* V *with constants* $\alpha(\varepsilon)$, $K(\varepsilon)$,

(ii) $R(F(t,\varepsilon)) \subseteq V(t)$ *for every* $t \in [t_0,\infty)$ *and every* $\varepsilon \in (0,\varepsilon_0]$,

(iii) $\|F(t,\varepsilon)\| \le \varepsilon\hat{K}$ *uniformly in* $[t_0,\infty)$ *for all* $\varepsilon \in (0,\varepsilon_0]$, *and*

(iv) $\alpha(\varepsilon) > \varepsilon K(\varepsilon)\hat{K}$, $\varepsilon \in (0,\varepsilon_0]$.

Then (4.2) *is eas with respect to* V, *and it holds that*

$$|x(t)| \le K(\varepsilon)e^{-\hat{\alpha}(\varepsilon)(t-\bar{t})}|x(\bar{t})|, \quad t_0 \le \bar{t} \le t < \infty,\ \varepsilon \in (0,\varepsilon_0],$$

for the solution of (4.2) *subject to the initial condition* $x(\bar{t}) = x^0 \in V(\bar{t})$. *Here,* $\hat{\alpha}(\varepsilon) = \alpha(\varepsilon) - \varepsilon K(\varepsilon)\hat{K}$.

Proof: Let $\phi(t,s,\varepsilon)$ denote the fundamental solution of (4.1) subject to the initial condition $\phi(s,s,\varepsilon) = I$. Due to assumption (i) we have

$$|\phi(t,s,\varepsilon)c| \le K(\varepsilon)|c|e^{-\alpha(\varepsilon)(t-s)}, \quad t_0 \le s \le t < \infty, \tag{4.3}$$

for $c \in V(s)$. The solution of (4.2) subject to the initial condition $x(\bar{t}) = x^0$ is uniquely determined as a solution of the integral equation

$$x(t) = \phi(t,\bar{t},\varepsilon)x^0 + \int_{\bar{t}}^{t} \phi(t,s,\varepsilon)F(s,\varepsilon)x(s)\mathrm{d}s, \quad t \ge \bar{t}.$$

Since $F(s,\varepsilon)x(s) \in V(s)$ by assumption (ii), (4.3) yields, for $x^0 \in V(\bar{t})$,

$$\begin{aligned} |x(t)| &\le K(\varepsilon)|x^0|e^{-\alpha(\varepsilon)(t-\bar{t})} + \int_{\bar{t}}^{t} K(\varepsilon)\|F(s,\varepsilon)\|\,|x(s)|e^{-\alpha(\varepsilon)(t-s)}\mathrm{d}s \\ &\le K(\varepsilon)|x^0|e^{-\alpha(\varepsilon)(t-\bar{t})} + \varepsilon K(\varepsilon)\hat{K}\int_{\bar{t}}^{t} e^{-\alpha(\varepsilon)(t-s)}|x(s)|\mathrm{d}s. \end{aligned} \tag{4.4}$$

The latter inequality will be multiplied by $e^{\alpha(\varepsilon)t}$. With the notation,

$$y(t) = \int_{\bar{t}}^{t} e^{\alpha(\varepsilon)s}|x(s)|\mathrm{d}s,$$

we obtain

$$y' \le K(\varepsilon)|x^0|e^{\alpha(\varepsilon)\bar{t}} + \varepsilon K(\varepsilon)\hat{K}y$$

or, equivalently,

$$y' - \varepsilon K(\varepsilon)\hat{K}y \le K(\varepsilon)|x^0|e^{\alpha(\varepsilon)\bar{t}}.$$

Multiplication by $e^{-\varepsilon K(\varepsilon)\hat{K}t}$ and integration yields

$$y(t) \le \frac{|x^0|}{\varepsilon\hat{K}}e^{\alpha(\varepsilon)\bar{t}}(e^{-\varepsilon K(\varepsilon)\hat{K}(\bar{t}-t)} - 1).$$

Introduce this estimate in (4.4) to obtain

$$|x(t)| \le K(\varepsilon)|x^0|e^{-(\alpha(\varepsilon)-\varepsilon K(\varepsilon)\hat{K})(t-\bar{t})}.$$

This gives the assertion.

We are now ready to prove our main result.

Theorem 3 *Let the assumptions of Theorem 2 be fulfilled. Moreover, assume that* QQ_1, $PP_1A_2^{-1}B_1$, $PQ_1A_2^{-1}B_1$, $(PQ_1)'$, $(PQ_1)''$, PP_1 *and* $(PP_1)'$ *are uniformly bounded on* $[t_0,\infty)$. *If* (2.2) *is eas, then* (3.1) *is so.*

Proof: (3.1) is equivalent to (3.6). Denote $X_\varepsilon = I+\varepsilon C_1$. The first two equations of (3.6) can now be written as

$$\begin{aligned} z' + (C_1 - (PP_1)')z - (PP_1)'PQ_1 y &= \varepsilon C_1 X_\varepsilon^{-1}(C_1 - (PP_1)')z \\ &\quad + \varepsilon X_\varepsilon^{-1}C_1' PQ_1 - \varepsilon C_1 X_\varepsilon^{-1}(PP_1)'y, \\ y' - (PQ_1)'z + (\tfrac{1}{\varepsilon}I - (PQ_1)')y &= 0. \end{aligned} \tag{4.5}$$

Here we have used the identity $X_\varepsilon^{-1} = I - \varepsilon C_1 X_\varepsilon^{-1}$. Moreover, $(I-PP_1)C_1 = 0$ and $X_\varepsilon^{-1}C_1'PQ_1 = X_\varepsilon^{-1}[(C_1PQ_1)' - C_1(PQ_1)'] = -X_\varepsilon^{-1}C_1(PQ_1)' = -C_1X_\varepsilon^{-1}(PQ_1)'$ such that the right-hand side of (4.5) belongs to

$$V(t) := R\begin{pmatrix} PP_1(t) & 0 \\ 0 & I \end{pmatrix}.$$

In view of Lemma 5 it is therefore advisable to consider the system

$$\begin{aligned} z' + (C_1 - (PP_1)')z - (PP_1)'PQ_1 y &= 0, \\ y' - (PQ_1)'z + (\tfrac{1}{\varepsilon}I - (PQ_1)')y &= 0 \end{aligned} \tag{4.6}$$

in a first step. Denote $M_\varepsilon(t) = (PQ_1)' - \frac{1}{\varepsilon}I$. We introduce the variable transformation

$$\binom{z}{y} = T(t,\varepsilon)\binom{\bar z}{\bar y}$$

with

$$T(t,\varepsilon) := \begin{pmatrix} I & 0 \\ -M_\varepsilon^{-1}(t)(PQ_1)' & I \end{pmatrix}, \quad T(t,\varepsilon)^{-1} = \begin{pmatrix} I & 0 \\ M^{-1}(t,\varepsilon)(PQ_1)' & I \end{pmatrix}. \tag{4.7}$$

Since $M_\varepsilon(t)^{-1} = -\varepsilon(I - \varepsilon(PQ_1)')^{-1}$, T and T^{-1} are bounded uniformly in t and ε. Therefore, $\binom{\bar z}{\bar y}$ have the same asymptotic properties as $\binom{z}{y}$. (4.6) leads to

$$\binom{\bar z'}{\bar y'} = \begin{pmatrix} (PP_1)' - C_1 & (PP_1)'PQ_1 \\ 0 & M_\varepsilon \end{pmatrix}\binom{\bar z}{\bar y} + F(t,\varepsilon)\binom{\bar z}{\bar y}, \tag{4.8}$$

where

$$F(t,\varepsilon) = \begin{pmatrix} -(PP_1)'PQ_1M_\varepsilon^{-1}(PQ_1)' & 0 \\ M_\varepsilon^{-1}(PQ_1)'((PP_1)' - C_1) - M_\varepsilon^{-1}(PQ_1)'(PP_1)'PQ_1M_\varepsilon^{-1}(PQ_1)' - [M_\varepsilon^{-1}(PQ_1)']' & M_\varepsilon^{-1}(PQ_1)'(PP_1)'PQ_1 \end{pmatrix}.$$

Under our assumptions,

$$\|F(t,\varepsilon)\| \le \varepsilon\hat K$$

uniformly in t. The main part of (4.8) reads

$$\binom{\bar z'}{\bar y'} = \begin{pmatrix} (PP_1)' - C_1 & (PP_1)'PQ_1 \\ 0 & M_\varepsilon \end{pmatrix}\binom{\bar y}{\bar z}. \tag{4.9}$$

Let again

$$V(t) := R \begin{pmatrix} PP_1(t) & 0 \\ 0 & I \end{pmatrix}.$$

Let $\bar{t} \in [t_0, \infty)$ and $\binom{z^0}{y^0} \in V(\bar{t})$ be given. The solution $\bar{y}$ of (4.9) fulfils

$$\bar{y}' + \frac{1}{\varepsilon}\bar{y} = (PQ_1)'\bar{y}$$

such that, for $t \geq \bar{t}$,

$$\bar{y}(t) = e^{-(t-\bar{t})/\varepsilon} y^0 + \int_{\bar{t}}^{t} e^{-(t-s)/\varepsilon}(PQ_1)'(s)\bar{y}(s)\mathrm{d}s$$

leading to the estimate

$$|\bar{y}(t)| \leq e^{-(t-\bar{t})/\varepsilon}|y^0| + \gamma \int_{\bar{t}}^{t} e^{-(t-s)/\varepsilon}|\bar{y}(s)|\mathrm{d}s,$$

where γ denotes a bound on $(PQ_1)'$. Similarly to the procedure of the proof of Lemma 5 one obtains

$$|\bar{y}(t)| \leq e^{-(1/\varepsilon - \gamma)(t-\bar{t})}|y^0|. \tag{4.10}$$

Let $\phi(t,s)$ denote the fundamental solution of $\bar{z}' = ((PP_1)' - C_1)\bar{z}$ subject to the initial condition $\phi(s,s) = I$. Since (2.2) is assumed to be eas, the estimate

$$|\phi(t,s)c| \leq \beta|c|e^{-\alpha(t-s)}, \quad t_0 \leq s \leq t < \infty,$$

is true for all $c \in R(PP_1(s))$ because of Theorem 2. (4.9) yields

$$\bar{z}(t) = \phi(t,\bar{t})z^0 - \int_{\bar{t}}^{t} \phi(t,s)(PP_1)'(s)PQ_1(s)\bar{y}(s)\mathrm{d}s.$$

Since $z^0 \in R(PP_1(\bar{t}))$ and $(PP_1)'(s)PQ_1(s)\bar{y}(s) = [(PP_1PQ_1)'(s) - PP_1(s)(PQ_1)'(s)]\bar{y}(s)$ $= -PP_1(s)(PQ_1)'(s)\bar{y}(s) \in R(PP_1(s))$, it holds that

$$\begin{aligned} |\bar{z}(t)| &\leq \beta|z^0|e^{-\alpha(t-\bar{t})} + \int_{\bar{t}}^{t} \beta|PP_1(s)(PQ_1)'(s)\bar{y}(s)|e^{-\alpha(t-s)}\mathrm{d}s, \\ |\bar{z}(t)| &\leq \beta|z^0|e^{-\alpha(t-\bar{t})} + \beta\delta \int_{\bar{t}}^{t} e^{-\alpha(t-s)}|\bar{y}(s)|\mathrm{d}s, \end{aligned}$$

where δ denotes a bound of $\|PP_1(PQ_1)'\|$. Finally, (4.10) yields

$$|\bar{z}(t)| \leq \beta|z^0|e^{-\alpha(t-\bar{t})} + \beta\delta|y^0|\frac{1}{\frac{1}{\varepsilon} - (\alpha+\gamma)}(e^{-\alpha(t-\bar{t})} - e^{-(\frac{1}{\varepsilon}-\gamma)(t-\bar{t})}).$$

If $\varepsilon < (\alpha+\gamma)^{-1}$,

$$|z(t)| \leq Ke^{-\alpha(t-\bar{t})}\max\{|z^0|, |y^0|\}, \quad t_0 \leq \bar{t} \leq t < \infty,$$

where $K = \max\{\beta, \frac{\beta\delta}{\frac{1}{\varepsilon}-(\alpha+\gamma)}\}$. Note that $K = \beta$, for small ε.
Now Lemma 5 applies since $R(F(t,\varepsilon)) \subseteq V(t)$ for all t and ε. For the solutions of (4.8) subject to the initial conditions $\bar{z}(\bar{t}) = z^0 \in R(PP_1(\bar{x}))$, $\bar{y}(t) = y^0 \in I\!R^m$, it holds that

$$\max\{|\bar{z}(t)|, |\bar{y}(t)|\} \leq Ke^{-\hat{\alpha}(t-\bar{t})} \max\{|\bar{z}(\bar{t})|, |\bar{y}(\bar{t})|\}$$

with K as above and $\hat{\alpha} = \alpha - \varepsilon K\hat{K} \approx \alpha$ for small ε.
Since the asymptotic behaviour is preserved by the transformation (4.7), a similar estimate holds for the solutions of (4.6). Applying Lemma 5 once again to (4.5) and Theorem 1 completes the proof.

Remark: The proof shows that the constant α describing the asymptotic decay of solutions for (2.2) and (3.1) coincides with the order $O(\varepsilon)$. Since $\|T(t,\varepsilon)\| = 1 + O(\varepsilon)$ and $\|T(t,\varepsilon)^{-1}\| = 1 + O(\varepsilon)$, the same is true for the stability constants K as far as the PP_1- and PQ_1-components are concerned. Since $\|A_{1,\varepsilon}^{-1}\| = O(\varepsilon^{-1})$, the stability constant for the nullspace component Qx behaves like $O(\varepsilon^{-1})$ V(cf.(2.6)). A closer look at (3.6) shows that this behaviour is due to the term QQ_1y' in the equation determining $v = Qx$. If the initial values for x in (3.1) are restricted to $PQ_1x(\bar{t}) = 0$, then $y(t) \equiv O(\varepsilon)$ and the stability constant remains within the order of K. In accordance with Definition 2 one should call such a notion "exponential asymptotic stability with respect ot $R(PP_1)$".

5 Conclusion

For regularized dae's with index 2 we have shown that a special type of asymptotic stability in the sense of Lyapunov is preserved. While this was known earlier for autonomous systems, our results are true for time-varying nullspaces, too. On the other hand, boundedness assumptions on certain projections and their derivatives up to second order were used. It is not clear whether these strong requirements are necessary. Moreover, there is strong evidence that similar statements can be given for certain other notions of asymptotic stability or regularizations. The investigation of nonlinear problems is completely open.

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