

# On the Poincaré series of Kac-Moody Lie algebras

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## Abstract

In this paper, we discuss the Poincaré series of Kac-Moody Lie algebras, especially for indefinite type. Firstly, we compute the Poincaré series of certain indefinite Kac-Moody Lie algebras whose Cartan matrices have the same type of  $2 \times 2$  principal sub-matrices. Secondly, we show that the Poincaré series of Kac-Moody Lie algebras satisfy certain interesting properties. Lastly we give some applications of the Poincaré series to other fields. Particularly we construct some counter examples to a conjecture of Victor Kac[5] and a conjecture of Chapavalov, Leites and Stekolshchik[16].

**Keywords:** Kac-Moody Lie algebra, Poincaré series, Cartan matrix, Flag manifold.

## 1 Introduction

To start with, we briefly review some concepts and results about Kac-Moody Lie algebras and their Poincaré series.

Let  $A = (a_{ij})$  be an  $n \times n$  integer matrix,  $A$  is called a Cartan matrix if it satisfies:

- (1) For each  $i$ ,  $a_{ii} = 2$ ;
- (2) For  $i \neq j$ ,  $a_{ij} \leq 0$ ;
- (3) If  $a_{ij} = 0$ , then  $a_{ji} = 0$ .

For each Cartan matrix  $A$ , there is an associated Kac-Moody Lie algebra  $g(A)$ , modulo its center which is generated by  $h_i, e_i, f_i, 1 \leq i \leq n$  over  $\mathbb{C}$ , with the defining relations:

- (1)  $[h_i, h_j] = 0$ ;
- (2)  $[e_i, f_i] = h_i, [e_i, f_j] = 0, i \neq j$ ;
- (3)  $[h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j$ ;
- (4)  $\text{ad}(e_i)^{-a_{ij}+1}(e_j) = 0$ ;
- (5)  $\text{ad}(f_i)^{-a_{ij}+1}(f_j) = 0$ .

For details see Kac[1] and Moody[2].

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Kac and Peterson[3][4][5] further constructed the Kac-Moody group  $G(A)$  with Lie algebra  $g(A)$ .

Let  $M(A) = (m_{ij})_{n \times n}$  be the Coxeter matrix of the Cartan matrix  $A$ . That is: for  $i = j, m_{ij} = 1$ ; for  $i \neq j, m_{ij} = 2, 3, 4, 6$  and  $\infty$  as  $a_{ij}a_{ji} = 0, 1, 2, 3$  and  $\geq 4$  respectively, then the Weyl group  $W(A)$  of  $g(A)$  is:

$$W(A) = \langle \sigma_1, \sigma_2, \dots, \sigma_n | (\sigma_i \sigma_j)^{m_{ij}} = 1, 1 \leq i \neq j \leq n \rangle.$$

Each element  $w \in W(A)$  has a decomposition  $w = \sigma_{i_1} \cdots \sigma_{i_k}, 1 \leq i_1, \dots, i_k \leq n$ . The length of  $w$  is defined as the least integer  $k$  in all of those decompositions of  $w$ , denoted by  $l(w)$ . The Poincaré series of  $g(A)$  is the power series  $P(A) = \sum_{w \in W(A)} t^{l(w)}$ . Hence  $P(A)$  only depends on the structure of Weyl group  $W(A)$  and the length function on it.

For the Kac-Moody Lie algebra  $g(A)$ , there is the Cartan decomposition  $g(A) = h \oplus \sum_{\alpha \in \Delta} g_\alpha$ , where  $h$  is the Cartan sub-algebra and  $\Delta$  is the root system. Let  $b = h \oplus \sum_{\alpha \in \Delta^+} g_\alpha$  be the Borel sub-algebra, then  $b$  corresponds to a Borel subgroup  $B(A)$  in the Kac-Moody group  $G(A)$ . The homogeneous space  $F(A) = G(A)/B(A)$  is called the complete flag manifold of  $G(A)$ . By Kumar[6]  $F(A)$  is a ind-variety.

The flag manifold  $F(A)$  admits a CW-decomposition of Schubert cells which is indexed by the elements of Weyl group  $W(A)$ . For each  $w \in W(A)$ , the complex dimension of Schubert variety  $X_w$  is  $l(w)$ . Therefore the Poincaré series of the flag manifold  $F(A)$  defined by Betti numbers is just the Poincaré series  $P(A)$  of  $g(A)$ . The Poincaré series  $P(A)$  is closely related to the topology of Kac-Moody group  $G(A)$  and its flag manifold  $F(A)$ . For flag manifolds over finite field  $F_q, q = p^k$  for a prime number  $p$ , by the Schubert decomposition, the number of elements in  $F(A)$  is just given by the value of Poincaré series  $P(A)$  at  $t = q$ . This fact can be further used to compute the orders of Kac-Moody group over  $F_q$  (by using a regularization procedure).

The cohomology of flag manifolds and Poincaré series of finite type and affine type Kac-Moody Lie algebras are extensively studied by Borel[7], Bott and Samelson[8], Bott[9], and Chevalley[10]. But for indefinite type, little is known.

For Kac-Moody Lie algebra  $g(A)$  of finite type, Poincaré series has the following form

$$P(A) = \prod_{i=1}^n \frac{t^{d_i} - 1}{t - 1} \quad (1.1)$$

where  $d_i$ 's are the degrees of basic invariants of  $g(A)$ .

Let  $I$  be a subset of  $S = \{1, 2, \dots, n\}$ , define the principal sub-matrix  $A_I = (a_{ij})_{i,j \in I}$ . It's obvious that  $A_I$  is also a Cartan matrix. Let  $W_I(A)$  be the subgroup of  $W(A)$  generated by  $\{\sigma_i | i \in I\}$ , then  $W_I(A)$  is the Weyl groups of  $g(A_I)$ . In the following we also denote  $P(A_I)$  by  $P_I(A)$ . The Poincaré series  $P(A)$  is also denoted by  $P_A(t)$  to emphasize the variable  $t$ .

According to the classification of Cartan matrix  $A = (a_{ij})_{n \times n}$ , Kac-Moody algebras are classified into three types:

- (1)  $g(A)$  is of finite type if  $A$  is positive definite.
- (2)  $g(A)$  is of affine type if  $A$  is positive semi-definite and has rank  $n - 1$ .
- (3)  $g(A)$  is of indefinite type otherwise.

Let  $\tilde{A}$  be the Extended Cartan matrix of the Cartan matrix  $A$  of finite type, then every non-twisted affine Kac-Moody Lie algebra  $g(\tilde{A})$  is the central extension of the infinite-dimensional Lie algebra  $g(A) \otimes \mathbb{C}[t, t^{-1}]$ . Bott[9] showed that the Poincaré series is

$$P(\tilde{A}) = P(A) \prod_{i=1}^n \frac{1}{1 - t^{d_i-1}} \quad (1.2)$$

where  $d_i$ 's are the degrees of basic invariants of  $g(A)$ .

For Kac-Moody Lie algebra  $g(A)$  of finite type,

$$\sum_{I \subset S} (-1)^{|I|} \frac{P(A)}{P_I(A)} = t^{D(A)} \quad (1.3)$$

where  $D(A)$  is the complex dimension of  $F(A)$ .

For Kac-Moody Lie algebra  $g(A)$  of affine or indefinite type,

$$\sum_{I \subset S} (-1)^{|I|} \frac{P(A)}{P_I(A)} = 0 \quad (1.4)$$

The Equations (1.3) and (1.4) are given in Steinberg[11] and Humphreys[12].

Theoretically, one can compute Poincaré series of any affine and indefinite Kac-Moody Lie algebras through Equation (1.3) and (1.4) by iterations. This reduces the computation of Poincaré series of Kac-Moody Lie algebras of affine and indefinite types to the finite case.

The paper proceeds as follows: In section 2, we give some concrete computation results of the Poincaré series of certain Kac-Moody Lie algebras of indefinite type; In section 3, we derive some properties satisfied by the Poincaré series. These properties are mainly derived from Equation (1.4) and verify a conjecture given by Gungormez and Karadayi[14]; In section 4, we show the computation of the Poincaré series for Kac-Moody Lie algebras can be used to compute the Poincaré series of the generalized flag manifolds of Kac-Moody groups. In section 5, we discuss the application of Poincaré series to graph theory and define the Poincaré-Coxeter invariants and the homotopy indices of a graph. In section 6, we construct some counter examples to a conjecture given by Kac[5]; In section 7, we derive the condition needed for the conjecture of Chapavalov, Leites and Stekolshchik[16] to be true and give some counter examples.

## 2 Computation of the Poincaré series for certain Kac-Moody Lie algebras

From the definition of Weyl group and Poincaré series, we see that Poincaré series are determined only by the products  $a_{ij}a_{ji}$ , for all  $i, j$ . If we change one  $a_{ij}a_{ji} > 4$  to 4, then the Coxeter matrix is unchanged. Therefore we have

**Proposition 1.** *Let  $A = (a_{ij})_{n \times n}$  be a Cartan matrix, define a Cartan matrix  $A' = (a'_{ij})_{n \times n}$ . For  $i > j$ ,  $(a'_{ij}, a'_{ji})$  is given by the product  $a_{ij}a_{ji}$  as follows,*

$a_{ij}a_{ji}$	0	1	2	3	$\geq 4$
$(a'_{ij}, a'_{ji})$	(0, 0)	(-1, -1)	(-2, -1)	(-3, -1)	(-4, -1)

then  $P(A) = P(A')$ .

Therefore for the computation of Poincaré series, we only need to consider Cartan matrix of form:

$$A = \begin{pmatrix} 2 & l(a_{21}) & \cdots & l(a_{n1}) \\ a_{21} & 2 & \cdots & l(a_{n2}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 2 \end{pmatrix} \quad (*),$$

where  $-4 \leq a_{ij} \leq 0$  and  $l(x) = \begin{cases} 0, & x = 0; \\ -1, & \text{otherwise.} \end{cases}$

**Theorem 1.** Let  $P_n$  be the Poincaré series of an  $n \times n$  Cartan matrix  $A$  in which all order 2 principal sub-matrices having the same type, and  $P_2$  denotes the Poincaré series of the order 2 principal sub-matrix, then

$$P_n = \frac{(1+t)P_2}{\frac{(n-1)(n-2)}{2}tP_2 - \frac{(n+1)(n-2)}{2}P_2 + \frac{n(n-1)}{2}(1+t)}$$

**Proof:** we prove the theorem by induction on  $n$ .

(i) For  $n = 2$ , Theorem 1 can be checked directly.

(ii) Suppose Theorem 1 is true for  $k < n$ , by Equation (1.4), we get

$$1 - \binom{n}{1} \frac{1}{P_1} + \binom{n}{2} \frac{1}{P_2} - \binom{n}{3} \frac{1}{P_3} + \cdots + (-1)^{n-1} \binom{n}{n-1} \frac{1}{P_{n-1}} = \frac{(-1)^{n-1}}{P_n}$$

By  $P_1 = 1 + t$ . and the induction assumption, we have

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \frac{\frac{(i-1)(i-2)}{2}tP_2 - \frac{(i+1)(i-2)}{2}P_2 + \frac{i(i-1)}{2}(1+t)}{(1+t)P_2} = \frac{(-1)^{n-1}}{P_n}$$

Since

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} i = (-1)^{n-1} n \text{ and } \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} i^2 = (-1)^{n-1} n^2.$$

We can show

$$(-1)^{n-1} \frac{(1+t)P_2}{\frac{(n-1)(n-2)}{2}tP_2 - \frac{(n+1)(n-2)}{2}P_2 + \frac{n(n-1)}{2}(1+t)} = \frac{(-1)^{n-1}}{P_n}.$$

So the theorem holds for  $k = n$ . □

By Theorem 1, we get

**Corollary 1.** Let  $A = (a_{ij})_{n \times n}$  be a Cartan matrix as in Theorem 1, then

$$P(A) = \begin{cases} \frac{P(A_2)}{\frac{(n-1)(n-2)}{2}t^3 - (n-2)t^2 - (n-2)t + 1} & \text{if } \forall i \neq j, a_{ij}a_{ji} = 1; \\ \frac{P(B_2)}{\frac{(n-1)(n-2)}{2}t^4 - (n-2)t^3 - (n-2)t^2 - (n-2)t + 1} & \text{if } \forall i \neq j, a_{ij}a_{ji} = 2; \\ \frac{P(G_2)}{\frac{(n-1)(n-2)}{2}t^6 - (n-2)(t^5 + t^4 + t^3 + t^2 + t) + 1} & \text{if } \forall i \neq j, a_{ij}a_{ji} = 3; \\ \frac{1+t}{1 - (n-1)t} & \text{if } \forall i \neq j, a_{ij}a_{ji} \geq 4. \end{cases}$$

The computation in the proof of Theorem 1 is just a sample. The same method can

be used to give a complete computation results for all the Poincaré series of finite and affine types (untwisted case and twisted case). For various special kinds of Cartan matrices of indefinite type, the method is also useful to get the various kinds of formulas for Poincaré series. But there is not a single simple formula, so we don't discuss it further.

### 3 Some properties of the Poincaré series

It is apparent that two different Cartan matrices may have the same Poincaré series. The following theorem shows that for Cartan matrix of form (\*) without element 0, the corresponding Poincaré series is determined by the set  $S(A) := \{a_{ij} | 1 \leq j < i \leq n\}$  (with multiplicity).

**Theorem 2.** *For a Cartan matrix  $A$  of form (\*) with  $a_{ij} \neq 0, \forall i, j$ , the corresponding Poincaré series is invariant under exchanging of any two elements  $a_{ij}$  and  $a_{i'j'}$  below diagonal.*

**Proof:** set

$$g(a) := \begin{cases} \frac{(1-t^2)(1-t^3)}{(1-t)^2}, & a = -1; \\ \frac{(1-t^2)(1-t^4)}{(1-t)^2}, & a = -2; \\ \frac{(1-t^2)(1-t^6)}{(1-t)^2}, & a = -3; \\ \frac{1+t}{1-t}, & a = -4. \end{cases}$$

In fact,  $g(a)$  is the Poincaré series of Cartan matrix  $\begin{pmatrix} 2 & -1 \\ a & 2 \end{pmatrix}$ .

To prove the theorem, we only need to prove

**Assertion:** For a Cartan matrix  $A = (a_{ij})_{n \times n}$  without element 0,  $P(A)$  is a symmetric rational function of  $\{g(a_{ij}) | 1 \leq j < i \leq n\}$ .

Now, we start to prove this assertion.

Now we prove the assertion by induction on  $n$ . Notice that, if there is no element 0 in  $A$ , then all principal sub-matrices with order  $\geq 3$  in  $A$  are of indefinite type.

(i) For  $n = 3$ , suppose  $A = (a_{ij})_{3 \times 3}$  without element 0, by Equation (1.4), we have

$$P(A) - 3 \frac{P(A)}{1+t} + \frac{P(A)}{g(a_{21})} + \frac{P(A)}{g(a_{31})} + \frac{P(A)}{g(a_{32})} = 1.$$

hence

$$P(A) = \frac{(1+t)g(a_{21})g(a_{31})g(a_{32})}{(t-2)g(a_{21})g(a_{31})g(a_{32}) + (t+1)(g(a_{21})g(a_{31}) + g(a_{21})g(a_{32}) + g(a_{31})g(a_{32}))}.$$

So  $P(A)$  is a symmetric rational function of  $g(a_{21}), g(a_{31}), g(a_{32})$ .

(ii) Suppose the assertion hold for all  $k$  with  $3 \leq k < n$ , and  $A = (a_{ij})_{n \times n}$  be a Cartan matrix without element 0. By the induction assumption, for any  $I \subsetneq S$ ,  $P_I(A)$  is a symmetric rational function of  $\{g(a_{ij}) | i, j \in I, j < i\}$ . We define an action of  $S_{\frac{n(n-1)}{2}}$  (permutation group of  $\frac{n(n-1)}{2}$ -elements) on the set of rational functions  $g(a_{ij})$ 's by the action of  $S_n$  on index  $(i, j)$ . Then we get an action of  $S_{\frac{n(n-1)}{2}}$  on the rational functions of  $g(a_{ij})$ 's.

Note the set

$$\{ P_I(A) \mid |I| = k, I \subsetneq S, k < n \}$$

is invariant under the  $S_{\frac{n(n-1)}{2}}$  action. Therefore, for any  $k < n$ ,  $\sum_{|I|=k, I \subsetneq S} \frac{1}{P_I(A)}$  is a symmetric rational function of  $\{g(a_{ij}) \mid 1 \leq j < i \leq n\}$ . And by equation

$$(-1)^{|S|} + \sum_{I \subsetneq S} (-1)^{|I|} \frac{P(A)}{P_I(A)} = \sum_{I \subset S} (-1)^{|I|} \frac{P(A)}{P_I(A)} = 0$$

we get that  $P(A)$  is a symmetric rational function of  $\{g(a_{ij}) \mid 1 \leq j < i \leq n\}$ . This finishes the proof.  $\square$

The following example shows that the condition “there is no element 0 in  $A$ ” can’t be discarded.

**Example 1:** For Cartan matrices

$$A = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -2 & 0 & 2 & -1 \\ -3 & -3 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -2 & -3 & -3 & 2 \end{pmatrix}$$

their Poincaré series are

$$P(A) = \frac{(t^3 + 1)(t^3 + t^2 + t + 1)(t^2 + t + 1)(t + 1)}{2t^9 + t^8 - 3t^5 - 3t^4 - 2t^3 - 2t^2 - t + 1}$$

and

$$P(B) = \frac{(t^5 + t^4 + t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)}{2t^8 - t^7 + t^6 - 2t^5 - t^4 - 2t^3 - 2t + 1}.$$

Therefore even if the sets  $S(A) = \{a_{ij} \mid 1 \leq j < i \leq n\}$  and  $S(B) = \{b_{ij} \mid 1 \leq j < i \leq n\}$  (with multiplicity) are same, the Poincaré series are different because  $A$  and  $B$  contain element 0. If we replace  $(0, 0)$  in  $A, B$  by  $(-4, -1)$ , we get the new Cartan matrices

$$A' = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -2 & -4 & 2 & -1 \\ -3 & -3 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -4 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -2 & -3 & -3 & 2 \end{pmatrix},$$

and their Poincaré series are

$$P(A') = P(B') = \frac{(t^5 + t^4 + t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)}{2t^8 - 2t^7 - 2t^5 - 3t^4 - 2t^3 - t^2 - 2t + 1}$$

Due to the symmetry in Theorem 2, we see that the number of Poincaré series corresponding to all Cartan matrices without element 0 is much smaller than the number it seems to be. Denote the number of Poincaré series corresponding to order  $n$  Cartan matrices without element 0 by  $K(n)$ , then through Theorem 2 and some Combinatorial computation, we get  $K(n) \leq \binom{\frac{n(n-1)}{2}+3}{3}$ . This motivates us to make the following

**Conjecture 1:**  $K(n) = \binom{\frac{n(n-1)}{2}+3}{3}$ .

The conjecture is verified in cases  $n = 3, 4, 5, 6$  by figuring out all the  $K(n)$  Poincaré series.

Besides the above, there is another interesting property. It's shown in Gungormez and Karadayi[14] that for Poincaré series of hyperbolic Kac-Moody Lie algebras, their explicit forms seem to be the ratio of the Poincaré series of a properly chosen finite Lie algebra and a polynomial of finite degree. In the following, we will see this result can be extended to all Kac-Moody algebras.

**Theorem 3.** *For any Kac-Moody Lie algebras  $g$  with Cartan matrix  $A$ , the Poincaré series can be written as a rational function whose numerator is the least common multiple of those Poincaré series of all finite type principal sub-matrices of  $A$ .*

**Proof:** by Equation (1.4), we have

$$\frac{(-1)^{n+1}}{P(A)} = \sum_{I \subsetneq S} \frac{(-1)^{|I|}}{P_I(A)}.$$

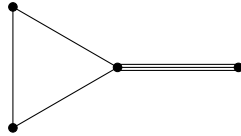
For those  $P_I(A)$ 's with affine or indefinite Cartan matrices  $A_I$ , substitute the item  $\frac{(-1)^I}{P_I(A)}$  by the same formula for  $A_I$ , do the same operation successively until the denominator of each item of the right side of the resulted formula is the Poincaré series of a finite type sub-matrix. This proves the theorem.  $\square$

This shows that the Poincaré series of a Kac-Moody Lie algebra is a rational function, and its numerator can be chosen to be the least common multiple of the Poincaré series of the finite type principal sub-matrix of  $A$ . By multiplying suitable polynomial factor if needed, the numerator can be the Poincaré series of a properly chosen finite type Lie algebra. Thus we have shown that for an infinite dimensional Kac-Moody Lie algebra  $g(A)$ , its Poincaré series can be written as the ratio of the Poincaré series  $P(A')$  of a properly chosen finite type Lie algebra  $g(A')$  and a denominator polynomial  $Q'$  of finite degree.

$$P(A) = \frac{P(A')}{Q'}$$

This proves a conjecture given by Gungormez and Karadayi[14].

**Example 2:** Take Cartan matrix  $A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$ , its Coxeter graph is

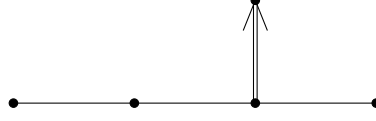


For  $A$  we see that all its finite type principal sub-matrices are  $A_1$ ,  $A_2$ ,  $G_2$ ,  $A_2 \oplus A_1$ , and  $A_1 \oplus A_1$ . Since the Poincaré series of  $G_2$  is the least common multiple of the Poincaré series of these Cartan matrices, the numerator of Poincaré series of  $A$  is just the Poincaré series of  $G_2$ .

By calculation, Poincaré series of  $A$  is:

$$\frac{(1+t)(1+t+t^2+t^3+t^4+t^5)}{2t^6 - t^5 - t^4 + t^3 - 2t + 1}$$

**Example 3:** For Cartan matrix corresponding to the Dynkin diagram as follows.



We see that all its finite principal sub-matrices are of type  $A_1, A_2, A_3, A_4, B_2, B_3, B_4, A_1 \oplus B_3, A_2 \oplus A_1 \oplus A_1, A_1 \oplus A_1 \oplus A_1, A_1 \oplus B_2, A_2 \oplus A_1$  and  $A_1 \oplus A_1$ . The least common multiple of their Poincaré series is  $\prod_{i=2,4,6,8,5} \frac{t^i - 1}{t - 1}$ , it is the Poincaré series of  $D_5$ . By calculation, Poincaré series of above Dynkin diagram is

$$\frac{(t^7 + t^6 + t^5 + t^4 + t^3 + t^2 + t + 1)(t^5 + t^4 + t^3 + t^2 + t + 1)(t^4 + t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)(t + 1)}{-t^{19} - t^{18} - t^{17} - t^{16} + t^{14} + t^{13} + 2t^{12} + 2t^{11} + 2t^{10} + t^9 + t^8 - t^7 - t^6 - 2t^5 - t^4 - t^3 + 1}$$

By the way, for case 1, 2 and 3 in Corollary 1, the numerators of Poincaré series are  $P(A_2), P(B_2)$  and  $P(G_2)$  respectively. For Cartan matrices in case 1, 2 and 3, their finite type principal sub-matrices are of type  $A_2, B_2$  and  $G_2$  respectively. And for case 4,  $1 + t$  is just the Poincaré series of  $A_1$ , the only finite type principal sub-matrix.

## 4 The Poincaré series of generalized flag manifolds

For each  $I \subset S$ , there is a parabolic subgroup  $G_I(A)$  corresponding to the Cartan matrix  $A_I$ . The inclusions  $B(A) \subset G_I(A) \subset G(A)$  induce a fibration

$$G_I(A)/B(A) \xrightarrow{i} G(A)/B(A) \xrightarrow{\pi} G(A)/G_I(A).$$

$G_I(A)/B(A) \cong F(A_I)$ , and  $G(A)/G_I(A)$  is called a generalized flag manifold. This gives a fibration  $F(A_I) \xrightarrow{i} F(A) \xrightarrow{\pi} G(A)/G_I(A)$ . The generalized flag manifold  $G(A)/G_I(A)$  also has a decomposition of Schubert cells which are indexed by the coset  $W(A)/W_I(A)$ . By Bernstein, Gel'fand and Gel'fand[13],  $W(A)/W_I(A)$  can be regarded as subset

$$W^I(A) = \{w \in W \mid \text{for all } w' \in W_I(A), l(w w') \geq l(w)\}.$$

of  $W(A)$ . The homomorphism  $i^* : H^*(F(A)) \rightarrow H^*(F(A_I))$  and  $\pi^* : H^*(G(A)/G_I(A)) \rightarrow H^*(F(A))$  have good properties. See [13] for details.

**Lemma 1:** 1.  $W(A) = W^I(A) \cdot W_I(A)$  and for each  $w \in W^I(A), w' \in W_I(A), l(w w') = l(w) + l(w')$ .

2. Schubert variety  $X_w, w \in W_I(A)$  forms an additive basis of  $H^*(F(A_I))$  and  $i^*$  is surjective. In fact, for each  $w \in W(A)$ , if  $w \in W_I(A)$ , then  $i^*(X_w) = X_w$ ; otherwise  $i^*(X_w) = 0$ . Here for  $w \in W_I(A)$ , we identify  $w \in W_I(A)$  with its image in  $W(A)$ .

3. Schubert variety  $X_w, w \in W^I(A)$  forms an additive basis of  $H^*(G(A)/G_I(A))$  and  $\pi^*$  is injective. In fact for each  $w \in W^I(A)$ ,  $\pi^*(X_w) = X_w$ ; Here we identify  $w$  with  $w W_I(A)$  for  $w \in W^I(A)$ .

As a consequence of the Lemma 1, we get

**Corollary 2:** The Leray-Serre spectral sequence of fibration  $F(A_I) \xrightarrow{i} F(A) \xrightarrow{\pi} G(A)/G_I(A)$  collapses at  $E_2$ -item and  $P(A) = P_I(A) \cdot P(G(A)/G_I(A))$ .

Hence if we have computed the Poincaré series of  $P(A)$  and  $P_I(A)$ , the Poincaré series of the generalized flag manifolds  $G(A)/G_I(A)$  is  $\frac{P(A)}{P_I(A)}$ .



## 5 Application to graph theory

The Poincaré series of Kac-Moody Lie algebras can be used to construct invariants of graphs.

**Definition 1:** Let  $\Gamma$  be an undirected graph without selfloop, suppose the set of vertices of  $\Gamma$  to be  $\{1, 2, \dots, n\}$  and the vertices are connected by edges with multiplicity. Let  $M(\Gamma)$  be the Coxeter matrix  $(m_{ij})_{n \times n}$  defined by  $m_{ij} = 1$  if  $i = j$ ;  $m_{ij} = 2, 3, 4, 6$  and  $\infty$  if the multiplicity of edge between vertices  $i, j$  is  $0, 1, 2, 3$  and  $\geq 4$  respectively. The Coxeter group  $W(\Gamma)$  of graph  $\Gamma$  is

$$W(\Gamma) = \langle \sigma_1, \dots, \sigma_n \mid (\sigma_i \sigma_j)^{m_{ij}} = 1, 1 \leq i, j \leq n \rangle.$$

The Poincaré-Coxeter invariant of  $\Gamma$  is defined as the Poincaré series of  $W(A)$ , denoted by  $P(\Gamma)$ .

According to the definition, the properties in the previous sections can be interpreted naturally as properties of the Poincaré-Coxeter invariants. For example,  $P(\Gamma)$  is a rational function of variable  $t$ ; If the multiplicity of an edge in  $\Gamma$  is great than 4, then changing the multiplicity to 4 doesn't alter the invariant; If any two vertices of  $\Gamma$  are connected by edges, then exchange of the multiplicities of edges doesn't alter the invariant;

The following definition is motivated by another work of the authors.

**Definition 2:** Let  $\Gamma$  be a graph as above, then the homotopy indices  $i_1, i_2, \dots, i_k, \dots$  are integers satisfying  $P(\Gamma) = \frac{1}{(1-t)^n} \prod_{k=1}^{\infty} (1-t^{2k-1})^{i_{2k-1}} (1-t^{2k})^{-i_{2k}}$ .

By results of the authors[15], the sequences  $i_k, k > 0$  is well defined and we have

The homotopy indices of the Coxeter graph of Kac-Moody groups of finite and affine types are tabulated as follows.

$\Gamma$	nonzero homotopy indices	$\tilde{\Gamma}$	nonzero homotopy indices
$A_n$	$i_k = 1, 2 \leq k \leq n+1$	$\tilde{A}_n$	$i_k = 1, k = n+1$
$B_n$	$i_k = 1, k = 2, 4, \dots, 2n$	$\tilde{B}_n$	$i_k = 1, k = 2, 3, 4, \dots, 2n-1, 2n;$
$D_{2n}$ $n \geq 2$	$i_k = 1, k = 2, 4, \dots,$ $4n-4, 4n-2, 2n$	$\tilde{D}_{2n}$ $n \geq 2$	$i_k = 1, k = 2, 3, 4, \dots, 2n-2;$ $i_k = 1, k = 2n, 2n+1, \dots, 4n-2;$ $i_{2n-1} = 2$
$D_{2n+1}$ $n \geq 2$	$i_k = 1, k = 2, 4, \dots,$ $4n-2, 4n, 2n+1$	$\tilde{D}_{2n+1}$ $n \geq 2$	$i_k = 1, k = 2, 3, 4, \dots, 2n-1;$ $i_k = 1, k = 2n+1, 2n+2, \dots, 4n;$
$G_2$	$i_k = 1, k = 2, 6$	$\tilde{E}_2$	$i_k = 1, k = 2, 5, 6$
$F_4$	$i_k = 1, k = 2, 6, 8, 12$	$\tilde{F}_4$	$i_k = 1, k = 2, 5, 6, 7, 8, 11, 12;$
$E_6$	$i_k = 1, k = 2, 5, 6, 8, 9, 12$	$\tilde{E}_6$	$i_k = 1, k = 2, 4, 6, 7, 9, 11, 12;$
$E_7$	$i_k = 1, k = 2, 6, 8, 10,$ $12, 14, 18$	$\tilde{E}_7$	$i_k = 1, k = 2, 5, 6, 7, 8, 9, 10, 11, 12, 13,$ $14, 17, 18$
$E_8$	$i_k = 1, k = 2, 8, 12, 14, 18,$ $20, 24, 30$	$\tilde{E}_8$	$i_k = 1, k = 2, 7, 8, 11, 12, 13, 14, 17, 18,$ $19, 20, 23, 24, 29, 30$

**Proposition 2:** The homotopy indices of  $\Gamma$  contain the same amount of information as the Poincaré-Coxeter invariant of graph  $\Gamma$ .

The homotopy indices of a Coxeter graph is important both for the research of combinatoric graph theory and for the research of algebraic topology of Kac-Moody groups. For example  $i_2$  is determined by the number of loops in  $\Gamma$ , see [5]. For more connection between

the homotopy indices and the rational homotopy groups of Kac-Moody groups, see [15].

So a natural question is

**Question 1:** Determine the homotopy indices  $i_1, i_2, \dots, i_k, \dots$  from the information of the graph  $\Gamma$ .

**Example 4:** For the graph  $\Gamma$  with 3 vertices and edges with multiplicities 2, 3, 4, the Poincaré-Coxeter invariant is

$$P(\Gamma) = \frac{(1-t^4)(1-t^6)}{(1-t)^2(-t^7-t^6-2t^5-t^4-2t^3-t+1)}.$$

Let  $g$  be the graded free Lie algebra over  $\mathbb{C}$  with generators  $x_1, x_3, y_3, x_4, x_5, y_5, x_6, x_7$  (set  $\deg x_i = \deg y_i = i$ ) and  $d_k$  be the dimension of the degree  $k$  homogeneous component of  $g$ . The homotopy indices of graph  $\Gamma$  be  $i_1 = 0, i_4 = d_4 - 1, i_6 = d_6 - 1$  and the other  $i_k = d_k$ .

## 6 Disproof of a conjecture of Victor Kac

In Kac[5], page 204, Victor Kac gave a conjecture on the Poincaré series of Kac-Moody Lie algebras of indefinite type.

To state the conjecture, we need the following definition.

**Definition 3:** An  $n \times n$  Cartan matrix  $A$  is called symmetrizable if there exists an invertible diagonal matrix  $D$  and a symmetric matrix  $B$  such that  $A = DB$ .  $g(A)$  is called a symmetrizable Kac-Moody Lie algebra if its Cartan matrix is symmetrizable.

Put  $\epsilon = 1$  or  $0$  according to  $A$  is symmetrizable or not and  $C(t) = P(A)(1-t)^n(1-t^2)^{-\epsilon}$ , Kac gave the following conjecture.

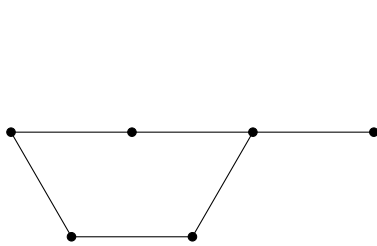
**Conjecture 2:**  $C(t) = \frac{1}{1-B(t)}$ , where  $B(t) = b_2t^2 + b_3t^3 + \dots +$  and  $b_i \geq 0$ .

We construct some counter examples to Kac's conjecture.

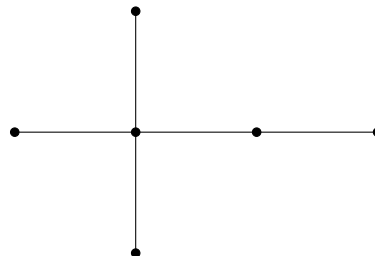
**Example 5:** For the Cartan matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

with Coxeter graphs as follows.



Coxeter graph of  $B$



Coxeter graph of  $A$

The Poincaré series are

$$P(A) = \frac{(1-t^2)(1-t^5)(1-t^6)(1-t^8)}{(1-t)^4(t^{16}-t^{15}-t^{11}+t^9-t^8+t^7+t^3-2t+1)}.$$

and

$$P(B) = \frac{(1-t^2)^2(1-t^4)(1-t^6)(1-t^8)}{(1-t)^5(t^{16}-t^{14}-t^{12}+t^8-t^7+2t^5+t^4-t^3-t^2-t+1)}.$$

$A$  is a symmetric matrix, so

$$C(t) = P(A) \cdot (1-t)^6 \cdot (1-t^2)^{-1}$$

From  $C(t) = \frac{1}{1-B(t)}$ ,  $B(t) = 1 - C(t)^{-1}$ , the Taylor expansion of

$$B(t) = 1 - \frac{(1-t)^4(t^{16}-t^{15}-t^{11}+t^9-t^8+t^7+t^3-2t+1)}{(1-t^2)(1-t^5)(1-t^6)(1-t^8)} \cdot \frac{1-t^2}{(1-t)^6}.$$

at  $t = 0$  is:

$$t^2 + t^3 + t^4 + t^7 + t^8 + t^9 - t^{15} - 2t^{16} - 2t^{17} - 3t^{18} - 3t^{19} + O(t^{20})$$

For Cartan matrix  $B$ , the Taylor expansion of

$$B(t) = 1 - \frac{(1-t)^5(t^{16}-t^{14}-t^{12}+t^8-t^7+2t^5+t^4-t^3-t^2-t+1)}{(1-t^2)^2(1-t^4)(1-t^6)(1-t^8)} \cdot \frac{1-t^2}{(1-t)^6}$$

at  $t = 0$  is:

$$2t^3 + t^5 - t^6 + 3t^7 - 2t^8 + 3t^9 - 2t^{10} + 5t^{11} - 4t^{12} + 4t^{13} - 5t^{14} + 8t^{15} - 8t^{16} + 6t^{17} - 10t^{18} + 10t^{19} + O(t^{20})$$

## 7 On the conjecture of Chapavalov, Leites and Stekolshchik

In their paper[16], page 208, Chapavalov, Leites and Stekolshchik made a conjecture on the Poincaré series of infinite dimensional Kac-Moody Lie algebras  $g(A)$ .

For a rational function  $P(t)$ , write  $P(t)$  as form  $P(t) = \frac{Q(t)}{R(t)}$ , define  $\deg P(t) = \deg Q(t) - \deg R(t)$ . It doesn't depend on how  $Q(t)$  and  $R(t)$  are chosen.

**Conjecture 3:** (Chapavalov-Leites-Stekolshchik) The Poincaré series of an infinite dimensional Kac-Moody Lie algebra  $g(A)$  with connected Coxeter graph satisfy  $0 \leq \deg P(A) \leq 1$ .

This conjecture is false for a finite type Lie algebra since in this case  $\deg P(A) =$  the complex dimension of  $F(A)$ .

The infinite dimensional Kac-Moody Lie algebras contains affine type and indefinite type. For affine type Conjecture 3 is true since in this case  $\deg P(A) = 0$ .

We need the following lemma for further discussion.

**Lemma 3:** For rational functions  $P_i(t), 1 \leq i \leq k$ ,  $\deg P_i(t) = d_i$ , suppose the sum of  $P_i(t), 1 \leq t \leq k$  be  $P(t)$ , then

1.  $\deg P(t) \leq \max_{1 \leq i \leq k} d_i$ .

2. If  $I = \{i | d_i = \max_{1 \leq t \leq k} d_t\}$  has only one element, then  $\deg P(t) = \max_{1 \leq i \leq k} d_i$ .

By using Lemma 3, we have

**Proposition 3:** The Poincaré series of an infinite dimensional Kac-Moody Lie algebra  $g(A)$  with connected Coxeter graph satisfy  $\deg P(A) \geq 0$ .

**Proof:** let  $A$  be a  $n \times n$  Cartan matrix of affine or indefinite type. For  $I \subsetneq S$ ,  $P_I(A)$  denotes the Poincaré series of  $A_I$ . We prove this proposition by induction on  $n$ . For  $n = 1$ , the proposition is obviously true. Assume for each Cartan matrix  $A'$  of affine or indefinite type with size  $< n$ , the proposition is true. By Equation (1.4), we have

$$-\frac{(-1)^n}{P(A)} = \sum_{I \subsetneq S} \frac{(-1)^{|I|}}{P_I(A)}$$

If  $I = \emptyset$ ,  $P_\emptyset(A) = 1$ ; If  $A_I(I \neq \emptyset)$  is of finite type, then  $\deg P_I(A) \geq 1$ . If  $A_I$  is of affine or indefinite type, then by induction assumption  $\deg P_I(A) \geq 0$ . Therefore for any  $I \subsetneq S$ , we have  $\deg P_I(A) \geq 0$ , so  $\deg \frac{(-1)^{|I|}}{P_I(A)} \leq 0$ . By Lemma 3,  $\deg \frac{(-1)^n}{P(A)} \leq 0$ , this shows  $\deg P(A) \geq 0$ .  $\square$

Let  $A$  be a Cartan matrix and  $I \subset S$ , denote by  $D_I$  the complex dimension of flag manifold  $F(A_I)$ . Put  $I_0 = S$ . For  $D = \infty$ , set  $t^D = 0$ .

By Equation (1.4)

$$\frac{(t^{D_{I_0}} + (-1)^{|I_0|+1})}{P(A)} = \sum_{I_1 \subsetneq S} \frac{(-1)^{|I_1|}}{P_{I_1}(A)}$$

We get

**Lemma 4:** 
$$\frac{1}{P(A)} = \sum_{S \supsetneq I_1} \frac{(-1)^{|I_1|}}{(t^{D_{I_0}} + (-1)^{|I_0|+1})} \frac{1}{P_{I_1}(A)}.$$

By induction we get

**Proposition 3:**

$$\frac{1}{P(A)} = \sum_{r=0}^{n-1} \sum_{S \supsetneq I_1 \supsetneq I_2 \supsetneq \dots \supsetneq I_r \supsetneq \emptyset} \left( \prod_{j=1}^r \frac{(-1)^{|I_j|}}{(t^{D_{I_{j-1}}} + (-1)^{|I_{j-1}|+1})} \right) \frac{1}{(t^{D_{I_r}} + (-1)^{|I_r|+1})} \quad (1.5)$$

Motivated by Equation (1.5), we give the following definition.

**Definition 4:** Let  $A$  be an  $n \times n$  Cartan matrix of affine or indefinite type,  $S = \{1, 2, \dots, n\}$ . A sequence  $T : S = I_0 \supsetneq I_1 \supsetneq I_2 \supsetneq \dots \supsetneq I_r \supsetneq \emptyset$  of subsets in  $S$  is called a chain of length  $r$ . If  $\dim g(A_{I_i}), 0 \leq i \leq r$  is infinite, the chain  $T$  is called an infinite chain. If  $\dim g(A_{I_i}), 0 \leq i \leq r-1$  is infinite and  $|I_r| = 1$ , the chain  $T$  is called a quasi-infinite chain. The set of all the chains (infinite chains and quasi-infinite chains respectively) is denoted by  $C(A)$  ( $C_\infty(A)$  and  $C_\infty^q(A)$  respectively).

Note  $C_\infty(A)$  and  $C_\infty^q(A)$  are disjoint.

Let  $D$  be the dimension function which send  $I \subset S$  to  $\dim F(A_I)$ . By Equation (1.5), we have

**Proposition 4:**  $1/P(A) = \sum_{T \in \mathcal{C}(A)} H_T$ , where for  $T : S = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r \supseteq \emptyset$ ,

$$H_T = \left( \prod_{j=1}^r \frac{(-1)^{|I_j|}}{(t^{D_{I_{j-1}}} + (-1)^{|I_{j-1}|+1})} \right) \frac{1}{(t^{D_{I_r}} + (-1)^{|I_r|+1})}.$$

This shows that the inverse power series  $\frac{1}{P(A)}$  of Poincaré series  $P(A)$  equals to the summation on the set of chains in  $S$ . The contribution of a chain is determined by the dimension function  $D$ .

**Theorem 4:** The Conjecture 3 is false for a Cartan matrix  $A$  if and only if for  $A$  both

$$\sum_{T \in \mathcal{C}_\infty(A)} H_T \text{ and } \sum_{T \in \mathcal{C}_\infty^q(A)} H_T \text{ equal to 0.}$$

**Proof:** The contribution of each chain  $T : S = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \emptyset$  to  $1/P(A)$  is  $H_T$ , with degree  $\leq 0$ . Note  $\deg H_T = 0$  if and only if  $T$  is a infinite chain and  $\deg H_T = -1$  if and only if  $T$  is a quasi-infinite chain. By using the fact  $H_T = (-1)^{-n-r-1}$  for infinite chain  $T$  and Lemma 3 we know: if  $\sum_{T \in \mathcal{C}_\infty(A)} H_T \neq 0$ , then  $\deg P(A) = 0$ . If  $\sum_{T \in \mathcal{C}_\infty(A)} H_T = 0$ , then the degree 0 items in summation cancel each other. A quasi-infinite chain  $T$  gives an item

$$H_T = \frac{(-1)^{|I_1|}}{(-1)^{|I_0|+1}} \frac{(-1)^{|I_2|}}{(-1)^{|I_1|+1}} \cdots \frac{(-1)^{I_r}}{(-1)^{|I_{r-1}|+1}} \frac{1}{t+1} = \frac{(-1)^{1-n-r}}{(1+t)}.$$

if

$$\sum_{T \in \mathcal{C}_\infty^q(A)} H_T = \sum_{T \in \mathcal{C}_\infty^q(A)} \frac{(-1)^{1-n-r}}{(1+t)} \neq 0.$$

By Lemma 3, we have  $\deg \frac{1}{P(A)} = -1$ , hence  $\deg P(A) = 1$ . □

By the way we get the following criteria for  $\deg P(A) = 0$  or 1. Let  $K_0(A) = \sum_{T \in \mathcal{C}_\infty(A)} H_T$  and  $K_1(A) = \sum_{T \in \mathcal{C}_\infty^q(A)} H_T$ , we have

**Proposition 5:** Let  $A$  be a Cartan matrix of affine or indefinite type with connected Coxeter graph. If  $K_0(A) \neq 0$ , the degree of the Poincaré series  $P(A)$  is 0, and the ratio of the coefficients of the highest power of  $t$  in the numerator and denominator polynomials of  $P(A)$  is  $1 : K_0(A)$ ; If  $K_0(A) = 0$ , but  $K_1(A) \neq 0$ , the degree of the Poincaré series  $P(A)$  is 1, and the ratio of the coefficients of the highest power of  $t$  in the numerator and denominator polynomials of  $P(A)$  is  $1 : K_1(A)$ .

**Remark:** Besides  $K_0(A)$  and  $K_1(A)$ , we can also construct invariants  $K_2(A), K_3(A), \dots, K_k(A), \dots$  of  $A$  as in obstruction theory, such that  $\deg P(A) = k$  if and only if  $K_0(A), K_1(A), \dots, K_{k-1}(A) = 0$ , but  $K_k(A) \neq 0$  and the ratio of the coefficients of the highest power of  $t$  in the numerator and denominator polynomials of  $P(A)$  is  $1 : K_k(A)$ . But things become more and more complicate as  $k$  increases, so we give up the explicit computation of  $K_k(A)$  for  $k > 1$ .

**Example 6:** For rank 3 Cartan matrix  $A$ , suppose the multiplicities of edges between three vertices be  $p_1, p_2, p_3$ , then the only infinite chains are  $S = \{1, 2, 3\}$  and  $S \supset S - \{i\}, p_i \geq 4$ , hence  $K_0(A) = 0$  if and only if there is exact one pair of vertices connected by edge with multiplicity  $\geq 4$ . In this case, there are 3 length 1 quasi-infinite chains  $S \supseteq \{i\}, 1 \leq i \leq 3$  and

2 length 2 quasi-infinite chains, so  $K_1(A) = -1$ . Therefore the Conjecture 3 is true for  $n = 3$ . The ratio of the coefficients of the highest power of  $t$  in the numerator and denominator polynomials of  $P(A)$  can be computed from  $K_0(A)$  and  $K_1(A)$ .

For  $n = 4$ , there are counter examples to the Conjecture 3.

**Example 7:** For Cartan matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -4 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -2 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & -1 \\ -1 & -4 & 2 & -1 \\ 0 & -2 & -2 & 2 \end{pmatrix}$$

Their Poincaré series are

$$P(A) = -\frac{(t^5 + t^4 + t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)(t + 1)}{t^5 + t^4 + t^3 + t^2 + t - 1}.$$

and

$$P(B) = -\frac{(t^5 + t^4 + t^3 + t^2 + t + 1)(t^3 + t^2 + t + 1)(t + 1)}{t^7 + t^6 + 2t^5 + 2t^4 + 2t^3 + 2t^2 + t - 1}.$$

with degree 4 and 2.

We don't have examples to show that the degree of Poincaré series can be arbitrary large and don't know if there exists any upper bound.

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