On Revenue Maximization with Sharp Multi-Unit Demands

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Abstract

We consider markets consisting of a set of indivisible items, and buyers that have *sharp* multi-unit demand. This means that each buyer i wants a specific number d_i of items; a bundle of size less than d_i has no value, while a bundle of size greater than d_i is worth no more than the most valued d_i items (valuations being additive). We consider the objective of setting prices and allocations in order to maximize the total revenue of the market maker. The pricing problem with sharp multi-unit demand buyers has a number of properties that the unit-demand model does not possess, and is an important question in algorithmic pricing. We consider the problem of computing a revenue maximizing solution for two solution concepts: competitive equilibrium and envy-free pricing.

For unrestricted valuations, these problems are NP-complete; we focus on a realistic special case of "correlated values" where each buyer i has a valuation $v_i q_j$ for item j, where v_i and q_j are positive quantities associated with buyer i and item j respectively. We present a polynomial time algorithm to solve the revenue-maximizing competitive equilibrium problem. For envy-free pricing, if the demand of each buyer is bounded by a constant, a revenue maximizing solution can be found efficiently; the general demand case is shown to be NP-hard.

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1 Introduction

The problems considered in this paper are motivated by applications illustrated by the following example. A publisher (e.g., a TV network) has some items (such as advertising slots) that are provided to potential customers (the advertisers). Each customer i has a demand which specifies the number of items that i needs. Given the demand requests from different customers, as well as the values that they are willing to pay, the problem that the publisher faces is how to allocate the items to customers at which prices. Demand is a practical consideration and has occurred in a number of applications. For instance, in TV (or radio) advertising [25], advertisers may request different lengths of advertising slots for their ads programs. In banner (or newspaper) advertising, advertisers may request different sizes or areas for their displayed ads, which may be decomposed into a number of base units. A remarkable application of our model is where advertisers choose to display their advertisement using rich medias (video, audio, animation) [4, 26] that would usually need a fixed number of positions while text ads would need only one position each. It has been formulated as a sharp-demand model in sponsored search in recent works [18, 12].

We study the economic problem as a two-sided market where the supply side is composed of m indivisible items and each item j has a parameter q_j, measuring the quality of the item. For example, in TV advertising, inventories of a commercial break are usually divided into slots of five seconds each, and every slot has an expected number of viewers. The other side of the market has n potential buyers where each buyer i has a demand d_i (the number of items that i requests) and a value v_i (the benefit to i for an item of unit quality). Thus, the valuation that i obtains from item j is given by $v_{ij} = v_i q_j$. The $v_i q_j$ valuation model has been considered by Edelman et al. [13] and Varian [28] in their seminal work for keywords advertising. We will focus on the *sharp* demand case, where every buyer requests exactly d_i items. This scenario captures some similarity but is still quite different from single-minded buyers (i.e., each one desires a fixed combination of items) and is distinct from the *relaxed* demand case, where every buyer requests at most d_i items. In the practical setting of the rich media advertisement, one slot can be sold as a single text ad, or several (three, as an example) slots for one rich media ad. To merge several slots into one single rich media slot ad, the chosen slots must be connected to each other. Though this does not in general hold, our solution has a consecutive property in terms of the quaility value, which will usually resolve the application issue.

Given the valuations and demands from the buyers, the market maker decides on a price vector $\mathbf{p} = (p_j)$ for all items and an allocation of items to buyers, as an output of the market. The question is one of which output the market maker should choose to achieve certain objectives. In this paper, we assume that the market maker would like to maximize his own revenue, which is defined to be the total payment collected from the buyers. While revenue maximization is a natural goal from the market maker's perspective, buyers may have their own objectives as well. We aim to model a "free market" where consumers are price takers; thus, in a robust solution concept, one has to consider the performance of the whole market and the interests of the buyers.

Competitive equilibrium provides such a solution concept that captures both market efficiency and fairness for the buyers. In a competitive equilibrium, every buyer obtains a best possible allocation that maximizes his own utility and every unallocated item is priced at zero (i.e., market clearance). Competitive equilibrium is one of the central solution concepts in economics and has been studied and applied in a variety of domains [24]. Combining the considerations from the two sides of the market, an ideal solution concept therefore would be revenue maximizing competitive equilibrium.

For sharp multi-unit demand buyers, when the valuations v_{ij} are arbitrary, even determining the existence of a competitive equilibrium is NP-complete (see Appendix A). For our correlated valuation $v_i q_j$ model, we have the following results.

Theorem 1. *For sharp multi-unit demand, a competitive equilibrium may not exist; even if an equilibrium is guaranteed to exist, a maximum equilibrium (in which each price is as high as it can be in any solution) may not exist. Further, there is a polynomial time algorithm that determines the existence of an equilibrium, and computes a revenue maximizing one if it does.*

While (revenue maximizing) competitive equilibrium has a number of nice economic properties and has been recognized as an elegant tool for the analysis of competitive markets, its possible non-existence largely ruins its applicability. Such non-existence is a result of the market clearance condition required in the equilibrium (i.e., unallocated items have to be priced at zero). In most applications, however, especially in advertising markets, market makers are able to manage the amount of supplies. For instance, in TV advertising, publishers can 'freely' adjust the length of a commercial break. Therefore, the market clearance condition becomes arguably unnecessary in those applications. This motivates the study of *envy-free pricing*(here envy-free pricing we mean envy-free item pricing [16]) which only requires the *fairness* condition in the competitive equilibrium, where no buyer can get a larger utility from any other allocation for the given prices. In contrast with competitive equilibrium, an envy-free solution always exists (a trivial one is obtained by setting all prices to ∞). Once again, taking the interests of both sides of the market into account, revenue maximizing envy-free pricing is a natural solution concept that can be applied in those marketplaces.

The study of algorithmic computation of revenue maximizing envy-free pricing was initiated by Guruswami et al. [21], where the authors considered two special settings with unit demand buyers and single-minded buyers and showed that a revenue maximizing envy-free pricing is NP-hard to compute. Because envy-free pricing has applications in various settings and efficient computation is a critical condition for its applicability, there is a surge of studies on its computational issues since the pioneering work of [21], mainly focusing on approximation solutions and special cases that admit polynomial time algorithms, e.g., [22, 2, 6, 3, 5, 9, 17, 14, 8, 19].

The NP-hardness result of [21] for unit demand buyers implies that we cannot hope for a polynomial time algorithm for general v_{ij} valuations in the multi-unit demand setting, even for the very special case when one has positive values for at most three items [8]. However, it does not rule out positive computational results for special, but important, cases of multi-unit demand. For $v_i q_j$ valuations with multi-unit demand, where the hardness reductions of [21, 8] does not apply, we have the following results.

Theorem 2. *There is a polynomial time algorithm that computes a revenue maximizing envy-free solution in the sharp multi-unit demand model with $v_i q_j$ valuations if the demand of every buyer is bounded by a constant. On the other hand, the problem is NP-hard if the sharp demand is arbitrary, even if there are only three buyers.*

For relaxed multi-unit demand, a standard technique can reduce the problem to the unit-demand version: each buyer i with demand d_i can be replaced by d_i copies of buyer i, each of whom requests one item. Note that under the sharp demand constraint, this trick is no longer applicable.

We summarize our results in the following table. Here, we have a complete overview of the existence and computation of competitive equilibrium and envy-free pricing with multi-unit demand buyers. Most of our results are positive, suggesting that competitive equilibrium and envy-free pricing are candidate solution concepts to be applicable in the domains where the valuations are correlated with respect to the quality of the items.

		Competitive equilibrium	Envy-free pricing
Unit demand (general values v_{ij})	existence	yes [27]	yes (trivial)
	max revenue computation	P [27, 11]	NP-hard [21]
Sharp multi-unit demand ($v_{ij} = v_i q_j$)	existence	not always (P decidable) (NP-hard for general v_{ij})	yes (trivial)
	max revenue computation	P (if one exists)	P (constant demand) NP-hard (arbitrary demand)

Table 1: Summary of previous work and our results.

Despite the recent surge in the studies of algorithmic pricing, multi-unit demand models have not received much attention. Most previous work has focused on two simple special settings: unit demand and single-minded buyers, but arguably multi-unit demand has much more applicability. While the relaxed demand model shares similar properties to unit demand (e.g., existence, solution structure, and computation), the sharp demand model has a number of features that unit demand does not possess.

- Existence of equilibrium. As discussed above, a competitive equilibrium may not exist in the sharp demand model. Further, even if a solutions exist, the solution space may not form a distributive lattice.

- Over-priced items. In unit-demand, the price p_j of any item j is always at most the value v_{ij} of the corresponding winner i. This no longer holds for sharp multi-unit demand. Specifically, even if $p_j > v_{ij}$, buyer i may still want to get j since his net profit from other items may compensate his loss from item j (see Example 3.3)[1]. This property enlarges the solution space and adds an extra challenge to finding a revenue maximizing solution.

Our Techniques. To compute a competitive equilibrium, we first find a candidate winner set, which can be proved to be an equilibrium winner set if a competitive equilibrium exists; then, with this set, we transform the computation of competitive equilibria to a linear program of exponential size, which can be solved by the ellipsoid algorithm in polynomial time. The situation becomes complicated when finding an optimal envy-free solution. Actually, we prove that it is NP-hard to compute an optimal envy-free solution even if there are only three buyers. Hence, our efforts focus on the special, yet very important bounded-demand case. To compute an optimal envy-free solution for bounded demand, certain candidate winner sets (the number of such sets is polynomial) are defined and found; and crucially, there is at least one optimal winner set in our selected candidate winner sets. For each candidate winner set, if the demand is bounded by a constant, we present a linear programming to characterize its optimal solution when the allocation is known. Finally, a dynamic programming algorithm is provided to find the allocation sets when a candidate winner set is fixed. Both the linear programming and the dynamic programming run in polynomial time.

1.1 Related Work

There are extensive studies on multi-unit demand in economics, see, e.g., [1, 15, 7]. Our study focuses on sharp demand buyers. An alternative model is when buyers have relaxed multi-unit demand (i.e., one can buy a subset of at most d_i items), where it is well known that the set of competitive equilibrium prices is non-empty and forms a distributive lattice [27, 20]. This immediately implies the existence of an equilibrium with maximum possible prices; hence, revenue is maximized. Demange, Gale, and Sotomayor [11] proposed a combinatorial dynamics which always converges to a revenue maximizing (or minimizing) equilibrium for unit demand; their algorithm can be easily generalized to relaxed multi-unit demand.

From an algorithmic point of view, the problem of revenue maximization in envy-free pricing was initiated by Guruswami et al. [21], who showed that computing an optimal envy-free pricing is APX-hard for unit-demand bidders and gave an $O(\log n)$ approximation algorithm. Briest [5] showed that given appropriate complexity assumptions, the unit-demand envy-free pricing problem in general cannot be approximated within $O(\log^{\epsilon} n)$ for some $\epsilon > 0$. Hartline and Yan [23] characterized optimal envy-free pricing for unit-demand and showed its connection to mechanism design. For the multi-unit demand setting, Chen et al. [9] gave an $O(\log D)$ approximation algorithm when there is a metric space behind all items, where D is the maximum demand, and Briest [5] showed that the problem is hard to approximate within a ratio of $O(n^{\epsilon})$ for some ϵ, unless $NP \subseteq \bigcap_{\epsilon>0} BPTIME(2^{n^{\epsilon}})$. It should be noticed that recent work by M, Feldman et al studies envy-free revenue maximization problem with budget but without demand constraints and present a 2-approximate mechanism for envy-free pricing problem [16]. Their model is special case of our model with all qualities equalling to 1. Another stream of research is on single-minded bidders, including, for example, [21, 3, 2, 6, 10, 14]. To the best of our knowledge, this paper is the first to study algorithmic computation of sharp multi-unit demand.

2 Preliminaries

We have a market with m indivisible items, $M = \{1, 2, \ldots, m\}$, where each item j has unit supply and a parameter $q_j > 0$, representing the quality or desirability of j. In the market, there are also n potential buyers, $N = \{1, 2, \ldots, n\}$, where each buyer i has a value $v_i > 0$, which gives the benefit that i obtains for each unit of quality. Hence, the valuation that buyer i has for item j is $v_{ij} = v_i \cdot q_j$. In addition, each buyer i has a demand request $d_i \in \mathbb{Z}^+$, which specifies the number of items that i would like to get. We assume that d_i is

[1]This phenomenon does occur in our real life. For example, in most travel packages offered by travel agencies, they could lose money for some specific programs; but their overall net profit could always be positive.

a sharp constraint, i.e., i gets either exactly d_i items[2] or nothing at all. Our model therefore defines a market with multi-unit demand buyers and unit supply items. For any subset of buyers $S \subseteq N$, we use $d(S) = \sum_{i \in S} d_i$ to denote the total demand of items by buyers in S.

An outcome of the market is a tuple $(\mathbf{p}, \mathbf{X})$, where

- $\mathbf{p} = (p_1, \ldots, p_m) \geq 0$ is a *price* vector, where p_j is the price charged for item j;
- $\mathbf{X} = (X_1, \ldots, X_n)$ is an *allocation* vector, where X_i is the set of items that i wins. If $X_i \neq \emptyset$, we say i is a winner and have $|X_i| = d_i$ due to the demand constraint; if $X_i = \emptyset$, i does not win any items and we say i is a loser. Further, since every item has unit supply, we require $X_i \cap X_{i'} = \emptyset$ for any $i \neq i'$.
- If $j \in X_i$, we use $i = b(j)$ to represent the buyer of $j \in M$.

Given an output $(\mathbf{p}, \mathbf{X})$, let $u_i(\mathbf{p}, \mathbf{X})$ denote the *utility* of i. That is, if $X_i \neq \emptyset$, then $u_i(\mathbf{p}, \mathbf{X}) = \sum\limits_{j \in X_i} (v_{ij} - p_j)$; if $X_i = \emptyset$, then $u_i(\mathbf{p}, \mathbf{X}) = 0$.

Definition 2.1 (Envy-freeness)**.** *We say a tuple* $(\mathbf{p}, \mathbf{X})$ *is an* envy-free *solution if every buyer is envy-free, where a buyer i is envy-free if the following conditions are satisfied:*

- *if $X_i \neq \emptyset$, then (i) $u_i(\mathbf{p}, \mathbf{X}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq 0$, and (ii) for any other subset of items T with $|T| = d_i$, $u_i(\mathbf{p}, \mathbf{X}) = \sum\limits_{j \in X_i} (v_{ij} - p_j) \geq \sum\limits_{j \in T} (v_{ij} - p_j)$;*
- *if $X_i = \emptyset$ (i.e., i wins nothing), then, for any subset of items T with $|T| = d_i$, $\sum\limits_{j \in T} (v_{ij} - p_j) \leq 0$.*

Envy-freeness captures fairness in the market — the utility of everyone is maximized at the corresponding allocation for the given prices. That is, if i wins a subset X_i, then i cannot obtain a higher utility from any other subset of the same size; if i does not win anything, then i cannot obtain a positive utility from any subset with size d_i. It is easy to see that an envy-free solution always exists (e.g., set all prices to be ∞ and allocate nothing to every buyer).

Another solution concept we will consider is competitive equilibrium, which requires that, besides envy-freeness, every unsold item must be priced at zero (or at any given reserve price). Such market clearance condition captures efficiency of the whole market. The formal definition is given below.

Definition 2.2 (Competitive equilibrium)**.** *We say a tuple* $(\mathbf{p}, \mathbf{X})$ *is a* competitive equilibrium *if it is envy-free, and for each item j, $p_j = 0$ if no-one wins j in the allocation* $\mathbf{X}$.

For a given output $(\mathbf{p}, \mathbf{X})$, the *revenue* collected by the market maker is defined as $\sum_{j=1}^m p_j$ (equivalently, $\sum_{i=1}^n \sum_{j \in X_i} p_j$). We are interested in revenue maximizing solutions, specifically, revenue maximizing competitive equilibrium (if one exists) and revenue maximizing envy-free pricing. The main objective of the paper is algorithmic computations of these two optimization problems.

To simplify the following discussions, we sort all buyers and items in non-increasing order of their unit values and qualities, respectively, i.e., $v_1 \geq v_2 \geq \cdots \geq v_n$ and $q_1 \geq q_2 \geq \cdots \geq q_m$. Let K be the number of distinct values in the set $\{v_1, \ldots, v_n\}$. Let $A_1, \ldots, A_K$ be a partition of all buyers where each A_k, $k = 1, 2, \ldots, K$, contains the set of buyers that have the kth largest value.

3 Computation of Competitive Equilibrium

It is well known that a competitive equilibrium always exists for unit demand buyers (even for general v_{ij} valuations) [27]; for our sharp multi-unit demand model, however, a competitive equilibrium may not exist, as the following example shows.

[2]By the nature of the solution concepts considered in the paper, it can be assumed without loss of generality that i will not get more than d_i items.

Example 3.1 (Competitive equilibrium need not exist)**.** *There are two buyers i_1, i_2 with values $v_{i_1} = 10$ and $v_{i_2} = 9$, and demands $d_{i_1} = 1$ and $d_{i_2} = 2$, respectively, and two items j_1, j_2 with unit quality $q_{j_1} = q_{j_2} = 1$. If i_1 wins an item, without loss of generality, say j_1, then j_2 is unsold and $p_{j_2} = 0$; by envy-freeness of i_1, we have $p_{j_1} = 0$. Thus, i_2 envies the bundle $\{j_1, j_2\}$. If i_2 wins both items, then $p_{j_1} + p_{j_2} \le v_{i_2 j_1} + v_{i_2 j_2} = 18$, implying that $p_{j_1} \le 9$ or $p_{j_2} \le 9$; thus, i_1 is not envy-free. Hence, there is no competitive equilibrium in the given instance.*

In the unit demand case, it is well-known that the set of equilibrium prices forms a distributive lattice; hence, there exist extremes which correspond to the maximum and the minimum equilibrium price vectors. In our multi-unit demand model, however, even if a competitive equilibrium exists, maximum equilibrium prices may not exist.

Example 3.2 (Maximum equilibrium need not exist)**.** *There are two buyers i_1, i_2 with values $v_{i_1} = 10, v_{i_2} = 1$ and demands $d_{i_1} = 2, d_{i_2} = 1$, and two items j_1, j_2 with unit quality $q_{j_1} = q_{j_2} = 1$. It can be seen that allocating the two items to i_1 at prices $(19, 1)$ or $(1, 19)$ are both revenue maximizing equilibria; but there is no equilibrium price vector which is at least both $(19, 1)$ and $(1, 19)$.*

3.1 Over-Priced Items

Because of the sharp multi-unit demand, an interesting and important property is that it is possible that some items are 'over-priced'; this is a significant difference between sharp multi-unit and unit demand models. Formally, in a solution $(\mathbf{p}, \mathbf{X})$, we say an item j is *over-priced* if there is a buyer i such that $j \in X_i$ and $p_j > v_i q_j$. That is, the price charged for item j is larger than its contribution to the utility of its winner.

Example 3.3 (Over-priced items in a revenue maximizing solution)**.** *There are two buyers i_1, i_2 with values $v_{i_1} = 20, v_{i_2} = 10$ and demands $d_{i_1} = 1$ and $d_{i_2} = 2$, and three items j_1, j_2, j_3 with qualities $q_{j_1} = 3, q_{j_2} = 2, q_{j_3} = 1$. We can see that the allocations $X_{i_1} = \{j_1\}, X_{i_2} = \{j_2, j_3\}$ and prices $(45, 25, 5)$ constitute a revenue maximizing envy-free solution with total revenue 75, where item j_2 is over-priced. If no items are over-priced, the maximum possible prices are $(40, 20, 10)$ with total revenue 70.*

We have the following characterization for over-priced items in an equilibrium solution.

Lemma 3.1. *For any given competitive equilibrium $(\mathbf{p}, \mathbf{X})$, the following claims hold:*

- *If there is any unallocated item, then there are no over-priced items.*
- *At most one winner can have over-priced items; further, that winner, say i, must be the one with the smallest value among all winners in the equilibrium $(\mathbf{p}, \mathbf{X})$. That is, for any other winner $i' \neq i$, we have $v_{i'} > v_i$.*

Proof. The first claim is obvious since any unallocated item j' is priced at 0; thus if there is a winner i and item $j \in X_i$ such that $p_j > v_i q_j$, then i would envy the subset $X_i \cup \{j'\} \setminus \{j\}$.

To prove the second claim, suppose there are two winners i, i' where $v_i \ge v_{i'}$, and suppose that i has over-priced item j. Since i' is envy-free, his own utility must be non-negative; we know there is an item $j' \in X_{i'}$ such that $v_{i'} q_{j'} \ge p_{j'}$. This implies that $v_i q_{j'} \ge p_{j'}$; thus, i would envy the subset $X_i \cup \{j'\} \setminus \{j\}$, a contradiction. □

3.2 Properties

We present some observations regarding envy-freeness and competitive equilibrium. Our first observation implies that a winner is envy-free if and only if he prefers each of his allocated items to any other item.

Lemma 3.2. *Given any solution $(\mathbf{p}, \mathbf{X})$ and any winner i, if i is envy-free then $v_{ij} - p_j \ge v_{ij'} - p_{j'}$ for any items $j \in X_i$ and $j' \notin X_i$. On the other hand, if i is not envy-free, then there is $j \in X_i$ and $j' \notin X_i$ such that $v_{ij} - p_j < v_{ij'} - p_{j'}$.*

Proof. If i is envy-free but (for $j \in X_i$ and $j' \notin X_i$) $v_{ij} - p_j < v_{ij'} - p_{j'}$, it is easy to see that i would envy subset $X_i \cup \{j'\} \setminus \{j\}$, a contradiction. If i is not envy-free, then there is a subset T of items with $|T| = d_i$ such that $\sum_{j \in X_i}(v_{ij} - p_j) < \sum_{j' \in T}(v_{ij'} - p_{j'})$. Since $|X_i| = |T|$, the inequality holds for at least one item, i.e., there is $j \in X_i$ and $j' \notin X_i$ such that $v_{ij} - p_j < v_{ij'} - p_{j'}$. □

Lemma 3.3. *For any envy-free solution* $(\mathbf{p}, \mathbf{X})$*, suppose there are two buyers* i, i' *with values* $v_i > v_{i'}$ *and two items* j *and* j' *that are allocated to* i *and* i' *respectively, i.e.,* $j \in X_i$ *and* $j' \in X_{i'}$*. Then* $q_j \geq q_{j'}$*.*

Proof. By the above Lemma 3.2, we have

$$\begin{aligned} v_i q_j - p_j &\geq v_i q_{j'} - p_{j'} \\ v_{i'} q_{j'} - p_{j'} &\geq v_{i'} q_j - p_j \end{aligned}$$

Adding the two inequalities together, we get $(v_i - v_{i'})(q_j - q_{j'}) \geq 0$, yielding the desired result. □

Lemma 3.3 implies that in any envy-free solution, the allocation of items is monotone in terms of their amount of qualities and the values of the winners, i.e., winners with larger values win items with larger qualities. However, it does not imply that the value of every winner is larger than or equal to the value of any loser. For instance, consider three buyers i_1, i_2, i_3 and two items j_1, j_2 with $q_{j_1} = 2$ and $q_{j_2} = 1$. The values and demands are $v_{i_1} = 1.3, v_{i_2} = 1, v_{i_3} = 0.9$ and $d_{i_1} = 1, d_{i_2} = 2, d_{i_3} = 1$. Then prices $p_{j_1} = 2.2, p_{j_2} = 0.9$ and allocations $X_{i_1} = \{j_1\}, X_{i_2} = \emptyset, X_{i_3} = \{j_2\}$ constitute a revenue maximizing envy-free solution. In this solution, $v_{i_2} > v_{i_3}$, but i_2 does not win any item (because of the sharp demand constraint) whereas i_3 wins item j_2.

Lemma 3.4. *If there is a competitive equilibrium* $(\mathbf{p}, \mathbf{X})$*, then for any winner* i*, item* $j \in X_i$ *and unallocated item* j'*, we have* $q_j \geq q_{j'}$*.*

Proof. Since item j' is not allocated to any buyer, its price $p_{j'} = 0$. By envy-freeness, we have $v_i q_j \geq v_i q_j - p_j \geq v_i q_{j'} - p_{j'} = v_i q_{j'}$, which implies that $q_j \geq q_{j'}$. □

By the above characterization, in any competitive equilibrium, all allocated items have larger qualities. Hence, by Lemmas 3.3 and 3.4, we know that if the set of winners is fixed in a competitive equilibrium, the allocation is determined implicitly as well. On the other hand, we observe that Lemma 3.4 does not hold if $(\mathbf{p}, \mathbf{X})$ is an (revenue maximizing) envy-free solution. For instance, consider two buyers i_1, i_2 with values $v_{i_1} = 10, v_{i_2} = 1$ and demand $d_{i_1} = 1, d_{i_2} = 10$, and twelve items $j_1, j_2, \ldots, j_{12}$ with qualities $q_{j_1} = 10, q_{j_2} = 5, q_{j_3} = \cdots = q_{j_{12}} = 1$. It can be seen that in the optimal envy-free solution, we set prices $p_{j_1} = 91, p_{j_2} = \infty, p_{j_3} = \cdots = p_{j_{12}} = 1$, and allocate $X_{i_1} = \{j_1\}, X_{i_2} = \{j_3, \ldots, j_{12}\}$, which generates total revenue $91 + 10 = 101$. In this solution, $q_{j_2} > q_{j_3} = \cdots = q_{j_{12}}$, but item j_2 is not allocated to any buyer.

Lemma 3.5. *Given an envy-free solution* $(\mathbf{p}, \mathbf{X})$*, a loser* ℓ *and any subset* T *of* d_ℓ *items, the following property cannot hold:*
A non-empty subset of items in T *are either allocated to winners with values smaller than* v_ℓ *or priced at 0; any other elements of* T *are allocated to winners having the same value* v_ℓ *as* ℓ*.*

Note that this is a result about envy-free prices, not just competitive equilibrium.

Proof. Let $(\mathbf{p}, \mathbf{X})$ be an envy-free pair of price and allocation vectors. Given the loser ℓ and T satisfying the conditions of the statement of the Lemma, we show how to construct a set T' of items that ℓ envies.

Let $T = T_0 \cup T_1 \cup \cdots \cup T_n$ be a partition of T where T_0 consists of items priced at 0 in $(\mathbf{p}, \mathbf{X})$ and for $i > 0$, $T_i = T \cap X_i$. Note that any non-empty T_i satisfies $v_i \leq v_\ell$, and if $T_0 = \emptyset$ then $T_i \neq \emptyset$ for some $i > 0$ with $v_i < v_\ell$.

Note that T_0 satisfies $\sum_{j \in T_0} v_\ell q_j - p_j \geq 0$, where the inequality is strict if T_0 is non-empty. Let $T'_0 = T_0$.

Consider any non-empty T_i (with $i > 0$). Let T'_i be the $|T_i|$ items $j \in X_i$ that maximize $v_i q_j - p_j$. We have $\sum_{j \in T'_i} v_i q_j - p_j \geq 0$. Hence $\sum_{j \in T'_i} v_\ell q_j - p_j \geq 0$, with strict inequality if $v_i < v_\ell$.

Summing these inequalities, we have $\sum_{i=0}^{n} \sum_{j \in T'_i} v_\ell q_j - p_j \geq 0$, and in fact the inequality is strict since at least one of the $n+1$ inequalities is strict. Let $T' = T'_0 \cup T'_1 \cup \cdots \cup T'_n$; $|T'| = |T| = d_\ell$ and we have shown that ℓ envies T'. □

3.3 Algorithm

Our main result of this section is the following.

Theorem 3.1. *There is a polynomial algorithm to determine the existence of a competitive equilibrium; and if one exists, it computes a revenue maximizing equilibrium.*

Thus, both the existence problem and the maximization problem become tractable, as a result of the correlated valuations $v_{ij} = v_i q_j$.

The algorithm, called MAX-CE, is divided into two steps. The first step is to compute a set of 'candidate' winners if an equilibrium exists. The second step is to calculate a 'candidate' equilibrium and verify if it is indeed a (revenue maximizing) equilibrium. Recall that A_k, $1 \le k \le K$ denotes all the buyers with the kth largest value.

MAX-CE STAGE 1.

1. Let $S^* \leftarrow \emptyset$ be the set of candidate winners
2. Let $k \leftarrow 1$ and let $D \leftarrow m$ be the number of ''available items''
3. While $k \le K$
   - If $d_i > D$ for every $i \in A_k$, let $k \leftarrow k+1$
   - Else
     - Let $S = \{i \mid i \in A_k,\ d_i \le D\}$
     - If $\sum_{i \in S} d_i > D$
       - **(a)** If there is $S' \subseteq S$ s.t. $\sum_{i \in S'} d_i = D$ let $S^* \leftarrow S^* \cup S'$, and goto MAX-CE STAGE 2
       - **(b)** Else, a competitive equilibrium does not exist, and return
     - Else $\sum_{i \in S} d_i \le D$
       - **(c)** Let $S^* \leftarrow S^* \cup S$, $\quad D \leftarrow D - \sum_{i \in S} d_i$, $\quad k \leftarrow k+1$
4. Goto MAX-CE STAGE 2

Note that in the above step 3(a) we check whether there is $S' \subseteq S$ such that $\sum_{i \in S'} d_i = D$; this is equivalent to solving a subset sum problem. However, in our instance, each demand satisfies $d_i \le m$. Hence, a dynamic programming approach can solve the problem in time $O(n^2 m)$. Hence, STAGE 1 runs in strongly polynomial time.

Lemma 3.6. *If an input to* MAX-CE *has a competitive equilibrium* $(\mathbf{p}, \mathbf{X})$*, then* STAGE 1 *will not return that an equilibrium does not exist at step 3(b).*

Proof. Let $(\mathbf{p}, \mathbf{X})$ be a competitive equilibrium of an input to MAX-CE. In this proof, when we refer to winning/losing buyers, or allocated/unallocated items, we mean with respect to $(\mathbf{p}, \mathbf{X})$. In particular, let W be the set of winners of $(\mathbf{p}, \mathbf{X})$.

Suppose that MAX-CE STAGE 1 exits on the k-th iteration of the loop. We claim that during the first $k-1$ iterations, all buyers added to S^* must be winners. To see this, suppose alternatively that at iteration $k' < k$, buyer ℓ is the first loser to be added to S^*. In that case, ℓ has d_ℓ items that satisfy the conditions of Lemma 3.5, contradicting envy-freeness. (Suppose that the winners found by the algorithm during the first $k'-1$ iterations are given their allocation in $(\mathbf{p}, \mathbf{X})$. At iteration k', the algorithm has more than d_ℓ available items, some of which are allocated to buyers with value less than ℓ, or are unallocated.)

At the final iteration k we must have $S \ne \emptyset$ (otherwise the algorithm will begin a new iteration). Since $\sum_{i \in S} d_i > D$, we have $S \backslash W \ne \emptyset$ (members of S have too much demand for them all to be able to win). Since there is no subset $S' \subseteq S$ such that $\sum_{i \in S'} d_i = D$, we have $\sum_{i \in S \cap W} d_i < D$. Hence, there are items that are not allocated to buyers in $S^* \cup (S \cap W)$. Let $i' \in S \setminus W$; we can find $d_{i'}$ items that satisfy the condition of Lemma 3.5, implying that a buyer in $S \backslash W$ is not envy-free, a contradiction. □

Lemma 3.7. *A revenue maximizing competitive equilibrium* $(\mathbf{p}, \mathbf{X})$ *can be converted to one with equal revenue whose winning set is* S^*.

Proof. Assume that the given instance has a competitive equilibrium $(\mathbf{p}, \mathbf{X})$ and that MAX-CE enters MAX-CE STAGE 2 at the kth iteration with the set of candidates S^*. Let W be the set of winners of $(\mathbf{p}, \mathbf{X})$, and let $W' = W \cap (A_1 \cup \cdots \cup A_{k-1})$ and $W'' = W \setminus W'$. Let $S^1 = S^* \cap (A_1 \cup \cdots \cup A_{k-1})$ and $S^2 = S^* \setminus S^1$ (note that $S^2 \subseteq A_k$). From the analysis of the above lemma and Lemma 3.5, we know that (i) $W' = S^1$, (ii) $W'' \subseteq A_k$, and (iii) $\sum_{i \in W''} d_i = \sum_{i \in S^2} d_i$. Thus, the only difference between S^* and W lies in the selection of buyers in A_k (this is due to possibly multiple choices in step 3(a) in MAX-CE STAGE 1). Due to envy-freeness, we have

$$\sum_{i \in W'' \setminus S^2} u_i(\mathbf{p}, \mathbf{X}) = \sum_{i \in W'' \setminus S^2} \sum_{j \in X_i} (v_i q_j - p_j) \geq 0 \geq \sum_{i \in S^2 \setminus W''} u_i(\mathbf{p}, \mathbf{X})$$

Since all buyers in $W'' \setminus S^2$ and $S^2 \setminus W''$ have the same value, we know that the above inequalities are tight. Thus, if we reassign the items in $\cup_{i \in W''} X_i$ to the buyers in S^2 and keep the same prices, the resulting output will still be an equilibrium. □

Given the above characterization, the second step of the algorithm MAX-CE is described as follows.

MAX-CE STAGE 2.

5. Allocation $\mathbf{X}^*$ is constructed as follows:
   - Let $X_i^* \leftarrow \emptyset$, for each buyer $i \notin S^*$
   - For each $i \in S^*$ in non-increasing order of v_i
     - allocate d_i of the remaining items to i in non-increasing order of q_j
6. Price $\mathbf{p}^*$ is computed according to the following linear program:

$$\begin{array}{lll} \max & \sum_{i \in S^*} \sum_{j \in X_i^*} p_j^* & \\ s.t. & p_j^* \geq 0 & \forall\ j \\ & p_j^* = 0 & \forall\ j \notin \cup_{i \in S^*} X_i^* \\ & v_i q_j - p_j^* \geq v_i q_{j'} - p_{j'}^* & \forall\ i \in S^*, j \in X_i^*, j' \notin X_i^* \\ & \sum_{j \in T} (v_i q_j - p_j^*) \leq 0 & \forall\ i \notin S^*,\ T \text{ with } |T| = d_i \end{array}$$

7. If the above linear program has a feasible solution, output the tuple $(\mathbf{p}^*, \mathbf{X}^*)$
8. Else, return that a competitive equilibrium does not exist

In the above LP, there are m variables where each item j has a variable p_j^*. The first two constraints ensure that the price vector is a set of feasible market clearing prices. The third condition guarantees that all winners are envy-free. The last condition says that for each loser i and any subset of items T with $T = |d_i|$, i cannot obtain a positive utility from T. Notice that it is possible that there are exponentially many combinations of T; thus the LP has an exponential number of constraints. However, observe that for any given solution $\mathbf{p}^*$, it is easy to verify if $\mathbf{p}^*$ is a feasible solution of the LP or find a violated constraint. In particular, for every loser i, we can order all items j in decreasing order of $v_i q_j - p_j^*$ and verify the subset T composed of the first d_i items; if i cannot obtain a positive utility from such T, then i is envy-free. Therefore, there is a separation oracle to the LP, and thus, the ellipsoid method can solve the LP in polynomial time. Hence, the total running time of MAX-CE is in polynomial time.

If the algorithm returns a tuple $(\mathbf{p}^*, \mathbf{X}^*)$, certainly it is a competitive equilibrium; further, it is a revenue maximizing equilibrium because of the objective function in the LP. It is therefore sufficient to show the following claim to complete the proof of Theorem 3.1.

Lemma 3.8. *If there exists a competitive equilibrium, then* STAGE 2 *will not claim that an equilibrium does not exist at step 8.*

Proof. If there is a competitive equilibrium $(\mathbf{p}, \mathbf{X})$, let W be the set of winners of the equilibrium. By Lemma 3.6, we know that MAX-CE will enter MAX-CE STAGE 2. By the above discussions, we know that W and S^* only differ in the last kth iteration of the main loop of MAX-CE STAGE 1 and replacing all winners in $W \cap A_k$ with $S^* \cap A_k$ gives an equilibrium as well. Further, by Lemma 3.3 and 3.4, the allocation of items to the winners in W is fixed. Hence, the equilibrium price vector $\mathbf{p}$ gives a feasible solution to the LP in the STAGE 2, which implies that the algorithm will not claim that an equilibrium does not exist. □

4 Computation of Envy-Free Pricing

In this section, we will ignore the market clearance condition (i.e. that unsold items are priced at 0) and only consider envy-freeness. We noted earlier that an envy-free solution always exists. Our main results are the following.

Theorem 4.1. *For the sharp multi-unit demand with $v_i q_j$ valuations, it is NP-hard to solve the revenue-maximizing envy-free pricing problem, even if there are only three buyers. However, if the demand of each buyer is bounded by a constant, then the revenue-maximizing envy-free pricing problem can be solved in polynomial time.*

We note that the correlated $v_i q_j$ valuations are crucial to derive an efficient computation when the demands are bounded by a constant; in contrast, for unit-demand, the envy-free pricing is NP-hard for general valuations v_{ij} even if every buyer is interested in at most three items [8].

4.1 Algorithm for Constant Demands

Throughout this subsection, let Δ be a constant where $d_i \leq \Delta$ for any buyer i, and again, buyers and items are sorted according to their values and qualities. For any tuple $(\mathbf{p}, \mathbf{X})$, we assume that all unsold items are priced at ∞. This assumption is without loss of generality for envy-freeness. We will first explore some important properties of an (optimal) envy-free solution, then at the end of the section present our algorithm.

4.1.1 Candidate Winner Sets

For a given set S of buyers, let $\max(S)$ and $\min(S)$ denote the buyer in S that has the largest and smallest index, respectively.

Definition 4.1 (Candidate winner set)**.** *Given a subset of buyers $S \neq \emptyset$, let $k = \max\{r | A_r \cap S \neq \emptyset\}$. We say S is a* candidate winner set *if the total demand of buyers in S is at most m, i.e., $d(S) \leq m$, and for any $i \in A_1 \cup \cdots \cup A_{k-1} \setminus S$, $d_i > \sum_{i' \in S:\ i' > i} d_{i'}$.*

The definition of candidate winner set is closely related to envy-freeness. Indeed, due to Lemma 3.5, the above definition defines a slightly larger set (than all possible sets of winners in envy-free solutions) as the inequality does not consider all the buyers completely in the same value category v_j. However, this definition makes it easier for us to describe and analyze the algorithm.

Proposition 4.1. *For any envy-free solution $(\mathbf{p}, \mathbf{X})$, let $S = \{i \mid X_i \neq \emptyset\}$ be the set of winners. Then S is either a candidate winner set or $S = \emptyset$.*

Proof. The claim follows directly from Lemma 3.5. □

FINDWINNERS(S): Input a set of buyers S

- Let $i_{\max} = \max(S)$ and assume $i_{\max} \in A_k$
- Initially let $W_S = S$
- For each buyer $j \in A_1 \cup \cdots \cup A_{k-1}$ in reverse order
  - If $j \notin S$ and $d_j \leq \sum_{i \in W_S:\ i > j} d_i$, let $W_S \leftarrow W_S \cup \{j\}$
- Return W_S

Proposition 4.2. *For any subset of buyers S, let W_S =*FindWinners*(S).*

- *If $d(W_S) \leq m$, then W_S is a candidate winner set and for any candidate winner set $S' \supseteq S$, $W_S \subseteq S'$.*
- *If $d(W_S) > m$, then there is no candidate winner set containing S.*

Proof. The proposition follows directly from the definition of candidate winner set. □

```
FindLoser(S):  Input a candidate winner set S
  • Let i_min = min(S) and assume i_min ∈ A_j
  • Initially let L_S = ∅, and α = ∞
  • For each k = j, j+1, ..., K
      – Let i_0 = arg min{d_i | i ∈ A_k \ S}
      – If d_{i_0} < α, let L_S ← L_S ∪ {i_0} and α ← d_{i_0}
  • Return L_S
```

Proposition 4.3. *For any given tuple $(\mathbf{p}, \mathbf{X})$ with winner set S, suppose that S is a candidate winner set and let L_S =*FindLoser*(S). If all losers in L_S are envy-free with respect to $(\mathbf{p}, \mathbf{X})$, then all other losers are envy-free as well.*

Proof. For any $i \in L_S$, if i is envy-free, then for any subset T of items with $|T| = d_i$, $\sum_{j \in T}(v_i q_j - p_j) \leq 0$. Hence, for any $v \leq v_i$ and T' with $|T'| \geq d_i$, we have

$$\sum_{j \in T'}(vq_j - p_j) = \frac{1}{\binom{|T'|-1}{d_i-1}} \sum_{T \subseteq T', |T| = d_i} \sum_{j \in T}(vq_j - p_j) \leq \frac{1}{\binom{|T'|-1}{d_i-1}} \sum_{T \subseteq T', |T| = d_i} \sum_{j \in T}(v_i q_j - p_j) \leq 0.$$

Hence, by the rules of FindLoser, we know that if all the losers in L_S are envy-free, all other losers in $A_j \cup \cdots \cup A_K$ are envy-free as well. On the other hand, for any loser $j \in A_1 \cup \cdots \cup A_{j-1}$, since S is a candidate winner set, we know that $d_j > \sum_{i \in S:\ i > j} d_i = \sum_{i \in S} d_i$. Since all unsold items are priced at ∞, we know that j is envy-free. Hence, all losers are envy-free. □

4.1.2 Bounding the Number of Candidate Winner Sets

We have the following bound on the number of candidate winner sets.

Lemma 4.1. *For any $k \in \{2, \ldots, K\}$ and $S \subseteq A_k$, suppose $d(S) \leq m$. Let*

$$\mathcal{C} = \left\{S \cup S' \mid S' \subseteq A_1 \cup \cdots \cup A_{k-1} \text{ and } S \cup S' \text{ is a candidate winner set}\right\}$$

Then $|\mathcal{C}| \leq \left\lfloor \frac{m}{d(S)} \right\rfloor$.

Proof. Let $a = d(S)$ and ℓ be the index of the buyer $\max(A_{k-1})$. We add buyers $\ell, \ell-1, \ell-2, \ldots, 1$ into S in sequence and maintain all the possible candidate winner sets. Let $\mathcal{C}_0 = \{S\}$. In general, we have constructed $\mathcal{C}_t$ containing all the candidate winner sets of $\{\ell, \ell-1, \ell-2, \ldots, \ell-t+1\} \cup S$. We order $\mathcal{C}_t = \{S_{t,1}, S_{t,2}, \ldots, S_{t,n_t}\}$ such that $d(S_{t,1}) \leq d(S_{t,2}) \leq \cdots \leq d(S_{t,n_t}) \leq m$. We should prove that $d(S_{t,i}) \geq i * d(S)$.

We now add $\ell - t$ into $\mathcal{C}_t$ to construct $\mathcal{C}_{t+1}$. Let $t_s = \max\{i : d(S_{t,i}) < d_{\ell-t}\}$ if $\{i : d(S_{t,i}) < d_{\ell-t}\} \neq \emptyset$, otherwise $t_s = 0$. Let $S_{t+1,j} = S_{t,j}$ for $j = 1, 2, \cdot, t_s$, $S_{t+1,j+t_s} = S_{t,j} \cup \{\ell - t\}$ for $j = 1, 2, \ldots, n_t$. Clearly that $d(S_{t+1,i}) \geq i * d(S)$ for $i \leq t_s$ by inductive hypothesis. And

$$d(S_{t+1,j+t_s}) = d(S_{t,j}) + d_{\ell-t} \geq j * d(S) + d(S_{t,t_s}) \geq (j + t_s) * d(S).$$

Let $n_{t+1} = \max\{i : d(S_{t+1,i}) \leq m\}$. Clearly the claim follows for $\ell - t$ and $\mathcal{C}_t$.

The lemma follows by the condition $m \geq d(S_{\ell,n_\ell}) \geq n_\ell * d(S)$. □

4.1.3 Optimal Winner Sets

Definition 4.2 (Optimal winner set). *A subset of buyers S is called an* optimal winner set *if there is a revenue maximizing envy-free solution* $(\mathbf{p}, \mathbf{X})$ *such that S is its set of winners.*

Proposition 4.4. *Let S be an optimal winner set and let $k = \max\{r \mid A_r \cap S \neq \emptyset\}$. For any $S' \subseteq A_k$, if $d(S') = d(S \cap A_k)$, then $S' \cup (S \backslash A_k)$ is an optimal winner set as well.*

Before proving the proposition, we first establish the following claim.

Claim 4.1. *Suppose there exists a revenue-maximizing envy-free solution* $(\mathbf{p}, \mathbf{X})$*, and let S be the winning set in* $(\mathbf{p}, \mathbf{X})$*, and let $k = \max\{r \mid A_r \cap S \neq \emptyset\}$. Then every buyer in A_k has utility zero.*

Proof. Of course, every loser in A_k has utility zero. To show that every winner in A_k has utility zero, we show that if such a winner has positive utility, then prices can be raised to the point where his utility becomes zero, while maintaining envy-freeness (contradicting the assumption that $(\mathbf{p}, \mathbf{X})$ maximizes revenue).

Let $i_{\max} = \max(S)$. Let

$$\delta = \frac{u_{i_{\max}}(\mathbf{p}, \mathbf{X})}{d_{i_{\max}}}.$$

We claim that $(\mathbf{p} + \delta, \mathbf{X})$ is an envy-free solution as well, where the price of each item is increased by δ.

Obviously we have $\delta \geq 0$, and the conclusion holds trivially if $\delta = 0$. Suppose $\delta > 0$. For the tuple $(\mathbf{p}+\delta, \mathbf{X})$, since all items have their prices increased by the same amount, all losers are still envy-free and all winners would not envy the items they don't get. Hence, we need only to check that each winner still gets a non-negative utility. For $i_{\max}$, we have $u_{i_{\max}}(\mathbf{p} + \delta, \mathbf{X}) = 0$. For any other winner $i \neq i_{\max}$, it holds that $v_i \geq v_{i_{\max}}$. Since i does not envy any item in $(\mathbf{p}, \mathbf{X})$, for any item $j' \in X_i$ and $j \in X_{i_{\max}}$, it holds that $v_i q_{j'} - p_{j'} \geq v_i q_j - p_j$, hence, $p_{j'} \leq v_i(q_{j'} - q_j) + p_j$. Then, we get

$$p_{j'} \leq \frac{\sum_{j \in X_{i_{\max}}} (v_i(q_{j'} - q_j) + p_j)}{d_{i_{\max}}} = v_i q_{j'} - \frac{\sum_{j \in X_{i_{\max}}} (v_i q_j - p_j)}{d_{i_{\max}}}.$$

This implies that

$$p_{j'} + \delta \leq v_i q_{j'} - \frac{\sum_{j \in X_{i_{\max}}} \left((v_i q_j - p_j) - (v_{i_{\max}} q_j - p_j)\right)}{d_{i_{\max}}} = v_i q_{j'} - \frac{\sum_{j \in X_{i_{\max}}} (v_i - v_{i_{\max}}) q_j}{d_{i_{\max}}} \leq v_i q_{j'}.$$

Hence, $u_i(\mathbf{p} + \delta, \mathbf{X}) = \sum_{j' \in X_i} (v_i q_{j'} - p_{j'} - \delta) \geq 0$. Therefore, $(\mathbf{p} + \delta, \mathbf{X})$ is an envy-free solution. □

We are now ready for the proof of Proposition 4.4.

Proof of Proposition 4.4. Since S is an optimal winner set, there is an optimal envy-free solution $(\mathbf{p}, \mathbf{X})$ such that $S = \{i \mid X_i \neq \emptyset\}$. We construct a new allocation $\mathbf{X}'$ with winner set $S' \cup (S \backslash A_k)$ as follows:

- For any $i \notin A_k$, $X'_i = X_i$.
- For any $i \in A_k \setminus S'$, $X'_i = \emptyset$.
- For all the buyers in S', allocate items in $\bigcup_{i \in S \cap A_k} X_i$ to them arbitrarily. (The allocation is feasible as $d(S') = d(S \cap A_k)$.)

Obviously, $(\mathbf{p}, \mathbf{X}')$ generates the same revenue as $(\mathbf{p}, \mathbf{X})$. We claim that $(\mathbf{p}, \mathbf{X}')$ is an envy-free solution (this implies our desired result that $S' \cup (S \backslash A_k)$ is an optimal winner set). For any buyer $i \notin A_k$, since prices are not changed, i is still envy-free.

Next we prove that all buyers $i \in A_k$ are envy-free in $(\mathbf{p}, \mathbf{X}')$. Let $J = \cup_{i \in S \cap A_k} X_i$ be the set of items allocated to buyers in A_k; we also have $J = \cup_{i \in S'} X'_i$. Suppose first that $|S \cap A_k| = |S'| = 1$; in this case $(\mathbf{p}, \mathbf{X})$ differs trivially from $(\mathbf{p}, \mathbf{X}')$, so $(\mathbf{p}, \mathbf{X}')$ is envy-free.

Alternatively, there is some buyer $\bar{i} \in A_k$ with $d_{\bar{i}} < d(S \cap A_k)$. We show that any item $j \in J$ allocated to $i \in A_k$ in $(\mathbf{p}, \mathbf{X}')$, affords zero utility to i, i.e. j satisfies $v_i q_j = p_j$. Let v be the value shared by all $i \in A_k$, i.e. $v = v_i$ for any $i \in A_k$. Since $(\mathbf{p}, \mathbf{X})$ is envy-free, we have using Claim 4.1 that $u_i(\mathbf{p}, \mathbf{X}) = 0$ for all $i \in A_k$, hence $\sum_{j \in J} v q_j - p_j = 0$. Suppose some $j \in J$ does not satisfy $v q_j - p_j = 0$. Arrange all $j \in J$ in descending order of $v q_j - p_j$. Any proper prefix P of this sequence satisfies $\sum_{j \in P} v q_j - p_j > 0$. Then buyer $\bar{i}$ envies this prefix. □

4.1.4 Maximizing Revenue for a Given Set of Winners and Allocated Items

Suppose that S is a candidate winner set and T is a subset of items, where $|T| = d(S)$. We want to know if there is an envy-free solution such that S is the set of winners and S wins items in T; if yes, we want to find one that maximizes revenue. This problem can be solved easily by a linear program with an exponential number of constraints for each $i \in S$. The following combinatorial algorithm does the same thing; the idea inside is critical to our main algorithm.

We will use the following notations: $S = \{i_1, i_2, \ldots, i_t\}$ with $i_1 < i_2 < \cdots < i_t$ and $T = \{j_1, j_2, \ldots, j_\ell\}$ with $j_1 < j_2 < \cdots < j_\ell$. Let $i_{b(s)}$ be the winner of j_s, $s = 1, 2, \ldots, \ell$.

MaxRevenue(S,T): Input a candidate winner set S and a subset of items T where $|T| = d(S)$

- Let $L_S = \text{FindLoser}(S)$.
- Allocation $\mathbf{X}$
  - Let $X_i \leftarrow \emptyset$, for each buyer $i \notin S$.
  - Allocate items in T to buyers in S according to the following rule (by Lemma 3.3): Buyers with smaller indices obtain items with smaller indices.
- Price $\mathbf{p}$
  - Let $Y = \emptyset$
  - For each item $j \notin T$, let $p_j = \infty$.
  - For each item $k \in X_{i_t}$, do the following
    - **(a)** Let J be the last 2Δ items with the largest indices in T. Run the following linear program (denoted by LP$^{(k)}$), which computes prices for items in $X_{i_{t-1}} \cup X_{i_t}$

$$
\begin{array}{llll}
\min & v_{i_{t-1}} q_k - p_k & & \\
s.t. & v_{i_{t-1}} q_k - p_k \geq v_{i_{t-1}} q_j - p_j & \forall\ j \in X_{i_t} & (1) \\
& \sum_{j \in X_{i_t}} (v_{i_t} q_j - p_j) = 0 & & (2) \\
& v_{i_{t-1}} q_j - p_j = v_{i_{t-1}} q_k - p_k & \forall\ j \in X_{i_{t-1}} & (3) \\
& v_{i_t} q_j - p_j \leq v_{i_t} q_{j'} - p_{j'} & \forall\ j \in X_{i_{t-1}}, j' \in X_{i_t} & (4) \\
& \sum_{j \in J'} (v_i q_j - p_j) \leq 0 & \forall\ i \in L_S,\ J' \subseteq J \text{ with } |J'| = d_i & (5) \\
& p_{j_s} = v_{b(s)}(q_{j_s} - q_{j_{s+1}}) + p_{j_{s+1}} & \forall j_s \in J - X_{i_t} - X_{i_{t-1}} & (6)
\end{array}
$$

    - **(b)** If the LP$^{(k)}$ in (a) has a feasible solution, let $Y \leftarrow Y \cup \{k\}$.
    - **(c)** For each item $j_s \in X_{i_1} \cup \cdots \cup X_{i_{t-2}}$ in the reverse order
      - $*$ let $p_{j_s} = v_{i_{b(s)}}(q_{j_s} - q_{j_{s+1}}) + p_{j_{s+1}}$
    - **(d)** Denote the price vector computed above by $\mathbf{p}^{(k)}$.
- If $Y = \emptyset$, return that there is no price vector $\mathbf{p}$ such that $(\mathbf{p}, \mathbf{X})$ is envy-free.
- Otherwise,
  - Let $k^* \in Y$ have the largest total revenue for which $(\mathbf{p}^{(\mathbf{k}^*)}, \mathbf{X})$ is an envy-free solution.
  - Output the tuple $(\mathbf{p}^{(\mathbf{k}^*)}, \mathbf{X})$.

Remark 4.1. *It should be noting that in LP^k, the objective function is equivalent to maximize p_k. By the pricing rule (2),(6) and (c) of* MaxRevenue(S,T)*, the total revenue $\sum_{j \in T} p_j$ obtained is a linear increasing function of p_k, hence maximizing p_k is equivalent to maximizing the total revenue.*

We establish the following properties:

Proposition 4.5. *Let $(\mathbf{p}, \mathbf{X})$ is computed in terms of LP$^{(k^*)}$ where $k^* \in X_{i_t}$ in* MaxRevenue(S,T)*. Let $i_{b(u)}$ be the winner of j_u. Use the convention $j_{\ell - d_{i_t} + 1} = k^*$. For $i = 1, 2, \ldots, l - d_{i_t}$, we have*

1. $v_{i_{b(i)}} q_{j_{i+1}} - p_{j_{i+1}} \geq 0$;

2. $\frac{p_{j_i}}{q_{j_i}} \geq \frac{p_{j_{i+1}}}{q_{j_{i+1}}}$;

3. $p_{j_i} \geq p_{j_{i+1}}$.

Proof. For the first inequality, consider the last case, $v_{i_{t-1}} q_{k^*} - p_{k^*} \geq 0$. Assume it does not hold. By Formula (1) in Algorithm MaxRevenue, $\sum_{j \in X_{i_t}} (v_{i_{t-1}} q_j - p_j) < 0$. Therefore, $\sum_{j \in X_{i_t}} (v_{i_t} q_j - p_j) < 0$, which contradicts Formula (2). Further, $v_{i_u} q_{k^*} - p_{k^*} \geq 0$ for all $u : 1 \leq u \leq t-1$. That is, all other buyers have nonnegative utility on item k^*. Now consider $s = 1, 2, \ldots, \ell - d_{i_t}$. By (6) and (c) in the algorithm, using the convention $j_{\ell - d_{i_t} + 1} = k^*$, item 1 holds as following

$$v_{b(s)} q_{j_{s+1}} - p_{j_{s+1}} \geq v_{b(s+1)} q_{j_{s+1}} - p_{j_{s+1}} = v_{b(s+1)} q_{j_{s+2}} - p_{j_{s+2}} \geq \cdots \geq v_{i_{t-1}} q_{k^*} - p_{k^*} \geq 0.$$

For the second inequality, by pricing rule (c), we know that

$$\frac{p_{j_i}}{q_{j_i}} \geq \frac{p_{j_{i+1}}}{q_{j_{i+1}}}$$

holds if and only if

$$\frac{v_{i_{b(i)}}(q_{j_i} - q_{j_{i+1}}) + p_{j_{i+1}}}{q_{j_i}} \geq \frac{p_{j_{i+1}}}{q_{j_{i+1}}}$$

which holds if and only if

$$(v_{i_{b(i)}} q_{j_{i+1}} - p_{j_{i+1}})(q_{j_i} - q_{j_{i+1}}) \geq 0,$$

which follows from the first inequality.

The third inequality follows immediately from the second one and the non-increasing ordering of q's. ☐

Lemma 4.2. *Suppose that S is a candidate winner set and T is a subset of items, where $|T| = d(S)$. Let $\mathbf{X}$ be the allocation computed in the procedure* MaxRevenue(S, T). *Then* MaxRevenue(S, T) *determines whether there exists a price vector $\mathbf{p}$ such that $(\mathbf{p}, \mathbf{X})$ is an envy-free solution, and if the answer is 'yes', it outputs one that maximizes total revenue given the allocation $\mathbf{X}$.*

Proof. Assume that there is a price vector $\mathbf{p'}$ such that $(\mathbf{p'}, \mathbf{X})$ is a revenue maximizing envy-free solution, with the winner set S and the sold item set T. In one direction, we prove that the algorithm given the input sets S and T returns a solution with at least the same total revenue. On another direction, we prove that the solution found by the Algorithm is an envy-free solution for the fixed sets S and T. By Remark 4.1, this envy-free solution must be an optimal one. The two parts together complete the proof.

For the first direction, let $S = \{i_1, i_2, \ldots, i_t\}$ with $v_{i_1} \geq v_{i_2} \geq \cdots \geq v_{i_t}$ and $T = \{j_1, j_2, \ldots, j_l\}$ with $q_{j_1} \geq q_{j_2} \geq \cdots \geq q_{j_l}$. By Claim 4.1, $\sum_{j \in X_{i_t}} (v_{i_t} q_j - p'_j) = 0$. Consider an item $k' = \arg\max_{k \in X_{i_t}} (v_{i_{t-1}} q_k - p'_k)$. Define a new price vector $\mathbf{p}$ as follows:

- For $j \in X_{i_t}$, $p_j = p'_j$.
- For $j \in X_{i_{t-1}}$, $p_j = v_{i_{t-1}}(q_j - q_{k'}) + p'_{k'}$.
- For $j \in X_{i_1} \cup \cdots \cup X_{i_{t-2}}$, p_j is defined according to Step (c) of the procedure MaxRevenue.

It is easy to see that the formulas (1), (2) and (3) of LP$^{(k')}$ are satisfied for price vector $\mathbf{p}$. By induction on the reverse order of items, we can show that $\mathbf{p'} \leq \mathbf{p}$. This implies that formula (4) of LP$_{k'}$ is satisfied as well. Further, since prices are monotonically increasing, all losers (in particular, those in L_S) are still envy-free, which implies formula (5) is satisfied. Formula (6) is automatically satisfied. Hence, $\mathbf{p}$ is a feasible solution of LP$^{(k')}$. Hence, there is a feasible solution in the above procedure MaxRevenue(S, T) for item k'; this implies that $Y \neq \emptyset$ in the course of the procedure.

In addition, again because of $\mathbf{p'} \leq \mathbf{p}$, the total revenue generated by $(\mathbf{p}, \mathbf{X})$ is at least that by $(\mathbf{p'}, \mathbf{X})$. By the objective of the linear program, we know that the revenue generated by the solution at LP$^{(k')}$ is at least that by $(\mathbf{p}, \mathbf{X})$ Therefore, MaxRevenue(S, T) computes a revenue no less than that of $(\mathbf{p}, \mathbf{X})$.

For the second direction, let $(\mathbf{p}, \mathbf{X})$ be the output of the procedure MaxRevenue(S, T). We need to show that $(\mathbf{p}, \mathbf{X})$ is an envy-free solution. Suppose $(\mathbf{p}, \mathbf{X})$ is computed in terms of $\mathrm{LP}^{(k^*)}$, where $k^* \in X_{i_t}$.

We first claim that all losers are envy-free. By Proposition 4.3, we need only to check if all the losers in L_S are envy-free for $(\mathbf{p}, \mathbf{X})$. Since $p_j = \infty, \forall j \notin T$, we only need to check that all the losers in L_S would not envy the items in T.

According to (5) in Step (a) of MaxRevenue(S, T), for any $i \in L_S$, we know that, for any buyer i, $\sum_{j \in T'} (v_i q_j - p_j) \le 0$ for any $T' \subseteq J$ with $|T'| = d_i$. Choose $T' = \{j_{\ell - d_{i_t} - d_i + 1}, j_{\ell - d_{i_t} - d_i + 2}, \cdots, j_{\ell - d_{i_t}}\} \subseteq J$ (as $d_i \le \Delta$). Let $j_{\max}$ be the largest index in T' such that $v_i q_{j_{\max}} - p_{j_{\max}} \le 0$. Then, by monotonicity of price-per-unit-quality in Proposition 4.5, we have

$$q_{j_1}\left(v_i - \frac{p_{j_1}}{q_{j_1}}\right) \le q_{j_2}\left(v_i - \frac{p_{j_2}}{q_{j_2}}\right) \le \cdots \le q_{j_{\max}}\left(v_i - \frac{p_{j_{\max}}}{q_{j_{\max}}}\right) \le 0,$$

and $v_i q_j - p_j > 0, \forall j \in \{j_{\max+1}, j_{\max+2}, \ldots, j_{\ell - d_{i_t}}\}$.

Hence, for every loser in L_S, its positive values in the set $\{v_i q_j - p_j \mid j \in T\}$ are contained in $\big\{v_i q_j - p_j \mid \{j_{\ell - d_{i_t} - d_i + 1}, j_{\ell - d_{i_t} - d_i + 2}, \ldots, j_\ell\} \subset J\big\}$. Therefore, the requirement (5) in Step (a) of MaxRevenue(S, T) would imply that for any $T' \subset T$ with $|T'| = d_i$, we have $\sum_{j \in T'} (v_i q_j - p_j) \le 0$, which means that i is envy-free. Hence, all the losers are envy-free for the tuple.

It remains to show that all winners are envy-free as well. Before doing this, by the pricing rule in subroutine (c), we can easily see that for any i_u and $j \in X_{i_u}$ with $u < t$, there exists item $j' \in X_{i_{u+1}}$ such that $p_j = v_{i_u}(q_j - q_{j'}) + p_{j'}$. We will use this particular property to show all winners are envy-free. Since $p_j = \infty$ for any $j \notin T$, it suffices to show that any winner would not envy the items of other winners. The claim follows from the following arguments.

- All winners get non-negative utility. Formula (2) guarantee that i_t gets nonnegative utility for X_{i_t}. For any winner $i_u < i_t$, none has over-priced item. It follows by the fact that, $\forall s \in J - X_{i_t}$, $p_{j_s} = v_{i_{b(s)}}(q_{j_s} - q_{j_{s+1}}) + p_{j_{s+1}}$ in the algorithm and $v_{i_{b(s)}} q_{j_{s+1}} - p_{j_{s+1}} \ge 0$ in Proposition 4.5.

- Buyer i_t would not envy items won by any other winner i_u, where $i_u < i_t$. We show this by induction. Formula (4) shows the base case hold (i.e., i_t would not envy items won by i_{t-1}). Then, for any item $j' \in X_{i_t}$ and any item $j \in X_{i_u}$, (notice that by the pricing rule, there exists $k \in X_{i_{u+1}}$ such that $p_j = v_{i_u}(c_j - c_k) + p_k$), we have

  $$v_{i_t} c_j - p_j = v_{i_t} c_j - (v_{i_u}(c_j - c_k) + p_k) = (v_{i_t} - v_{i_u})(c_j - c_k) + v_{i_t} c_k - p_k \le v_{i_t} c_k - p_k \le v_{i_t} c_{j'} - p_{j'},$$

  where the first inequality follows from $v_{i_t} - v_{i_u} \le 0$ and $c_j - c_k \ge 0$, and the second inequality follows from the induction hypothesis.

- For any i_u, $i_u < i_t$, i_u would not envy items won by i_t. Again, the proof is by induction. The base case $i_u = i_{t-1}$, for any item $j \in X_{i_{t-1}}$ and item $j' \in X_{i_t}$, it holds that

  $$v_{i_{t-1}} c_j - p_j = v_{i_{t-1}} c_j - (v_{i_{t-1}}(c_j - c_{k^*}) + p_{k^*}) = v_{i_{t-1}} c_{k^*} - p_{k^*} \ge v_{i_{t-1}} c_{j'} - p_{j'},$$

  where the first equality follows from formula (3) and the inequality follows from formula (1). Hence, the base case holds. Next for any $j \in X_{i_u}$ and item $j' \in X_{i_t}$, (notice by pricing rule, there exists $k \in X_{i_{u+1}}$ such that $p_j = v_{i_u}(c_j - c_k) + p_k$), we have

  $$v_{i_u} c_j - p_j = v_{i_u} c_j - (v_{i_u}(c_j - c_k) + p_k) = v_{i_u} c_k - p_k = (v_{i_u} - v_{i_{u+1}})(c_k - c_{j'}) + v_{i_u} c_{j'} + (v_{i_{u+1}}(c_k - c_{j'}) - p_k).$$

  Since $v_{i_u} - v_{i_{u+1}} \ge 0$ and $c_k - c_{j'} \ge 0$, and by the induction hypothesis, $v_{i_{u+1}} c_k - p_k \ge v_{i_{u+1}} c_{j'} - p_{j'}$, it holds that $v_{i_u} c_j - p_j \ge v_{i_u} c_{j'} - p_{j'}$.

- Every winner in $S \backslash \{i_t\}$ would not envy the items won by other winner in $S \backslash \{i_t\}$. Use the convention $j_{\ell - d_{i_t} + 1} = k^*$, recall $\forall u, 1 \le u \le \ell - d_{i_t}$, $p_{j_u} = v_{i_{b(u)}}(c_{j_u} - c_{j_{u+1}}) + p_{j_{u+1}}$, then for $1 \le s < s' \le \ell - d_{i_t}$,

  $$p_{j_s} - p_{j_{s'}} = \sum_{u=s}^{s'-1} (p_{j_u} - p_{j_{u+1}}) = \sum_{u=s}^{s'-1} v_{i_{b(u)}}(c_{j_u} - c_{j_{u+1}}) \le v_{i_{b(s)}} \sum_{u=s}^{s'-1} (c_{j_u} - c_{j_{u+1}}) = v_{i_{b(s)}}(c_{j_s} - c_{j_{s'}}).$$

Rewrite $p_{j_s} - p_{j_{s'}} \leq v_{i_{b(s)}}(c_{j_s} - c_{j_{s'}})$ as $v_{i_{b(s)}}c_{j_s} - p_{j_s} \geq v_{i_{b(s)}}c_{j_{s'}} - p_{j_{s'}}$, which means buyer with smaller index would not envy items won by buyer with larger index. Similarly, noting that

$$p_{j_s} - p_{j_{s'}} = \sum_{u=s}^{s'-1} v_{i_{b(u)}}(c_{j_u} - c_{j_{u+1}}) \geq v_{i_{b(s')}} \sum_{u=s}^{s'-1}(c_{j_u} - c_{j_{u+1}}) = v_{i_{b(s')}}(c_{j_s} - c_{j_{s'}}).$$

Rewrite $p_{j_s} - p_{j_{s'}} \geq v_{i_{b(s')}}(c_{j_s} - c_{j_{s'}})$ as $v_{i_{b(s')}}c_{j_s} - p_{j_s} \leq v_{i_{b(s')}}c_{j_{s'}} - p_{j_{s'}}$, which means buyer with larger index would not envy items won by buyer with smaller index. In all, every winner in $S\backslash\{i_t\}$ would not envy the items won by other winner in $S\backslash\{i_t\}$.

Therefore, we know that the tuple $(\mathbf{p}, \mathbf{X})$ is an envy-free solution. □

Observe that the computation of Step (a) of MaxRevenue does not depend on the whole set T. In fact, we only need to know the last 2Δ items with largest indices in T to check whether Y is empty or not. Therefore, whether MaxRevenue(S, T) will output a tuple only depends on the last 2Δ items in T. The prices for those 2Δ items are determined in one of the linear programs there. Suppose that the last 2Δ items in T are J and let $j_{\min} = \min\{j \in J\}$, then if MaxRevenue$(S, T)$ output a tuple $(\mathbf{p}, \mathbf{X})$, we can re-choose any other set $Z \subseteq \{1, 2, 3, \ldots, j_{\min} - 1\}$ with $|Z| = \ell - 2\Delta$ and run MaxRevenue$(S, Z \cup J)$, which would always output an envy-free tuple $(\mathbf{p}', \mathbf{X}')$ as well. Similarly, if MaxRevenue(S, T) claims that there is no tuple $(\mathbf{p}, \mathbf{X})$ which is an envy-free solution, then MaxRevenue$(S, Z \cup J)$ also claims that no tuple exists. These observations are critical in our main algorithm Max-EF.

4.1.5 Only the Winner Set is Known

Suppose that we are given a candidate winner set $S = \{i_1, i_2, \ldots, i_t\}$ and a set of items $J = \{j_1, \ldots, j_{2\Delta}\}$ with $i_1 < i_2 < \cdots < i_t$ and $j_1 < \cdots < j_{2\Delta}$. Assume that $\ell = d(S) > 2\Delta$. Let $Y = \{1, 2, \ldots, j_1 - 1\}$ denote the set of items that have indices smaller than j_1. Our objective is to pick a subset $Z \subseteq Y$ with $|Z| = \ell - 2\Delta$ such that the revenue given by MaxRevenue$(S, Z \cup J)$ is as large as possible. By Steps (a) and (c) of MaxRevenue, for the given set of winners S, the prices of the items in J are already fixed (no matter which Z is chosen). Hence, to maximize revenue from MaxRevenue$(S, Z \cup J)$, it suffices to maximize revenue (or equivalently, prices) from the items in Z. To this end, we use the approach of dynamic programming to find an optimal solution.

Consider any subset $Z = \{z_1, z_2, \ldots, z_{\ell-2\Delta}\} \subseteq Y$ with $z_1 < z_2 < \cdots < z_{\ell-2\Delta}$; denote $z_{\ell-2\Delta+1} = j_1$. Suppose MaxRevenue$(S, Z \cup J)$ will output a tuple $(\mathbf{p}, \mathbf{X})$. As we already know that each z_j will be allocated to which winner by MaxRevenue$(S, Z \cup J)$, let $w_j = v_i$ if $z_j \in X_i$, for $j = 1, 2, \ldots, \ell - 2\Delta$; further, let $w_0 = 0$. An important observation is that the values of all w_j's are independent to the selection of Z. By the pricing rule in MaxRevenue$(S, Z \cup J)$, it holds that $p_{z_j} = w_j(q_{z_j} - q_{z_{j+1}}) + p_{z_{j+1}}$, for $j = 1, 2, \ldots, \ell - 2\Delta$. Hence, we have

$$\begin{aligned}
\sum_{j=1}^{\ell-2\Delta} p_{z_j} &= \sum_{j=1}^{\ell-2\Delta}\left(\sum_{u=j}^{\ell-2\Delta}(p_{z_u} - p_{z_{u+1}}) + p_{j_1}\right) \\
&= \sum_{j=1}^{\ell-2\Delta}\sum_{u=j}^{\ell-2\Delta}\big((q_{z_u} - q_{z_{u+1}})w_u\big) + (\ell - 2\Delta)p_{j_1} \\
&= \sum_{j=1}^{\ell-2\Delta}(j \cdot q_{z_j}w_j - j \cdot q_{z_{j+1}}w_j) + (\ell - 2\Delta)p_{j_1} \\
&= \left[\sum_{j=1}^{\ell-2\Delta}\big(j \cdot w_j - (j-1) \cdot w_{j-1}\big)q_{z_j}\right] - \Big[(\ell - 2\Delta)(q_{j_1}w_{\ell-2\Delta} - p_{j_1})\Big] \\
&\triangleq R_1 - R_2,
\end{aligned}$$

where R_1 and R_2 are the first and second term of the difference, respectively. By the rule of MaxRevenue, the allocation of $z_{\ell-2\Delta}$ (thus, the value $w_{\ell-2\Delta}$) and the price p_{j_1} are fixed. Hence, to maximize $\sum_{j=1}^{\ell-2\Delta} p_{z_j}$, it suffices to maximize R_1. For any α, β with $1 \le \alpha \le \beta \le j_1 - 1$, let $opt(\alpha, \beta)$ denote the optimal value of the following problem, denoted by $DLP(\alpha, \beta)$, which picks α items from the first β items to maximize a given objective (recall that w_j is defined above for $j = 1, \ldots, \ell - 2\Delta$).

$$\begin{aligned} \max \quad & \sum_{j=1}^{\alpha} \big(j \cdot w_j - (j-1) \cdot w_{j-1}\big) q_{z_j} \\ \text{s.t.} \quad & z_1 < z_2 < \cdots < z_\alpha, \{z_1, z_2, \ldots, z_\alpha\} \subseteq \{1, 2, \ldots, \beta\}. \end{aligned}$$

The problem that maximizes R_1 is exactly $DLP(\ell - 2\Delta, j_1 - 1)$, which can be solved by the following dynamic programming.

SOLVE-DLP

1. `Compute` $opt(1,1), opt(1,2), \ldots, opt(1, j_1 - 1)$.

2. `Compute`

$$opt(\alpha, \beta+1) = \begin{cases} \max\Big\{opt(\alpha,\beta), opt(\alpha-1,\beta) + (\alpha \cdot w_\alpha - (\alpha-1) w_{\alpha-1}) q_{\beta+1}\Big\} & \texttt{if } \beta+1 \ge \alpha \\ 0 & \texttt{Otherwise} \end{cases}$$

3. `Find a subset` Z^* `that maximizes` $opt(\ell - 2\Delta, j_1 - 1)$.

4. `Return the output of` MAXREVENUE$(S, Z^* \cup J)$.

The following claim is straightforward from the definition of $DLP(\alpha, \beta)$ and the above dynamic programming.

Proposition 4.6. *Given a candidate winner set S and a subset J of 2Δ items, the above* SOLVE-DLP *picks in polynomial time a subset $Z \subseteq Y$ with $|Z| = \ell - 2\Delta$ such that the revenue given by* MAXREVENUE$(S, Z \cup J)$ *is the maximum if we guessed S and J correctly.*

4.1.6 Algorithm

In this subsection, we will present our main algorithm MAX-EF. The algorithm has two stages: STAGE 1 is to select the set of possible winners (candidate winners) who will be allocated items, and STAGE 2 is designed to calculate all the 'candidate' maximum revenue and presents an optimal envy-free solution and maximum revenue.

The algorithm is described as follows.

MAX-EF STAGE 1.

1. Initialize $D = \emptyset$ (denote the collection of candidate winner sets).
2. Find $S \subseteq A_1$ such that $d(S) = \max\{d(S') \mid d(S') \leq m, S' \subseteq A_1\}$, let $D \leftarrow \{S\}$.
3. For $k = 2, \ldots, K$
   - Sort $A_1 \cup A_2 \cup \cdots \cup A_k$ in the decreasing order of their values.
   - For each d such that $1 \leq d \leq m$
     - Let $S = \texttt{argmax}_S\{d(S)|d(S) \leq d, S \subset A_k\}$.
     - Let $S_{0,1} = S$, $n_0 = 1$ and $\mathcal{C}_0 = \{S_{0,1}\}$.
     - Let $\ell = |A_1 \cup A_2 \cup \cdots \cup A_{k-1}|$.
     - For $t = 1, 2, \ldots, \ell$ do:
       * In general, we have constructed $\mathcal{C}_t$ containing all the candidate winner sets of $\{\ell - t + 1, \ell - t + 2, \ldots, \ell\} \cup S$.
       * We order $\mathcal{C}_t = \{S_{t,1}, S_{t,2}, \ldots, S_{t,n_t}\}$ such that $d(S_{t,1}) \leq d(S_{t,2}) \leq \cdots \leq d(S_{t,n_t}) \leq m$.
       * We now add $\ell - t$ into $\mathcal{C}_t$ to construct $\mathcal{C}_{t+1}$.
         · Let $t_s = \max\{i : d(S_{t,i}) < d_{\ell-t}\}$ if $\{i : d(S_{t,i}) < d_{\ell-t}\} \neq \emptyset$, otherwise $ts = 0$.
         · Let $S_{t+1,j} = S_{t,j}$ for $j = 1, 2, \cdots, t_s$.
         · Let $S_{t+1,j+t_s} = S_{t,j} \cup \{\ell - t\}$ for $j = 1, 2, \ldots, n_t$.
         · Let $n_{t+1} = \max\{i \leq t_s + n_t : d(S_{t+1,i}) \leq m\}$.
         · Let $\mathcal{C}_{t+1} = \{S_{t+1,i} : i \leq n_{t+1}, d(S_{t+1,i}) \leq m\}$.
     - $D \leftarrow D \cup \mathcal{C}_\ell$.
4. return D

STAGE 1 of MAX-EF is designed to select candidate winner sets one of which contains exactly the winners in an optimal envy-free solution. For each $1 \leq k \leq K \leq n$ and $1 \leq d \leq m$ the problem is of one discussed in Lemma 4.1. It constructs $\mathcal{C}$, consisting of up to $\frac{m}{d}$ subsets of total size $O(\frac{m*n}{d})$ in time $O(\frac{m*n^2}{d})$. The total time complexity then adds up to $O(m * n^3 \log m)$. Hence, MAX-EF runs in strongly polynomial time.

Proposition 4.7. *There is an optimal winner set contained in the set D.*

Proof. Now suppose there is an optimal winner set W, if $W \subseteq A_1$, then by Proposition 4.4, the set S selected in above algorithm is an optimal winner set and we are done. Otherwise, let $i_{\max} = \max(W)$; suppose $i_{\max} \in A_{k^*}$, where $k^* \geq 2$, and let $w^* = d(W \cap A_{k^*})$. Now consider the k^*th and w^*th round of the for loop. There exists $T \subseteq A_{k^*}$ such that $d(T) = w^*$. By Proposition 4.4, we know that $(W \backslash (W \cap A_k)) \cup T$ is an optimal winner set. By the procedure of the algorithm and Proposition 4.2, the algorithm would find all the candidate winner sets with the form $C \cup T$ where $C \subseteq A_1 \cup \cdots \cup A_{k-1}$. Hence, $(W \backslash (W \cap A_k)) \cup T \in D$. □

Max-EF stage 2.

5. For each candidate winner set $S \in D$
   - Let $\ell = d(S)$
   - If $\ell \le 2\Delta$
     - For any set $J \subseteq \{1, 2, \ldots, m\}$ with $|J| = \ell$
       * Run MaxRevenue(S, J).
       * If it outputs a tuple $(\mathbf{p}, \mathbf{X})$, let $R^{S,J} \leftarrow \sum_{i=1}^{n} \sum_{j \in X_i} p_j$
       * Else, let $R^{S,J} \leftarrow 0$.
   - Else $\ell > 2\Delta$
     - For any set $J \subseteq \{\ell - 2\Delta + 1, \ell - 2\Delta + 2, \ldots, m\}$ with $|J| = 2\Delta$
       * Let $j_{\min} \leftarrow \min\{j \in J\}$
       * Choose any $Z \leftarrow \{z_1, \ldots, z_{\ell-2\Delta}\} \subseteq \{1, 2, \ldots, j_{\min} - 1\}$, where $z_1 > z_2 > \cdots > z_{\ell-2\Delta}$.
       * Run MaxRevenue$(S, J \cup Z)$
       * If it outputs a tuple
         · run Solve-DLP on S and J to get a tuple $(\mathbf{p}, \mathbf{X})$
         · let $R^{S,J} \leftarrow \sum_{i=1}^{n} \sum_{j \in X_i} p_j$
       * Else, let $R^{S,J} \leftarrow 0$
6. Output a tuple $(\mathbf{p}, \mathbf{X})$ which gives the maximum $R^{S,J}$.

Since MaxRevenue and Solve-DLP takes polynomial time, and $|D| \le nm \log m$, we know stage 2 of Max-EF runs in polynomial time.

Proof of Theorem 4.1. Since Max-EF takes polynomial time, we only need to check that Max-EF will output an optimal envy-free solution. By the above analysis, we know that Max-EF will output an envy-free solution. Since there is an optimal winner $S \in D$, there exists an optimal envy-free solution $(\mathbf{p}, \mathbf{X})$ such that $S = \{i | X_i \neq \emptyset\}$. W.l.o.g. suppose that the items in $T = \bigcup_{i=1}^{n} X_i$ are allocated to S by the rules of allocation of MaxRevenue(S, T) (otherwise, there exists $i > i'$ and $j < j'$ such that $j \in X_i$ and $j' \in X_{i'}$, if $v_i = v_{i'}$, then $v_i q_j - p_j \ge v_i q_{j'} - p_{j'}$ and $v_{i'} q_j - p_j \le v_{i'} q_{j'} - p_{j'}$, hence $v_i q_j - p_j = v_i q_{j'} - p_{j'}$, then exchanging the allocation j and j' without changing their prices would still make everyone envy-free. If $v_i < v_{i'}$, then by Lemma 3.3, we have $q_j = q_{j'}$, then exchanging allocation j and j' and their prices would still make everyone envy-free). If $d(S) \le 2\Delta$, then by the argument of Lemma 4.2, we know $R^{S,T} \ge \sum_{i=1}^{n} \sum_{j \in X_i} p_j$. Similarly if $d(S) > 2\Delta$, let J be the 2Δ largest values in T, by the argument of Lemma 4.2 and Proposition 4.6, we know $R^{S,J} \ge \sum_{i=1}^{n} \sum_{j \in X_i} p_j$. Therefore, the output $(\mathbf{p}, \mathbf{X})$ of Max-EF is an optimal envy-free solution. □

4.2 Proof of Hardness

We next prove the NP-hardness result that is part of Theorem 4.1, that envy-free revenue maximization with $v_i q_j$ valuations is NP-hard.

We reduce from the exact cover by 3-sets problem (X3C): Given a ground set $A = \{a_1, a_2, \ldots, a_{3n}\}$ and collection $T = \{S_1, S_2, \ldots, S_m\}$ where each $S_i \subset A$ and $|S_i| = 3$, we are asked if there are n elements of T that cover all elements in A. We assume that $n \le m \le 2n - 1$; it is easy to see that the problem still remains NP-complete (as we can add dummy elements x, y, z to A and subset $\{x, y, z\}$ to T to balance the sizes of A and T).

Given an instance of X3C, we construct a market with 3 buyers and $n+m$ items as follows. Let $M = 3nm+1$, $L = \sum_{i=1}^{3n} M^i$. Note that $L < 3nM^{3n}$, whose binary representation is of size polynomial in m and n. Consider

m values $R_i = \sum_{a_j \in S_i} M^j$, for $i = 1, 2, \ldots, m$, and rearranging if necessary, let $R_1 \geq R_2 \geq \cdots \geq R_m$ be a non-increasing order of these values. The valuations and demands of buyers are

$$\begin{aligned} d_1 = n, &\quad v_1 = 3 \\ d_2 = 2n, &\quad v_2 = \frac{3n+1}{n+1} \\ d_3 = n, &\quad v_3 = 2 \end{aligned}$$

The qualities of items are defined as follows: Let $q_j = L$, for $j = 1, 2, \ldots, n$, and $q_{n+j} = R_j$, for $j = 1, 2, \ldots, m$. Obviously, the unit values and qualities are in non-increasing order, and the construction is polynomial.

Consider the winner set in an optimal envy-free solution $(\mathbf{p}, \mathbf{X})$. Since $n \leq m \leq 2n - 1$, the possible winner sets are $\{1\}$, $\{2\}$, $\{3\}$, and $\{1, 3\}$. There is no envy-free solution where $\{2\}$ or $\{3\}$ is the winner set, since buyer 1 would be envious. It remains to consider $\{1\}$ and $\{1, 3\}$. If the winner set is $\{1\}$, then the optimal revenue is $v_1 \cdot \left(\sum_{i=1}^{n} q_i\right) = 3nL$ where buyer 1 gets the first n items. If the winner set is $\{1, 3\}$, it is not difficult to see that in the optimal envy-free solution $(\mathbf{p}, \mathbf{X})$, it holds that $X_1 = \{1, 2, \ldots, n\}$. Suppose that $X_3 = \{j_1, j_2, \ldots, j_n\} \subset \{n+1, n+2, \ldots, n+m\}$ where $j_1 > j_2 > \cdots > j_n$. Applying the characterizations of optimal envy-freeness MaxRevenue(S, T) and Lemma 4.2 in Section 4.1.4, in the optimal solution $(\mathbf{p}, \mathbf{X})$ with $X_1 = \{1, 2, \ldots, n\}$ and $X_3 = \{j_1, j_2, \ldots, j_n\}$, we should prove the following claim

Claim 4.2.

$$v_1 q_k - p_k = v_1 q_j - p_j \quad \forall k, j \in X_3$$

proof of Claim 4.2. According to MaxRevenue(S, T), there exists $k^* : n+1 \leq k^* \leq m+n$ such that $(\mathbf{p}, \mathbf{X})$ is the optimal solution of the following linear program(denoted by $LP^{(k^*)}$).

$$\begin{array}{llll} \min & v_1 q_{k^*} - p_{k^*} & & \\ s.t. & v_1 q_{k^*} - p_{k^*} \geq v_1 q_j - p_j & \forall\, j \in X_3 & (1^*) \\ & \sum\limits_{j \in X_3} (v_3 q_j - p_j) = 0 & & (2^*) \\ & v_1 q_j - p_j = v_1 q_{k^*} - p_{k^*} & \forall\, j \in X_1 & (3^*) \\ & v_3 q_j - p_j \leq v_3 q_{j'} - p_{j'} & \forall\, j \in X_1, j' \in X_3 & (4^*) \\ & \sum_{j \in X_1 \cup X_3} (v_2 q_j - p_j) \leq 0 & & (5^*) \end{array}$$

Please note that the last set of equations (6^*) in the original LP are not needed since they are empty under the current restriction of three buyers. We first prove all the inequalities in (1^*) must be equalities. Suppose it is not true. Then there exists $\ell \in X_3$ such that

$$v_1 q_{k^*} - p_{k^*} > v_1 q_\ell - p_\ell.$$

Set $a_j = v_1 q_j - p_j$, $j \in X_3$. From (2^*), it follows that $\sum\limits_{j \in X_3} a_j = (v_1 - v_3) \sum\limits_{j \in X_3} q_j$. Take the average

$$\bar{a} = \frac{\sum\limits_{j \in X_3} a_j}{|X_3|} = \frac{(v_1 - v_3) \sum\limits_{j \in X_3} q_j}{|X_3|}$$

We introduce the price vector $\mathbf{p}' = (p'_1, p'_2, \cdots, p'_n, p'_{j_1}, p'_{j_2}, \cdots, p'_{j_n})$ such that $\forall j \in X_3$: $p'_j = v_1 q_j - \bar{a}$ and $\forall j \in X_1$: $p'_j = v_1(q_j - q_{k^*}) + p'_{k^*}$. If we can prove that $(\mathbf{p}', \mathbf{X})$ is still a feasible solution for LP^{k^*}, then $p'_{k^*} > p_{k^*}$ (due to $a_{k^*} > \bar{a}$ by (1^*)). It results in a smaller objective value than $v_1 q_{k^*} - p_{k^*}$, a contradiction to the optimality of $(\mathbf{p}, \mathbf{X})$.

First, (1^*) (2^*) (3^*) follows directly from definition of $\mathbf{p}'$. We need only to check (4^*) and (5^*). From $p'_{k^*} > p_{k^*}$, $\forall\, j \in X_1$ $p'_j = v_1(q_j - q_{k^*}) + p'_{k^*} > v_1(q_j - q_{k^*}) + p_{k^*} = p_j$. We have $\forall j \in X_1$: $p'_j > p_j$. Hence, the inequality (5*) holds. To see inequality (4*), notice

$$\begin{aligned}v_3 q_j - p'_j &= v_3 q_j - v_1(q_j - q_{k^*}) - p'_{k^*} \\ &= v_3 q_j - v_1(q_j - q_{j'}) - p'_{j'} \\ &= (v_3 - v_1)(q_j - q_{j'}) + v_3 q_{j'} - p'_{j'} \\ &\leq v_3 q_{j'} - p'_{j'}, \quad \forall j \in X_1, j' \in X_3.\end{aligned}$$

Claim 4.2 is proven. □

By Claim 4.2 and the above condition (3*), we have

$$v_1 q_i - p_i = v_1 q_j - p_j, \ \forall\, i \in X_1, j \in X_3 \tag{1}$$

By the above condition (2*),

$$\sum_{j \in X_3} p_j = v_3 \cdot \sum_{k=1}^{n} q_{j_k}. \tag{2}$$

Combining (1) and (2), the total revenue is

$$R = \sum_{i=1}^{n} p_i + \sum_{j \in X_3} p_j = v_1 \cdot \sum_{i=1}^{n} q_i + (2v_3 - v_1) \cdot \sum_{k=1}^{n} q_{j_k}.$$

Since buyer 2 is envy-free, we have

$$v_2 \cdot \Big(\sum_{i=1}^{n} q_i + \sum_{k=1}^{n} q_{j_k} \Big) - R = (v_2 - v_1) \cdot \sum_{i=1}^{n} q_i + (v_1 + v_2 - 2v_3) \cdot \sum_{k=1}^{n} q_{j_k} \leq 0.$$

Therefore, computing the maximum revenue when the winner set is $\{1, 3\}$ is equivalent to solving the following program:

$$\begin{aligned}\max \quad & R = v_1 \cdot \sum_{i=1}^{n} q_i + (2v_3 - v_1) \cdot \sum_{k=1}^{n} q_{j_k} \\ \text{s.t.} \quad & (v_2 - v_1) \cdot \sum_{i=1}^{n} q_i + (v_1 + v_2 - 2v_3) \cdot \sum_{k=1}^{n} q_{j_k} \leq 0 \\ & j_1 > j_2 > \cdots > j_n, \ j_k \in \{n+1, n+2, \ldots, n+m\}, k = 1, 2, \ldots, n.\end{aligned} \tag{3}$$

Considering $v_1 = 3$, $v_2 = \frac{3n+1}{n+1}$, $v_3 = 2$, and $q_i = L$, $i = 1, 2, \ldots, n$, the program (3) is equivalent to

$$\begin{aligned}\max \quad & R = 3nL + \sum_{k=1}^{n} q_{j_k} \\ \text{s.t.} \quad & \sum_{k=1}^{n} q_{j_k} \leq L \\ & j_1 > j_2 > \cdots > j_n, \ j_k \in \{n+1, n+2, \ldots, n+m\}, k = 1, 2, \ldots, n.\end{aligned} \tag{4}$$

It is not difficult to see that the maximum revenue (i.e., the optimal value of the above program) is $(3n+1)L$ if and only if there is a positive answer to the instance of X3C. This completes the proof.

5 Conclusions

In this paper, multi-unit demand models of the matching market are studied and their competitive equilibrium solutions and envy-free solutions are considered. For the sharp demand model, a strongly polynomial time algorithm is presented to decide whether a competitive equilibrium exists or not and if one exists, to compute one that maximizes the revenue. In contrast, the revenue maximization problem for envy-free solutions is shown to be NP-hard. In a special case when the sharp demands of all players are bounded by a constant, a polynomial time algorithm is provide to solve the (envy-free) revenue maximization problem if the demand of each buyer is bounded by a constant number.

The sharp demand model is related to interesting applications such as sponsored search market for rich media ad pricing. Our work serves a modest step toward an efficient algroithmic solution. Our models may be further investigated to deal much more complicated settings of application problems.

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A Hardness for General Valuations

Theorem A.1. *It is NP-complete to determine the existence of a competitive equilibrium for general valuations in the sharp demand model (even when all demands are 3, and valuations are 0/1).*

Proof. We reduce from exact cover by 3-sets (X3C): Given a ground set $A = \{a_1, \ldots, a_{3n}\}$ and a collection of subsets $S_1, \ldots, S_m \subset A$ where $|S_i| = 3$ for each i, we are asked whether there are n subsets that cover all elements in A. Given an instance of X3C, we construct a market with $3n + 3$ items and $9n + m + 1$ buyers as follows. Every element in A corresponds to an item; further, we introduce another three items $B = \{b_1, b_2, b_3\}$. For each subset S_i, there is a buyer with value $v_{ij} = 1$ if $j \in S_i$ and $v_{ij} = 0$ otherwise; further, for every possible subset $\{x, y, z\}$ where $x \in A$ and $y, z \in B$, there is a buyer with value $v_{ij} = 1$ if $j \in \{x, y, z\}$ and $v_{ij} = 0$ otherwise; finally, there is a buyer with value $v_{ij} = 1$ if $j \in B$ and $v_{ij} = 0$ otherwise. The demand of every buyer is 3.

We claim that there is a positive answer to the X3C instance if and only if there is a competitive equilibrium in the constructed market. Assume that there is $T \in \{S_1, \ldots, S_m\}$ with $|T| = n$ that covers all elements in A. Then we allocate items in A to the buyers in T and allocate B to the buyer who desires B, and set all prices to be 1. It can be seen that this defines a competitive equilibrium.

On the other hand, assume that there is a competitive equilibrium $(\mathbf{p}, \mathbf{X})$. We first claim that all items are allocated out in the equilibrium. Otherwise, there must exist an item $a_j \in A$ that is not allocated to any buyer. (If all unallocated items just belonged to B, then all 3 items in B would be unallocated, contradicting envy-freeness of the buyer who values B.) Then we have $p_{a_j} = 0$. Consider the buyers who desire subsets $\{a_j, b_1, b_2\}, \{a_j, b_1, b_3\}, \{a_j, b_2, b_3\}$. They do not win since a_j is not sold. Due to envy-freeness, we have

$$\begin{aligned} p_{b_1} + p_{b_2} &\geq 3 \\ p_{b_1} + p_{b_3} &\geq 3 \\ p_{b_2} + p_{b_3} &\geq 3 \end{aligned}$$

This implies that $p_{b_1} + p_{b_2} + p_{b_3} \geq 4.5$. Hence, the buyer who desires B cannot afford the price of B and at least one item in B, say b_1, is not allocated out. Thus $p_{b_1} = 0$ and $p_{b_2} + p_{b_3} \geq 4.5$. This contradicts envy-freeness of the buyer who gets b_2 and b_3.

Now since all items in A are allocated out, because of the construction of the market, we have to allocate all items in A to n buyers and allocate B to one buyer; the former gives a solution to the X3C instance. ☐