Employee Retention and Job Assignment Strategies of Entrepreneurial Firms under Uncertainty in Employee Capability

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Abstract

We study the employee retention and job assignment strategy of growth-oriented entrepreneurial firms in which the employee's capability is unknown to both the firm and the employee. As the employee performs his task, both the firm and the employee update their common belief about the employee's capability based on the noisy profit stream from the employee's performance. The firm seeks to dismiss low-capability employees while high-capability employees seek to leave the firm for higher compensation. We model this situation as a real options game between the firm and the employee, and we obtain a Markov perfect equilibrium (MPE) characterized by the employment termination strategies of the two players. In stark contrast to conventional real options models, a higher rate of learning can *hurt* both players when the employee's capability is sufficiently uncertain. This suggests that firms should assign employees with highly uncertain capabilities to tasks with high noise levels.

OR/MS subject classifications: Decision analysis: inference. Games/group decisions: stochastic. Probability: diffusion.

Keywords: Entrepreneurial operations, employee retention, real options game, Bayesian sequential decision.

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1 Introduction

Employee retention is one of the biggest concerns for growth-oriented entrepreneurial firms (Hendricks 2006). Those that retain good employees can harness their valuable tacit knowledge (Tansky 2006), save significant time and money on hiring and training (Yoo et al. 2011), and achieve faster growth (Baron et al. 2001; Baron and Hannan 2002). Despite its importance, little is known about the retention challenges concerning entrepreneurial firms (Cardon and Stevens 2004). In particular, when the employee's capability is uncertain, the problem of employee retention becomes complex because the entrepreneur has an incentive to dismiss a low-capability employee while a high-capability employee has an incentive to leave the company in search of higher compensation. To mitigate the uncertainty, the entrepreneur can attempt to learn the employee's capability by assigning the employee to more informative tasks that produce less noisy outcomes (Pastorino 2004), but it is not clear whether this is a good, let alone an optimal, policy. In this paper, we examine a game between an entrepreneurial firm and an employee when the employee's capability is uncertain in order to provide insights into the impact of learning on the equilibrium strategies and expected payoffs. In particular, we investigate whether the firm should strive to reduce the noise of the employee's performance.

In a resource-constrained entrepreneurial firm undergoing a phase of rapid growth, payroll is often the largest cost. Hence, it is important for each employee to contribute to the firm's profit. Facing tight capital constraints, an entrepreneurial firm must learn quickly about the employee's capability to contribute to the firm's profit so it can exert effort to retain high-capability employees and dismiss those with low capability. This, however, presents unique challenges due to a number of salient features of entrepreneurial firms. First, unlike those of established firms, employees hired by entrepreneurial firms come from a highly diverse pool of workforce (Barber et al. 1999) so that a given employee's capability is highly uncertain. Moreover, the employees of an entrepreneurial firm most often work on novel tasks whose outcomes are inherently more noisy. Hence, the firms must observe the employee's performance for a long duration to learn the employee's capability. Second, due to the resource constraints, the entrepreneurial firm often cannot offer competitive compensation or long-term stability to the most capable employees. Hence, the employees who learn of their high capability develop the incentive to quit and seek higher compensation in other firms or by launching their own companies. Finally, the entrepreneurial firms lack a formal departmentalized structure, and the employees observe the same information as the entrepreneur regarding the opportunities and challenges of the firm (Quinn and Cameron 1983): employees directly observe the impact of their own performance on the firm's profit. Consequently, an employee and the firm learning about the employee's capability is contemporaneous. This lack of structure eliminates any organizational buffer between the firm and the employee, causing constant tension between the two throughout the learning process. Incorporating these three salient features, we address the following research questions:

1. How does the employee's threat of quitting influence the firm's dismissal strategy? Similarly, how does the firm's threat of dismissal impact the employee's quitting strategy? When both firm and the employee are free to choose their termination strategies, what are the equilibrium strategies and profits?

2. What is the impact of the rate of learning on the equilibrium strategies and profits? Given that the firm is able to control the rate of learning through job assignment, should the firm increase the rate of learning?

In the model that we study, an employee generates a profit stream, which is observable to both the entrepreneur (firm) and the employee; in return, the firm pays the employee a fixed salary per unit time and a fixed proportion of the profit he generates. We assume that the employee is one of two types: a high-capability employee who contributes highly to the firm's profit, or a low-capability employee who contributes less. The employee's profit stream is modeled as a Brownian motion with (i) drift that is perfectly correlated with the employee's capability and (ii) volatility (the noise level) of the profit generated by the employee. The noise level depends only on the employee's tasks and is independent of the employee's capability. For example, if the task is to generate sales in an established market, the noise level will be low; if the task is to generate sales through new product development, the noise level will be high.

The firm and the employee share the same prior and posterior belief regarding the employee's capability. Due to the noise in the profit stream, both must observe the employee's performance over time to update their common belief about the employee's capability. At any point in time during the learning process, either player can unilaterally terminate the employment relation: the firm can dismiss an employee if he is deemed to be of low-capability, whereas an employee who learns of his high capability can quit in search of higher compensation. After termination of employment, the firm expects a profit stream without the employee, whereas the employee expects an outside option represented by the present value of the lifetime income stream that is perfectly correlated with his capability.

Restricting our attention to Markov strategies, we first analyze the best responses of each player. We find that the firm's strategy is to dismiss the employee when the posterior belief that the employee has high capability falls below a lower threshold, and the employee's strategy is to quit when the posterior belief exceeds an upper threshold. In particular, when the employee's threat of quitting increases (when the upper threshold decreases), the firm's best response is to expedite the dismissal (increase the lower threshold). Similarly, when the firm's threat of dismissal increases (when the lower threshold increases), the employee's best response is to quit sooner (decrease the upper threshold). This occurs because the increased threat of separation (due to quitting or dismissal) from the opponent decreases the value of waiting; in turn, a lower value of waiting induces an incentive for earlier termination of employment. We then characterize the focal point of the player's strategies by obtaining a unique Pareto-dominant Markov perfect equilibrium (MPE), the subgame perfect equilibrium in Markov strategies (Dutta and Rustichini 1993; Maskin and Tirole 2001).

Next, we investigate the impact of the noise level associated with the employee's task on the equilibrium payoffs. We find that, if the belief about the employee's capability is sufficiently weak (i.e. if the posterior is sufficiently far away from the two thresholds), the equilibrium payoffs to both the firm and the employee increase in the noise level. Because the noise level is inversely proportional to the rate of learning, this result implies that a higher rate of learning can *hurt* both players. This occurs for the following reason: a higher rate of learning accelerates the time evolution of the posterior (Kwon and Lippman 2010) and expedites termination of the employment relationship, and the shortened duration of employment deprives both players of the opportunity to take advantage of the option value of waiting. This result is in stark contrast to the conventional results of real options models under incomplete information, in which the real option value decreases in the noise (e.g. Decamps et al. 2005 and Kwon and Lippman 2010). Moreover, while there do exist well-known examples of games with incomplete and *asymmetric* information in which acquisition of additional information hurts all players (see, for example, Kreps 1988, p. 41), our model presents an example of an incomplete information game in which a similar result holds despite the information being *symmetric*. Finally, our results lead to a counterintuitive managerial implication: the firm should position employees with highly uncertain capability into tasks with high noise levels.

The paper is organized as follows. We first review related literature in Section 2. In Section 3, we introduce the model of a real options game between an entrepreneurial firm and its employee, present each player's best responses, and characterize the MPE strategies and payoffs. We address the impact of the learning rate on the MPE payoffs and strategies in Section 4, and we conclude in Section 5. All proofs appear in the Appendix.

2 Related Literature

We first motivate our context relative to the employee retention literature, with particular focus upon the economic and the organizational dimensions. We then highlight our theoretical contributions on the novel application of the real options game and novel insights regarding the impact of learning.

Among the most significant hurdles that growth-oriented entrepreneurial firms face in retaining their employees is their less-than-competitive compensation packages. Due to their resource constraint and their uncertain profit streams, entrepreneurial firms are restricted in the salary that they can offer to their employees in contrast to their established firm counterparts. Although they can offer a larger portion of compensation in performance-based format to attract talented employees (Balkin and Gomez-Mejia 1987), such "at-risk" payments are not sufficient for retaining employees who are offered greater non-risky compensation packages by large firms (Graham 2002). Another difficulty faced by entrepreneurial firms is high job mobility among inexperienced and young workers, a characteristic of the employees of high-growth entrepreneurial firms. Johnson (1978) formulates a job shopping model in which a worker sequentially experiences jobs to learn about his own capability. In a similar vein, Viscusi (1980) studies a model of a Bayesian worker who learns about a job's attributes and finds that a worker will prefer jobs with uncertain prospects.

Employee retention and job assignment policy has been widely studied in the established firm settings. For example, empirical organizational studies find that the organizational climate of a firm – i.e. its degree of trust, conflict, rewards equity – impacts employee's behavior (Glick 1985), turnover (Huselid 1995), and firm performance (Burton et al. 2004). In particular, employee retention is also influenced by individual traits of the employees such as their level of job satisfaction, (Mitchell et al. 2001), relationships with co-workers (Griffeth et al. 2000), and fit with the organization (Chatman 1991). Using a theoretical model, Pastorino (2004) studies a firm's experimentation with employees with uncertain capabilities through job assignments with varying degrees of informativeness and finds that the firm should assign employees to decreasing degrees of informativeness over time. Although the model of Pastorino (2004) bears some similarity to ours, its main focus is on established firms with long-running employment relations. Unlike these studies, we explicitly model the organizational climate of growth-oriented entrepreneurial firms.

On the theoretical front, our paper lies in the domain of real options games, where there is rich literature in various contexts. For a comprehensive review of real options game models, see Azevedo and Paxson (2010). While most real options game models address competition between two or more rivalrous players to win limited resources or limited investment opportunities, our paper addresses a game between two non-rivalrous players in an employment relation.

The Bayesian framework of our model is based on the work of Shiryaev (1967) who studies the single player decision theoretic problem of minimizing the cost of errors with two hypotheses on the drift of a one-dimensional Brownian motion. The framework has been utilized by Ryan and Lippman (2003), who consider when to stop (abandon) a project with unknown profitability; Decamps et al. (2005), who employ Shiryaev's framework to study the optimal time to invest in an asset with an unknown underlying value; Kwon and Lippman (2010) who study an expansion and exit decision regarding a pilot project with unknown profitability which can be in one of two states. Shiryaev's framework has been also applied to dynamic game models under incomplete information. Bolton and Harris (1999) study a free-rider problem arising from information externality when many agents face the same uncertainty and experimentation, and Bergemann and Välimäki (2000) examine a multi-agent learning model of two sellers who compete with price and many buyers who experiment with a new product with unknown quality.

Finally, in the context of real options decisions, the impact of uncertainty has been of particular interest in the economics literature. The value function generally increases in the uncertainty under conventional situations of real options (Dixit 1992). In a model of a firm which has an option to enter and exit an industry, Dixit (1989) obtains the comparative statics of the optimal entry and exit thresholds with respect to the uncertainty (volatility) in the profit stream. Alvarez (2003) proves a general comparative statics result for the optimal policy and the optimal return with respect to the uncertainty for a class of optimal stopping problems which often arise in economic decisions. Kwon (2010) shows that an embedded option can result in non-trivial comparative statics of the optimal policy with respect to the uncertainty. Kwon and Lippman (2010) examine the impact of uncertainty on the time-to-decision. While these papers address the effect of uncertainty on the optimal policy and the optimal return in decision theoretic models, our paper addresses the impact of the rate of learning in a game setting. We report non-trivial results arising from the strategic interaction between the players.

3 The Model

Consider a growth-oriented entrepreneurial firm, where each employee plays a critical role in contributing to the firm's profit. We assume that each employee generates profit independently of the other employees, and hence focus on the interaction between the entrepreneur (firm) and a single employee who was hired at time $t = 0$. Let X_t denote the cumulative marginal profit contribution earned by an employee from time 0 through t. The firm's cumulative profit is a stochastic process $X = \{X_t : t \geq 0\}$ given by a Brownian motion:

$$X_t = \mu t + \sigma B_t\,,$$

where μ is the drift (the expected profit per unit time earned by the employee), σ is the constant volatility, and $B \equiv \{B_t : t \geq 0\}$ is a Wiener process. For the duration of employment, the firm pays the employee a fixed and unalterable wage s per unit time and a proportion λ of the profit he generates. Moreover, we assume that the firm and the employees are risk-neutral with a common discount rate $\alpha > 0$.

The magnitude of the drift μ represents the capability of the employee. The value of μ is unknown to both the firm and the employee, but it is commonly known to be either h for a high-capability employee or ℓ for a low-capability employee; of course, $h > \ell$. The constant volatility σ, on the other hand, represents the noise level of the profit stream related to the nature of the job or task at hand: it is independent of the employee's capability. For example, if the employee's job is to make frequent sales to a known market, his capability to make sales will be known after a short time, in which case the value of σ is low. On the other hand, if the job is to work on an R&D project or to create a new profit stream from a novel product or service, the profit stream is inherently more noisy and will mask the capability of the employee, resulting in a high σ. In fact, the ratio $(h-\ell)/\sigma$ has the meaning of the signal-to-noise ratio (SNR) of the observed performance of the employee (Bolton and Harris 1999, Bergemann and Välimäki 2000), and we interpret it as the *learning rate* regarding the employee's capability, with low (high) σ corresponding to faster (slower) learning rate.

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be the probability space on which X_t, μ, and B_t are measurable. We let $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ denote the filtration generated by the observable cumulative profit process $X = \{X_t : t \geq 0\}$. The two players have a common prior $p_0 \equiv \mathbb{P}(\{\mu = h\}|\mathcal{F}_0)$, the initial probability that the employee is of high-capability. Moreover, both players observe X and update the common posterior probability denoted by $P_t \equiv \mathbb{P}(\{\mu = h\}|\mathcal{F}_t)$. From Bayes rule (Peskir and Shiryaev 2006, pp. 288-289), we can derive the following expression of P_t in terms of the observable process X:

$$P_t \quad = \quad \left(1 + \frac{1-p_0}{p_0}\exp\left\{-\frac{h-\ell}{\sigma^2}\cdot\left[X_t - \frac{h+\ell}{2}t\right]\right\}\right)^{-1}. \tag{1}$$

Note: The time-evolution (stochastic differential equation) of P_t is given by

$$dP_t = \frac{h-\ell}{\sigma} P_t(1-P_t)d\tilde{B}_t \, ,$$

$$\text{where} \qquad \tilde{B}_t \equiv \frac{1}{\sigma}\left(X_t - \int_0^t E[\mu|\mathcal{F}_s]ds\right) = \frac{1}{\sigma}[X_t - \int_0^t (P_s h + (1-P_s)\ell)ds] \, ,$$

is a Wiener process constructed purely from the observable process X (Liptser and Shiryayev, 1977). Note that the speed of Bayesian updating is proportional to $[(h-\ell)/\sigma]P_t(1-P_t)$. For the sake of convenience, we call the factor $(h-\ell)/\sigma$ the *rate of learning* although the actual speed of learning involves both factors $(h-\ell)/\sigma$ and $P_t(1-P_t)$.

3.1 Real Options Game

Both the firm and the employee simultaneously observe and update their belief P_t about the employee's capability, and either player can unilaterally terminate the employment relation at any point in time to seek an outside option. If the firm believes that $\mu = \ell$, it will dismiss the employee to avoid loss in profit and to obtain its outside option u, which is the present value of the absence of the employee. The outside option u can include the expected net present value of hiring another employee from the general pool of workers. The employee can voluntarily quit to search for an outside option whose compensation is higher than s and the proportion $\lambda \in [0,1)$ of the profit that he generates. The employee's outside option is given by a random variable W, which represents the present value of the employee's lifetime income stream. The random variable W is perfectly correlated with the employee's capability:

$$W = \begin{cases} w_h & \text{if} \quad \mu = h \\ w_\ell & \text{if} \quad \mu = \ell \end{cases}$$

with $w_h > w_\ell$. This assumption is reasonable because a worker capable of contributing highly to a given firm's profit is likely to do the same for another firm's profit in the same industry (e.g. Freeman 1977, Johnson 1978, and Gonzalez and Shi 2010).

Each player's strategy is represented by a stopping time at which the employment is terminated. Let τ_i denote the stopping time for player $i \in \{f, e\}$ where $i = f$ for the firm and $i = e$ for the employee. Given the strategy profile $S \equiv (\tau_f, \tau_e)$ determined by τ_f and τ_e, the stopping time of termination of employment is

$\tau_S \equiv \tau_f \wedge \tau_e$. Let $E^p[\cdot] \equiv E[\cdot|P_0 = p]$ denote the expected value conditional on the initial belief $P_0 = p$. Then, the expected payoff to the firm is

$$\begin{aligned} V_f(p;S) &= E^p[\int_0^{\tau_S}((1-\lambda)\mu - s)e^{-\alpha t}dt + \int_0^{\tau_S}\sigma e^{-\alpha t}dB_t + ue^{-\alpha\tau_S}], \\ &= \frac{1-\lambda}{\alpha}[ph + (1-p)\ell] - \frac{s}{\alpha} + E^p[e^{-\alpha\tau_S}g_f(P_{\tau_S})], \qquad (2) \\ \text{where} \quad & g_f(p) = u - \frac{1-\lambda}{\alpha}[ph + (1-p)\ell] + \frac{s}{\alpha} \quad, \qquad (3) \end{aligned}$$

and the firm's objective is to find τ_f that maximizes Eq. (2) given the employee's strategy τ_e. Similarly, the payoff to the employee is

$$\begin{aligned} V_e(p;S) &= E^p[\int_0^{\tau_S}(\lambda\mu + s)e^{-\alpha t}dt + We^{-\alpha\tau_S}] = \frac{s}{\alpha} + \frac{\lambda}{\alpha}[ph + (1-p)\ell] + E^p[e^{-\alpha\tau_S}g_e(P_{\tau_S})], \qquad (4) \\ \text{where} \quad & g_e(p) = p\left(w_h - \frac{\lambda h}{\alpha}\right) + (1-p)\left(w_\ell - \frac{\lambda\ell}{\alpha}\right) - \frac{s}{\alpha} \quad, \qquad (5) \end{aligned}$$

and the employee's objective is to find τ_e that maximizes Eq. (4) given the firm's strategy τ_f.

Defining $\alpha' = \alpha/(1-\lambda)$, $s' = s'/(1-\lambda)$, $w'_h = w_h - \lambda h/\alpha$, and $w'_\ell = w_\ell - \lambda\ell/\alpha$, and re-expressing $g_f(\cdot)$ and $g_e(\cdot)$ in terms of the primed parameters, one can remove the explicit dependence of $g_f(\cdot)$ and $g_e(\cdot)$ on λ. Because the maximization problem depends only on the functions $g_f(\cdot)$ and $g_e(\cdot)$, we can take $\lambda = 0$ for the remainder of the paper without loss of generality. Furthermore, as we are interested in the regime of model parameters in which the firm wants to dismiss the low-capability employee and to retain the one who is of high-capability, we assume $(h-s)/\alpha > u > (\ell - s)/\alpha$. Similarly, as the employee wants to quit only if he is of high-capability, we assume $w_h > s/\alpha > w_\ell$.

3.2 Best Responses

Because the posterior process P_t is a Markov process and $g_f(\cdot)$ and $g_e(\cdot)$ do not depend on the calendar time, it suffices to restrict our attention to stationary Markov strategies (Oksendal 2003, p.220). A stationary policy can be represented by the thresholds with respect to P_t. For example, the firm may have a stationary policy of dismissing the employee when P_t falls below a threshold θ_f. In other words, the firm terminates the employment at the first exit time of P_t from the set $(\theta_f, 1]$. In general, the stopping times τ_i for $i \in \{f, e\}$ for stationary policies can be expressed as $\tau_i = \inf\{t \geq 0 : P_t \notin C_i\}$ (the first exit time of P_t from C_i) for some open sets $C_i \subset [0,1]$. Then the player i's objective is to find the open set C_i^* that maximizes $V_i(p; C_f, C_e)$ given the opponent's ($-i$'s) strategy C_{-i}. As we will show below, this is equivalent to finding the optimal thresholds with respect to P_t.

In order to obtain the best response to the opponent's strategy, we need to utilize optimal stopping theory. The most direct way to find the optimal solution is to construct a candidate value function $V_i(p; C_f, C_e)$ which is a return function to a candidate policy C_i, and to verify that it satisfies a number of sufficient conditions as laid out by Theorem 10.4.1 of Oksendal (2003). One of the conditions stipulates that $\mathcal{A} V_i(p; C_f, C_e) = 0$ where $\mathcal{A}$ is the characteristic differential operator for P_t (Peskir and Shiryaev 2006) given by

$$\mathcal{A} \equiv -\alpha + \frac{1}{2}\left(\frac{h-\ell}{\sigma}\right)^2 p^2(1-p)^2 \partial_p^2 .$$

Here the term $-\alpha$ replaces the term ∂_t from the time-dependent characteristic operator (Oksendal, 2003) because the payoff from Markov strategies is time-invariant except for the discount factor $e^{-\alpha t}$. The positive fundamental solutions to the equation $\mathcal{A} f(p) = 0$ are given by

$$\begin{aligned} \phi(p) &= p^{\frac{1}{2}(1-\gamma)}(1-p)^{\frac{1}{2}(1+\gamma)}, \\ \psi(p) &= p^{\frac{1}{2}(1+\gamma)}(1-p)^{\frac{1}{2}(1-\gamma)}, \\ \text{where} \qquad \gamma &\equiv \sqrt{1 + \frac{8\alpha\sigma^2}{(h-\ell)^2}} . \end{aligned} \tag{6}$$

Note that $\phi(\cdot)$ is convex decreasing while $\psi(\cdot)$ is convex increasing. Then the value function $V_i(p; C_f, C_e)$ is given by a linear combination of $\phi(\cdot)$ and $\psi(\cdot)$. The following lemma provides the necessary conditions for the best response.

Lemma 1 *Suppose that there exists a best response C_f^* to a given strategy C_e. Then $C_f^* = (\theta_f, 1]$ for some θ_f which depends on C_e. Similarly, suppose that there exists a best response C_e^* to a given strategy C_f. Then $C_e^* = [0, \theta_e)$ for some θ_e which depends on C_f.*

Lemma 1 is intuitive and straightforward. The firm does not want to dismiss a high-capability employee, so it waits until the posterior P_t is sufficiently low before dismissing him. Similarly, the employee wants to quit the job only if he is sufficiently optimistic about his capability, so he waits until the posterior P_t is sufficiently high. The lemma enables us to characterize the stopping times τ_i for $i \in \{r, e\}$ in terms of a pair of thresholds θ_f and θ_e as follows:

$$\begin{aligned}\tau_f &= \inf\left\{t > 0 : P_t \leq \theta_f\right\}, \\ \tau_e &= \inf\left\{t > 0 : P_t \geq \theta_e\right\}.\end{aligned}$$

Thus, if $\theta_e > \theta_f$, then the interval (θ_f, θ_e) is the region of continued employment. A higher θ_f corresponds to earlier dismissal by the entrepreneur, and a lower θ_e corresponds to earlier quitting by the employee. For simplicity of notation, from here on we let the strategy profile S be represented by a pair of thresholds (θ_f, θ_e) instead of a pair of open sets (C_f, C_e). The next two propositions establish one player's best response and expected payoff given the other player's strategy.

Proposition 1 *Suppose the firm's strategy is given by* θ_f.

(i) If $g_e(\theta_f) < 0$*, then the employee has a unique best response* θ_e *which satisfies both* $\theta_f < \theta_e$ *and the equation*

$$a_1 \phi(\theta_f) + a_2 \psi(\theta_f) = g_e(\theta_f), \tag{7}$$

$$\text{where} \qquad \begin{aligned} a_1 &= \frac{\psi(\theta_e)}{2\gamma}\left[\left(w_h - \frac{s}{\alpha}\right)\frac{\gamma-1}{1-\theta_e} + \left(w_\ell - \frac{s}{\alpha}\right)\frac{\gamma+1}{\theta_e}\right], \quad (8) \\ a_2 &= \frac{\phi(\theta_e)}{2\gamma}\left[\left(w_h - \frac{s}{\alpha}\right)\frac{\gamma+1}{1-\theta_e} + \left(w_\ell - \frac{s}{\alpha}\right)\frac{\gamma-1}{\theta_e}\right]. \quad (9) \end{aligned}$$

The employee's expected payoff is given by

$$V_e(p; \theta_f, \theta_e) = \begin{cases} \frac{s}{\alpha} + a_1 \phi(p) + a_2 \psi(p) & \text{for } p \in (\theta_f, \theta_e) \\ \frac{s}{\alpha} + g_e(p) & \text{otherwise} \end{cases}. \tag{10}$$

(ii) If $g_e(\theta_f) \geq 0$*, then the best response of the employee is to quit immediately (*$\theta_e \leq \theta_f$*), and* $V_e(p; \theta_f, \theta_e) = s/\alpha + g_e(p)$ *for all* $p \in [0, 1]$.

Proposition 2 *Suppose the employee's strategy is given by* θ_e.

(i) If $g_f(\theta_e) < 0$, then the firm's unique best response θ_f satisfies both $\theta_f < \theta_e$ and the equation

$$b_1\phi(\theta_e) + b_2\psi(\theta_e) = g_f(\theta_e)\,, \tag{11}$$

$$\text{where} \qquad \begin{aligned} b_1 &= \frac{\psi(\theta_f)}{2\gamma}\left[\left(u+\frac{s-h}{\alpha}\right)\frac{\gamma-1}{1-\theta_f} + \left(u+\frac{s-\ell}{\alpha}\right)\frac{\gamma+1}{\theta_f}\right], \\ b_2 &= \frac{\phi(\theta_f)}{2\gamma}\left[\left(u+\frac{s-h}{\alpha}\right)\frac{\gamma+1}{1-\theta_f} + \left(u+\frac{s-\ell}{\alpha}\right)\frac{\gamma-1}{\theta_f}\right]. \end{aligned}$$

The firm's expected payoff is given by

$$V_f(p;\theta_f,\theta_e) = \begin{cases} \frac{1}{\alpha}[ph+(1-p)\ell-s] + b_1\phi(p) + b_2\psi(p) & \textit{for } p \in (\theta_f,\theta_e) \\ \frac{1}{\alpha}[ph+(1-p)\ell-s] + g_f(p) & \textit{otherwise} \end{cases}. \tag{12}$$

(ii) If $g_f(\theta_e) \geq 0$, then the best response of the firm is to dismiss the employee immediately ($\theta_f \geq \theta_e$), and $V_f(p;\theta_f,\theta_e) = \frac{1}{\alpha}[ph+(1-p)\ell-s] + g_f(p)$ for all $p \in [0,1]$.

If the employee's strategy is to never quit ($\theta_e = 1$), the model reduces to a decision-theoretic one where only the firm has the option to dismiss the employee. From Eq. (2), the firm's problem is to find the optimal τ to maximize $E^p[e^{-\alpha\tau}g_f(P_\tau)]$. The best response of the firm θ_f is consistent with the optimal policy proposed by Ryan and Lippman (2003). Similarly, if $\theta_f = 0$, then only the employee has the option to quit, and the employee's problem is to find τ to maximize $E^p[e^{-\alpha\tau}g_e(P_\tau)]$. The best response of the employee is θ_e, again consistent with the optimal policy that can be derived from Ryan and Lippman (2003). Hence, the solutions to the decision-theoretic models are special cases of the real options game model.

Next, we investigate the sensitivity of the best response θ_i to the opponent's strategy θ_{-i}.

Proposition 3 *(i) The employee's best response θ_e is non-increasing in θ_f.*

(ii) The firm's best response θ_f is non-increasing in θ_e.

The proposition can be understood as follows. If θ_f is increased, then the dismissal of the employee happens earlier. Because there is always a non-zero probability that an employee is of low-capability, earlier dismissal decreases the overall expected payoff to the employee. A decrease in the payoff induces the employee to quit and seek his outside option earlier. Thus, an increase in θ_f decreases θ_e. Similarly, suppose that

θ_e is increased. Then the employee quits earlier, which decreases the firm's overall expected payoff because there is always a non-zero probability that the employee is of high-capability. A decrease in the payoff induces the firm to dismiss the current employee earlier and seek its outside option earlier. Therefore, a decrease in θ_e increases in θ_f.

In the game-theoretic model, the thresholds can be interpreted as the level of threat of termination of employment. A higher threshold θ_f implies a higher likelihood of the employee's dismissal. Hence, we say that the firm's threat of dismissal is higher when θ_f is higher. Likewise, a lower threshold θ_e implies a higher likelihood of the employee's quitting, and hence we say that the employee's threat of quitting is higher when θ_e is lower. Thus, by Proposition 3, a player's threat increases in the opponent's threat.

3.3 Markov Perfect Equilibrium of the Real Options Game

We now obtain the MPE strategies and profits for the real options game model. We limit ourselves to MPEs because we want to focus on strategies that depend only on the current value of the posterior and that are subgame perfect. The necessary condition for an MPE is that the equilibrium strategies θ_f and θ_e are best responses to each other. According to the strategy profiles of Propositions 1-2, there can be multiple MPEs. In particular, the strategy profile (θ_f, θ_e) with $\theta_f = 1$ and $\theta_e = 0$ is always an MPE, and it leads to immediate termination of employment. We call an equilibrium strategy profile (θ_f^*, θ_e^*) *degenerate* if $\theta_f^* \geq \theta_e^*$. Here we are interested in delineating the conditions under which there exists an MPE that is *non-degenerate*, i.e. one that satisfies $\theta_e^* > \theta_f^*$.

Proposition 4 *(i) Suppose the inequality*

$$\left(\frac{s - w_\ell \alpha}{w_h \alpha - s}\right)\left(\frac{h - s - u\alpha}{u\alpha + s - \ell}\right) > 1 \tag{13}$$

holds. Then, there exists an MPE with a strategy profile (θ_f^*, θ_e^*) that satisfies $\theta_f^* < \theta_e^*$. Moreover, a unique Pareto-dominant MPE exists, and it is characterized by the highest ratio θ_e^*/θ_f^* of the two thresholds among all MPEs.

(ii) If $\left(\frac{s - w_\ell \alpha}{w_h \alpha - s}\right)\left(\frac{h - s - u\alpha}{u\alpha + s - \ell}\right)$ is sufficiently small, an MPE strategy profile (θ_f^*, θ_e^*) satisfying $\theta_f^* < \theta_e^*$ does not exist.

The inequality (13) can be intuitively understood. The numerator of the ratio $(s-w_\ell\alpha)/(w_h\alpha-s)$ corresponds to the employee's loss from quitting when he is a low-capability employee while the denominator corresponds to his gain from quitting when he is a high-capability employee. Hence, a larger value of $(s-w_\ell\alpha)/(w_h\alpha-s)$ signifies greater loss than gain from quitting, which induces the employee to quit later, i.e. the threshold θ_e is higher. Similarly, the numerator of the ratio $(h-s-u\alpha)/(u\alpha+s-\ell)$ corresponds to the firm's loss when dismissing a high-capability employee while the denominator corresponds to its gain when dismissing a low-capability employee. Hence if the ratio $(h-s-u\alpha)/(u\alpha+s-\ell)$ is larger, the firm has the incentive to dismiss the employee later, i.e. the threshold θ_f is lower.

While non-uniqueness of equilibria in economic games is commonplace, we conjecture, based upon extensive numerical study, that the non-degenerate MPE is unique if the inequality (13) is satisfied. (See the comment in the proof of Proposition 4 in Appendix.) Even if a unique non-degenerate MPE does not exist, the inequality (13) also guarantees a unique Pareto-dominant MPE that is non-degenerate. In contrast, if the left-hand-side of the inequality (13) is sufficiently small, then the only MPE is the degenerate MPE in which both players terminate the employment (dismiss or quit) immediately.

4 The Impact of Noise Level

In this section, restricting our attention to the unique Pareto-dominant MPE, we investigate the impact of the rate of learning on the MPE strategies and payoffs. In particular, letting θ_f^*, θ_e^*, V_f^*, and V_e^* denote the MPE thresholds of the firm and the employee and the MPE payoffs to the firm and the employee respectively, we examine the comparative statics of θ_f^*, θ_e^*, V_f^* and V_e^* with respect to σ, and we compare and contrast the results with those of conventional real options problems under incomplete information.

As a benchmark, we consider the decision-theoretic model described in the comments that follow Proposition 2. The decision-theoretic model exhibits the following three characteristics regarding the noise level σ: (1) the payoff function $V(p)$ decreases in σ, (2) θ_e decreases in σ while θ_f increases in σ, and (3) in the limit of small values of σ, θ_e approaches 1 while θ_f approaches 0. These characteristics are typical of real options problems under incomplete information (see Bolton and Harris 1999, Bergemann and Välimäki 2000, Ryan and Lippman 2003 and Kwon and Lippman 2010). The three characteristics can be explained as follows. First, as shown by Bolton and Harris (1999) and Bergemann and Välimäki (2000), $(h-\ell)/\sigma$ has the meaning of the signal-to-noise ratio (SNR) of the observed performance of the employee, and it is interpreted as the rate of learning about the employee's capability. Thus, if the firm is able to discern the employee's capability quicker (which happens with low σ), then its payoff is higher. Moreover, if the payoff is higher, then the value of waiting is also higher, resulting in the expansion of the continued employment region. Hence, the upper threshold θ_e decreases in σ while the lower threshold θ_f increases in σ. Finally, in the limit of small values of σ, because the learning rate is extremely high, the firm can very quickly learn about the employee's capability. Hence, it makes sense for both players to impose a very stringent criterion for an action, i.e., to wait until P_t reaches a value very close to 1 or 0.

For the remainder of this section, we examine the limits of small and large values of the noise level σ and derive analytical results and insights on the MPE strategies and profits.

4.1 Small Noise Levels

The noise level σ is small when the outcomes of employee's tasks are relatively predictable (e.g. sales in known markets). In the following Proposition, we establish the uniqueness of a non-degenerate MPE when σ is sufficiently small and characterize the small-σ behaviors of the thresholds.

Proposition 5 *Suppose the inequality (13) holds. Then, in the small-σ limit, there exists a unique non-degenerate MPE with a strategy profile (θ_f^*, θ_e^*) which satisfies $\theta_e^* \uparrow \theta_e^0 < 1$ and $\theta_f^* \downarrow \theta_f^0 > 0$ as $\sigma \to 0$.*

The proposition states that, in the small σ limit, θ_e^* decreases in σ while θ_f^* increases in σ. In contrast to the characteristics of the decision-theoretic models discussed in the beginning of this section, we obtain $\theta_e^* \to \theta_e^0 \neq 1$ and $\theta_f^* \to \theta_f^0 \neq 0$ as $\sigma \to 0$. This is due to the strategic interaction between the firm and the employee. For instance, if θ_f were set to strictly 0, then θ_e^* would converge to 1 in the limit $\sigma \to 0$; similarly, if θ_e were set to strictly 1, then θ_f^* would converge to 0 in the limit $\sigma \to 0$. In light of Proposition 3, due to the presence of mutual threat of employment termination, both players' expected payoffs (values of waiting before termination) are diminished, and both are induced to terminate earlier. Hence, the MPE thresholds do not converge to extreme values (0 or 1) even as $\sigma \to 0$.

Next, we inspect the comparative statics of the MPE payoffs $V_e^*(p) \equiv V_e(p; \theta_f^*, \theta_e^*)$ and $V_f^*(p) \equiv V_f(p; \theta_f^*, \theta_e^*)$ with respect to σ.

Proposition 6 *For all $p \in (\theta_f^0, \theta_e^0)$, the MPE payoffs $V_e^*(p)$ and $V_f^*(p)$ increase in σ for sufficiently small values of σ.*

Proposition 6 is in stark contrast to conventional results from decision-theoretic real options models under incomplete information. Intuitively, one would expect that the noise σ is always detrimental to the expected payoff. The counterintuitive result of Proposition 6 is again a consequence of the strategic interaction between the firm and the employee. When the employee's capability is uncertain, the noise in the performance delays both the firm and the employees' termination decision. Specifically, an increase in σ will delay the firm's decision to dismiss the employee due to its slow learning rate and hence the payoff to the employee increases; see the discussions on time-to-decision in Kwon and Lippman (2010). Likewise, an increase in σ will delay the employee's quitting decision due to his slow learning rate and hence the payoff to the firm also increases. In other words, having more performance noise prolongs the employment relationship which could be potentially favorable. Of course, the players do not know which one actually benefits from the prolonged employment when the employee's capability is highly uncertain. Nevertheless, the expected returns to both players increase when the employment is prolonged because they can avoid the downside risk by terminating the relation at their choice.

4.2 Large Noise Levels

The noise level σ is large when the outcomes of employee's tasks are relatively unpredictable (e.g. R&D or new product development). In the next proposition, we characterize the large-σ behaviors of the MPE strategies.

Proposition 7 *In the large-σ limit, $\theta_e^* \downarrow \theta_e^\infty$ and $\theta_f^* \uparrow \theta_f^\infty$ as $\sigma \to \infty$, where $\theta_e^\infty \equiv (s/\alpha - w_\ell)/(w_h - w_\ell)$ and $\theta_f^\infty \equiv (u\alpha + s - \ell)/(h - \ell)$.*

Note that both players continue the employment relation if $P_t \in (\theta_f^\infty, \theta_e^\infty)$ even in the limit $\sigma \to \infty$. If σ is very large, then the rate of learning is extremely slow, and the convergence of P_t to either threshold would take a very long time. Hence, the decisions of both players are made as if the belief will never be updated. It follows that the players will either almost immediately terminate the employment or wait for a very long time before taking an action. For example, if $P_t \in (\theta_f^\infty, \theta_e^\infty)$, then $g_e(P_t) < 0$ and $g_f(P_t) < 0$, and therefore neither player has an incentive to terminate the employment relationship immediately.

Next, we investigate the comparative statics of the MPE payoffs with respect to σ.

Proposition 8 *Suppose* $p \in (\theta_f^\infty, \theta_e^\infty)$ *and let*

$$\hat{p} \equiv \frac{\sqrt{\theta_e^\infty \theta_f^\infty}}{\sqrt{\theta_e^\infty \theta_f^\infty} + \sqrt{(1-\theta_e^\infty)(1-\theta_f^\infty)}}.$$

In the limit $\sigma \to \infty$,

(i) $V_e^*(p) \uparrow s/\alpha$ *for* $p \leq \hat{p}$ *and* $V_e^*(p) \downarrow s/\alpha$ *for* $p > \hat{p}$;

(ii) $V_f^*(p) \uparrow [ph + (1-p)\ell - s]/\alpha$ *for* $p \geq \hat{p}$ *and* $V_f^*(p) \downarrow [ph + (1-p)\ell - s]/\alpha$ *for* $p < \hat{p}$.

The limiting values of the payoff functions $V_e^*(p)$ and $V_f^*(p)$ can be explained as follows. Suppose P_t is within the interval $(\theta_f^\infty, \theta_e^\infty)$. Then the process P_t will take a very long time to exit the region (θ_f^*, θ_e^*) in the limit $\sigma \to \infty$ (Kwon and Lippman, 2010). Hence, due to discounting, $V_e^*(p) \to s/\alpha$ and $V_f^*(p) \to [ph + (1-p)\ell - s]/\alpha$.

We also note that the direction of convergence of the payoff functions depends on the value of p. If $p \in (\hat{p}, \theta_e^\infty)$, then $V_e^*(p)$ decreases in σ because σ slows down the desired confirmation that he is of high-capability. If $p \in (\theta_f^\infty, \hat{p}]$, then an increase in σ is good for the employee because it delays the firm's decision to dismiss him, so $V_e^*(p)$ increases in σ. Similarly, if $p \in [\hat{p}, \theta_e^\infty)$, then $V_f^*(p)$ increases in σ because an increase in σ delays the employee's decision to quit. If $p \in (\theta_f^\infty, \hat{p})$, then an increase in σ delays the desired confirmation that the employee is of low-capability, so $V_f^*(p)$ decreases in σ.

4.3 Summary and Discussion

In subsections 4.1 and 4.2, we find that, for certain values of p, an increase in σ increases the payoffs. We also find that both payoff functions decrease in σ for sufficiently small values of p and for sufficiently large values of p. For a numerical illustration of the σ-dependence of the payoff functions, see Figures 1 and 2.

The figures illustrate that the payoffs increase in σ when p is far from both thresholds (as explained at the end of Sec. 4.1) and decrease in σ when p is close to either threshold. The σ-dependence of $V_e^*(p)$ and $V_f^*(p)$ for p close to θ_e^* and θ_f^* has the following explanation. In decision-theoretic real options problems under incomplete information, an increase in σ slows down the learning rate and decreases the value function. The same intuition applies to $V_e^*(p)$ when p is close to θ_e^* so that $V_e^*(p)$ decreases in σ when p is close to θ_e^*. It follows that θ_e^* decreases in σ. A decrease in θ_e^* adversely affects the payoff to the firm if p is close to θ_e^*, so $V_f^*(p)$ also decreases in σ for p close to θ_e^*. A similar argument explains why $V_e^*(p)$ and $V_f^*(p)$ decrease in σ when p is close to θ_f^*.

In summary, the comparative statics of the payoffs depend on both the magnitude of σ and the value of p. From Propositions 5, 6, 7, and 8, we obtain the following result:

Theorem 1 *(i) If $p \in (\theta_f^0, \theta_f^\infty] \cup [\theta_e^\infty, \theta_e^0)$, then the MPE payoffs increase in σ for sufficiently small σ and decrease in σ for sufficiently large σ.*

(ii) Suppose $p \in (\theta_f^\infty, \hat{p})$, where $\hat{p}$ is given by Proposition 8. Then the MPE payoffs increase in σ for sufficiently small σ. In the large-σ limit, $V_e^*(p)$ increases in σ while $V_f^*(p)$ decreases in σ.

(iii) Suppose $p \in (\hat{p}, \theta_e^\infty)$. Then the MPE payoffs increase in σ for sufficiently small σ. In the large-σ limit, $V_e^*(p)$ decreases in σ while $V_f^*(p)$ increases in σ.

The non-trivial comparative statics of the payoffs reflects the convex-concave-convex property of the payoff functions $V_e^*(p)$ and $V_f^*(p)$. Alvarez (2003) proves that the value function of a real options problem monotonically increases in the volatility of the underlying asset if the value function is convex. Because the payoff function of our model is not purely convex or concave, its comparative statics has non-monotone dependence on σ.

5 Conclusions

In this paper, we examine a game between an entrepreneurial firm and its employee, who simultaneously learn about the employee's capability. The firm observes the employee's performance to identify and dismiss a low-capability employee while a high-capability employee seeks to leave the firm in search of higher compensation. Our goal is to shed light on the firm's job assignment policy when an employee's capability is highly uncertain.

We find that the firm's optimal policy is to dismiss the employee when the belief that the employee is of high-capability falls below a lower threshold while the employee's optimal policy is to quit when that belief exceeds an upper threshold. The mutual threat of termination of employment tends to induce each player to terminate the employment relation earlier. We obtain the unique Pareto-dominant MPE and investigate the impact of the learning rate on the equilibrium strategies and profits. We find that, in stark contrast to the conventional real options models under incomplete information, the payoffs to both the firm and the employee can decrease in the learning rate. This is because lower learning rate helps to prolong the duration of an employment relation, and consequently it increases the chance for the both players to take advantage of the opponent while retaining the option to unilaterally terminate the employment at any point in time.

Our main result (Theorem 1) provides practical and yet counterintuitive managerial insights for entrepreneurial firms in their high growth phase: firms should place employees with highly uncertain capabilities in tasks with noisy outcomes (high σ) to reduce the mutual learning rate.

While our model is motivated by the strategic interactions inside entrepreneurial firms, our findings can be extended to other situations in which the following key components exists: (i) partnership (employment) can be beneficial, (ii) mutual learning about the contribution, and (iii) conflicting incentives to terminate the agreement.

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Appendix

Proof of Lemma 1. We first consider the firm's best response. We note that

$$\mathcal{A}g_f(p) = -\alpha g_f(p)\,.$$

Because $g_f(\cdot)$ is a decreasing function and because of the inequalities $(h-s)/\alpha > u > (\ell - s)/\alpha$, there is $y \in (0,1)$ such that $\mathcal{A}g_f(p) > 0$ for $p > y$ and $\mathcal{A}g_f(p) < 0$ for $p < y$. By the argument of Oksendal (2003), p. 215, the best response of the firm must contain $(y,1]$ and cannot have a component disconnected from $(y,1]$. Thus, the best response is $(\theta_f, 1]$ for some $\theta_f < y$ which depends on C_e. Using an analogous argument, we can show that the employee's best response is $[0, \theta_e)$ for some θ_e which depends on C_f. ■

Proof of Proposition 1. To prove this Proposition, we solve the optimal stopping time problem of the employee to obtain $\sup_{\tau_e} E^p[e^{-\alpha\tau_e\wedge\tau_f} g_e(P_{\tau_e\wedge\tau_f})]$. Let's assume that $\theta_f < \theta_e$. Under the firm's strategy θ_f, the domain of the employee's value function is restricted to $[\theta_f, 1]$. To prove the existence of the best response, we only need to find θ_e and the solution $f(\cdot)$ to $\mathcal{A}f(p) = 0$ for $p \in (\theta_f, \theta_e)$ which is continuous in $[\theta_f, 1]$ and which satisfies the smooth-pasting condition $f'(\theta_e) = g'_e(\theta_e)$. The solution $f(\cdot)$ also has to satisfy $f(p) \geq g_e(p)$ for all $p \in [\theta_f, 1]$, $f(\theta_f) = g_e(\theta_f)$, and $\mathcal{A}f(p) \leq 0$ for all $p \in (\theta_e, 1]$. In the end, $f(p)$ is identified as $\sup_{\tau_e} E^p[e^{-\alpha\tau_e\wedge\tau_f} g_e(P_{\tau_e\wedge\tau_f})]$. (See Oksendal 2003, Theorem 10.4.1.).

From $\mathcal{A}f(p) = 0$ for $p \in (\theta_f, \theta_e)$ and the continuity, $f(\cdot)$ must have the following form:

$$f(p) = \begin{cases} a_1\phi(p) + a_2\psi(p) & \text{for } p \in [\theta_f, \theta_e] \\ g_e(p) & \text{for } p \in \{\theta_f\} \cup [\theta_e, 1] \end{cases}$$

for some coefficients a_1 and a_2. The coefficients a_1 and a_2 are determined by the conditions $f(\theta_e) = g_e(\theta_e)$ and $f'(\theta_e) = g'_e(\theta_e)$, and they are given by Eqs. (8) and (9). Equation (7) is derived from the condition $f(\theta_f) = g_e(\theta_f)$.

We first establish that a unique solution θ_e to Eq. (7) exists if $g_e(\theta_f) < 0$. Define $\beta_f \equiv \theta_f/(1-\theta_f)$, $\beta_e \equiv \theta_e/(1-\theta_e)$, and $\eta \equiv \beta_e/\beta_f$. Note that β_f (β_e) is strictly increasing in θ_f (θ_e) and that $\eta > 1$ if $\theta_f < \theta_e$. From Eq. (7) we obtain the following equation for η:

$$\frac{s/\alpha - w_\ell}{(w_h - s/\alpha)\beta_f} = \frac{j(\eta,\gamma)}{j(\eta^{-1},\gamma)} \tag{14}$$

$$\text{where} \qquad j(\eta,\gamma) = \eta^{(\gamma+1)/2}(\gamma-1) + \eta^{-(\gamma-1)/2}(\gamma+1) - 2\gamma. \tag{15}$$

It is straightforward to prove that $j(\eta,\gamma)$ takes a minimum value of 0 at $\eta = 1$ and is strictly positive for $\eta \neq 1$. Since we are interested in $\eta > 1$, $j(\eta,\gamma)$ and $j(\eta^{-1},\gamma)$ are strictly positive. Let

$$m(\eta,\gamma) \equiv \frac{j(\eta,\gamma)}{j(\eta^{-1},\gamma)}. \tag{16}$$

After some algebra, we obtain

$$\partial_\eta m(\eta,\gamma) = \frac{(\gamma^2-1)}{[j(\eta^{-1},\gamma)]^2\eta}(\eta^{\gamma/2} - \eta^{-\gamma/2})[\eta^{\gamma/2} - \eta^{-\gamma/2} - \gamma(\eta^{1/2} - \eta^{-1/2})] .$$

It is straightforward to prove that $\eta^{\gamma/2} - \eta^{-\gamma/2} - \gamma(\eta^{1/2} - \eta^{-1/2}) > 0$ for all $\eta > 1$ and $\gamma > 1$ because its derivative with respect to η is strictly positive for $\eta > 1$. It follows that $\partial_\eta m(\eta,\gamma) > 0$ for all $\eta > 1$ and $\gamma > 1$. We also note that $j(\eta,\gamma)/j(\eta^{-1},\gamma) \to 1$ in the limit $\eta \downarrow 1$ and $j(\eta,\gamma)/j(\eta^{-1},\gamma) \to \infty$ in the limit $\eta \to \infty$. Hence, $j(\eta,\gamma)/j(\eta^{-1},\gamma)$ monotonically increases in η if $\eta > 1$, and it can take any value in $(1,\infty)$. We conclude that there is a unique value of $\eta \in (1,\infty)$ which satisfies Eq. (14) given β_f, w_h, w_ℓ, and s/α as long as $\frac{s/\alpha - w_\ell}{(w_h - s/\alpha)\beta_f} > 1$, which is equivalent to the condition $g_e(\theta_f) < 0$.

We observe that $\theta_f < \theta_e$ can be satisfied only if $g_e(\theta_f) < 0$. If $g_e(\theta_f) \geq 0$, then there is no solution $\eta > 1$ that satisfies Eq. (14). Thus, the employee's best response is to quit immediately. This proves statement (ii).

Next, we prove $f(p) \geq g_e(p)$. We first inspect the sign of a_1 and a_2. Suppose that both a_1 and a_2 have the same sign, either positive or negative. Since $\phi(\cdot)$ and $\psi(\cdot)$ are both convex, $f(p) = a_1\phi(p) + a_2\psi(p)$ is either strictly convex or strictly concave, and it cannot intersect with a linear function $g_e(p)$ twice (at θ_e and θ_f) if $f'(\theta_e) = g_e'(\theta_e)$ is satisfied. Hence, a_1 and a_2 must have opposite signs. If $a_1 > 0$ and $a_2 < 0$, then $f(\cdot)$ is monotonically decreasing, which contradicts the condition $f(\theta_f) = g_e(\theta_f) < g_e(\theta_e) = f(\theta_e)$. Thus, $a_1 < 0$ and $a_2 > 0$.

Since $f(\cdot)$ cannot be strictly convex or concave in the interval $[\theta_f, \theta_e]$, $f(\cdot)$ must be concave-convex from the functional form of $\phi(\cdot)$ and $\psi(\cdot)$, i.e., $f(p)$ must be concave for $p < p_I$ and convex for $p > p_I$ for some inflection point $p_I \in (\theta_f, \theta_e)$. It follows that $f(p) - g_e(p)$ is concave-convex which vanishes at $p \in \{\theta_f, \theta_e\}$ with a vanishing first derivative at $p = \theta_e$. The only way this is possible is if $f'(p) - g_e'(p)$ is positive at θ_f, turns negative once somewhere in the interval (θ_f, θ_e), and approaches zero as $p \to \theta_e$. It follows that $f(p) \geq g_e(p)$ for all $p \in [\theta_f, 1]$.

Now we confirm the inequality $\mathcal{A} f(p) \leq 0$ for $p \in (\theta_f, 1]$. Because $\mathcal{A} f(p) = 0$ for $p \in (\theta_f, \theta_e)$, we only need to check the interval $[\theta_e, 1]$. From $\mathcal{A} f(p) = 0$ and $\partial_p^2 f(p) > 0$ for $p \uparrow \theta_e$, we find that $f(\theta_e) > 0$ or $g_e(\theta_e) > 0$. That implies that $\mathcal{A} g_e(p) = -\alpha g_e(p) < 0$ for $p > \theta_e$ since $g_e(\cdot)$ is increasing. This proves that the solution θ_e to Eq. (7) is the best response threshold. ■

Proof of Proposition 2. Proof is analogous to Proof of Proposition **1**. ■

Proof of Proposition 3. Suppose that the given threshold θ_f of the firm increases. This implies that the firm will dismiss the employee earlier even if the employee would like to stay. Let $\theta'_f > \theta_f$ and $\tau_f = \inf\{t \geq 0 : P_t \leq \theta_f\}$, $\tau'_f = \inf\{t \geq 0 : P_t \leq \theta'_f\}$. Because $\tau_f \geq \tau'_f$, the employee's optimal expected gain from stopping satisfies the following:

$$\sup_{\tau \geq 0} E^p[e^{-\alpha\tau\wedge\tau_f} g_e(P_{\tau\wedge\tau_f})] = \sup_{0\leq\tau\leq\tau_f} E^p[e^{-\alpha\tau} g_e(P_\tau)] \geq \sup_{0\leq\tau\leq\tau'_f} E^p[e^{-\alpha\tau} g_e(P_\tau)] = \sup_{\tau\geq 0} E^p[e^{-\alpha\tau\wedge\tau'_f} g_e(P_{\tau\wedge\tau'_f})] \,.$$

Hence, the payoff to the employee decreases in θ_f. If the payoff $f(p)$ to the employee decreases, then the best response upper threshold θ_e decreases because (θ_f, θ_e) is identified as $\{p : f(p) > g_e(p)\}$. This proves that θ_e decreases in θ_f. We can use a similar argument to prove that θ_f also decreases in θ_e. ■

Proof of Proposition 4.

(i) Suppose θ_e and θ_f are best responses to each other. Then they must satisfy Eq. (14). Similarly, from Eq. (11) which is the condition that θ_f is the best response to θ_e, the following condition has to be satisfied:

$$\left(\frac{h - s - u\alpha}{u\alpha + s - \ell}\right) \beta_e = \frac{j(\eta, \gamma)}{j(\eta^{-1}, \gamma)} \,, \tag{17}$$

where $j(\eta, \gamma)$ is given by Eq. (15).

The thresholds θ_e and θ_f are completely determined if and only if $\beta_e = \theta_e/(1-\theta_e)$ and $\beta_f = \theta_f/(1-\theta_f)$ are determined. Hence, it suffices to determine the values of $\beta_e\beta_f$ and $\eta \equiv \beta_e/\beta_f$. From the ratio of Eqs. (14) to (17), the value of $\beta_e\beta_f$ is given by

$$\left(\frac{s - w_\ell\alpha}{w_h\alpha - s}\right)\left(\frac{u\alpha + s - \ell}{h - s - u\alpha}\right) = \beta_e\beta_f \,. \tag{18}$$

There is a value of $\beta_e\beta_f \in (0, \infty)$ which satisfies this equation because the left-hand-side is positive. Next, by multiplying Eqs. (14) and (17), we obtain

$$\left(\frac{s-w_\ell\alpha}{w_h\alpha-s}\right)\left(\frac{h-s-u\alpha}{u\alpha+s-\ell}\right)=k(\eta,\gamma)\,, \tag{19}$$

where the right-hand-side

$$k(\eta,\gamma)\equiv\frac{1}{\eta}\left[\frac{j(\eta,\gamma)}{j(\eta^{-1},\gamma)}\right]^2 \tag{20}$$

takes the value of 1 in the limit $\eta\downarrow 1$ and ∞ in the limit $\eta\to\infty$. Hence, there exists at least one value of $\eta>1$ which satisfies this equation as long as the left-hand-side is larger than 1.

[Note: Equation (13) is actually a necessary and sufficient condition for an MPE with $\theta_f<\theta_e$ if the function $k(\eta,\gamma)$ is strictly larger than 1 for all $\eta>1$ and $\gamma>1$. Our numerical study indicates that $k(\eta,\gamma)$ is larger than 1 for all values of $\eta\in(1,100)$ and $\gamma\in(1,100)$, so we speculate that Eq. (13) is a necessary and sufficient condition for an MPE with $\theta_f<\theta_e$. Moreover, the same numerical study shows that $k(\eta,\gamma)$ is a strictly increasing function of η. From the apparent monotonicity of $k(\cdot,\gamma)$, we further speculate that there is a unique value of η that satisfies Eq. (19) and that the MPE is unique.]

Next, note that there are a finite number of values of η which satisfy Eq. (19) because the function $k(\eta,\gamma)$ is continuously differentiable and $k(\eta,\gamma)\to\infty$ in the limit $\eta\to\infty$. Thus, there are a finite number of MPEs. Let n be the total number of MPEs, and let $S_i=(\theta_{r,i},\theta_{e,i})$ denote the ith MPE strategy profile with the firm's threshold $\theta_{r,i}$ and the employee's threshold $\theta_{e,i}$. In particular, we index S_i in such a way that $\theta_{e,i}>\theta_{e,j}$ if $i<j$. Then $\theta_{r,1}<\theta_{r,2}$ must be true because the firm's best response θ_f must decrease in the strategy θ_e of the employee by Propositions 1 and 2. Hence, $\theta_{r,1}<\theta_{r,2}<\theta_{e,2}<\theta_{e,1}$ must be satisfied. From the proof of Proposition 1, we note that $V_e(p;\theta_{r,2},\theta_{e,2})\le V_e(p;\theta_{r,1},\theta_{e,2})$ because $\theta_{r,2}>\theta_{r,1}$. By the property of MPEs, we have

$$V_e(p;S_2)=V_e(p;\theta_{r,2},\theta_{e,2})\le V_e(p;\theta_{r,1},\theta_{e,2})\le V_e(p;\theta_{r,1},\theta_{e,1})=V_e(p;S_1)\,.$$

Similarly, we can show that $V_f(p;S_2)\le V_f(p;S_1)$. We can repeat the same argument for all i between 2 and n and conclude that $V_e(p;S_1)\ge V_e(p;S_i)$ and $V_f(p;S_1)\ge V_f(p;S_i)$. Thus, S_1, which has the highest ratio $\theta_{e,i}/\theta_{r,i}$, is the Pareto-dominant MPE.

(ii) From Eq. (19) and the fact that $j(\eta,\gamma)$ is strictly positive for $\eta>1$, we know that $k(\eta,\gamma)$ is strictly positive. Also, because $\lim_{\eta\downarrow 1}k(\eta,\gamma)=1$ and $\lim_{\eta\to\infty}k(\eta,\gamma)=\infty$, there exists $\inf_{\eta>1}k(\eta,\gamma)=c\in(0,1]$.

Hence, if $(\frac{s-w_\ell\alpha}{w_h\alpha-s})(\frac{h-s-u\alpha}{u\alpha+s-\ell}) \leq c$, then Eq. (13) cannot be satisfied by any value of $\eta > 1$. ■

Proof of Proposition 5. Since $\gamma \downarrow 1$ as $\sigma \to 0$, it suffices to study the limits of small values of $\delta \equiv \gamma - 1$. We note that

$$j(\eta, 1+\delta) = (x-1-\ln x)\delta + \frac{1}{2}(\ln x)(x-1+\frac{1}{2}\ln x)\delta^2 + O(\delta^3)$$

in the limit $\delta \to 0$. Hence, Eq. (19) reduces to

$$\left(\frac{s-w_\ell\alpha}{w_h\alpha-s}\right)\left(\frac{h-s-u\alpha}{u\alpha+s-\ell}\right) = \frac{(\eta-1-\ln\eta)^2}{\eta(\eta^{-1}-1+\ln\eta)^2} + O(\delta)\,.$$

Let

$$k(\eta) \equiv \frac{(\eta-1-\ln\eta)^2}{\eta(\eta^{-1}-1+\ln\eta)^2}\,.$$

We note that $\lim_{\eta\to 1} k(\eta) = 1$ and $\lim_{\eta\to\infty} k(\eta) = \infty$. Hence, if $(\frac{s-w_\ell\alpha}{w_h\alpha-s})(\frac{h-s-u\alpha}{u\alpha+s-\ell}) > 1$, there exists a finite value of $\eta \in (1,\infty)$ which satisfies Eq. (19) in the limit $\sigma \to 0$. It means that $\theta_e^* \not\to 1$ and $\theta_f^* \not\to 0$ in the limit $\sigma \to 0$.

Now we prove that there is a unique MPE in the small-σ limit by showing that $k(\eta)$ is strictly increasing for $\eta > 1$. We note that

$$\frac{dk(\eta)}{d\eta} = \frac{(\eta-1-\ln\eta)}{(1-\eta+\eta\ln\eta)^3}k_1(\eta)$$

where $k_1(\eta) \equiv \eta(\ln\eta)^2 + (\eta^2-1)\ln\eta - 3\eta^2 + 6\eta - 3$. We note that $k_1(1) = 0$ and that its first and second derivatives vanishes at $\eta = 1$, and its third derivative is zero at $\eta = 1$ but strictly positive for $\eta > 1$. Consequently, $d^2k_1/d\eta^2$, $dk_1/d\eta$, and $k_1(\eta)$ are all strictly positive and increasing for $\eta > 1$. It follows that $k(\eta)$ is strictly increasing for all $\eta > 1$.

Next, we prove that θ_e^* decreases in σ while θ_f^* increases in σ by showing that η^* decreases in σ in the small-σ limit. To do so, we simply need to show $\lim_{\gamma\to 1^+}\partial_\gamma m(\eta^*,\gamma) > 0$ and use the expression

$$\frac{d\eta^*}{d\gamma} = -\left(\frac{2m(\eta^*,\gamma)}{\eta^*}\right)\frac{\partial_\gamma m(\eta^*,\gamma)}{\partial_\eta k(\eta^*,\gamma)}\,, \tag{21}$$

which is derived by applying the implicit function theorem on Eq. (19), since we already know that $dk(\eta)/d\eta > 0$. After some algebra,

$$\lim_{\gamma\to 1^+}\partial_\gamma m(\eta,\gamma) = \frac{(\eta-1-\ln\eta)\eta\ln\eta}{(1-\eta+\eta\ln\eta)^3}\cdot m_1(\eta),$$

where $$m_1(\eta) = -2(\eta-1)^2 + \frac{1}{2}(\eta^2-1)\ln\eta + \eta(\ln\eta)^2\,.$$

We note that $m_1(1) = 0$, and its first and second derivatives vanishes at $\eta = 1$ and its third derivative is strictly positive for $\eta > 1$ from the property of $\eta^2 - 2\eta \ln \eta - 1$. Consequently, $d^2 m_1/d\eta^2$, $dm_1/d\eta$, and $m_1(\eta)$ are all increasing and strictly positive for $\eta > 1$. It follows that $\lim_{\gamma \to 1^+} \partial_\gamma m(\eta, \gamma) > 0$ for all $\eta > 1$. Therefore, $d\eta^*/d\sigma < 0$ is true, and θ_e^* decreases in σ while θ_f^* increases in σ from Eq. (18). ■

Proof of Proposition 6. Let $\theta_f^*(\sigma)$ and $\theta_e^*(\sigma)$ denote the MPE thresholds when the volatility is σ. Similarly, let $V_e^*(p;\sigma)$ and $V_f^*(p;\sigma)$ denote the MPE payoffs when the volatility is σ. For $p \in (\theta_f^*(\sigma), \theta_e^*(\sigma))$, we study the limiting values $\lim_{\sigma \to 0} V_e^*(p;\sigma)$ and $\lim_{\sigma \to 0} V_f^*(p;\sigma)$. From Eq. (10),

$$V_e^*(p;\sigma) = \frac{s}{\alpha} + a_1 \phi(p) + a_2 \psi(p)$$

where a_1 and a_2 are given by Eqs. (8) and (9). For small values of σ,

$$a_1 \phi(p) + a_2 \psi(p) = p w_h + (1-p) w_\ell - \frac{s}{\alpha} + O(\sigma^2) = g_e(p) + O(\sigma^2)\,.$$

Hence, $\lim_{\sigma \to 0} V_e^*(p;\sigma) = s/\alpha + g_e(p)$. However, $V_e^*(p;\sigma) > s/\alpha + g_e(p)$ for $\sigma > 0$ by the property of the best response of the employee. Thus, we conclude

$$V_e^*(p;\sigma) > \lim_{\sigma \to 0} V_e^*(p;\sigma)\,.$$

Using the same procedure, we can show that

$$V_f^*(p;\sigma) > \lim_{\sigma \to 0} V_f^*(p;\sigma)\,.$$

These inequalities imply that $V_e^*(p;\sigma)$ and $V_f^*(p;\sigma)$ decrease as $\sigma \to 0$ for $p \in (\theta_f^*(\sigma), \theta_e^*(\sigma))$. ■

Proof of Proposition 7. In the large-γ limit, we find that

$$m(\eta, \gamma) = \eta + O(\gamma^{-1})\,, \qquad k(\eta, \gamma) = \eta + O(\gamma^{-1})\,.$$

From Eqs. (17) and (18), we find that $\beta_e^* \equiv \theta_e^*/(1-\theta_e^*)$ and $\beta_f^* \equiv \theta_f^*/(1-\theta_f^*)$ are given by

$$\beta_e^* = \frac{s - w_\ell \alpha}{w_h \alpha - s} + O(\gamma^{-1})\,, \qquad \beta_f^* = \frac{u\alpha + s - \ell}{h - s - u\alpha} + O(\gamma^{-1})\,.$$

So we obtain $\beta_e^* \to \beta_e^\infty$, $\beta_f^* \to \beta_f^\infty$, $\theta_e^* \to \theta_e^\infty$ and $\theta_f^* \to \theta_f^\infty$. The $O(\gamma^{-1})$ terms can be obtained directly from

Eqs. (7) and (11):

$$\beta_e^* = \beta_e^\infty(\frac{\gamma+1}{\gamma-1}) + \frac{2g_e(\theta_f^\infty)\sqrt{\beta_e^\infty}}{(w_h - s/\alpha)\sqrt{\theta_f^\infty(1-\theta_f^\infty)}}\left(\frac{\beta_e^\infty}{\beta_f^\infty}\right)^{-\gamma/2} + o((\beta_e^\infty/\beta_f^\infty)^{-\gamma/2}), \tag{22}$$

$$\beta_f^* = \beta_f^\infty(\frac{\gamma-1}{\gamma+1}) + \frac{2g_f(\theta_e^\infty)\sqrt{\beta_f^\infty}}{(u + s/\alpha - h/\alpha)\sqrt{\theta_e^\infty(1-\theta_e^\infty)}}\left(\frac{\beta_e^\infty}{\beta_f^\infty}\right)^{-\gamma/2} + o((\beta_e^\infty/\beta_f^\infty)^{-\gamma/2}). \tag{23}$$

Hence, $\theta_e^* \downarrow (s/\alpha - w_\ell)/(w_h - w_\ell)$ and $\theta_f^* \uparrow (u\alpha + s - \ell)/(h - \ell)$ as $\gamma \to \infty$. ■

Proof of Proposition 8. To prove this Proposition, we need to inspect the large-γ behaviors of $a_1\phi(p) + a_2\psi(p)$ and $b_1\phi(p) + b_2\psi(p)$ [see Propositions 1 and 2 for Eqs. (10) and (12)]. We insert Eq. (22) into Eqs. (8) and (9) to obtain the following:

$$a_1\phi(p) = g_e(\theta_f^\infty)\sqrt{\frac{p(1-p)}{\theta_f^\infty(1-\theta_f^\infty)}}\left(\frac{p/(1-p)}{\beta_f^\infty}\right)^{-\gamma/2} + o\left[\left(\frac{p/(1-p)}{\beta_f^\infty}\right)^{-\gamma/2}\right],$$

$$a_2\psi(p) = 2(\frac{s}{\alpha} - w_\ell)\sqrt{\frac{p(1-p)}{\beta_e^\infty}}\frac{1}{\gamma}\left(\frac{\beta_e^\infty}{p/(1-p)}\right)^{-\gamma/2} + o\left[\left(\frac{\beta_e^\infty}{p/(1-p)}\right)^{-\gamma/2}\right].$$

Note that $\frac{p/(1-p)}{\beta_f^\infty} > 1$ and $\frac{\beta_e^\infty}{p/(1-p)} > 1$ because $\theta_f^\infty < p < \theta_e^\infty$. Both $a_1\phi(p)$ and $a_2\psi(p)$ converge to zero as $\gamma \to \infty$, but $a_1\phi(p)$ converges to zero more slowly than $a_2\psi(p)$ if and only if $p/(1-p) \le \sqrt{\beta_e^\infty\beta_f^\infty}$. Hence, $a_1\phi(p) + a_2\psi(p) \uparrow 0$ for $p/(1-p) \le \sqrt{\beta_e^\infty\beta_f^\infty}$ and $a_1\phi(p) + a_2\psi(p) \downarrow 0$ for $p/(1-p) > \sqrt{\beta_e^\infty\beta_f^\infty}$ because $a_1\phi(p) < 0 < a_2\psi(p)$. Following an analogous procedure and using Eq. (23), we can show that $b_1\phi(p) + b_2\psi(p) \uparrow 0$ for $p/(1-p) \ge \sqrt{\beta_e^\infty\beta_f^\infty}$ and $b_1\phi(p) + b_2\psi(p) \downarrow 0$ for $p/(1-p) < \sqrt{\beta_e^\infty\beta_f^\infty}$. ■

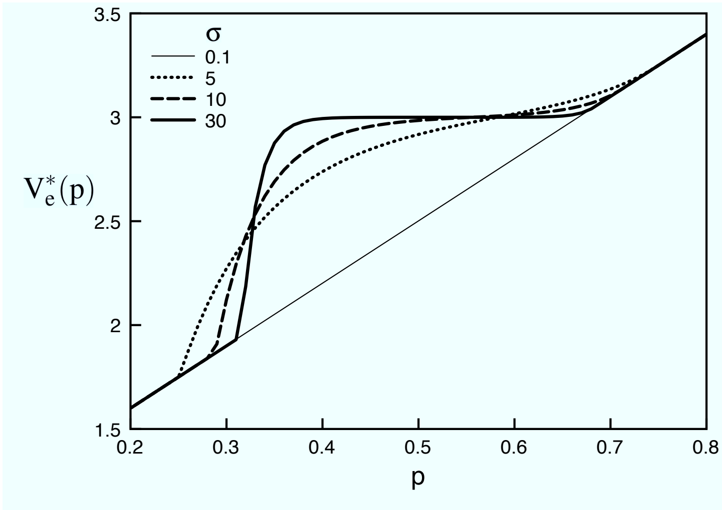

Figure 1: $V_e^*(p)$ for σ =0.1, 5, 10, 30 when $\alpha = 1$, $h = 6$, $\ell = 3$, $s = 3$, $w_h = 4$, $w_\ell = 1$, and $u = 1$.

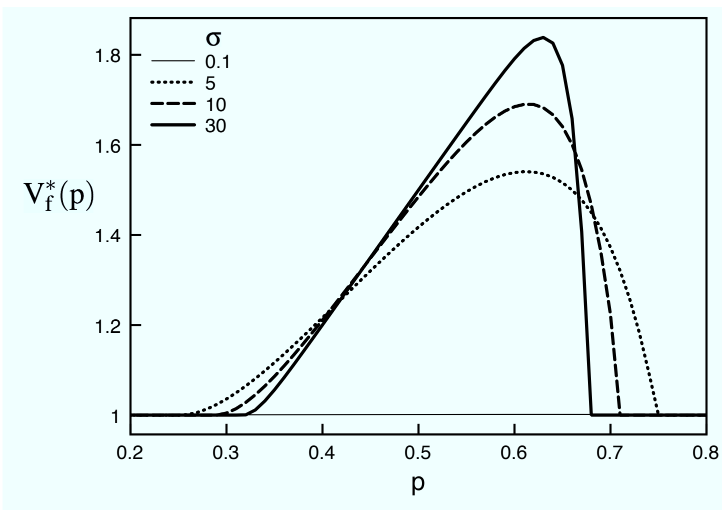

Figure 2: $V_f^*(p)$ for σ =0.1, 5, 10, 30 when $\alpha = 1$, $h = 6$, $\ell = 3$, $s = 3$, $w_h = 4$, $w_\ell = 1$, and $u = 1$.