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Perturbations of rotating cosmological black holes

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Abstract

Charged, rotating black hole solutions of Einstein's gravitational equations are investigated in the presence of a cosmological constant. A pair of wave equations governing the electromagnetic and gravitational perturbations are derived.

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I. Introduction

Recent observations[1] support the presence of a cosmological term in the gravitational equations of Einstein. If this is the case then an exact description of cosmological black holes will not be by a solution of the short Einstein equations but by a solution of the equations including the cosmological constant. Indeed, the most general stationary black hole state due to Carter does contain the cosmological constant.

The Carter space-time with cosmological constant Λ, mass m, rotation parameter a and electric charge e has the metric[2]

$$\begin{aligned} ds^2 &= \frac{1}{(1+\alpha)^2\zeta\overline{\zeta}}\left\{\mathbf{\Delta}(du - a\sin^2\vartheta d\phi)^2 - \Delta_\vartheta \sin^2\vartheta\left[adu - (r^2+a^2)d\phi\right]^2\right\} \\ &\quad -\zeta\overline{\zeta}\left(\frac{dr^2}{\mathbf{\Delta}} + \frac{d\vartheta^2}{\Delta_\vartheta}\right) \end{aligned} \tag{1}$$

where

$$\mathbf{\Delta} = \left(r^2+a^2\right)\left(1-\alpha\frac{r^2}{a^2}\right) - 2\mathrm{m}r + \mathrm{e}^2, \qquad \Delta_\vartheta = 1 + \alpha\cos^2\vartheta \tag{2}$$

and

$$\alpha = \frac{\Lambda a^2}{3}, \qquad \zeta = r - ia\cos\vartheta. \tag{3}$$

In the special case when the mass m, the electric charge e and the rotation parameter a all vanish, the metric (1) describes the de Sitter space time, to which the metric tends for large values of r. Another limit is that with a vanishing cosmological constant Λ, in which case the metric (1) describes the Kerr-Newman space-time.

In the generic case, the condition $\mathbf{\Delta} = 0$ has up to four distinct roots. The number of roots clearly depends on the values of the parameters. The equilibrium solutions have charge e equal to mass m up to an overall sign. For a positive cosmological constant, $\Lambda > 0$, this is no longer the condition of extremality, as the equation $(r-\mathrm{m})^2 + a^2 - \alpha r^2\left(\frac{r^2}{a^2}+1\right) = 0$ then has real roots.

The main purpose of the present paper is to present the equations governing the coupled electromagnetic and gravitational perturbations of rotating cosmological black holes. These equations are separable and they generalize the master equations for the perturbations of Kerr-Newman black holes.[6]

II. Analytic extension

The metric (1) is singular on the horizon hypersurface given by the largest root of $\mathbf{\Delta} = 0$. One can transform to a better coordinate system by

$$\begin{aligned} dt &= du + (1+\alpha)\tfrac{r^2+a^2}{\mathbf{\Delta}}dr \\ d\varphi &= d\phi + (1+\alpha)\tfrac{a}{\mathbf{\Delta}}dr \end{aligned} \tag{4}$$

which is a generalization of the inverse Boyer-Lindquist transformation. In the new coordinates, the metric acquires the null form

$$\begin{aligned} ds^2 = & \frac{1}{(1+\alpha)^2}\left[1 - \tfrac{2\mathrm{m}r-\mathrm{e}^2}{\zeta\overline{\zeta}} - \alpha(\tfrac{r^2}{a^2} + \sin^2\vartheta)\right]\left(dt - a\sin^2\vartheta d\varphi\right)^2 \\ & + \frac{2}{1+\alpha}\left(dt - a\sin^2\vartheta d\varphi\right)\left(dr + \frac{\Delta_\vartheta}{1+\alpha}a\sin^2\vartheta d\varphi\right) \\ & - \zeta\overline{\zeta}\left(\frac{d\vartheta^2}{\Delta_\vartheta} + \frac{\Delta_\vartheta}{(1+\alpha)^2}\sin^2\vartheta d\varphi^2\right) \end{aligned} \tag{5}$$

with the four-potential

$$A = -\tfrac{\mathrm{e}r}{(1+\alpha)^2\zeta\overline{\zeta}}\left(dt - a\sin^2\vartheta d\varphi\right). \tag{6}$$

The coordinates t and r run from $-\infty$ to ∞ while ϑ and φ are coordinates on a distorted 2-sphere such that φ is periodic with period 2π and ϑ ranges from 0 to π.

The singularities are conveniently explored by introducing the null tetrad[3]

$$\begin{aligned} D &\equiv \ell^a\tfrac{\partial}{\partial x^a} = \tfrac{\partial}{\partial r} \\ \Delta &\equiv n^a\tfrac{\partial}{\partial x^a} = \tfrac{1}{2}(1+\alpha)^2\left[\tfrac{2\mathrm{m}r-\mathrm{e}^2}{\zeta\overline{\zeta}} - 1 + \alpha(\tfrac{r^2}{a^2} + \sin^2\vartheta)\right]\tfrac{\partial}{\partial r} + (1+\alpha)\tfrac{\partial}{\partial t} \\ \delta &\equiv m^a\tfrac{\partial}{\partial x^a} = \tfrac{1}{2^{1/2}\overline{\zeta}}\left[\Delta_\vartheta^{1/2}\left(\tfrac{\partial}{\partial\vartheta} - ia\sin\vartheta\tfrac{\partial}{\partial r}\right) + \tfrac{1+\alpha}{\Delta_\vartheta^{1/2}}\left(\tfrac{i}{\sin\vartheta}\tfrac{\partial}{\partial\varphi} + ia\sin\vartheta\tfrac{\partial}{\partial t}\right)\right] \\ \overline{\delta} &\equiv \overline{m}^a\tfrac{\partial}{\partial x^a}. \end{aligned} \tag{7}$$

In the NP notation, the Maxwell tensor components are

$$\begin{aligned}
\Phi_0 &\equiv F_{ab}\ell^a m^b = 0 \\
\Phi_1 &\equiv \tfrac{1}{2}F_{ab}\left(\ell^a n^b + \overline{m}^a m^b\right) = \frac{\mathrm{e}}{2^{1/2}\zeta^2} \\
\Phi_2 &\equiv F_{ab}\overline{m}^a n^b = \frac{iea\sin\vartheta}{\zeta^3}\Delta_\vartheta^{1/2}.
\end{aligned} \tag{8}$$

The significant components of the Weyl curvature C_{abcd} are

$$\begin{aligned}
\Psi_2 &= \frac{\mathrm{e}^2 - \mathrm{m}\overline{\zeta}}{\zeta^3\overline{\zeta}} \\
\Psi_3 &= -3i\Delta_\vartheta^{1/2} a\sin\vartheta\frac{\mathrm{m}\overline{\zeta} - \mathrm{e}^2}{2^{1/2}\zeta^4\overline{\zeta}} \\
\Psi_4 &= 3\Delta_\vartheta a^2\sin^2\vartheta\frac{\mathrm{m}\overline{\zeta} - \mathrm{e}^2}{\zeta^5\overline{\zeta}}
\end{aligned} \tag{9}$$

and the remaining two components vanish, $\Psi_0 \equiv -C_{abcd}\ell^a m^b \ell^c m^d = 0$ and $\Psi_1 \equiv -C_{abcd}\ell^a n^b \ell^c m^d = 0$. The curvature singularities are at $r = 0$ and $\vartheta = \pi/2$.

The null coordinates $(r, \vartheta, t, \varphi)$ cover a part of the manifold extending beyond the hypersurface $\mathbf{\Delta} = 0$. A different extension is given by the twin transformation

$$\begin{aligned}
d\widetilde{t} &= du - (1+\alpha)\tfrac{r^2+a^2}{\mathbf{\Delta}}dr \\
d\widetilde{\varphi} &= d\phi - (1+\alpha)\tfrac{a}{\mathbf{\Delta}}dr.
\end{aligned} \tag{10}$$

The analytic extension of the space-time can be traversed by transiting directly from the coordinate patches $(r, \vartheta, t, \varphi)$ to $\left(r, \vartheta, \widetilde{t}, \widetilde{\varphi}\right)$ by use of the combined transformations (4) and (10) on each of the domains of the space-time lying in between the zeroes of the function Δ. The complete analytic extension of the uncharged Kerr-de Sitter space-time has been briefly discussed in Ref. 4. There is a one-to-one correspondence between the structure of the charged and the uncharged metrics. The diagrammatic representation on Fig. 6.5 in Ref. 4 of the complete manifold remains valid for the charged Carter space-time.

III. Black hole perturbations

We choose the perturbed tetrad such that the spinor o^A is one of the four principal spinors of the Weyl curvature:

$$\Psi_0 = 0. \tag{11}$$

The quantity Ψ_0 transforms under the infinitesimal dyad transformation

$$o^A \to o^A + b\iota^A, \qquad \iota^A \to \iota^A, \tag{12}$$

as follows:

$$\Psi_0 \to \Psi_0 + 4b\Psi_1. \tag{13}$$

Here b is an arbitrary but small complex multiplier function such that higher powers of b are negligible. Since the curvature quantity Ψ_1 itself is small, the spinor o^A remains a principal spinor of the curvature under the transformations (12). We use this gauge freedom to eliminate all coupling terms from the wave equation

$$\Box_1 \phi = 0 \tag{14}$$

where the wave operator is

$$\begin{aligned}
\Box_s \;=\; & \boldsymbol{\Delta}^{-s} \tfrac{\partial}{\partial r} \boldsymbol{\Delta}^{s+1} \tfrac{\partial}{\partial r} + \tfrac{1}{\sin\vartheta} \tfrac{\partial}{\partial\vartheta} \Delta_\vartheta \sin\vartheta \tfrac{\partial}{\partial\vartheta} \\
& + \tfrac{1+\alpha}{\Delta_\vartheta \sin^2\vartheta} \left[(1+\alpha) \tfrac{\partial}{\partial\varphi} + 2is \left(\Delta_\vartheta - \alpha \sin^2\vartheta \right) \cos\vartheta \right] \tfrac{\partial}{\partial\varphi} \\
& + 2\,(1+\alpha)\, a \left(\tfrac{1+\alpha}{\Delta_\vartheta} \tfrac{\partial}{\partial t} - \tfrac{\partial}{\partial r} \right) \tfrac{\partial}{\partial\varphi} \\
& + \tfrac{(1+\alpha)^2}{\Delta_\vartheta} a^2 \sin^2\vartheta \tfrac{\partial^2}{\partial t^2} - 2\,(1+\alpha)\,(r^2+a^2) \tfrac{\partial^2}{\partial r \partial t} \\
& - 2\,(1+\alpha) \left[(s+2) r + \left(\Delta_\vartheta + s\alpha \sin^2\vartheta \right) ia\cos\vartheta \right] \tfrac{\partial}{\partial t} \\
& - 12 \tfrac{\alpha s}{a^2} r^2 + \left(1 - \alpha - 6\alpha\cos^2\vartheta \right) s \\
& - \tfrac{s^2 \cos^2\vartheta}{\sin^2\vartheta} \Delta_\vartheta \left(1 - \tfrac{\alpha}{\Delta_\vartheta} \sin^2\vartheta \right)^2 ,
\end{aligned} \tag{15}$$

and the electromagnetic perturbation function ϕ is defined by

$$\Phi_0 = \mathrm{e}\phi. \tag{16}$$

The second wave equation can be obtained by essentially repeating the steps made in Ref. 6 for deriving the master equation of the gravitational perturbation components. In the resulting equation, each term on the left contains the small function

$$\psi = \frac{\sigma}{\zeta^2} \tag{17}$$

and each term on the right contains either a ϕ or the first-order spin coefficient $\kappa \equiv \ell_{a;b} m^a \ell^b$. When inserting the unperturbed values of the operators and factors, neither the ψ terms, nor the ϕ terms cancel but those with κ do. We get the wave equation

$$\Box_2 \psi = \frac{1}{\bar{\zeta}^2} J\phi. \tag{18}$$

Thus the solution ϕ of Eq. (14) will provide the source function $J\phi$ for the equation for ψ. The source term is a functional of the field ϕ containing up to second derivatives, with the operator

$$\begin{aligned} J \;=\; & \left[(\mathrm{e}^2 - \mathrm{m}\zeta) + \tfrac{\alpha}{a^2}\zeta\bar{\zeta}^3\right] \\ & \times \left[\Delta_\vartheta^{1/2}\left(\tfrac{\partial}{\partial\vartheta} - \tfrac{\cos\vartheta}{\sin\vartheta}\right) + i\tfrac{1+\alpha}{\Delta_\vartheta^{1/2}}\left(a\sin\vartheta\tfrac{\partial}{\partial t} + \tfrac{1}{\sin\vartheta}\tfrac{\partial}{\partial\varphi}\right) + \tfrac{\alpha}{\Delta_\vartheta^{1/2}}\sin\vartheta\cos\vartheta\right]\tfrac{\partial}{\partial r} \\ & -\tfrac{1}{\zeta}\left[\mathrm{e}^2 + \left(\mathrm{m} - \tfrac{\alpha}{a^2}\bar{\zeta}^3\right)(2\bar{\zeta} - \zeta)\right] \\ & \times \left[\Delta_\vartheta^{1/2}\left(\tfrac{\partial}{\partial\vartheta} - \tfrac{\cos\vartheta}{\sin\vartheta}\right) + i\tfrac{1+\alpha}{\Delta_\vartheta^{1/2}}\left(a\sin\vartheta\tfrac{\partial}{\partial t} + \tfrac{1}{\sin\vartheta}\tfrac{\partial}{\partial\varphi}\right) + \tfrac{\alpha}{\Delta_\vartheta^{1/2}}\sin\vartheta\cos\vartheta\right] \\ & +\tfrac{1}{\zeta}\left[\mathrm{e}^2 - \left(\mathrm{m} - \tfrac{\alpha}{a^2}\bar{\zeta}^3\right)(\zeta + 2\bar{\zeta})\right] ia\sin\vartheta\Delta_\vartheta^{1/2}\tfrac{\partial}{\partial t}. \end{aligned} \tag{19}$$

One can treat these equations by expanding the homogeneous solutions in quasi-normal modes with energy ω and helicity[7] m:

$$\Phi_0 = \int d\omega \sum_{l,m} R(r)\, S_l^m(\vartheta)\, e^{i(m\varphi - \omega t)}. \tag{20}$$

Here the radial function $R(r)$ and the angular function $S_l^m(\vartheta)$ satisfy the ordinary differential equations, respectively,

$$\begin{aligned} & \left\{\boldsymbol{\Delta}^{-s}\tfrac{\partial}{\partial r}\boldsymbol{\Delta}^{s+1}\tfrac{\partial}{\partial r} + 2i(1+\alpha)\left[(r^2+a^2)\omega - am\right]\tfrac{\partial}{\partial r}\right. \\ & \left. +2i\omega(1+\alpha)(s+2)r - 12\tfrac{\alpha s}{a^2}r^2 - K\right\} R = 0 \end{aligned} \tag{21}$$

$$\begin{aligned}&\left\{\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\Delta_\vartheta\sin\vartheta\frac{\partial}{\partial\vartheta}\right.\\&-\frac{1}{\Delta_\vartheta}\left[\frac{(1+\alpha)m+s\Delta_\vartheta\cos\vartheta}{\sin\vartheta}-(1+\alpha)\,\omega a\sin\vartheta-\alpha s\cos\vartheta\sin\vartheta\right]^2\\&\left.+\left(1-\alpha-6\alpha\cos^2\vartheta\right)s-2\,(1+\alpha)\,(s+1)\,\omega a\cos\vartheta+K\right\}S_l^m=0\end{aligned}\tag{22}$$

and K is the separation constant of the kernel of operator $\Box_s$.

In the present coordinate system, the metric (1) remains regular on the null hypersurfaces $\boldsymbol{\Delta}=0$. The radial equation (21) has a singularity on the horizon (the outer solution of $\boldsymbol{\Delta}=0$), but, as was shown in Ref. 8, one can choose the boundary conditions for the wave equations (21) and (23) on the horizon such that the perturbations ϕ and ψ are regular.

In terms of a solution of Eqs. (14) and (18), the perturbation functions σ and Φ_0 are available from the respective simple relations (17) and (16). One can next compute the spin coefficient κ by a method quite similar to that described in Ref. 6.

IV. Conclusions

Equations (14) and (18) generalize the perturbation equations of Ref. 6 for black holes in the presence of a cosmological constant. The nature of these equations is quite unprecedented in general relativity. Their new feature is that they refer to the perturbations of a well-defined geometrical object in an electrovacuum (or vacuum) space-time. This object is a principal null congruence. In contrast, the first known master equation for the Kerr black hole[7] – also a linear hyperbolic equation – describes the perturbations of the component Ψ_0 of the Weyl curvature. The latter quantity is *not* uniquely defined, since it changes with tetrad rotations unless those are restricted to (12).

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