# ON MAXIMAL TENSOR PRODUCTS AND QUOTIENT MAPS OF OPERATOR SYSTEMS

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Abstract. We introduce quotient maps in the category of operator systems and show that the maximal tensor product is projective with respect to them. Whereas, the maximal tensor product is not injective, which makes the $(\text{el}, \max)$-nuclearity distinguish a class in the category of operator systems. We generalize Lance's characterization of $C^*$-algebras with the WEP by showing that $(\text{el}, \max)$-nuclearity is equivalent to the weak expectation property. Applying Werner's unitization to the dual spaces of operator systems, we consider a class of completely positive maps associated with the maximal tensor product and establish the duality between quotient maps and complete order embeddings.

## 1. Introduction

Kadison characterized the unital subspaces of a real continuous function algebra on a compact set [Ka]. As for its noncommutative counterpart, Choi and Effros gave an abstract characterization of the unital involutive subspaces of $B(H)$ [CE]. The former is called a real function system or a real ordered vector space with an Archimedean order unit while the latter is termed an operator system. Ever since the work by Choi and Effros, the notion of operator systems has been a useful tool in studying the local structures and the functorial aspects of $C^*$-algebras.

Recently, the fundamental and systematic developments in the theory of operator systems have been carried out through a series of papers [PT, PTT, KPTT1, KPTT2]. Perhaps, the reference [PT] is the cornerstone for this program. Under the naive definition of the quotient and that of the tensor product of real function systems, the order unit sometimes fails to be Archimedean. See for example, [A, p.67] and [PTT, Remark 3.12]. The Archimedeanization process introduced in [PT] helps remedy this problem. In order to apply this idea to the noncommutative situation, the Archimedeanization of a matrix ordered $*$-vector space with a matrix order unit is introduced in [PTT]. Based on the matricial Archimedeanization process, the tensor products and the quotients of operator systems are defined and studied in [KPTT1] and [KPTT2] respectively.

Based on these developments, a simple and generalized proof of the celebrated Choi-Effros-Kirchberg approximation theorem for nuclear $C^*$-algebras has been given in [HP].

In this paper, we continue to study the tensor products in [KPTT1] and the quotients in [KPTT2]. With the same spirit as in [HP], we give a simple and generalized proof of the classical Lance's theorem [L1, L2].

In section 3, we introduce the notion of *complete order quotient maps* which can be regarded as quotient maps in the category of operator systems and show that the maximal tensor product is projective under this definition.

2000 *Mathematics Subject Classification.* 46L06, 46L07, 47L07.

*Key words and phrases.* operator system, tensor product, quotient map, unitization.

The minimal tensor product is injective functorially, while the maximal tensor product need not be injective. This misbehavior distinguishes a class which is called $(\mathrm{el}, \max)$-nuclear operator systems [KPTT2]. In the category of $C^*$-algebras, Lance characterized this tensorial property by the factorization property for the inclusion map into the second dual through $B(H)$. In [KPTT2], it is proved that this factorization property for an operator system implies its $(\mathrm{el}, \max)$-nuclearity and it is asked whether the converse holds. In section 4, we answer this question in the affirmative and deduce Lance's theorem as a corollary.

The order unit of the dual spaces of operator systems cannot be considered in general. However, the dual spaces of finite dimensional operator systems have a non-canonical Archimedean order unit [CE]. This enables the duality between tensor products and mapping spaces to work in the proofs of the Choi-Effros-Kirchberg theorem for operator systems [HP] and Lance's theorem for operator systems in section 4. Not only is the finite dimensional assumption restrictive, but also the matrix order unit norm on the dual spaces of finite dimensional operator systems is irrelevant to the matrix norm given by the standard dual of operator spaces.

To get rid of the finite dimensional assumption and to reflect the operator space dual norm, we apply Werner's unitization of matrix ordered operator spaces [W] to the dual spaces of operator systems. We consider the completely positive maps associated with the maximal tensor products and prove their factorization property in section 5. Finally, we establish the duality between complete order quotient maps and complete order embeddings in section 6.

## 2. PRELIMINARIES

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. As in [KPTT1], an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ is defined as a family of cones $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ satisfying:

- (T1) $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+\}_{n=1}^\infty, 1_\mathcal{S} \otimes 1_\mathcal{T})$ is an operator system denoted by $\mathcal{S} \otimes_\tau \mathcal{T}$,
- (T2) $M_n(\mathcal{S})^+ \otimes M_m(\mathcal{T})^+ \subset M_{mn}(\mathcal{S} \otimes_\tau \mathcal{T})^+$ for all $n, m \in \mathbb{N}$, and
- (T3) If $\varphi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive maps, then $\varphi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is a unital completely positive map.

By an operator system tensor product, we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, denoted $\mathcal{S} \otimes_\tau \mathcal{T}$. We call an operator system tensor product $\tau$ functorial, if the following property is satisfied:

- (T4) For any operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1, \mathcal{T}_2$ and unital completely positive maps $\varphi : \mathcal{S}_1 \to \mathcal{T}_1, \psi : \mathcal{S}_2 \to \mathcal{T}_2$, the map $\varphi \otimes \psi : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2$ is unital completely positive.

An operator system structure is defined on two fixed operator systems, while the functorial operator system tensor product can be thought of as the bifunctor on the category consisting of operator systems and unital completely positive maps.

For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = \{[p_{i,j}]_{i,j} \in M_n(\mathcal{S} \otimes \mathcal{T}) : \forall \varphi \in S_k(\mathcal{S}), \psi \in S_m(\mathcal{T}), [(\varphi \otimes \psi)(p_{i,j})]_{i,j} \in M_{nkm}^+\}$$

and let $\iota_\mathcal{S} : \mathcal{S} \to B(H)$ and $\iota_\mathcal{T} : \mathcal{T} \to B(K)$ be unital completely order isomorphic embeddings. Then the family $\{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+\}_{n=1}^\infty$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ rising from the embedding $\iota_\mathcal{S} \otimes \iota_\mathcal{T} : \mathcal{S} \otimes \mathcal{T} \to B(H \otimes K)$. We call the operator system $(\mathcal{S}\otimes\mathcal{T}, \{M_n(\mathcal{S}\otimes_{\min}\mathcal{T})\}_{n=1}^{\infty}, 1_{\mathcal{S}}\otimes 1_{\mathcal{T}})$ the minimal tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S}\otimes_{\min}\mathcal{T}$.

The mapping $\min : \mathcal{O}\times\mathcal{O}\to\mathcal{O}$ sending $(\mathcal{S},\mathcal{T})$ to $\mathcal{S}\otimes_{\min}\mathcal{T}$ is an injective, associative, symmetric and functorial operator system tensor product. The positive cone of the minimal tensor product is the largest among all possible positive cones of operator system tensor products [KPTT1, Theorem 4.6]. For $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, we have the completely order isomorphic inclusion

$$\mathcal{A}\otimes_{\min}\mathcal{B}\subset\mathcal{A}\otimes_{\mathrm{C^*}\min}\mathcal{B}$$

[KPTT1, Corollary 4.10].

For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$D_n^{\max}(\mathcal{S},\mathcal{T}) = \{\alpha(P\otimes Q)\alpha^* : P\in M_k(\mathcal{S})^+, Q\in M_l(\mathcal{T})^+, \alpha\in M_{n,kl},\ k,l\in\mathbb{N}\}.$$

Then it is a matrix ordering on $\mathcal{S}\otimes\mathcal{T}$ with order unit $1_{\mathcal{S}}\otimes 1_{\mathcal{T}}$. Let $\{M_n(\mathcal{S}\otimes_{\max}\mathcal{T})^+\}_{n=1}^{\infty}$ be the Archimedeanization of the matrix ordering $\{D_n^{\max}(\mathcal{S},\mathcal{T})\}_{n=1}^{\infty}$. Then it can be written as

$$M_n(\mathcal{S}\otimes_{\max}\mathcal{T})^+ = \{X\in M_n(\mathcal{S}\otimes\mathcal{T}) : \forall\varepsilon>0, X+\varepsilon I_n\in D_n^{\max}(\mathcal{S},\mathcal{T})\}.$$

We call the operator system $(\mathcal{S}\otimes\mathcal{T}, \{M_n(\mathcal{S}\otimes_{\max}\mathcal{T})^+\}_{n=1}^{\infty}, 1_{\mathcal{S}}\otimes 1_{\mathcal{T}})$ the maximal operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S}\otimes_{\max}\mathcal{T}$.

The mapping $\max : \mathcal{O}\times\mathcal{O}\to\mathcal{O}$ sending $(\mathcal{S},\mathcal{T})$ to $\mathcal{S}\otimes_{\max}\mathcal{T}$ is an associative, symmetric and functorial operator system tensor product. The positive cone of the maximal tensor product is the smallest among all possible positive cones of operator system tensor products [KPTT1, Theorem 5.5]. For $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, we have the completely order isomorphic inclusion

$$\mathcal{A}\otimes_{\max}\mathcal{B}\subset\mathcal{A}\otimes_{\mathrm{C^*}\max}\mathcal{B}$$

[KPTT1, Theorem 5.12].

The inclusion $\mathcal{S}\otimes\mathcal{T}\subset I(\mathcal{S})\otimes_{\max}\mathcal{T}$ for the injective envelope $I(\mathcal{S})$ of $\mathcal{S}$ induces the operator system structure on $\mathcal{S}\otimes\mathcal{T}$, which is denoted by $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T}$. Here the injective envelope $I(\mathcal{S})$ can be replaced by any injective operator system containing $\mathcal{S}$. The mapping $\mathrm{el} : \mathcal{O}\times\mathcal{O}\to\mathcal{O}$ sending $(\mathcal{S},\mathcal{T})$ to $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T}$ is a left injective functorial operator system tensor product [KPTT1, Theorems 7.3, 7.5].

An operator system $\mathcal{S}$ is called (el, max)-nuclear if $\mathcal{S}\otimes_{\mathrm{el}}\mathcal{T} = \mathcal{S}\otimes_{\max}\mathcal{T}$ for any operator system $\mathcal{T}$. An operator system $\mathcal{S}$ is (el, max)-nuclear if and only if $\mathcal{S}\otimes_{\max}\mathcal{T}\subset\mathcal{S}_2\otimes_{\max}\mathcal{T}$ for any inclusion $\mathcal{S}\subset\mathcal{S}_2$ and any operator system $\mathcal{T}$ [KPTT2, Lemma 6.1]. We say that the operator system $\mathcal{S}$ has the weak expectation property (in short, WEP) if the inclusion map $\iota : \mathcal{S}\hookrightarrow\mathcal{S}^{**}$ can be factorized as $\Psi\circ\Phi = \iota$ for unital completely positive maps $\Phi : \mathcal{S}\to B(H)$ and $\Psi : B(H)\to\mathcal{S}^{**}$. If $\mathcal{S}$ has the WEP, then it is (el, max)-nuclear [KPTT2, Theorem 6.7].

Given an operator system $\mathcal{S}$, we call $\mathcal{J}\subset\mathcal{S}$ a kernel, provided that it is the kernel of a unital completely positive map from $\mathcal{S}$ to another operator system. The kernel can be characterized in an intrinsic way: $\mathcal{J}$ is a kernel if and only if it is the intersection of a closed two-sided ideal in $C_u^*(\mathcal{S})$ with $\mathcal{S}$ [KPTT2, Corollary 3.8]. If we define a family of positive cones $M_n(\mathcal{S}/\mathcal{J})^+$ on $M_n(\mathcal{S}/\mathcal{J})$ by

$$M_n(\mathcal{S}/\mathcal{J})^+ = \{[x_{i,j}+J]_{i,j} : \forall\varepsilon>0, \exists k_{i,j}\in J,\ \varepsilon I_n\otimes 1_{\mathcal{S}} + [x_{i,j}+k_{i,j}]_{i,j}\in M_n(\mathcal{S})^+\},$$

then $(\mathcal{S}/\mathcal{J}, \{M_n(\mathcal{S}/\mathcal{J})^+\}_{n=1}^{\infty}, 1_{\mathcal{S}/\mathcal{J}})$ satisfies all the conditions of an operator system [KPTT2, Proposition 3.4]. We call it the quotient operator system. With this definition, the first isomorphism theorem is proved: If $\varphi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive map with $\mathcal{J} \subset \ker \varphi$, then the map $\tilde{\varphi} : \mathcal{S}/\mathcal{J} \to \mathcal{T}$ given by $\tilde{\varphi}(x + \mathcal{J}) = \varphi(x)$ is a unital completely positive map [KPTT2, Proposition 3.6].

Since the kernel $\mathcal{J}$ in an operator system $\mathcal{S}$ is a closed subspace, the operator space structure of $\mathcal{S}/\mathcal{J}$ can be interpreted in two ways, one as the operator space quotient and the other as the operator space structure induced by the operator system quotient. The two matrix norms can be different. For a concrete example, see [KPTT2, Example 4.4].

## 3. Projectivity of maximal tensor product

We show that the maximal tensor product is projective functorially in the category of operator systems. To this end, we first define the quotient maps in the category of operator systems.

**Definition 3.1.** For operator systems $\mathcal{S}$ and $\mathcal{T}$, we let $\Phi : \mathcal{S} \to \mathcal{T}$ be a unital completely positive surjection. We call $\Phi : \mathcal{S} \to \mathcal{T}$ *a complete order quotient map* if for any $Q$ in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take an element $P$ in $M_n(\mathcal{S})$ so that it satisfies

$$P + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \quad \text{and} \quad \Phi_n(P) = Q.$$

The key point of the above definition is that the lifting $P$ depends on the choice of $\varepsilon > 0$. A slight modification of [PT, Theorem 2.45] implies the following proposition that justifies the above terminology.

**Proposition 3.2.** *For operator systems $\mathcal{S}$ and $\mathcal{T}$, we suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive surjection. Then $\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map if and only if the induced map $\tilde{\Phi} : \mathcal{S}/\ker\Phi \to \mathcal{T}$ is a unital complete order isomorphism.*

*Proof.* $\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map

$\Leftrightarrow \forall Q \in M_n(\mathcal{T})^+, \forall \varepsilon > 0, \exists P \in M_n(\mathcal{S}), P + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+$ and $\Phi_n(P) = Q$

$\Leftrightarrow \forall Q \in M_n(\mathcal{T})^+, \exists P \in M_n(\mathcal{S}), P + \ker\Phi_n \in M_n(\mathcal{S}/\ker\Phi)^+$ and $\tilde{\Phi}_n(P + \ker\Phi_n) = Q$

$\Leftrightarrow$ the induced map $\tilde{\Phi} : \mathcal{S}/\ker\Phi \to \mathcal{T}$ is a complete order isomorphism.

□

Recall that for operator spaces $V$ and $W$, the linear map $\Phi : V \to W$ is called a complete quotient map if $\Phi_n : M_n(V) \to M_n(W)$ is a quotient map for each $n \in \mathbb{N}$, that is, $\Phi_n$ maps the open unit ball of $M_n(V)$ onto the open unit ball of $M_n(W)$.

**Proposition 3.3.** *For operator systems $\mathcal{S}$ and $\mathcal{T}$, we suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive surjection. If $\Phi : \mathcal{S} \to \mathcal{T}$ is a complete quotient map, then it is a complete order quotient map.*

*Proof.* We choose $Q \in M_n(\mathcal{T})^+$ and $\varepsilon > 0$. We then have

$$-\frac{1}{2}\|Q\| I_n \otimes 1_{\mathcal{T}} \leqslant Q - \frac{1}{2}\|Q\| I_n \otimes 1_{\mathcal{T}} \leqslant \frac{1}{2}\|Q\| I_n \otimes 1_{\mathcal{T}}.$$

There exists an element $P$ in $M_n(\mathcal{S})$ such that

$$\Phi_n(P) = Q - \frac{1}{2}\|Q\| I_n \otimes 1_{\mathcal{T}} \qquad \text{and} \qquad \|P\| \leqslant \frac{1}{2}\|Q\| + \varepsilon.$$

By considering $(P + P^*)/2$ instead, we may assume that $P$ is self-adjoint. It follows that

$$P + \frac{1}{2}\|Q\| I_n \otimes 1_{\mathcal{S}} + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \qquad \text{and} \qquad \Phi_n(P + \frac{1}{2}\|Q\| I_n \otimes 1_{\mathcal{S}}) = Q.$$

□

The following theorem says that the maximal tensor product is projective functorially in the category of operator systems.

**Theorem 3.4.** *For operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}$ and a complete order quotient map $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$, the linear map $\Phi \otimes \mathrm{id}_{\mathcal{T}} : \mathcal{S}_1 \otimes_{\max} \mathcal{T} \to \mathcal{S}_2 \otimes_{\max} \mathcal{T}$ is a complete order quotient map.*

*Proof.* We choose an element $z$ in $M_n(\mathcal{S}_2 \otimes_{\max} \mathcal{T})^+$ and $\varepsilon > 0$. Then we can write

$$z + \varepsilon I_n \otimes 1_{\mathcal{S}_2} \otimes 1_{\mathcal{T}} = \alpha P_2 \otimes Q \alpha^*, \qquad P_2 \in M_p(\mathcal{S}_2)^+, Q \in M_q(\mathcal{T})^+, \alpha \in M_{n,pq}.$$

We may assume that $\|Q\|, \|\alpha\| \leqslant 1$. There exists an element $P_1$ in $M_p(\mathcal{S}_1)$ such that

$$\Phi_p(P_1) = P_2 \qquad \text{and} \qquad P_1 + \varepsilon I_p \otimes 1_{\mathcal{S}_1} \in M_p(\mathcal{S}_1)^+.$$

It follows that

$$(\Phi \otimes \mathrm{id}_{\mathcal{T}})_n(\alpha P_1 \otimes Q \alpha^* - \varepsilon I_n \otimes 1_{\mathcal{S}_1} \otimes 1_{\mathcal{T}}) = \alpha P_2 \otimes Q \alpha^* - \varepsilon I_n \otimes 1_{\mathcal{S}_2} \otimes 1_{\mathcal{T}} = z$$

and

$$\begin{aligned}&(\alpha P_1 \otimes Q \alpha^* - \varepsilon I_n \otimes 1_{\mathcal{S}_1} \otimes 1_{\mathcal{T}}) + 2\varepsilon I_n \otimes 1_{\mathcal{S}_1} \otimes 1_{\mathcal{T}} \\ =&\alpha(P_1 + \varepsilon I_p \otimes 1_{\mathcal{S}_1}) \otimes Q \alpha^* + (\varepsilon I_n \otimes 1_{\mathcal{S}_1} \otimes 1_{\mathcal{T}} - \varepsilon \alpha((I_p \otimes 1_{\mathcal{S}_1}) \otimes Q)\alpha^*) \\ \in& M_n(\mathcal{S}_1 \otimes_{\max} \mathcal{T})^+.\end{aligned}$$

□

Suppose that we are given an operator system $\mathcal{S}$ and a unital $C^*$-algebra $\mathcal{A}$ such that $\mathcal{S}$ is an $\mathcal{A}$-bimodule. Moreover, we assume that $1_{\mathcal{A}} \cdot s = s$ for $s \in \mathcal{S}$. We call such an $\mathcal{S}$ an operator $\mathcal{A}$-system [Pa, Chapter 15] provided that $a \cdot 1_{\mathcal{S}} = 1_{\mathcal{S}} \cdot a$ and

$$[a_{i,j}] \cdot [s_{i,j}] \cdot [a_{i,j}]^* = [\sum_{k,l=1}^{n} a_{i,k} \cdot s_{k,l} \cdot a_{j,l}^*] \in M_n(\mathcal{S})^+, \qquad [a_{i,j}] \in M_n(\mathcal{A}), [s_{i,j}] \in M_n(\mathcal{S})^+.$$

The maximal tensor product $\mathcal{A} \otimes_{\max} \mathcal{S}$ is an operator $\mathcal{A}$-system [KPTT1, Theorem 6.7].

The converse of Proposition 3.3 does not hold in general since the operator space structure induced by the operator system quotient by a kernel can be different from the operator space quotient by it [KPTT2, Example 4.4]. However, the converse holds in the following special situation. Although the following theorem overlaps with [KPTT2, Corollary 5.15], we include it here because the proof is so elementary.

**Theorem 3.5.** *Suppose that $\mathcal{S}$ is an operator system and $\mathcal{A}$ is a unital $C^*$-algebra with its norm closed ideal $\mathcal{I}$. Then the canonical map*

$$\pi \otimes \mathrm{id}_{\mathcal{S}} : \mathcal{A} \otimes_{\max} \mathcal{S} \to \mathcal{A}/\mathcal{I} \otimes_{\max} \mathcal{S}$$

*is a complete quotient map.*

*Proof.* By the nuclearity of matrix algebras, it is sufficient to show that the canonical map $\pi \otimes \mathrm{id}_{\mathcal{S}} : \mathcal{A} \otimes_{\max} \mathcal{S} \to \mathcal{A}/\mathcal{I} \otimes_{\max} \mathcal{S}$ is a quotient map. We suppose that

$$\|(\pi \otimes \mathrm{id}_{\mathcal{S}})(z)\|_{\mathcal{A}/\mathcal{I}\otimes_{\max}\mathcal{S}} < 1$$

for some $z \in \mathcal{A} \otimes \mathcal{S}$. Then we have

$$\begin{pmatrix} (1-\varepsilon)1_{\mathcal{A}/\mathcal{I}} \otimes 1_{\mathcal{S}} & (\pi \otimes \mathrm{id}_{\mathcal{S}})(z) \\ (\pi \otimes \mathrm{id}_{\mathcal{S}})(z)^* & (1-\varepsilon)1_{\mathcal{A}/\mathcal{I}} \otimes 1_{\mathcal{S}} \end{pmatrix} \in M_2(A/\mathcal{I} \otimes_{\max} \mathcal{S})^+$$

for $\varepsilon = 1 - \|(\pi \otimes \mathrm{id}_{\mathcal{S}})(z)\|_{\mathcal{A}/\mathcal{I}\otimes_{\max}\mathcal{S}}$. By Theorem 3.4, we can find an element $\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$ in $M_2(\mathcal{I} \otimes \mathcal{S})$ such that

$$\begin{pmatrix} 1_{\mathcal{A}} \otimes 1_{\mathcal{S}} + \omega_{11} & z + \omega_{12} \\ z^* + \omega_{21} & 1_{\mathcal{A}} \otimes 1_{\mathcal{S}} + \omega_{22} \end{pmatrix} \in M_2(\mathcal{A} \otimes_{\max} \mathcal{S})^+.$$

Since we have $(\mathcal{I} \otimes \mathcal{S})_h = \mathcal{I}_h \otimes \mathcal{S}_h$ [CE], $\omega_{11}$ can be written as $\omega_{11} = \sum_{i=1}^n a_i \otimes s_i$ for $a_i \in I_h$ and $s_i \in \mathcal{S}_h$. Let $a_i = a_i^+ - a_i^-$ for $1 \leqslant i \leqslant n$ and $a_i^+, a_i^- \in \mathcal{I}^+$. We have

$$\omega_{11} = \sum_{i=1}^n a_i^+ \otimes s_i + a_i^- \otimes (-s_i) \leqslant \sum_{i=1}^n \|s_i\|(a_i^+ + a_i^-) \otimes 1_{\mathcal{S}}.$$

Hence, we can find elements $a$ and $d$ in $\mathcal{I}^+$ such that

$$\begin{pmatrix} (1_{\mathcal{A}} + a) \otimes 1_{\mathcal{S}} & z + \omega_{12} \\ z^* + \omega_{21} & (1_{\mathcal{A}} + d) \otimes 1_{\mathcal{S}} \end{pmatrix} \in M_2(\mathcal{A} \otimes_{\max} \mathcal{S})^+.$$

Since $\mathcal{A} \otimes_{\max} \mathcal{S}$ is an operator $\mathcal{A}$-system [KPTT1, Theorem 6.7], we have

$$\begin{aligned} &\begin{pmatrix} 1_{\mathcal{A}} \otimes 1_{\mathcal{S}} & (1_{\mathcal{A}} + a)^{-\frac{1}{2}} \cdot (z + \omega_{12}) \cdot (1_{\mathcal{A}} + d)^{-\frac{1}{2}} \\ (1_{\mathcal{A}} + d)^{-\frac{1}{2}} \cdot (z^* + \omega_{21}) \cdot (1_{\mathcal{A}} + a)^{-\frac{1}{2}} & 1_{\mathcal{A}} \otimes 1_{\mathcal{S}} \end{pmatrix} \\ &= \begin{pmatrix} (1_{\mathcal{A}} + a)^{-\frac{1}{2}} & 0 \\ 0 & (1_{\mathcal{A}} + d)^{-\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} (1_{\mathcal{A}} + a) \otimes 1_{\mathcal{S}} & z + \omega_{12} \\ z^* + \omega_{21} & (1_{\mathcal{A}} + d) \otimes 1_{\mathcal{S}} \end{pmatrix} \cdot \begin{pmatrix} (1_{\mathcal{A}} + a)^{-\frac{1}{2}} & 0 \\ 0 & (1_{\mathcal{A}} + d)^{-\frac{1}{2}} \end{pmatrix} \\ &\in M_2(\mathcal{A} \otimes_{\max} \mathcal{S})^+. \end{aligned}$$

It follows that

$$\|(1_{\mathcal{A}} + a)^{-\frac{1}{2}} \cdot (z + \omega_{12}) \cdot (1_{\mathcal{A}} + d)^{-\frac{1}{2}}\|_{\mathcal{A}\otimes_{\max}\mathcal{S}} \leqslant 1$$

and

$$(\pi \otimes \mathrm{id}_{\mathcal{S}})((1_{\mathcal{A}} + a)^{-\frac{1}{2}} \cdot (z + \omega_{12}) \cdot (1_{\mathcal{A}} + d)^{-\frac{1}{2}}) = (\pi \otimes \mathrm{id}_{\mathcal{S}})(z).$$

□

## 4. The equivalence of the (el, max)-nuclearity and the WEP

As we have seen in the previous section, the maximal tensor product is projective. However, the maximal tensor product need not be injective. This misbehavior distinguishes a class which is called (el, max)-nuclear operator systems [KPTT2]. In the category of $C^*$-algebras, Lance characterized this tensorial property by the factorization property for the inclusion map into the second dual through $B(H)$ [L1, L2]. In [KPTT2], it is proved that this factorization property for an operator system implies its (el, max)-nuclearity and it is asked whether the converse holds. In this section, we answer this question in the affirmative, independent of Lance's original theorem.

**Theorem 4.1.** *Let $\mathcal{S}$ be an operator system. The following are equivalent:*

(i) *we have*

$$\mathcal{S} \otimes_{\max} \mathcal{T} \subset \mathcal{S}_2 \otimes_{\max} \mathcal{T}$$

*for any inclusion* $\mathcal{S} \subset \mathcal{S}_2$ *and any operator system* $\mathcal{T}$*;*

(ii) *we have*

$$\mathcal{S} \otimes_{\max} E \subset \mathcal{S}_2 \otimes_{\max} E$$

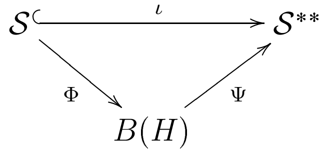
*for any inclusion* $\mathcal{S} \subset \mathcal{S}_2$ *and any finite dimensional operator system* $E$*;*

(iii) *we have*

$$\mathcal{S} \otimes_{\max} E \subset B(H) \otimes_{\max} E$$

*for any inclusion* $\mathcal{S} \subset B(H)$ *and any finite dimensional operator system* $E$*;*

(iv) *there exist unital completely positive maps* $\Phi : \mathcal{S} \to B(H)$ *and* $\Psi : B(H) \to \mathcal{S}^{**}$ *such that* $\Psi \circ \Phi = \iota$ *for the canonical inclusion* $\iota : \mathcal{S} \hookrightarrow \mathcal{S}^{**}$*.*

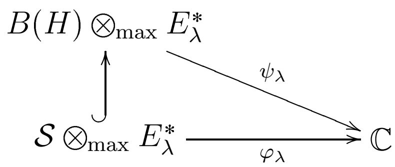



*Proof.* Clearly, (i) implies (ii) and (ii) implies (iii). The direction (iv) $\Rightarrow$ (i) follows from [KPTT2, Theorem 6.7].

(iii) $\Rightarrow$ (iv). Suppose that an operator system $\mathcal{S}$ acts on a Hilbert space $H$. Considering the bidual of the inclusion $\mathcal{S} \subset B(H)$ and the universal representation of $B(H)$, we may assume that the inclusions $\mathcal{S} \subset \mathcal{S}^{**} \subset B(H)$ are given such that the second inclusion is weak*-WOT homeomorphic. Let $\{P_\lambda\}$ be the family of projections on the finite dimensional subspaces of $H$ directed by the inclusions of their ranges. We put

$$E_\lambda = P_\lambda \mathcal{S} P_\lambda \qquad \text{and} \qquad \Phi_\lambda := P_\lambda \cdot P_\lambda : \mathcal{S} \to E_\lambda.$$

Since each $E_\lambda$ is a finite dimensional operator system, there exists a non-canonical Archimedean order unit on its dual space $E_\lambda^*$ [CE, Corollary 4.5]. In other words, the dual space $E_\lambda^*$ is an operator system. By [KPTT1, Lemma 5.7], a functional $\varphi_\lambda$ on $\mathcal{S} \otimes_{\max} E_\lambda^*$ corresponding to the compression $\Phi_\lambda : \mathcal{S} \to E_\lambda$ in the standard way is positive. By assumption, $\mathcal{S} \otimes_{\max} E_\lambda^*$ is an operator subsystem of $B(H) \otimes_{\max} E_\lambda^*$. By Krein's theorem, $\varphi_\lambda$ extends to a positive functional $\psi_\lambda$ on $B(H) \otimes_{\max} E_\lambda^*$.




Let $\Psi_\lambda : B(H) \to E_\lambda$ be a completely positive map corresponding to $\psi_\lambda$ in the standard way. Since $\Phi_\lambda = \Psi_\lambda \circ \iota$ for the inclusion $\iota : \mathcal{S} \subset B(H)$, $\Psi_\lambda$ is a unital completely positive map. We take a state $\omega_\lambda$ on $E_\lambda$ and define a unital completely positive map $\theta_\lambda : E_\lambda \to B(H)$ by

$$\theta_\lambda(x) = x + \omega_\lambda(x)(I - P_\lambda), \qquad x \in E_\lambda.$$

Let $\Psi : B(H) \to B(H)$ be a point-weak* cluster point of $\{\theta_\lambda \circ \Psi_\lambda\}$. We may assume that $\theta_\lambda \circ \Psi_\lambda$ converges to $\Psi$ in the point-weak* topology. For $\xi, \eta \in H$ and $x \in \mathcal{S}$, we have

$$\langle \Psi(x)\xi, \eta \rangle = \lim_\lambda \langle (P_\lambda x P_\lambda + \omega_\lambda(P_\lambda x P_\lambda)(I - P_\lambda))\xi, \eta \rangle = \langle x\xi, \eta \rangle.$$

It follows that $\Psi|_{\mathcal{S}} = \iota$. Since $\Phi_\lambda$ is surjective and $\Psi_\lambda$ is an extension of $\Phi_\lambda$, for each $x \in B(H)$ there exists an element $x_\lambda$ in $\mathcal{S}$ such that $\Psi_\lambda(x) = \Phi_\lambda(x_\lambda) = P_\lambda x_\lambda P_\lambda$. For $\xi, \eta \in H$ and $x \in B(H)$, we have

$$\langle \Psi(x)\xi, \eta\rangle = \lim_\lambda \langle (P_\lambda x_\lambda P_\lambda + \omega_\lambda(P_\lambda x_\lambda P_\lambda)(I - P_\lambda))\xi, \eta\rangle = \lim_\lambda \langle x_\lambda \xi, \eta\rangle.$$

It follows that $\Psi(x)$ belongs to the WOT-closure of $\mathcal{S}$ which coincides with $\mathcal{S}^{**}$ because the inclusion $\mathcal{S}^{**} \subset B(H)$ is weak*-WOT homeomorphic. □

As a corollary, we deduce the following theorem of Lance. The proof is similar to that of [HP, Corollary 3.3].

**Corollary 4.2** (Lance)**.** *Let $\mathcal{A}$ be a unital $C^*$-algebra. Then we have $\mathcal{A} \otimes_{\max} \mathcal{B} \subset \mathcal{A}_2 \otimes_{\max} \mathcal{B}$ for any inclusion $\mathcal{A} \subset \mathcal{A}_2$ and any unital $C^*$-algebra $\mathcal{B}$ if and only if $\mathcal{A}$ has the weak expectation property.*

*Proof.* By Theorem 4.1, it will be enough to prove that if $\mathcal{A} \otimes_{\max} \mathcal{B} \subset B(H) \otimes_{\max} \mathcal{B}$ for any unital $C^*$-algebra $\mathcal{B}$, then $\mathcal{A} \otimes_{\max} \mathcal{T} \subset B(H) \otimes_{\max} \mathcal{T}$ for any operator system $\mathcal{T}$. Due to [KPTT1, Theorem 6.4] and [KPTT1, Theorem 6.7], we obtain the following commutative diagram, which yields the conclusion:

$$\begin{array}{ccccc} \mathcal{A} \otimes_{\max} \mathcal{T} & = & \mathcal{A} \otimes_{\mathrm{c}} \mathcal{T} & \hookrightarrow & \mathcal{A} \otimes_{C^*\max} C_u^*(\mathcal{T}) \\ \downarrow & & & & \downarrow \\ B(H) \otimes_{\max} \mathcal{T} & = & B(H) \otimes_{\mathrm{c}} \mathcal{T} & \hookrightarrow & B(H) \otimes_{C^*\max} C_u^*(\mathcal{T}) \end{array}$$

□

Examples of nuclear operator systems which are not unitally completely order isomorphic to any unital $C^*$-algebra have been constructed in [KW, HP]. These also provide examples of operator systems with the WEP which are not unitally completely order isomorphic to any unital $C^*$-algebra.

## 5. Completely positive maps associated with maximal tensor products

The order unit of the dual spaces of operator systems cannot be considered in general. However, the dual spaces of finite dimensional operator systems have a non-canonical Archimedean order unit. This enables the duality between tensor products and mapping spaces to work in the proofs of the Choi-Effros-Kirchberg theorem for operator systems [HP] and the Lance theorem for operator systems in the previous section. Not only is the finite dimensional assumption restrictive, but also the matrix order unit norm on the dual spaces of finite dimensional operator systems is irrelevant to the matrix norm given by the standard dual of operator spaces. To get rid of the finite dimensional assumption and to reflect the operator space dual norm, we apply Werner's unitization of matrix ordered operator spaces [W] to the dual spaces of operator systems.

Let $V$ be a matrix ordered operator space. We give the involution and the matrix order on $V \oplus \mathbb{C}$ as follows:

(1) $(x + a)^* = x^* + \bar{a}, \qquad x \in V, a \in \mathbb{C}$

(2) $X + A \in M_n(V \oplus \mathbb{C})^+$ iff

$$A \in M_n^+ \quad \text{and} \quad \varphi((A + \varepsilon I_n)^{-\frac{1}{2}} X (A + \varepsilon I_n)^{-\frac{1}{2}}) \geqslant -1$$

for any $\varepsilon > 0$ and any positive contractive functional $\varphi$ on $M_n(V)$.

We denote by $\widetilde{V}$ the space $V\oplus\mathbb{C}$ with the above involution and matrix order and call it the unitization of $V$ [W, Definition 4.7]. The unitization $\widetilde{V}$ of a matrix ordered operator space $V$ is an operator system and the canonical inclusion $\iota : V \hookrightarrow \widetilde{V}$ is a completely contractive complete order isomorphism onto its range [W, Lemma 4.8]. However it need not be completely isomorphic. We apply Werner's unitization of matrix ordered operator spaces to the dual spaces of operator systems. In this case, the canonical inclusion $\iota : \mathcal{S}^* \hookrightarrow \widetilde{\mathcal{S}^*}$ is 2-completely isomorphic [Kar, H, KPTT1].

**Lemma 5.1.** *Suppose that $\mathcal{S}$ is an operator system and $\widetilde{\mathcal{S}^*}$ is the unitization of the dual space $\mathcal{S}^*$. Let $f : \mathcal{S} \to M_n$ be a self-adjoint linear map and $A \in M_n^+$. Then $f+A$ belongs to $M_n(\widetilde{\mathcal{S}^*})^+$ if and only if we have $f_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}$ and $x \in M_m(\mathcal{S})_1^+$.*

*Proof.* ⇒) The element $f + A$ belongs to $M_n(\widetilde{\mathcal{S}^*})^+$ if and only if

$$\varphi((A+\varepsilon I_n)^{-\frac{1}{2}} f (A+\varepsilon I_n)^{-\frac{1}{2}}) \geqslant -1$$

for any $\varepsilon > 0$ and any positive contractive functional $\varphi$ on $M_n(\mathcal{S}^*)$. For $x \in M_m(\mathcal{S})_1^+$ and $\xi \in (\ell^2_{mn})_1$, the map

$$\varphi_{x,\xi} : f \in M_n(\mathcal{S}^*) = CB(\mathcal{S}, M_n) \mapsto \langle f_m(x)\xi|\xi\rangle \in \mathbb{C}$$

is a positive contractive functional on $M_n(\mathcal{S}^*)$. It follows that

$$\begin{aligned} &\langle (I_m \otimes (A+\varepsilon I_n)^{-\frac{1}{2}}) f_m(x) (I_m \otimes (A+\varepsilon I_n)^{-\frac{1}{2}})\xi|\xi\rangle \\ =&\varphi_{x,\xi}((A+\varepsilon I_n)^{-\frac{1}{2}}\ f\ (A+\varepsilon I_n)^{-\frac{1}{2}}) \\ \geqslant& -1. \end{aligned}$$

Hence, we have $f_m(x) \geqslant -I_m \otimes A$.

⇐) Put $\Omega = \{\varphi_{x,\xi} : m \in \mathbb{N}, x \in M_m(\mathcal{S})_1^+, \xi \in (\ell^2_{mn})_1\}$ where $\varphi_{x,\xi}$ defined as above. Let $\Gamma_1$ be a weak*-closed convex hull of $\Omega$ and $\Gamma_2$ a weak*-closed cone generated by $\Omega$. We want $\Gamma_1 = (M_n(\mathcal{S}^*)^*)_1^+$. Here, $(M_n(\mathcal{S}^*)^*)_1^+$ denotes the set of positive contractive functionals on $M_n(\mathcal{S}^*)$. If this were not the case, we could choose $\varphi_0 \in (M_n(\mathcal{S}^*)^*)_1^+/\Gamma_1$. By the Krein-Smulian theorem [C, Theorem 5.12.1], we have $\mathbb{R}^+ \cdot \Gamma_1 = \Gamma_2$, thus $\varphi_0$ does not belong to $\Gamma_2$. By the Hahn-Banach separation theorem, there exists $f_0 \in M_n(\mathcal{S}^*)_{sa}$ which separates $\Gamma_2$ and $\varphi_0$ strictly. We have $\Gamma_2(f_0) = \{0\}$ or $[0,\infty)$ or $(-\infty, 0]$. If $\varphi_{x,\xi}(f_0) = 0$ for all $\varphi_{x,\xi} \in \Omega$, then we have $f_0 = 0$. We may assume that $\Gamma_2(f_0) = [0,\infty)$. Then we have $f_0 \in M_n(\mathcal{S}^*)^+$ which is a contradiction since $f_0$ separates $\Gamma_2$ and $\varphi_0$ strictly. The conclusion follows from $\Gamma_1 = (M_n(\mathcal{S}^*)^*)_1^+$ and

$$\varphi_{x,\xi}((A+\varepsilon I_n)^{-\frac{1}{2}}\ f\ (A+\varepsilon I_n)^{-\frac{1}{2}}) \geqslant -1.$$

□

**Proposition 5.2.** *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a finite rank map for operator systems $\mathcal{S}$ and $\mathcal{T}$. Then $\Phi$ is completely positive if and only if it belongs to $(\widetilde{\mathcal{S}^*} \otimes_{\min} \mathcal{T})^+$.*

*Proof.* ⇒) The finite rank map $\Phi$ can be regarded as an element in $\mathcal{S}^* \otimes \mathcal{T} \subset \widetilde{\mathcal{S}^*} \otimes \mathcal{T}$. For a positive element $x$ in $M_n(\mathcal{S})$, the evaluation map $\mathrm{ev}_x : \mathcal{S}^* \to M_n$ defined by

$$\mathrm{ev}_x(f) = f_n(x), \qquad f \in \mathcal{S}^*$$

is completely positive. For a completely positive map $g : \mathcal{T} \to M_m$, we have

$$(\mathrm{ev}_x \otimes g)(\Phi) = (g \circ \Phi)_n(x) \in M_{mn}^+, \qquad (\mathrm{ev}_x \otimes g : \mathcal{S}^* \otimes \mathcal{T} \to M_{mn}).$$

For a completely positive map $f : \widetilde{\mathcal{S}^*} \to M_n$, its restriction $f|_{\mathcal{S}^*}$ belongs to $CP(\mathcal{S}^*, M_n) = M_n(\mathcal{S}^{**})^+$. It is the point-norm limit of evaluation maps $\mathrm{ev}_x$ for $x \in M_n(\mathcal{S})^+$. Thus, we have

$$(f \otimes g)(\Phi) = (f|_{\mathcal{S}^*} \otimes g)(\Phi) \in M_{mn}^+.$$

In other words, $\Phi$ belongs to $(\widetilde{\mathcal{S}^*} \otimes_{\min} \mathcal{T})^+$.

$\Leftarrow$) For $x \in M_n(\mathcal{S})_1^+$, we define an evaluation map $\mathrm{ev}_x : \widetilde{\mathcal{S}^*} \to M_n$ by

$$\mathrm{ev}_x(f + \lambda) = f_n(x) + \lambda I_n.$$

By Lemma 5.1, the evaluation map $\mathrm{ev}_x : \widetilde{\mathcal{S}^*} \to M_n$ is completely positive.

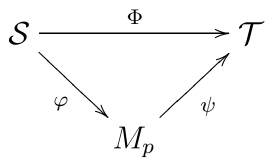
From

$$(g \circ \Phi)_n(x) = (\mathrm{ev}_x \otimes g)(\Phi) \in M_{mn}^+$$

we see that $\Phi : \mathcal{S} \to \mathcal{T}$ is completely positive. □

By Proposition 5.2, finite rank completely positive maps from $\mathcal{S}$ to $\mathcal{T}$ correspond to the elements in $\mathcal{S}^* \otimes \mathcal{T} \cap (\widetilde{\mathcal{S}^*} \otimes_{\min} \mathcal{T})^+$. Our next goal is to study completely positive maps corresponding to the elements in $\mathcal{S}^* \otimes \mathcal{T} \cap (\widetilde{\mathcal{S}^*} \otimes_{\max} \mathcal{T})^+$.

**Theorem 5.3.** *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a finite rank map. Then $\Phi$ belongs to $(\widetilde{\mathcal{S}^*} \otimes_{\max} \mathcal{T})^+$ if and only if for any $\varepsilon > 0$, there exist a factorization $\Phi = \psi \circ \varphi$ and a positive semidefinite matrix $A \in M_p$ such that $\varphi : \mathcal{S} \to M_p$ is a self-adjoint map with $\varphi_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}, x \in M_m(\mathcal{S})_1^+$ and $\psi : M_p \to \mathcal{T}$ is a completely positive map with $\psi(A) = \varepsilon 1_{\mathcal{T}}$.*




*Proof.* $\Rightarrow$) Let $\Phi \in (\widetilde{\mathcal{S}^*} \otimes_{\max} \mathcal{T})^+$. For any $\varepsilon > 0$, we can write

$$\Phi + \varepsilon 1_{\widetilde{\mathcal{S}^*}} \otimes 1_{\mathcal{T}} = \alpha((\varphi + A \cdot 1_{\widetilde{\mathcal{S}^*}}) \otimes Q)\alpha^*$$

for $\alpha \in M_{1,pq}, \varphi \in M_p(\mathcal{S}^*), A \in M_p, Q \in M_q(\mathcal{T})^+$ and $\varphi + A \cdot 1_{\widetilde{\mathcal{S}^*}} \in M_p(\widetilde{\mathcal{S}^*})^+$. It follows that

$$\Phi(x) = \alpha\varphi(x) \otimes Q\alpha^* \quad \text{and} \quad \varepsilon 1_{\mathcal{T}} = \alpha(A \otimes Q)\alpha^*.$$

By Lemma 5.1, we have

$$\varphi_n(x) \geqslant -I_n \otimes A, \qquad n \in \mathbb{N}, x \in M_n(\mathcal{S})_1^+.$$

We define a completely positive map $\psi : M_p \to \mathcal{T}$ by $\psi(B) = \alpha(B \otimes Q)\alpha^*$. Then we have

$$\Phi(x) = \alpha\varphi(x) \otimes Q\alpha^* = \psi(\varphi(x)) \quad \text{and} \quad \psi(A) = \alpha(A \otimes Q)\alpha^* = \varepsilon 1_{\mathcal{T}}.$$

$\Leftarrow$) We put

$$Q = [\psi(e_{i,j})]_{1 \leqslant i,j \leqslant p} \in M_p(\mathcal{T})^+ \quad \text{and} \quad \alpha = [e_1, \cdots, e_p] \in M_{1,p^2}.$$

Because

$$\psi([b_{i,j}]) = \psi(\sum_{i,j=1}^p b_{i,j}e_{i,j}) = \sum_{i,j=1}^p b_{i,j}Q_{i,j} = \alpha([b_{i,j}] \otimes Q)\alpha^*,$$

we can write

$$\Phi(x) = \alpha\varphi(x) \otimes Q\alpha^* \quad \text{and} \quad \varepsilon 1_{\mathcal{T}} = \alpha(A \otimes Q)\alpha^*.$$

By Lemma 5.1, we have $\varphi + A \in M_n(\widetilde{\mathcal{S}^*})^+$. It follows that

$$\Phi + \varepsilon 1_{\widetilde{\mathcal{S}^*}} \otimes 1_{\mathcal{T}} = \alpha((\varphi + A \cdot 1_{\widetilde{\mathcal{S}^*}}) \otimes Q)\alpha^* \in (\widetilde{\mathcal{S}^*} \otimes_{\max} \mathcal{T})^+$$

□

Let $\Phi : \mathcal{S} \to \mathcal{T}$ be a completely positive map factoring through a matrix algebra in a completely positive way. By the proof of Theorem 5.3, $\Phi$ corresponds to an element in the subcone

$$\{\alpha(\varphi \otimes Q)\alpha^* : \alpha \in M_{1,pq}, \varphi \in M_p(\mathcal{S}^*)^+, Q \in M_q(\mathcal{T})^+\}$$

of the cone $\mathcal{S}^* \otimes \mathcal{T} \cap (\widetilde{\mathcal{S}^*} \otimes_{\max} \mathcal{T})^+$.

**Theorem 5.4.** *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a completely positive map for operator systems $\mathcal{S}$ and $\mathcal{T}$. The map*

$$\mathrm{id}_{\mathcal{R}} \otimes \Phi : \mathcal{R} \otimes_{\min} \mathcal{S} \to \mathcal{R} \otimes_{\max} \mathcal{T}$$

*is completely positive for any operator system $\mathcal{R}$ if and only if we have*

$$\Phi|_E \in (\widetilde{E^*} \otimes_{\max} \mathcal{T})^+$$

*for any finite dimensional operator subsystem $E$ of $\mathcal{S}$.*

*Proof.* $\Rightarrow$) By Proposition 5.2, we can regard the inclusion $\iota : E \subset \mathcal{S}$ as an element in $(\widetilde{E^*} \otimes_{\min} \mathcal{S})^+$. By assumption, we see that $\Phi|_E = (\mathrm{id}_{\widetilde{E^*}} \otimes \Phi)(\iota)$ belongs to $(\widetilde{E^*} \otimes_{\max} \mathcal{T})^+$.

$\Leftarrow$) We choose an element

$$z = \sum_{i=1}^{n} x_i \otimes y_i \in (\mathcal{R} \otimes_{\min} \mathcal{S})_1^+.$$

Let $E$ be a finite dimensional operator subsystem of $\mathcal{S}$ which contains $\{y_i : 1 \leqslant i \leqslant n\}$. By Theorem 5.3, there exist a factorization $\Phi|_E = \psi \circ \varphi$ and a positive semidefinite matrix $A \in M_n$ such that $\varphi : E \to M_n$ is a self-adjoint map with $\varphi_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}, x \in M_m(\mathcal{S})_1^+$ and $\psi : M_n \to \mathcal{T}$ is a completely positive map with $\psi(A) = \varepsilon 1_{\mathcal{T}}$. Let $\mathcal{R}$ be a concrete operator system acting on a Hilbert space $H$ and $P$ the projection onto the finite dimensional subspace of $H$. The compression $P\mathcal{R}P$ is the operator subsystem of a matrix algebra $M_p$ for $p = \mathrm{rank}P$. From the commutative diagram

$$\begin{array}{ccc} M_p \otimes_{\min} E & \xrightarrow{\mathrm{id}_{M_p} \otimes \varphi} & M_p \otimes_{\min} M_n \\ \uparrow & & \uparrow \\ P\mathcal{R}P \otimes_{\min} E & \xrightarrow{\mathrm{id}_{P\mathcal{R}P} \otimes \varphi} & P\mathcal{R}P \otimes_{\min} M_n \end{array}$$

we see that

$$(\mathrm{id}_{\mathcal{R}} \otimes \varphi)(z) \geqslant -1_{\mathcal{R}} \otimes A.$$

From the commutative diagram

$$\begin{array}{ccc} \mathcal{R} \otimes_{\min} \mathcal{S} & \xrightarrow{\mathrm{id}_{\mathcal{R}} \otimes \Phi} & \mathcal{R} \otimes_{\max} \mathcal{T} \\ \uparrow & & \uparrow \mathrm{id}_{\mathcal{R}} \otimes \psi \\ \mathcal{R} \otimes_{\min} E & \xrightarrow{\mathrm{id}_{\mathcal{R}} \otimes \varphi} & \mathcal{R} \otimes_{\min} M_n = \mathcal{R} \otimes_{\max} M_n \end{array}$$

we also see that

$$\begin{aligned} (\mathrm{id}_{\mathcal{R}} \otimes \Phi)(z) &= (\mathrm{id}_{\mathcal{R}} \otimes \psi) \circ (\mathrm{id}_{\mathcal{R}} \otimes \varphi)(z) \\ &\geqslant -\varepsilon(\mathrm{id}_{\mathcal{R}} \otimes \psi)(1_{\mathcal{R}} \otimes A) \\ &= -\varepsilon 1_{\mathcal{R}} \otimes 1_{\mathcal{T}} \end{aligned}$$

in $\mathcal{R} \otimes_{\max} \mathcal{T}$. Since the choice of $\varepsilon > 0$ is arbitrary, we conclude that the map

$$\mathrm{id}_{\mathcal{R}} \otimes \Phi : \mathcal{R} \otimes_{\min} \mathcal{S} \to \mathcal{R} \otimes_{\max} \mathcal{T}$$

is positive.

$\square$

## 6. Duality

We establish the duality between complete order embeddings and complete order quotient maps.

**Theorem 6.1.** *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are operator systems with complete norms and $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive surjection. Then $\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map if and only if its dual map $\Phi^* : \mathcal{T}^* \to \mathcal{S}^*$ is a complete order embedding.*

*Proof.* $\Rightarrow$) Let $\Phi_n^*(f) \in M_n(\mathcal{S}^*)^+ = CP(\mathcal{S}, M_n)$ for $f \in M_n(\mathcal{T}^*)$. We choose a positive element $y$ in $M_m(\mathcal{T})$. For any $\varepsilon > 0$, there exists an element $x$ in $M_m(\mathcal{S})$ such that

$$\Phi_m(x) = y \qquad \text{and} \qquad x + \varepsilon I_m \otimes 1_{\mathcal{S}} \in M_m(\mathcal{S})^+.$$

We have

$$(f \circ \Phi)_m(x) + \varepsilon(f \circ \Phi)_m(I_m \otimes 1_{\mathcal{S}}) = (\Phi_n^*(f))_m(x + \varepsilon I_n \otimes 1_{\mathcal{S}}) \in M_{mn}^+.$$

Since the choice of $\varepsilon > 0$ is arbitrary, we have

$$f_m(y) = (f \circ \Phi)_m(x) \in M_{mn}^+.$$

It follows that $f : \mathcal{T} \to M_n$ is completely positive.

$\Leftarrow$) We put

$$C_n := \{y \in M_n(\mathcal{T}) : \forall \varepsilon > 0, \exists x \in M_n(\mathcal{S}), x + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \text{ and } \Phi_n(x) = y\}.$$

For $y \in C_n$, we have

$$y + \varepsilon I_n \otimes 1_{\mathcal{T}} = \Phi_n(x + \varepsilon I_n \otimes 1_{\mathcal{S}}) \in M_n(\mathcal{T})^+,$$

thus $C_n \subset M_n(\mathcal{T})^+$. The map $\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map if and only if $C_n = M_n(\mathcal{T})^+$ holds for all $n \in \mathbb{N}$. It is easy to check that $C_n$ is a cone. We have the inclusions of the cones

$$\Phi_n(M_n(\mathcal{S})^+) \subset C_n \subset M_n(\mathcal{T})^+.$$

Suppose that $y_k \in C_n$ converges to $y \in M_n(\mathcal{T})$. By the open mapping theorem, there exists $M > 0$ such that $\|\widetilde{\Phi}_n^{-1} : M_n(\mathcal{T}) \to M_n(\mathcal{S})/\mathrm{Ker}\Phi_n\| \leqslant M$. We choose $y_{k_0}$ so that $\|y - y_{k_0}\| < \varepsilon/M$. There exist $x, x'$ in $M_n(\mathcal{S})$ such that

$$\Phi_n(x) = y_{k_0},\ x + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \quad \text{and} \quad \Phi_n(x') = y - y_{k_0},\ \|x'\| < \varepsilon.$$

Replacing $x'$ by $(x' + x'^*)/2$, we may assume that $x'$ is self-adjoint. It follows that

$$y = y_{k_0} + (y - y_{k_0}) = \Phi_n(x + x') \quad \text{and} \quad x + x' + 2\varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+.$$

Hence, the cone $C_n$ is closed. We assume $C_n \subsetneq M_n(\mathcal{T})^+$ and choose $y_0 \in M_n(\mathcal{T})^+/C_n$. By the Hahn-Banach separation theorem, there exists a self-adjoint functional $f$ on $M_n(\mathcal{T})$ such that $f(C_n) = [0, \infty)$ and $f(y_0) < 0$. Even though the functional $f : M_n(\mathcal{T}) \to \mathbb{C}$ is not positive, we have $\Phi_n^*(f)(x) = f \circ \Phi_n(x) \geqslant 0$ for all $x \in M_n(\mathcal{S})^+$ because $\Phi_n(M_n(\mathcal{S})^+)$ is a subcone of $C_n$. Hence, we see that the dual map $\Phi^* : \mathcal{T}^* \to \mathcal{S}^*$ is not a complete order embedding. $\square$

**Lemma 6.2.** *Suppose that $f : \mathcal{S} \to M_n$ is a self-adjoint linear map for an operator system $\mathcal{S}$. Then we have $f_m(x) \geqslant -I_{mn}$ for all $m \in \mathbb{N}, x \in M_m(S)_1^+$ if and only if the self-adjoint linear map $\widetilde{f} : \mathcal{S} \to M_n$ defined by*

$$\widetilde{f}(x) = (f(1_{\mathcal{S}}) + 2I_n)^{-\frac{1}{2}} f(x) (f(1_{\mathcal{S}}) + 2I_n)^{-\frac{1}{2}}$$

*is completely contractive.*

*Proof.* $\Rightarrow$) We choose a contractive element $x$ in $M_m(\mathcal{S})$. Then we have

$$0 \leqslant \begin{pmatrix} I_m \otimes 1_{\mathcal{S}} & x \\ x^* & I_m \otimes 1_{\mathcal{S}} \end{pmatrix} \leqslant 2 \begin{pmatrix} I_m \otimes 1_{\mathcal{S}} & 0 \\ 0 & I_m \otimes 1_{\mathcal{S}} \end{pmatrix}.$$

By assumption, we have

$$0 \leqslant \begin{pmatrix} I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn} & f_m(x) \\ f_m(x)^* & I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn} \end{pmatrix}.$$

Multiplying both sides by $(I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn})^{-\frac{1}{2}} \oplus (I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn})^{-\frac{1}{2}}$ from the left and from the right, we see that $\|\widetilde{f}_m(x)\| \leqslant 1$.

$\Leftarrow$) We choose an element $x$ in $M_m(\mathcal{S})_1^+$. Then we have

$$\|x - \frac{1}{2} I_m \otimes 1_{\mathcal{S}}\| \leqslant \frac{1}{2}.$$

By assumption, we have

$$\begin{aligned} &\|(I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn})^{-\frac{1}{2}} f_m(x - \frac{1}{2} I_m \otimes 1_{\mathcal{S}})(I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn})^{-\frac{1}{2}}\| \\ =&\|\widetilde{f}_m(x - \frac{1}{2} I_m \otimes 1_{\mathcal{S}})\| \\ \leqslant&\frac{1}{2}, \end{aligned}$$

thus

$$-\frac{1}{2}(I_m \otimes f(1_{\mathcal{S}}) + 2I_{mn}) \leqslant f_m(x - \frac{1}{2} I_m \otimes 1_{\mathcal{S}}).$$

It follows that $f_m(x) \geqslant -I_{mn}$. $\square$

**Lemma 6.3.** *Let $\mathcal{S}$ be an operator subsystem of an operator system $\mathcal{T}$ and $A$ a positive semidefinite $n \times n$ matrix. Suppose that $f : \mathcal{S} \to M_n$ is a self-adjoint linear map satisfying $f_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}, x \in M_m(\mathcal{S})_1^+$. Then $f : \mathcal{S} \to M_n$ extends to a self-adjoint linear map $F : \mathcal{T} \to M_n$ satisfying $F_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}, x \in M_m(\mathcal{T})_1^+$.*

*Proof.* We define a self-adjoint linear map $g : \mathcal{S} \to M_n$ by

$$g(x) = (A + \varepsilon I_n)^{-\frac{1}{2}} f(x)(A + \varepsilon I_n)^{-\frac{1}{2}}$$

for $0 < \varepsilon < 1$. Then we have

$$g_m(x) \geqslant -(I_m \otimes A + \varepsilon I_{mn})^{-\frac{1}{2}}(I_m \otimes A)(I_m \otimes A + \varepsilon I_{mn})^{-\frac{1}{2}} \geqslant -I_{mn}$$

for all $m \in \mathbb{N}, x \in M_m(\mathcal{S})_1^+$. By Lemma 6.2, the self-adjoint linear map $\widetilde{g} : \mathcal{S} \to M_n$ defined by

$$\widetilde{g}(x) = (g(1_{\mathcal{S}}) + 2I_n)^{-\frac{1}{2}} g(x)(g(1_{\mathcal{S}}) + 2I_n)^{-\frac{1}{2}}$$

is completely contractive. By the Wittstock extension theorem, $\widetilde{g}$ extends to a complete contraction $\widetilde{G} : \mathcal{T} \to M_n$. By considering $\frac{1}{2}(\widetilde{G} + \widetilde{G}^*)$ instead, we may assume that $\widetilde{G}$ is self-adjoint. We put

$$G(x) = (g(1_{\mathcal{S}}) + 2I_n)^{\frac{1}{2}} \widetilde{G}(x)(g(1_{\mathcal{S}}) + 2I_n)^{\frac{1}{2}}$$

and

$$F(x) = (A + \varepsilon I_n)^{\frac{1}{2}} G(x)(A + \varepsilon I_n)^{\frac{1}{2}}.$$

Then $F : \mathcal{T} \to M_n$ (respectively, $G : \mathcal{T} \to M_n$) is a self-adjoint extension of $f : \mathcal{S} \to M_n$ (respectively, $g : \mathcal{S} \to M_n$). The self-adjoint linear map $\widetilde{G}$ can be written as

$$\widetilde{G}(x) = (G(1_{\mathcal{S}}) + 2I_n)^{-\frac{1}{2}} G(x)(G(1_{\mathcal{S}}) + 2I_n)^{-\frac{1}{2}}.$$

By using Lemma 6.2 again, we see that

$$F_m(x) = (I_m \otimes A + \varepsilon I_{mn})^{\frac{1}{2}} G_m(x)(I_m \otimes A + \varepsilon I_{mn})^{\frac{1}{2}} \geqslant -I_m \otimes A - \varepsilon I_{mn}$$

for all $m \in \mathbb{N}, x \in M_m(\mathcal{T})_1^+$. The extension $F : \mathcal{T} \to M_n$ depends on the choice of $\varepsilon$. However the norm of $F$ is uniformly bounded as can be seen from

$$\begin{aligned} \|F\| &\leqslant \|(A + \varepsilon I_n)^{\frac{1}{2}}((A + \varepsilon I_n)^{-\frac{1}{2}} f(1_{\mathcal{S}})(A + \varepsilon I_n)^{-\frac{1}{2}} + 2I_n)^{\frac{1}{2}}\|^2 \\ &= \|f(1_{\mathcal{S}}) + 2(A + \varepsilon I_n)\| \\ &\leqslant \|f(1_{\mathcal{S}}) + 2A\| + 2. \end{aligned}$$

Since the range space is finite dimensional, we can consider the point-norm cluster point of $\{F_\varepsilon : 0 < \varepsilon < 1\}$. □

Let $T : V \to W$ be a completely contractive and completely positive map for matrix ordered operator spaces $V$ and $W$. Then its unitization $\widetilde{T} : \widetilde{V} \to \widetilde{W}$ defined by

$$\widetilde{T}(x + \lambda 1_{\widetilde{V}}) = T(x) + \lambda 1_{\widetilde{W}}, \qquad x \in V, \lambda \in \mathbb{C}$$

is a unital completely positive map [W, Lemma 4.9].

**Theorem 6.4.** *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive map for operator systems $\mathcal{S}$ and $\mathcal{T}$. Then $\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order embedding if and only if the unitization of its dual map $\widetilde{\Phi^*} : \widetilde{\mathcal{T}^*} \to \widetilde{\mathcal{S}^*}$ is a complete order quotient map.*

*Proof.* $\Rightarrow$) Let $f + A \in M_n(\widetilde{\mathcal{S}^*})^+$. By Lemma 5.1, we have $f_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}$ and $x \in M_m(\mathcal{S})_1^+$. We can regard $\mathcal{S}$ as an operator subsystem of $\mathcal{T}$. By Lemma 6.3, there exists a self-adjoint extension $F : \mathcal{T} \to M_n$ such that $F_m(x) \geqslant -I_m \otimes A$ for all $m \in \mathbb{N}$ and $x \in M_m(\mathcal{T})_1^+$. By Lemma 5.1 again, we have

$$\Phi_n^*(F + A) = f + A \qquad \text{and} \qquad F + A \in M_n(\widetilde{\mathcal{T}^*})^+.$$

$\Leftarrow$) Suppose that $\Phi_n(x)$ belongs to $M_n(\mathcal{T})^+$ for $x \in M_n(\mathcal{S})$. Let $f : M_n(\mathcal{S}) \to \mathbb{C}$ be a positive functional. For any $\varepsilon > 0$, there exists a self-adjoint linear functional $F : M_n(\mathcal{T}) \to \mathbb{C}$ such that

$$\Phi_n^*(F) = f \qquad \text{and} \qquad F + \varepsilon I_n \otimes 1_{\widetilde{\mathcal{T}^*}} \in M_n(\widetilde{\mathcal{T}^*})^+.$$

We have

$$f(x) = F \circ \Phi(x) \geqslant -\varepsilon n^2 \|x\|.$$

It follows that $x \in M_n(\mathcal{S})^+$. If $\Phi_n(x) = 0$, then $x \in M_n(\mathcal{S})^+ \cap -M_n(\mathcal{S})^+ = \{0\}$. ☐

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