THE INTEGRAL COHOMOLOGY RING OF $E_8/T^1 \cdot E_7$

MASAKI NAKAGAWA*

Abstract. We determine the integral cohomology ring of the homogeneous space $E_8/T^1 \cdot E_7$ by the Borel presentation and a method due to Toda. Then using the Gysin exact sequence associated with the circle bundle $S^1 \to E_8/E_7 \to E_8/T^1 \cdot E_7$, we also determine the integral cohomology of E_8/E_7.

1. Introduction

Let G be a compact connected Lie group and H a centralizer of a toral subgroup. Then the homogeneous space G/H is called a *generalized flag manifold*.[1] It is a natural generalization of a (full or complete) flag manifold G/T, where T is a maximal torus of G, since the centralizer of a maximal torus is T itself. It is also expressed in the form $G_{\mathbb{C}}/P$, where $G_{\mathbb{C}}$ denotes the complexification of G and P a parabolic subgroup of $G_{\mathbb{C}}$. Furthermore, it is realized as an orbit of the coadjoint representation of G on $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$ of G. From these realizations, a generalized flag manifold G/H has many interesting geometric properties. In fact, G/H admits a complex structure, a Kähler structure and a symplectic structure, and therefore plays an important role in algebraic topology, differential geometry, and algebraic geometry.

In algebraic topology, it is a classical problem to determine the integral cohomology ring $H^*(G/H;\mathbb{Z})$ of a generalized flag manifold G/H, and many mathematicians tried to find the method to compute it. Here, we briefly review the history of determination of the cohomology ring of G/H; The first notable result is due to Borel. In his thesis [4], Borel described the rational cohomology ring $H^*(G/H;\mathbb{Q})$ of G/H (as of all homogeneous spaces G/H, with H of maximal rank in G) in terms of the rings of invariants of the Weyl groups of G and H (see also §3). This description of $H^*(G/H;\mathbb{Q})$ is called the *Borel presentation*. However, Borel's result does not hold in general when G or H has torsion. Using the Morse theory, Bott showed that G/H is free of torsion, i.e., $H^*(G/H;\mathbb{Z})$ is a free $\mathbb{Z}$-module, and its odd Betti numbers vanish ([7, Theorem A]).[2] Furthermore, there is an algorithm for constructing the ring $H^*(G/H;\mathbb{Z})$ in terms of the Cartan integers ([9, Theorem III"]). So the problem of computing the integral cohomology ring of a generalized flag manifold G/H could be solved in principle. However, in practice, it becomes unmanageable to carry out the above algorithm. After that, another new technique was introduced by Baum [2]. He used the Eilenberg-Moore spectral sequence converging to

2000 *Mathematics Subject Classification.* Primary 57T15; Secondary 57T10.

Key words and phrases. cohomology, Lie groups, homogeneous spaces, flag manifolds.

* Partially supported by the Grant-in-Aid for Scientific Research (C) 21540104, Japan Society for the Promotion of Science.

[1]Generalized flag manifolds exhaust all compact, simply connected homogeneous Kähler manifolds as was shown by Wang [45] (see also [5], [6, 13.5]). For this reason, they are sometimes called *Kählerian C-spaces*.

[2]This fact also follows from the cell decomposition of $G/H \cong G_{\mathbb{C}}/P$ by *Schubert cells* ([12]).

$H^*(G/H;k)$, where k is a field or the integers. Under certain hypothesis, he showed that the Eilenberg-Moore spectral sequence of the fibration $G/H \longrightarrow BH \longrightarrow BG$ collapses, where BH (resp. BG) denotes the classifying space of H (resp. G). However, this collapse theorem does not give the ring structure of $H^*(G/H;k)$ unless we solve the so-called *extension problem* of a spectral sequence (see [33, Chapter 8]). Although these general methods are quite useful when we discuss general properties of the cohomology ring of G/H, it seems difficult to apply them directly to some homogeneous spaces of exceptional Lie groups, especially of E_8. Using the Borel presentation of the rational cohomology ring $H^*(G/H;\mathbb{Q})$ and the results on mod p cohomology rings $H^*(G;\mathbb{Z}/p\mathbb{Z})$ for all primes p, Toda initiated the research for computing the integral cohomology ring of G/H with H torsion free ([42]). Along the line of his idea, the integral cohomology rings of various flag manifolds have been computed explicitly ([43], [46], [27], [26], [47], [36]).[3] Toda's method is quite useful in practical computations. For instance, based on Toda's result, Kono and Ishitoya computed the mod 2 cohomology ring $H^*(E_8/T;\mathbb{Z}/2\mathbb{Z})$ explicitly ([31]),[4] and Totaro computed the $\mathbb{Z}_{(2)}$-cohomology ring $H^*(E_8/A_8;\mathbb{Z}_{(2)})$, where A_8 denotes the subgroup of E_8 isomorphic to $SU(9)/(\mathbb{Z}/3\mathbb{Z})$. Using these results, Totaro succeeded in computing the *torsion index*[5] of E_8 ([44]). Finally, we should mention another direction towards determination of $H^*(G/H;\mathbb{Z})$; As mentioned before, there is a description of $H^*(G/H;\mathbb{Z})$ in terms of the cell decomposition of $G/H \cong G_{\mathbb{C}}/P$ by Schubert cells. This originated in the work of Ehresmann [19] on the cell decomposition of a complex Grassman manifold, and was later extended to general flag manifolds by Chevalley [12]. This description of $H^*(G/H;\mathbb{Z})$ is called the *Schubert presentation*. In this description, the additive structure of $H^*(G/H;\mathbb{Z})$ is described in terms of the so-called *Schubert classes* indexed by certain subset of the Weyl group of G (see [3, §5]). As for the multiplicative structure, we have to compute the *structure constants* for the multiplication of Schubert classes (see for example [40]). This is one of the main problems of the *Schubert calculus*.[6] In this direction, many authors studied the Chow rings of generalized flag manifolds ([13], [25], [38], [39]).[7] Recently Duan developed extensively a multiplicative rule of Schubert classes which is a generalization of the Littlewood-Richardson rule of a complex Grassmann manifold ([15]). Furthermore, he and Zhao computed the integral cohomology rings of the above flag manifolds independently of Toda's method ([16], [17], [18]).[8]

Until recently none of these methods have been successful in computing the *integral* cohomology rings of homogeneous spaces of the exceptional Lie group E_8.[9] The group E_8 contains a closed connected subgroup $T^1 \cdot E_7$ whose local type is $T^1 \times E_7$, where T^1 is a one dimensional torus (see [27, §2]). It is obtained as the centralizer of a certain one dimensional torus (see 2.1). Hence the homogeneous space $E_8/T^1 \cdot E_7$

[3]Some of these results contain minor errors that are corrected in [16], [29].

[4]Their result also contains an error that was corrected in [44].

[5]The torsion index is an integer associated to any compact connected Lie group defined by Grothendieck [22].

[6]In the case of a complex Grassman manifold, the multiplicative rule of Schubert classes is known as the *Littlewood-Richardson rule* (see [20]).

[7]The Chow ring $A(G/H)$ of G/H is canonically isomorphic to $H^*(G/H;\mathbb{Z})$ via the cycle map ([22, §6]), [21, Chapter 19]).

[8]The relations between their results and ours are revealed in [29] via the *divided difference operators* due to Bernstein-Gelfand-Gelfand [3] and Demazure [14].

[9]Partial result was obtained by Nikolenko and Semenov [38].

is a generalized flag manifold. In this paper, using the above method due to Borel and Toda, we compute the integral cohomology ring of $E_8/T^1\cdot E_7$ explicitly.[10]

The motivation of the current work is not only the determination of the integral cohomology ring $H^*(E_8/T^1\cdot E_7;\mathbb{Z})$ itself, but the cohomology of the irreducible symmetric space $EIX = E_8/S^3\cdot E_7$ (see [23, TABLE V], [27, §2]), as well as the integral cohomology ring of the full flag manifold E_8/T. The symmetric space EIX is a *quaternionic Kähler manifold* and $E_8/T^1\cdot E_7$ is the *twister space* of EIX ([41]). Using the Gysin exact sequence associated with the 2-sphere bundle $S^2 \longrightarrow E_8/T^1\cdot E_7 \longrightarrow EIX$, we can compute the integral cohomology ring of EIX.[11] Using Theorem 4.7 of this paper, we also compute $H^*(E_8/T;\mathbb{Z})$, which is the only remaining case among G/T's for G simple, in our forthcoming paper ([37]). In [30], the result of $H^*(E_8/T;\mathbb{Z})$ will be used to compute the Chow ring $A(\mathrm{E}_8)$ of the complex algebraic group E_8 (For the Chow rings of complex algebraic groups, see [22], [28], [32]). Moreover, the homogeneous space $E_8/T^1\cdot E_7$ is a *generating variety* of E_8 in the sense of Bott [8] and its integral cohomology ring is needed to compute the Pontrjagin ring $H_*(\Omega E_8;\mathbb{Z})$, where ΩE_8 denotes the based loop space of E_8.

The paper is organized as follows: In §2, we compute the rings of invariants of the Weyl groups of E_8 and $T^1\cdot E_7$ which are needed for the computations of the rational cohomology ring of $E_8/T^1\cdot E_7$. In §3, using these results and the Borel presentation of a rational cohomology ring, we compute the rational cohomology ring of $E_8/T^1\cdot E_7$. In §4, by investigating the integral cohomology ring of E_8/T in low degrees, we determine the integral cohomology ring of $E_8/T^1\cdot E_7$ explicitly (Theorem 4.7). Furthermore, in §5, using the Gysin exact sequence associated with the circle bundle $S^1 \longrightarrow E_8/E_7 \longrightarrow E_8/T^1\cdot E_7$, we also determine the integral cohomology of E_8/E_7 (Corollary 5.1).

Acknowledgments. We thank Mamoru Mimura for many useful suggestions and advice during the preparation of the manuscript. Thanks are also due to Haibao Duan and Xhezi Zhao, who informed us their result on the Schubert presentation of $H^*(E_8/T^1\cdot E_7;\mathbb{Z})$. Finally, we thank Tetsu Nishimoto and Shizuo Kaji for valuable discussions. The main part of this manuscript was obtained around 2007. I would like to apologize for not posting it sooner.

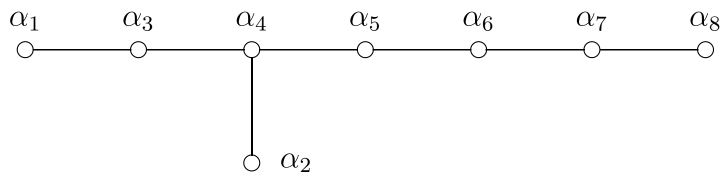
2. Rings of invariants of Weyl groups

In this section, we compute the rings of invariants of the Weyl groups of E_8 and $T^1\cdot E_7$ over $\mathbb{Q}$ explicitly that are needed for the computation of the rational cohomology ring of $E_8/T^1\cdot E_7$.

2.1. **Notations.** Let T be a fixed maximal torus of E_8. According to [10], the Dynkin diagram of E_8 is as follows:

[10] Duan and Zhao also computed $H^*(E_8/T^1\cdot E_7;\mathbb{Z})$ in terms of Schubert classes ([16, Theorem 7].

[11] The real cohomology rings of quaternionic Kähler manifolds are studied in [35].



where α_i $(1 \leq i \leq 8)$ are the simple roots.[12] As usual we may regard each root as an element of $H^1(T;\mathbb{Z})$[13], and therefore an element of $H^2(BT;\mathbb{Z})$ via the negative *transgression* (see [6, §10]).

Let C_8 be the centralizer of a one dimensional torus determined by $\alpha_i = 0$ $(i \neq 8)$ in $\mathfrak{t}$. Then as shown in [27, §2],

$$C_8 = T^1 \cdot E_7 \quad \text{and} \quad T^1 \cap E_7 \cong \mathbb{Z}/2\mathbb{Z}.$$

The Weyl groups of E_8 and C_8 are respectively given as follows:

$$W(E_8) = \langle s_i \ (1 \leq i \leq 8) \rangle, \quad W(C_8) = \langle s_i \ (i \leq i \leq 7) \rangle,$$

where s_i denotes the simple reflection corresponding to the simple root α_i.

Let $\{\omega_i\}_{1 \leq i \leq 8}$ be the fundamental weights corresponding to the system of the simple roots $\{\alpha_i\}_{1 \leq i \leq 8}$. We also regard each weight as an elememt of $H^2(BT;\mathbb{Z})$. Then $\{\omega_i\}_{1 \leq i \leq 8}$ forms a basis of $H^2(BT;\mathbb{Z})$ and $H^*(BT;\mathbb{Z}) = \mathbb{Z}[\omega_1, \omega_2, \ldots, \omega_8]$. The action of s_i's on $\{\omega_i\}_{1 \leq i \leq 8}$ is given as follows:

$$s_i(\omega_k) = \begin{cases} \omega_i - \displaystyle\sum_{j=1}^{8} \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)} \omega_j & \text{if} \quad k = i, \\ \omega_k & \text{if} \quad k \neq i, \end{cases}$$

where $(\,\cdot\,|\,\cdot\,)$ denotes the $W(E_8)$-invariant inner product on $\mathfrak{t}^*$, the dual of $\mathfrak{t}$.

Now we introduce the elements of $H^2(BT;\mathbb{Z})$ as follows (Throughout this paper, $\sigma_i(x_1, \ldots, x_n)$ denotes the i-th elementary symmetric function in the variables $x_1, \ldots, x_n$):

$$\begin{aligned} t_8 &= \omega_8, \\ t_7 &= s_8(t_8) = \omega_7 - \omega_8, \\ t_6 &= s_7(t_7) = \omega_6 - \omega_7, \\ t_5 &= s_6(t_6) = \omega_5 - \omega_6, \\ t_4 &= s_5(t_5) = \omega_4 - \omega_5, \\ t_3 &= s_4(t_4) = \omega_2 + \omega_3 - \omega_4, \\ t_2 &= s_3(t_3) = \omega_1 + \omega_2 - \omega_3, \\ t_1 &= s_1(t_2) = -\omega_1 + \omega_2, \end{aligned}$$

[12] In topology, a *root* α is a real linear form on $\mathfrak{t}$, the Lie algebra of T ([1, Chapter 4]. It is customary that $2\pi\sqrt{-1}\alpha$, regarded as a complex linear form on $\mathfrak{t}_{\mathbb{C}}$, the complexification of $\mathfrak{t}$, is a *root* in the theory of complex Lie algebras.

[13] Let $\exp : \mathfrak{t} \longrightarrow T$ be the exponetial map. The *integrer lattice* (or *unit lattice*) Γ is defined as the inverse image of the unit e of T under exp ([1, 4.11]). A real linear form α on $\mathfrak{t}$ is said to be *integral* if it takes integral values on Γ ([6, 1.2]). Then we may identify the integral real linear forms on $\mathfrak{t}$ with $\mathrm{Hom}(\Gamma, \mathbb{Z}) \cong H^1(T;\mathbb{Z})$. Notice that each root takes integral values on Γ by definition.

$$c_i = \sigma_i(t_1, \dots, t_8),$$
$$t = \omega_2 = \frac{1}{3}c_1.$$

Then t and t_i's span $H^2(BT;\mathbb{Z})$, since each ω_i is an integral linear combination of t and t_i's, and we have the following isomorphism:

$$H^*(BT;\mathbb{Z}) = \mathbb{Z}[t_1, \dots, t_8, t]/(c_1 - 3t).$$

The elements $\{t_i\}_{1\le i\le 8}$ and t behave nicely under the action of the Weyl group $W(E_8)$ of E_8. In fact, the action of s_i $(1 \le i \le 8)$ on $\{t_i\}_{1\le i\le 8}$ and t is given by Table 1, where blanks indicate the trivial action.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
t_1	t_2	$t - t_2 - t_3$						
t_2	t_1	$t - t_1 - t_3$	t_3					
t_3		$t - t_1 - t_2$	t_2	t_4				
t_4				t_3	t_5			
t_5					t_4	t_6		
t_6						t_5	t_7	
t_7							t_6	t_8
t_8								t_7
t		$2t - t_1 - t_2 - t_3$						

Table 1.

Next we introduce a basis of $H^2(BT;\mathbb{Q})$ which behaves nicely under the action of the Weyl group $W(C_8)$ of C_8:

$$u = t_8,$$
$$\tau = t - \frac{3}{2}u,$$
$$\tau_i = t_i - \frac{1}{2}u \ (1 \le i \le 7).$$

Then we have

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[u, \tau, \tau_1, \dots, \tau_7]/(\bar{c}_1 - 3\tau) = \mathbb{Q}[u, \tau_1, \dots, \tau_7]$$

for $\bar{c}_1 = \tau_1 + \cdots + \tau_7$. The action of s_i $(1 \le i \le 7)$ on $\{\tau_i\}_{1\le i\le 7}$ and τ is given by Table 2, where blanks also indicate the trivial action.

Since $E_7 \cap T = T'$ is a maximal torus of E_7, we have a commutative diagram of natural maps:

$$(2.1) \qquad \begin{array}{ccccc} E_7/T' & \xrightarrow{\sim} & C_8/T & \xrightarrow{i} & E_8/T \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_0} \\ BT' & \xrightarrow{g} & BT & \xrightarrow{=} & BT. \end{array}$$

Since E_8 is 2-connected, $\iota_0^* : H^2(BT;\mathbb{Z}) \longrightarrow H^2(E_8/T;\mathbb{Z})$ is an isomorphism. Under this isomorphism, we denote the ι_0^*-images of t_i $(1 \le i \le 8)$ and t by the same letters. Thus we have the generators t_i $(1 \le i \le 8)$ and t of $H^2(E_8/T;\mathbb{Z})$ with a relation

	s_1	s_2	s_3	s_4	s_5	s_6	s_7
τ_1	τ_2	$\tau-\tau_2-\tau_3$					
τ_2	τ_1	$\tau-\tau_1-\tau_3$	τ_3				
τ_3		$\tau-\tau_1-\tau_2$	τ_2	τ_4			
τ_4				τ_3	τ_5		
τ_5					τ_4	τ_6	
τ_6						τ_5	τ_7
τ_7							τ_6
τ		$-\tau+\tau_4+\tau_5+\tau_6+\tau_7$					
u							

TABLE 2.

$c_1 = 3t$. We donote the generators t_i $(1 \leq i \leq 7)$ and x in [46, §1] by t'_i $(1 \leq i \leq 7)$ and t' respectively. Then, by a similar argument to that in [46, §1], we have

$$g^*(t_i) = t'_i \ (1 \leq i \leq 7),\ g^*(t_8) = 0,\ g^*(t) = t' \tag{2.2}$$

under the homomorphism $g^* : H^2(BT;\mathbb{Z}) \longrightarrow H^2(BT';\mathbb{Z})$.

2.2. **Ring of invariants of $W(C_8)$.** In this subsection, we compute the ring of invariants of the Weyl group $W(C_8)$ explicitly. First we recall the rational invariant forms for E_7 given in [46, §2] (see also [34, 2.2]). We put

$$x'_i = 2t'_i - t' \ (1 \leq i \leq 7) \quad \text{and} \quad x'_8 = t'.$$

Then the set

$$S' = \{\pm\,(x'_i + x'_j)\ (1 \leq i < j \leq 8)\} \subset H^2(BT';\mathbb{Q})$$

is invariant under the action of $W(E_7)$. Thus we have $W(E_7)$-invariant forms

$$I'_n = \sum_{y \in S'} y^n \in H^{2n}(BT';\mathbb{Q})^{W(E_7)} \quad (n \geq 0). \tag{2.3}$$

Then direct computation using the method as in [46, §2] yields the following results:

$$\begin{aligned}
I'_2 &= -2^5 \cdot 3(c'_2 - 4t'^2),\\
I'_6 &\equiv 2^8 \cdot 3^2({c'_3}^2 + 8c'_6) \mod (t', \mathfrak{a}'_6),\\
I'_8 &\equiv 2^{12} \cdot 5(2{c'_4}^2 - 3c'_3c'_5) \mod (t', \mathfrak{a}'_8),\\
I'_{10} &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7({c'_5}^2 - 4c'_3c'_7) \mod (t', \mathfrak{a}'_{10}),\\
I'_{12} &\equiv 2^{15} \cdot 3^2 \cdot 5(-54{c'_6}^2 + 18c'_5c'_7 - c'_3c'_4c'_5) \mod (t', \mathfrak{a}'_{12}),\\
I'_{14} &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2{c'_7}^2 + 2c'_3c'_4c'_7 - c'_3c'_5c'_6) \mod (t', \mathfrak{a}'_{14}),\\
I'_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126c'_5c'_6c'_7 - 5c'_3c'_4c'_5c'_6) \mod (t', \mathfrak{a}'_{18}),
\end{aligned} \tag{2.4}$$

where $c'_i = \sigma_i(t'_1, \ldots, t'_7)$ and $\mathfrak{a}'_i$ denotes the ideal of $H^*(BT';\mathbb{Q})$ generated by I'_j for $j < i$ with $j \in \{2, 6, 8, 10, 12, 14, 18\}$. We also recall the following result:

Proposition 2.1 ([46], Lemma 2.1, [34], 2.2)**.** *The ring of invariants of the Weyl group $W(E_7)$ over $\mathbb{Q}$ is given as follows*:

$$H^*(BT';\mathbb{Q})^{W(E_7)} = \mathbb{Q}[I'_2, I'_6, I'_8, I'_{10}, I'_{12}, I'_{14}, I'_{18}].$$

TABLE 2 shows that the action of $W(C_8)$ on $\tau, \tau_1, \ldots, \tau_7$ is the same as that of $W(E_7)$ on $t', t'_1, \ldots, t'_7$. Therefore if we put

$$\chi_i = 2\tau_i - \tau \ (1 \le i \le 7) \quad \text{and} \quad \chi_8 = \tau,$$

the set

$$\Sigma = \{\pm(\chi_i + \chi_j) \ (1 \le i < j \le 8)\} \subset H^2(BT; \mathbb{Q})$$

is invariant under the action of $W(C_8)$. We define $W(C_8)$-invariant forms J_n $(n \ge 0)$ as

$$J_n = \sum_{y \in \Sigma} y^n \in H^{2n}(BT; \mathbb{Q})^{W(C_8)}.$$

Then, by Proposition 2.1, we have

Lemma 2.2. *The ring of invariants of the Weyl group $W(C_8)$ over $\mathbb{Q}$ is given as follows*:

$$H^*(BT; \mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}].$$

2.3. **Ring of invariants of $W(E_8)$.** In this subsection, we compute the ring of invariants of the Weyl group $W(E_8)$ over $\mathbb{Q}$ explicitly. According to Chevalley [11], this ring of invariants is generated by 8 algebraically independent polynomials (basic invariants) $f_1, \ldots, f_8$ of degrees $2, 8, 12, 14, 18, 20, 24, 30$ ([24, p.59]). We will give these invariant polynomials explicitly.

We put

$$\xi_i = 2t_i - \frac{2}{3}t \ (1 \le i \le 8) \quad \text{and} \quad \xi_9 = -\frac{2}{3}t.$$

Then the action of s_i $(1 \le i \le 8)$ on $\{\xi_i\}_{1 \le i \le 9}$ is given by TABLE 3, where $\eta = (\xi_1 + \xi_2 + \xi_3)/3$.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
ξ_1	ξ_2	$\xi_1 - 2\eta$						
ξ_2	ξ_1	$\xi_2 - 2\eta$	ξ_3					
ξ_3		$\xi_3 - 2\eta$	ξ_2	ξ_4				
ξ_4		$\xi_4 + \eta$		ξ_3	ξ_5			
ξ_5		$\xi_5 + \eta$			ξ_4	t_6		
ξ_6		$\xi_6 + \eta$				ξ_5	ξ_7	
ξ_7		$\xi_7 + \eta$					ξ_6	ξ_8
ξ_8		$\xi_8 + \eta$						ξ_7
ξ_9		$\xi_9 + \eta$						

TABLE 3.

From TABLE 3, we see that the set

$$S = \{\pm\,(\xi_i - \xi_j)\ (1 \le i < j \le 9),\ \pm\,(\xi_i + \xi_j + \xi_k)\ (1 \le i < j < k \le 9)\} \subset H^2(BT; \mathbb{Q})$$

is invariant under the action of $W(E_8)$.[14] Thus we have $W(E_8)$-invariant forms[15]

$$I_n = \sum_{y \in S} y^n \in H^{2n}(BT; \mathbb{Q})^{W(E_8)} \quad (n \ge 0).$$

[14] In fact, S is an orbit of $2\omega_8$ under the action of $W(E_8)$. Since ω_8 is equal to the highest root $\tilde{\alpha}$ ([10]), it turns out that S is 2 times the root system of E_8.

[15] For a compact connected Lie group G, the set of roots is nothing but the set of non-zero weights of the adjoint representation of G on $\mathfrak{g}$. Therefore I_n is equal to the n-th Chern character of the adjoint representation of E_8 up to the multiple of 2.

Let us compute I_n's in the following way; We put

$$s_n = \xi_1^n + \cdots + \xi_9^n, \quad d_n = \sigma_n(\xi_1, \ldots, \xi_9).$$

Then s_n's and d_n's are related to each other by the Newton formula:

$$s_n = \sum_{i=1}^{n-1} (-1)^{i-1} s_{n-i} d_i + (-1)^{n-1} n d_n \ \ (d_n = 0 \ \text{ for } \ n > 9). \tag{2.5}$$

Note that $s_0 = 9$ and

$$s_1 = d_1 = \sum_{i=1}^{9} \xi_i = \sum_{i=1}^{8} \left(2t_i - \frac{2}{3}t\right) - \frac{2}{3}t = 2(c_1 - 3t) = 0.$$

Then

$$\begin{aligned}
\sum_{n\geq 0} \frac{I_n}{n!} &= \sum_{1\leq i<j\leq 9} e^{\xi_i - \xi_j} + \sum_{1\leq i<j\leq 9} e^{-\xi_i + \xi_j} + \sum_{1\leq i<j<k\leq 9} e^{\xi_i + \xi_j + \xi_k} + \sum_{1\leq i<j<k\leq 9} e^{-\xi_i - \xi_j - \xi_k} \\
&= \left(\sum_{i=1}^{9} e^{\xi_i}\right)\left(\sum_{j=1}^{9} e^{-\xi_j}\right) - 9 + \frac{1}{3}\left(\sum_{i=1}^{9} e^{3\xi_i} + \sum_{i=1}^{9} e^{-3\xi_i}\right) \\
&- \frac{1}{2}\left\{\left(\sum_{i=1}^{9} e^{\xi_i}\right)\left(\sum_{i=1}^{9} e^{2\xi_i}\right) + \left(\sum_{i=1}^{9} e^{-\xi_i}\right)\left(\sum_{i=1}^{9} e^{-2\xi_i}\right)\right\} \\
&+ \frac{1}{6}\left\{\left(\sum_{i=1}^{9} e^{\xi_i}\right)^3 + \left(\sum_{i=1}^{9} e^{-\xi_i}\right)^3\right\} \\
&= \left(\sum_{n\geq 0} \frac{s_n}{n!}\right)\left(\sum_{m\geq 0} \frac{(-1)^m s_m}{m!}\right) - 9 + \frac{1}{3}\left(\sum_{n\geq 0} \frac{3^n s_n}{n!} + \sum_{n\geq 0} \frac{(-1)^n 3^n s_n}{n!}\right) \\
&- \frac{1}{2}\left\{\left(\sum_{n\geq 0} \frac{s_n}{n!}\right)\left(\sum_{m\geq 0} \frac{2^m s_m}{m!}\right) + \left(\sum_{n\geq 0} \frac{(-1)^n s_n}{n!}\right)\left(\sum_{m\geq 0} \frac{(-1)^m 2^m s_m}{m!}\right)\right\} \\
&+ \frac{1}{6}\left\{\left(\sum_{n\geq 0} \frac{s_n}{n!}\right)^3 + \left(\sum_{n\geq 0} \frac{(-1)^n s_n}{n!}\right)^3\right\}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
I_n &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} s_i s_{n-i} + 2 \cdot 3^{n-1} s_n - \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} s_i s_{n-i} \\
&+ \frac{1}{3} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} s_i s_j s_{n-i-j}
\end{aligned} \tag{2.6}$$

for n even. On the other hand, since $\xi_i = 2t_i - \frac{2}{3}t$ $(1 \le i \le 8)$ and $\xi_9 = -\frac{2}{3}t$, we have

$$\sum_{n=0}^{9} d_n = \prod_{i=1}^{9}(1+\xi_i) = \left(1 - \frac{2}{3}t\right)\prod_{i=1}^{8}\left(1 - \frac{2}{3}t + 2t_i\right)$$
$$= \left(1 - \frac{2}{3}t\right)\sum_{i=0}^{8}\left(1 - \frac{2}{3}t\right)^{8-i} 2^i c_i$$

and therefore

$$d_n = \sum_{i=0}^{n}\left\{\binom{8-i}{n-i} + \binom{8-i}{n-i-1}\right\}\left(-\frac{2}{3}t\right)^{n-i} 2^i c_i \tag{2.7}$$

for $1 \le n \le 9$. Using (2.6), (2.5) and (2.7), I_n can be expressed as a polynomial in t and c_i $(2 \le i \le 8)$, and we obtain the following results:

Lemma 2.3.

(i) $I_2 = -2^5 \cdot 3 \cdot 5(c_2 - 4t^2)$,

(ii) $I_8 \equiv 2^{14} \cdot 3 \cdot 5\{-18c_8 - 3c_3c_5 + 2c_4^2 + t(12c_7 - 3c_3c_4) + t^2(-6c_6 + 3c_3^2) + 12t^3c_5 + 2t^4c_4 - 12t^5c_3 + 14t^8\} \mod \tilde{\mathfrak{a}}_8$,

(iii) $I_{12} \equiv 2^{18} \cdot 3^5 \cdot 7\left(c_6^2 - \frac{5}{3}c_5c_7 + \frac{5}{54}c_3c_4c_5 - \frac{1}{6}c_3^2c_6 + \frac{1}{24}c_3^4\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{12})$,

(iv) $I_{14} \equiv 2^{20} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11\left(c_7^2 - \frac{1}{2}c_3c_5c_6 + \frac{1}{3}c_3c_4c_7 + \frac{1}{6}c_4c_5^2\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{14})$,

(v) $I_{18} \equiv -2^{23} \cdot 3^2 \cdot 5 \cdot 7 \cdot 13\left(c_3^6 - 7c_3^4c_6 + \frac{29}{9}c_3^3c_4c_5 + 182c_3^2c_5c_7 + 75c_3c_5^3 - \frac{476}{3}c_3c_4c_5c_6 - 24c_5c_6c_7\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{18})$,

(vi) $I_{20} \equiv 2^{27} \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 17 \cdot 41\left(\frac{1}{144}c_5^4 - \frac{1}{18}c_3c_5^2c_7 - \frac{1}{54}c_3^2c_4c_5^2 - \frac{1}{27}c_3^3c_4c_7 + \frac{1}{18}c_3^3c_5c_6\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{20})$,

(vii) $I_{24} \equiv 2^{32} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 19 \cdot 199\left(\frac{31}{8640}c_3^5c_4c_5 + \frac{1}{480}c_3^4c_5c_7 + \frac{337}{25920}c_3^3c_5^3 - \frac{71}{4320}c_3^3c_4c_5c_6 + \frac{31}{240}c_3^2c_5c_6c_7 + \frac{31}{480}c_3c_5^3c_6 - \frac{22}{135}c_3c_4c_5^2c_7 - \frac{1}{120}c_4c_5^4\right)$ $\mod (t, c_8, \tilde{\mathfrak{a}}_{20})$,

(viii) $I_{30} \equiv 2^{38} \cdot 3^4 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 61\left(-\frac{599}{51840}c_3^5c_4c_5c_6 + \frac{47}{34560}c_3^5c_5^3 + \frac{1519}{25920}c_3^4c_5c_6c_7 + \frac{6293}{7290}c_3^3c_4c_5^2c_7 - \frac{32537}{25920}c_3^3c_5^3c_6 + \frac{189919}{466560}c_3^2c_4c_5^4 + \frac{2012}{1215}c_3c_4c_5^2c_6c_7 - \frac{16693}{25920}c_3c_5^4c_7 - \frac{223}{6480}c_4c_5^4c_6 - \frac{1}{1728}c_5^6\right) \mod (t, c_8, \tilde{\mathfrak{a}}_{20})$, *where* $c_i = \sigma_i(t_1, \ldots, t_8)$ *and* $\tilde{\mathfrak{a}}_i$ *denotes the ideal of* $H^*(BT;\mathbb{Q})$ *generated by* I_j *for* $j < i$ *with* $j \in \{2, 8, 12, 14, 18, 20, 24, 30\}$.

Next consider the following elements of $H^*(BT;\mathbb{Q})^{W(C_8)} \subset H^*(BT;\mathbb{Q})$:

$$
\begin{aligned}
\tilde{I}_{20} &= 9u^{20} + 45u^{14}v + 12u^{10}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2, \\
\tilde{I}_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2 \\
&\quad +30u^2v^2w + 5v^4, \\
\tilde{I}_{30} &= -9u^{30} - 24u^{24}v - 12u^{20}w + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 \\
&\quad +120u^8v^2w - 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3,
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
u &= t_8, \\
v &= \frac{1}{46080}J_6 - \frac{273}{640}u^6, \\
w &= \frac{1}{15482880}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10}.
\end{aligned}
\tag{2.9}
$$

Remark 2.4. *By the classical result of Borel* [4]*, the rational cohomology ring of* E_8/C_8 *is described in terms of the rings of invariants of the Weyl groups*:

$$H^*(E_8/C_8;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^{W(C_8)}/(H^+(BT;\mathbb{Q})^{W(E_8)})$$

(*see* §*3*). *Under this isomorphism, the above elements* u, v, w *represent the rational cohomology classes in* $H^*(E_8/C_8;\mathbb{Q})$. *Moreover, the integral cohomology ring of* E_8/C_8 *is torsion free* ([7]), *and hence, is contained in the rational cohomology ring*: $H^*(E_8/C_8;\mathbb{Z}) \hookrightarrow H^*(E_8/C_8;\mathbb{Q})$. *Then the elements* u, v, w *are in fact the integral cohomology classes in* $H^*(E_8/C_8;\mathbb{Z})$ (*For details, see 4.2*).

We wish to find the relations among $W(C_8)$-invariants J_n, $W(E_8)$-invariants I_n and $\tilde{I}_{20}$, $\tilde{I}_{24}$, $\tilde{I}_{30}$. For this purpose, we consider the ring of invariants:

$$A = H^*(BT;\mathbb{Q})^{\langle s_1, s_3, \ldots, s_7\rangle}.$$

Then A is a subalgebra of $H^*(BT;\mathbb{Q})$ containing both $H^*(BT;\mathbb{Q})^{W(C_8)}$ and $H^*(BT;\mathbb{Q})^{W(E_8)}$. More explicitly, we have

$$A = \mathbb{Q}[u, t, c_2, \ldots, c_7]. \tag{2.10}$$

In fact, we can show (2.10) as follows; Putting $\tilde{c}_i = \sigma_i(t_1, \ldots, t_7)$, we have

$$c_n = \tilde{c}_n + u\tilde{c}_{n-1} \ (1 \leq n \leq 8),$$

since

$$\sum_{n=0}^{8} c_n = \prod_{i=1}^{8}(1 + t_i) = (1 + u)\prod_{i=1}^{7}(1 + t_i) = (1 + u)\sum_{n=0}^{7}\tilde{c}_n.$$

Conversely, one can express

$$\tilde{c}_n = c_n - uc_{n-1} + u^2c_{n-2} - \cdots + (-1)^n u^n \ (1 \leq n \leq 7).$$

In particular, the following relation holds:

$$c_8 = uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^6c_2 + u^7c_1 - u^8. \tag{2.11}$$

Therefore, by Table 1, we have

$$\begin{aligned}
A &= H^*(BT;\mathbb{Q})^{\langle s_1,s_3,\ldots,s_7\rangle} \\
&= \mathbb{Q}[t_1,t_2,\ldots,t_7,u]^{\langle s_1,s_3,\ldots,s_7\rangle} \\
&= \mathbb{Q}[u,\tilde{c}_1,\tilde{c}_2,\ldots,\tilde{c}_7] \\
&= \mathbb{Q}[u,c_1,c_2,\ldots,c_7] \\
&= \mathbb{Q}[u,t,c_2,\ldots,c_7],
\end{aligned}$$

which has shown (2.10).

Denote by

$$\mathfrak{a}_i \subset A \text{ (resp. } \mathfrak{b}_i \subset H^*(BT;\mathbb{Q})^{W(C_8)}),$$

the ideal of A (resp. of $H^*(BT;\mathbb{Q})^{W(C_8)}$) generated by I_j's for $j < i$ where $j \in \{2,8,12,14,18,20,24,30\}$.

The remainder of this section is devoted to proving the next lemma:

Lemma 2.5. *In $H^*(BT;\mathbb{Q})^{W(C_8)} = \mathbb{Q}[u,J_2,J_6,J_8,J_{10},J_{12},J_{14},J_{18}]$, we have*

(i)

$$\begin{aligned}
I_2 &= 5J_2 + 120u^2, \\
I_8 &= 2^2\cdot 3J_8 + (\mathit{decomp.}), \\
I_{12} &= -2^2\cdot 7J_{12} + (\mathit{decomp.}), \\
I_{14} &= \frac{2^3\cdot 3\cdot 5^2}{29}J_{14} + (\mathit{decomp.}), \\
I_{18} &= -\frac{2^4\cdot 3\cdot 5^2\cdot 7\cdot 13}{1229}J_{18} + (\mathit{decomp.}),
\end{aligned}$$

where (decomp.) means decomposable elements of $\mathbb{Q}[u,J_2,J_6,J_8,J_{10},J_{12},J_{14},J_{18}]$.

(ii)

$$\begin{aligned}
I_{20} &\equiv 2^{27}\cdot 3^2\cdot 5^2\cdot 11\cdot 17\cdot 41\tilde{I}_{20} \mod \mathfrak{b}_{20}, \\
I_{24} &\equiv 2^{32}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 19\cdot 199\tilde{I}_{24} \mod \mathfrak{b}_{20}, \\
I_{30} &\equiv 2^{35}\cdot 3^4\cdot 5^5\cdot 7\cdot 11\cdot 13\cdot 61\tilde{I}_{30} \mod \mathfrak{b}_{20}.
\end{aligned}$$

Suppose Lemma 2.5 for the moment. Then we have the following (see also [34, 2.3]):

Lemma 2.6. *The ring of invariants of the Weyl group $W(E_8)$ over $\mathbb{Q}$ is given as follows*:

$$H^*(BT;\mathbb{Q})^{W(E_8)} = \mathbb{Q}[I_2,I_8,I_{12},I_{14},I_{18},I_{20},I_{24},I_{30}].$$

Proof. By Lemma 2.5, I_i ($i = 2,8,12,14,18,20,24,30$) are algebraically independent. On the other hand, by the result of Borel [4], the ring of invariants $H^*(BT;\mathbb{Q})^{W(E_8)}$ is isomorphic to the rational cohomology ring $H^*(BE_8;\mathbb{Q})$ of the classifying space BE_8, and it is known that

$$H^*(BE_8;\mathbb{Q}) \cong \mathbb{Q}[y_4,y_{16},y_{24},y_{28},y_{36},y_{40},y_{48},y_{60}]$$

with deg $(y_i) = i$. Therefore we have the required result. □

Proof of Lemma 2.5 (i). Since $\tau_i = t_i - \frac{1}{2}u$ $(1 \le i \le 7)$ and $\tau = t - \frac{3}{2}u$, we have

$$\tau_i \equiv t_i \ (1 \le i \le 7), \ \tau \equiv t \mod (u).$$

Therefore, putting $\bar{c}_i = \sigma_i(\tau_1, \dots, \tau_7)$ $(1 \le i \le 7)$, we obtain

$$c_n \equiv \bar{c}_n \ (1 \le n \le 7), \ c_8 \equiv 0 \mod (u). \tag{2.12}$$

Since J_n has the same expression as I'_n by simply replacing t', c'_i with τ, $\bar{c}_i$ (see 2.2), we have, by (2.4) and (2.12),

$$\begin{aligned}
J_2 &= -2^5 \cdot 3(\bar{c}_2 - 4\tau^2) \\
&\equiv -2^5 \cdot 3c_2 \mod (t, u), \\
J_6 &\equiv 2^8 \cdot 3^2(\bar{c}_3^2 + 8\bar{c}_6) \mod (\tau, \bar{\mathfrak{a}}_6) \\
&\equiv 2^8 \cdot 3^2(c_3^2 + 8c_6) \mod (t, u, \bar{\mathfrak{a}}_6), \\
J_8 &\equiv 2^{12} \cdot 5(2\bar{c}_4^2 - 3\bar{c}_3\bar{c}_5) \mod (\tau, \bar{\mathfrak{a}}_8) \\
&\equiv 2^{12} \cdot 5(2c_4^2 - 3c_3c_5) \mod (t, u, \bar{\mathfrak{a}}_8), \\
J_{10} &\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(\bar{c}_5^2 - 4\bar{c}_3\bar{c}_7) \mod (\tau, \bar{\mathfrak{a}}_{10}) \\
&\equiv 2^{12} \cdot 3^2 \cdot 5 \cdot 7(c_5^2 - 4c_3c_7) \mod (t, u, \bar{\mathfrak{a}}_{10}), \\
J_{12} &\equiv 2^{15} \cdot 3^2 \cdot 5(-54\bar{c}_6^2 + 18\bar{c}_5\bar{c}_7 - \bar{c}_3\bar{c}_4\bar{c}_5) \mod (\tau, \bar{\mathfrak{a}}_{12}) \\
&\equiv 2^{15} \cdot 3^2 \cdot 5(-54c_6^2 + 18c_5c_7 - c_3c_4c_5) \mod (t, u, \bar{\mathfrak{a}}_{12}), \\
J_{14} &\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2\bar{c}_7^2 + 2\bar{c}_3\bar{c}_4\bar{c}_7 - \bar{c}_3\bar{c}_5\bar{c}_6) \mod (\tau, \bar{\mathfrak{a}}_{14}) \\
&\equiv 2^{16} \cdot 3 \cdot 7 \cdot 11 \cdot 29(2c_7^2 + 2c_3c_4c_7 - c_3c_5c_6) \mod (t, u, \bar{\mathfrak{a}}_{14}), \\
J_{18} &\equiv 2^{21} \cdot 5 \cdot 1229(-126\bar{c}_5\bar{c}_6\bar{c}_7 - 5\bar{c}_3\bar{c}_4\bar{c}_5\bar{c}_6) \mod (\tau, \bar{\mathfrak{a}}_{18}) \\
&\equiv 2^{21} \cdot 5 \cdot 1229(-126c_5c_6c_7 - 5c_3c_4c_5c_6) \mod (t, u, \bar{\mathfrak{a}}_{18}),
\end{aligned} \tag{2.13}$$

where $\bar{\mathfrak{a}}_i$ denotes the ideal of $H^*(BT;\mathbb{Q})^{W(C_8)}$ generated by J_j's for $j < i$ with $j \in \{2, 6, 8, 10, 12, 14, 18\}$.

Now we prove the last formula of (i). Since $I_{18} \in H^*(BT;\mathbb{Q})^{W(E_8)} \subset H^*(BT;\mathbb{Q})^{W(C_8)} = \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]$, we can put

$$I_{18} = \alpha_{18}J_{18} + (\text{decomp.}) \tag{2.14}$$

for some $\alpha_{18} \in \mathbb{Q}$. On the other hand, by using (2.10) and (2.13), we have

$$\begin{aligned}
A/(t, u, \bar{\mathfrak{a}}_{18}) &= A/(t, u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}) \\
&= \mathbb{Q}[u, t, c_2, c_3, c_4, c_5, c_6, c_7] \\
&\qquad \Bigg/ \begin{pmatrix} t, \ u, \ c_2, \ c_3^2 + 8c_6, \ c_4^2 - \frac{3}{2}c_3c_5, \ c_5^2 - 4c_3c_7, \\ c_6^2 - \frac{1}{3}c_5c_7 + \frac{1}{54}c_3c_4c_5, \ c_7^2 + c_3c_4c_7 - \frac{1}{2}c_3c_5c_6 \end{pmatrix} \\
&= \mathbb{Q}[c_3, c_4, c_5, c_6, c_7] \\
&\qquad \Bigg/ \begin{pmatrix} c_3^2 + 8c_6, \ c_4^2 - \frac{3}{2}c_3c_5, \ c_5^2 - 4c_3c_7, \\ c_6^2 - \frac{1}{3}c_5c_7 + \frac{1}{54}c_3c_4c_5, \ c_7^2 + c_3c_4c_7 - \frac{1}{2}c_3c_5c_6 \end{pmatrix}.
\end{aligned}$$

We consider (2.14) in the ring $A/(t, u, \bar{\mathfrak{a}}_{18})$. Then, by Lemma 2.3 (v) and (2.13), we have

$$I_{18} \equiv -2^{25} \cdot 3 \cdot 5^3 \cdot 7 \cdot 13(-126c_5c_6c_7 - 5c_3c_4c_5c_6),$$
$$J_{18} \equiv 2^{21} \cdot 5 \cdot 1229(-126c_5c_6c_7 - 5c_3c_4c_5c_6).$$

Therefore $\alpha_{18} = -2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 13/1229$. A similar tedious computation gives the other formulas. ☐

Before proceeding the proof of Lemma 2.5 (ii), we need the following lemma:

Lemma 2.7. *Explicit forms of $W(C_8)$-invariants J_6, J_{10} are given as follows*:

$$\begin{aligned}
J_6 = {} & 2^8 \cdot 3\{24\bar{c}_6 + 3\bar{c}_3^2 - 4\bar{c}_2\bar{c}_4 - 2\bar{c}_2^3 + (-12\bar{c}_5 - 6\bar{c}_2\bar{c}_3)\tau + (31\bar{c}_2^2 + 16\bar{c}_4)\tau^2 \\
& + 12\bar{c}_3\tau^3 - 136\bar{c}_2\tau^4 + 188\tau^6\}, \\
J_{10} = {} & 2^{12} \cdot 3\{105\bar{c}_5^2 - 420\bar{c}_3\bar{c}_7 + 90\bar{c}_2\bar{c}_3\bar{c}_5 - 60\bar{c}_2\bar{c}_4^2 + 300\bar{c}_2^2\bar{c}_6 + 15\bar{c}_2^2\bar{c}_3^2 \\
& - 20\bar{c}_2^3\bar{c}_4 - 2\bar{c}_2^5 + (-30\bar{c}_2^3\bar{c}_3 - 330\bar{c}_2^2\bar{c}_5 + 480\bar{c}_2\bar{c}_7 + 90\bar{c}_2\bar{c}_3\bar{c}_4 - 210\bar{c}_4\bar{c}_5)\tau \\
& + (270\bar{c}_2^2\bar{c}_4 - 210\bar{c}_2\bar{c}_3^2 - 150\bar{c}_3\bar{c}_5 - 2220\bar{c}_2\bar{c}_6 + 75\bar{c}_2^4 + 345\bar{c}_4^2)\tau^2 \\
& + (480\bar{c}_2{}^2\bar{c}_3 - 570\bar{c}_3\bar{c}_4 + 2070\bar{c}_2\bar{c}_5 - 660\bar{c}_7)\tau^3 \\
& + (-1050\bar{c}_2\bar{c}_4 + 4080\bar{c}_6 - 950\bar{c}_2^3 + 705\bar{c}_3^2)\tau^4 + (-2250\bar{c}_2\bar{c}_3 - 3420\bar{c}_5)\tau^5 \\
& + (1580\bar{c}_4 + 5165\bar{c}_2^2)\tau^6 + 2820\bar{c}_3\tau^7 - 12360\bar{c}_2\tau^8 + 10868\tau^{10}\},
\end{aligned}$$

where $\bar{c}_i = \sigma_i(\tau_1, \dots, \tau_7)$.

Proof. Since J_n has the same expression as I'_n by replacing t', c'_i with $\tau, \bar{c}_i$ (see 2.2), we have to compute the $W(E_7)$-invariant forms I'_6, I'_{10} explicitly. But this can be done from the data in [46, §2]. ☐

In order to show Lemma 2.5 (ii), we need to describe the elements v, w (see (2.9)) in the ring A, in other words, in terms of t, u, c_i $(2 \leq i \leq 7)$. First will rewrite J_6, J_{10} in terms of t, u, c_i $(2 \leq i \leq 7)$; Since $u = t_8$ and $\tau_i = t_i - \frac{1}{2}u$ $(1 \leq i \leq 7)$, we have

$$\begin{aligned}
\left(1 + \frac{1}{2}u\right)\sum_{n=0}^{7} \bar{c}_n &= \left(1 + \frac{1}{2}u\right)\prod_{i=1}^{7}(1 + \tau_i) = \left(1 + \frac{1}{2}u\right)\prod_{i=1}^{7}\left(1 - \frac{1}{2}u + t_i\right) \\
&= \prod_{i=1}^{8}\left(1 - \frac{1}{2}u + t_i\right) = \sum_{i=0}^{8}\left(1 - \frac{1}{2}u\right)^{8-i} c_i,
\end{aligned}$$

and hence

$$\bar{c}_n + \frac{1}{2}u\bar{c}_{n-1} = \sum_{i=0}^{n}\binom{8-i}{n-i}\left(-\frac{1}{2}\right)^{n-i} c_i u^{n-i} \quad (1 \leq n \leq 7).$$

From this, we obtain

$$
(2.15)\quad
\begin{aligned}
\bar{c}_1 &= 3t - \frac{9}{2}u,\\
\bar{c}_2 &= c_2 - 12tu + \frac{37}{4}u^2,\\
\bar{c}_3 &= c_3 - \frac{7}{2}c_2u + \frac{87}{4}tu^2 - \frac{93}{8}u^3,\\
\bar{c}_4 &= c_4 - 3c_3u + \frac{11}{2}c_2u^2 - 24tu^3 + \frac{163}{16}u^4,\\
\bar{c}_5 &= c_5 - \frac{5}{2}c_4u + 4c_3u^2 - \frac{21}{4}c_2u^3 + \frac{297}{16}tu^4 - \frac{219}{32}u^5,\\
\bar{c}_6 &= c_6 - 2c_5u + \frac{11}{4}c_4u^2 - \frac{13}{4}c_3u^3 + \frac{57}{16}c_2u^4 - \frac{45}{4}tu^5 + \frac{247}{64}u^6,\\
\bar{c}_7 &= c_7 - \frac{3}{2}c_6u + \frac{7}{4}c_5u^2 - \frac{15}{8}c_4u^3 + \frac{31}{16}c_3u^4 - \frac{63}{32}c_2u^5 + \frac{381}{64}tu^6\\
&\quad -\frac{255}{128}u^7.
\end{aligned}
$$

By Lemma 2.7 and (2.15), we can express J_6, J_{10} (and hence v, w) in terms of t, u, c_i $(2 \leq i \leq 7)$. In fact, we need only the expressions modulo certain elements (see (2.17) below); By Lemma 2.3 (ii) and (2.11), we have

$$
\begin{aligned}
I_8 &\equiv 2^{14}\cdot 3\cdot 5(2c_4^2 - 3c_3c_5) \mod (t, c_8, \mathfrak{a}_8),\\
c_8 &= uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^6c_2 + u^7c_1 - u^8\\
&\equiv uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 - u^8 \mod (t, \mathfrak{a}_8),
\end{aligned}
$$

and hence

$$
(2.16)\quad
\begin{aligned}
c_4^2 &\equiv \frac{3}{2}c_3c_5 \mod (t, c_8, \mathfrak{a}_{12}),\\
u^8 &\equiv uc_7 - u^2c_6 + u^3c_5 - u^4c_4 + u^5c_3 \mod (t, c_8, \mathfrak{a}_8).
\end{aligned}
$$

Therefore, by (2.9), Lemma 2.7, (2.15) and (2.16), we obtain

$$
(2.17)\quad
\begin{aligned}
v &= \frac{1}{46080}J_6 - \frac{273}{640}u^6\\
&\equiv \frac{2}{5}c_6 + \frac{1}{20}c_3^2 - \frac{1}{2}c_5u + \frac{1}{3}c_4u^2 - \frac{1}{2}c_3u^3 \mod (t, \mathfrak{a}_8),\\
w &= \frac{1}{15482880}J_{10} - \frac{55}{24}u^4v - \frac{666919}{645120}u^{10}\\
&\equiv \frac{1}{12}c_5^2 - \frac{1}{3}c_3c_7 + \left(\frac{1}{2}c_3c_6 - \frac{1}{6}c_4c_5\right)u - \frac{1}{6}c_3c_5u^2 + \left(-c_7 + \frac{1}{3}c_3c_4\right)u^3\\
&\quad -\frac{1}{2}c_3^2u^4 + \frac{1}{3}c_4u^6 + \frac{1}{2}c_3u^7 \mod (t, c_8, \mathfrak{a}_{12}).
\end{aligned}
$$

Under these preparations, we will prove Lemma 2.5 (ii).

Proof of Lemma 2.5 (ii). First note that

$$
\begin{aligned}
H^*(BT;\mathbb{Q})^{W(C_8)} &= \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]\\
&= \mathbb{Q}[u, I_2, v, I_8, w, I_{12}, I_{14}, I_{18}]
\end{aligned}
$$

by (i) and (2.9). Since $I_{20} \in H^*(BT;\mathbb{Q})^{W(E_8)} \subset H^*(BT;\mathbb{Q})^{W(C_8)}$, we can put

$$(*) \quad I_{20} \equiv 2^{27}\cdot 3^3\cdot 5^2\cdot 11\cdot 17\cdot 41(\lambda_1 u^{20}+\lambda_2 u^{14}v+\lambda_3 u^8 v^2+\lambda_4 u^2 v^3 + \lambda_5 u^{10}w+\lambda_6 u^4 vw+\lambda_7 w^2) \mod \mathfrak{b}_{20}$$

for some $\lambda_i \in \mathbb{Q}$. In order to determine the coefficients λ_i, we need the following lemma, which is directly verified by making use of (2.10) and Lemma 2.3:

Lemma 2.8.

$$\begin{aligned} A/(t,c_8,\mathfrak{a}_{20}) &= A/(t,I_2,c_8,I_8,I_{12},I_{14},I_{18}) \\ &= \mathbb{Q}[u,c_3,c_4,c_5,c_6,c_7]/J, \end{aligned}$$

where J is the ideal generated by

$$u^8-uc_7+u^2c_6-u^3c_5+u^4c_4-u^5c_3,$$
$$c_4^2-\frac{3}{2}c_3c_5,$$
$$c_6^2-\frac{5}{3}c_5c_7+\frac{5}{54}c_3c_4c_5-\frac{1}{6}c_3^2c_6+\frac{1}{24}c_3^4,$$
$$c_7^2-\frac{1}{2}c_3c_5c_6+\frac{1}{3}c_3c_4c_7+\frac{1}{6}c_4c_5^2,$$
$$c_3^6-7c_3^4c_6+\frac{29}{9}c_3^3c_4c_5+182c_3^2c_5c_7+75c_3c_5^3-\frac{476}{3}c_3c_4c_5c_6-24c_5c_6c_7.$$

In particular, $A/(t,c_8,\mathfrak{a}_{20})$ has a basis $\{u^ic_3^jc_4^kc_5^lc_6^mc_7^n\ (0\le i\le 7,\ 0\le j\le 5,\ 0\le l,\ 0\le k,m,n\le 1)\}$ as a $\mathbb{Q}$-vector space.

Now we consider the relation $(*)$ in the ring $A/(t,c_8,\mathfrak{a}_{20})$. By Lemma 2.3 (vi), we have

$$I_{20}\equiv 2^{27}\cdot 3^3\cdot 5^2\cdot 11\cdot 17\cdot 41\left(\frac{1}{144}c_5^4-\frac{1}{18}c_3c_5^2c_7-\frac{1}{54}c_3^2c_4c_5^2-\frac{1}{27}c_3^3c_4c_7+\frac{1}{18}c_3^3c_5c_6\right).$$

On the other hand, using (2.17) and Lemma 2.8, we can rewrite each monomial in the right hand side of $(*)$. For example, we have

$$\begin{aligned} w^2 \equiv{}& \frac{1}{144}c_5^4-\frac{1}{18}c_3c_5^2c_7-\frac{1}{54}c_3^2c_4c_5^2-\frac{1}{27}c_3^3c_4c_7+\frac{1}{18}c_3^3c_5c_6-\frac{1}{12}u^7c_3^2c_7 \\ &+\frac{1}{12}u^7c_3c_5^2+\frac{1}{24}u^6c_3^3c_5+\frac{1}{3}u^6c_3c_5c_6-\frac{5}{9}u^6c_3c_4c_7+\cdots. \end{aligned}$$

Then, using the second half of Lemma 2.8, the coefficients in $(*)$ are obtained as follows:

$$\lambda_1=3,\ \lambda_2=15,\ \lambda_3=20,\ \lambda_4=\frac{10}{3},\ \lambda_5=4,\ \lambda_6=10,\ \lambda_7=1.$$

Thus we have obtained

$$\begin{aligned} I_{20} &\equiv 2^{27}\cdot 3^3\cdot 5^2\cdot 11\cdot 17\cdot 41\left(3u^{20}+15u^{14}v+20u^8v^2+\frac{10}{3}u^2v^3+4u^{10}w \right.\\ &\quad \left. +10u^4vw+w^2\right) \\ &\equiv 2^{27}\cdot 3^2\cdot 5^2\cdot 11\cdot 17\cdot 41(9u^{20}+45u^{14}v+60u^8v^2+10u^2v^3+12u^{10}w \\ &\quad +30u^4vw+3w^2) \\ &\equiv 2^{27}\cdot 3^2\cdot 5^2\cdot 11\cdot 17\cdot 41\tilde{I}_{20} \mod \mathfrak{b}_{20}. \end{aligned}$$

Putting

$$(**)\quad I_{24} \equiv 2^{32}\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 199\,(\mu_1 u^{24}+\mu_2 u^{18}v+\mu_3 u^{12}v^2+\mu_4 u^6 v^3 + \mu_5 v^4+\mu_6 u^{14}w+\mu_7 u^8 vw+\mu_8 u^2v^2w+\mu_9 u^4w^2) \mod \mathfrak{b}_{20}$$

for some $\mu_i \in \mathbb{Q}$, we will proceed quite similarly. By Lemma 2.3 (vii), we have

$$I_{24} \equiv 2^{32}\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 199\left(\frac{31}{8640}c_3^5c_4c_5+\frac{1}{480}c_3^4c_5c_7+\frac{337}{25920}c_3^3c_5^3 -\frac{71}{4320}c_3^3c_4c_5c_6+\frac{31}{240}c_3^2c_5c_6c_7+\frac{31}{480}c_3c_5^3c_6-\frac{22}{135}c_3c_4c_5^2c_7-\frac{1}{120}c_4c_5^4\right).$$

On the other hand, in $A/(t, c_8, \mathfrak{a}_{20})$, we have, for example,

$$v^4 \equiv \frac{31}{8640}c_3^5c_4c_5+\frac{1}{480}c_3^4c_5c_7+\frac{337}{25920}c_3^3c_5^3-\frac{71}{4320}c_3^3c_4c_5c_6+\frac{31}{240}c_3^3c_5c_6c_7 +\frac{31}{480}c_3c_5^3c_6-\frac{22}{135}c_3c_4c_5^2c_7-\frac{1}{120}c_4c_5^4+\frac{11}{16}u^7c_3^4c_5-u^7c_3^2c_5c_6 -\frac{11}{18}u^7c_3c_4c_5^2+\frac{9}{160}u^6c_3^4c_6-\frac{619}{480}u^6c_3^3c_4c_5+\cdots.$$

Then using the second half of Lemma 2.8, the coefficients in $(**)$ are obtained as follows:

$$\mu_1=\frac{11}{5},\ \mu_2=12,\ \mu_3=21,\ \mu_4=12,\ \mu_5=1,\ \mu_6=\frac{21}{5},\ \mu_7=12,\ \mu_8=6,\ \mu_9=\frac{9}{5}.$$

Thus we have obtained

$$\begin{aligned} I_{24} &\equiv 2^{32}\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 19\cdot 199\left(\frac{11}{5}u^{24}+12u^{18}v+21u^{12}v^2+12u^6v^3 +v^4+\frac{21}{5}u^{14}w+12u^8vw+6u^2v^2w+\frac{9}{5}u^4w^2\right)\\ &\equiv 2^{32}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 19\cdot 199(11u^{24}+60u^{18}v+105u^{12}v^2+60u^6v^3 + 5v^4+21u^{14}w+60u^8vw+30u^2v^2w+9u^4w^2)\\ &\equiv 2^{32}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 19\cdot 199\tilde{I}_{24} \mod \mathfrak{b}_{20}.\end{aligned}$$

Finally, we can also put

$$(***)\quad I_{30} \equiv 2^{38}\cdot 3^4\cdot 5^5\cdot 7\cdot 11\cdot 13\cdot 61(\nu_1u^{30}+\nu_2u^{24}v+\nu_3u^{18}v^2+\nu_4u^{12}v^3 +\nu_5u^6v^4+\nu_6v^5+\nu_7u^{20}w+\nu_8u^{14}vw+\nu_9u^8v^2w+\nu_{10}u^2v^3w +\nu_{11}u^{10}w^2+\nu_{12}u^4vw^2+\nu_{13}w^3) \mod \mathfrak{b}_{20}$$

for some $\nu_i \in \mathbb{Q}$. Then, by Lemma 2.3 (viii), we have

$$I_{30} \equiv 2^{38}\cdot 3^4\cdot 5^5\cdot 7\cdot 11\cdot 13\cdot 61\left(-\frac{599}{51840}c_3^5c_4c_5c_6+\frac{47}{34560}c_3^5c_5^3+\frac{1519}{25920}c_3^4c_5c_6c_7 +\frac{6293}{7290}c_3^3c_4c_5^2c_7-\frac{32537}{25920}c_3^3c_5^3c_6+\frac{189919}{466560}c_3^2c_4c_5^4+\frac{2012}{1215}c_3c_4c_5^2c_6c_7-\frac{16693}{25920}c_3c_5^4c_7 -\frac{223}{6480}c_4c_5^4c_6-\frac{1}{1728}c_5^6\right).$$

On the other hand, in $A/(t, c_8, \mathfrak{a}_{20})$, we have, for example,

$$\begin{aligned}
v^5 \equiv &\frac{31}{11520}c_3^5c_4c_5c_6 - \frac{47}{69120}c_3^5c_5^3 + \frac{1}{640}c_3^4c_5c_6c_7 + \frac{1993}{5760}c_3^3c_5^3c_6 - \frac{91}{360}c_3^3c_4c_5^2c_7 \\
&- \frac{1279}{11520}c_3^2c_4c_5^4 - \frac{49}{135}c_3c_4c_5^2c_6c_7 + \frac{7}{1440}c_4c_5^4c_6 + \frac{299}{1920}c_3c_5^4c_7 + \cdots, \\
w^3 \equiv &\frac{1}{162}c_3^5c_4c_5c_6 - \frac{5}{81}c_3^4c_5c_6c_7 + \frac{365}{648}c_3^3c_5^3c_6 - \frac{1043}{2916}c_3^3c_4c_5^2c_7 - \frac{1079}{5832}c_3^2c_4c_5^4 \\
&- \frac{226}{243}c_3c_4c_5^2c_6c_7 + \frac{2}{81}c_4c_5^4c_6 + \frac{431}{1296}c_3c_5^4c_7 + \frac{1}{1728}c_5^6 + \cdots.
\end{aligned}$$

Then using the second half of Lemma 2.8, the coefficients in $(***)$ are obtained as follows:

$$\begin{aligned}
&\nu_1 = -\frac{9}{8},\ \nu_2 = -3,\ \nu_3 = 0,\ \nu_4 = -5,\ \nu_5 = -\frac{35}{2},\ \nu_6 = -2,\ \nu_7 = -\frac{3}{2} \\
&\nu_8 = \frac{9}{2},\ \nu_9 = 15,\ \nu_{10} = -5,\ \nu_{11} = -\frac{3}{2},\ \nu_{12} = 3,\ \nu_{13} = -1.
\end{aligned}$$

Thus we have obtained

$$\begin{aligned}
I_{30} \equiv &2^{38}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 61\left(-\frac{9}{8}u^{30} - 3u^{24}v - 5u^{12}v^3 - \frac{35}{2}u^6v^4 - 2v^5\right. \\
&\left.-\frac{3}{2}u^{20}w + \frac{9}{2}u^{14}vw + 15u^8v^2w - 5u^2v^3w - \frac{3}{2}u^{10}w^2 + 3u^4vw^2 - w^3\right) \\
\equiv &2^{35}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 61(-9u^{30} - 24u^{24}v - 40u^{12}v^3 - 140u^6v^4 - 16v^5 \\
&- 12u^{20}w + 36u^{14}vw + 120u^8v^2w - 40u^2v^3w - 12u^{10}w^2 + 24u^4vw^2 - 8w^3) \\
\equiv &2^{35}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13\cdot 61\tilde{I}_{30} \mod \mathfrak{b}_{20}.
\end{aligned}$$

Consequently, we have established Lemma 2.5. □

3. Rational cohomology ring of $E_8/T^1\cdot E_7$

With the above results, we will compute the rational cohomology ring of E_8/C_8. We begin by recalling the classical results of Borel [4]; Let G be a compact connected Lie group, H a closed connected subgroup of G of maximal rank and T a commnon maximal torus. Consider the fibration

$$G/H \xrightarrow{\iota} BH \xrightarrow{\rho} BG.$$

Since $H^*(BG;\mathbb{Q})$ is a polynomial ring generated by elements of even degrees and $H^*(G/H;\mathbb{Q})$ has vanishing odd dimensional part (Hirsch formula [4]), the Serre spectral sequence with rational coefficients for this fibration collapses. In particular, we have the following description of the rational cohomology ring of G/H:

$$\begin{aligned}
H^*(G/H;\mathbb{Q}) \overset{\iota^*}{\underset{\sim}{\longleftarrow}} &H^*(BH;\mathbb{Q})/(\rho^*H^+(BG;\mathbb{Q})) \\
&\cong H^*(BT;\mathbb{Q})^{W(H)}/(H^+(BT;\mathbb{Q})^{W(G)}),
\end{aligned}$$

where $H^+ = \oplus_{i>0}H^i$ and $(H^+(BT;\mathbb{Q})^{W(G)})$ means the ideal of $H^*(BT;\mathbb{Q})^{W(H)}$ generated by $H^+(BT;\mathbb{Q})^{W(G)}$.

We apply this result to the fibration:

$$E_8/C_8 \xrightarrow{\iota} BC_8 \xrightarrow{\rho} BE_8.$$

Then, using Lemmas 2.2, 2.6, 2.5 and (2.9), we have

$$\begin{aligned} H^*(E_8/C_8;\mathbb{Q}) &\cong H^*(BT;\mathbb{Q})^{W(C_8)}/(H^+(BT;\mathbb{Q})^{W(E_8)}) \\ &\cong \mathbb{Q}[u, J_2, J_6, J_8, J_{10}, J_{12}, J_{14}, J_{18}]/(I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}) \\ &\cong \mathbb{Q}[u, I_2, v, I_8, w, I_{12}, I_{14}, I_{18}]/(I_2, I_8, I_{12}, I_{14}, I_{18}, I_{20}, I_{24}, I_{30}) \\ &\cong \mathbb{Q}[u, v, w]/(\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}). \end{aligned}$$

Thus we have obtained the following:

Lemma 3.1. *The rational cohomology ring of $E_8/T^1 \cdot E_7$ is given as follows*:

$$H^*(E_8/T^1\cdot E_7;\mathbb{Q}) = \mathbb{Q}[u, v, w]/(\tilde{I}_{20}, \tilde{I}_{24}, \tilde{I}_{30}),$$

where $\deg u = 2,\ \deg v = 12,\ \deg w = 20$*,* $\tilde{I}_{20}, \tilde{I}_{24}$ *and* $\tilde{I}_{30}$ *are given by* (*2.8*).

4. Integral cohomology ring of $E_8/T^1\cdot E_7$

4.1. **Integral cohomology ring of E_8/T in low degrees.** Consider the fibration

$$E_7/T' \cong C_8/T \xrightarrow{i} E_8/T \xrightarrow{p} E_8/C_8.$$

Since $H^*(E_8/C_8;\mathbb{Z})$ and $H^*(E_7/T';\mathbb{Z})$ have no torsion and vanishing odd dimensional part by Bott [7], the Serre spectral sequence with the integral coefficients for the above fibration collapses and the following sequence

$$\mathbb{Z} \to H^*(E_8/C_8;\mathbb{Z}) \xrightarrow{p^*} H^*(E_8/T;\mathbb{Z}) \xrightarrow{i^*} H^*(C_8/T;\mathbb{Z}) \cong H^*(E_7/T';\mathbb{Z}) \to \mathbb{Z}$$

is co-exact[16], that is,

p^* is injective, i^* is surjective and

$\operatorname{Ker} i^* = (p^*H^+(E_8/C_8;\mathbb{Z})),$ the ideal generated by $p^*H^+(E_8/C_8;\mathbb{Z})$.

Therefore we will obtain some information about the generators of $H^*(E_8/C_8;\mathbb{Z})$ by considering $\operatorname{Ker} i^*$. In order to investigate $\operatorname{Ker} i^*$, we will determine $H^*(E_8/T;\mathbb{Z})$ up to degrees ≤ 36. First we need a simple lemma concerning the action of the cohomology operations:

Lemma 4.1. *For the elements t and $c_i = \sigma_i(t_1, \ldots, t_8)$ in $H^*(BT;\mathbb{Z})$, we have*

(i)
$$\begin{aligned} &Sq^2(c_2) \equiv c_3 + tc_2, \\ &Sq^4(c_3) \equiv c_5 + tc_4 + c_2c_3, \\ &Sq^8(c_5 + tc_4) \equiv tc_8 + c_2c_7 + c_3c_6 + c_4c_5 + tc_4^2 + t^2c_7 + t^3c_6 + t^2c_2c_5 + t^2c_3c_4, \\ &Sq^{14}(c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8) \equiv (c_8 + t^2c_6 + t^4c_4 + t^8)(c_7 + tc_6) \mod 2. \end{aligned}$$

(ii)
$$\begin{aligned} &\mathcal{P}^1(c_2 + 2t^2) \equiv c_4 + c_2^2 + t^4, \\ &\mathcal{P}^3(c_4 + 2t^4) \equiv c_5^2 + 2c_4c_6 + 2c_3c_7 + 2c_2c_8 + c_3^2c_4 + c_2c_4^2 + c_2^2c_6 + 2c_2c_3c_5 \\ &\qquad + 2t^{10} \mod 3. \end{aligned}$$

(iii)
$$\begin{aligned} &\mathcal{P}^1(c_2 + t^2) \equiv c_6 + 2c_3^2 + 4c_2c_4 + 2c_2^3 + 2tc_5 + tc_2c_3 + 4t^2c_4 + 4t^2c_2^2 + 3t^3c_3 \\ &\qquad + t^4c_2 + 2t^6 \mod 5. \end{aligned}$$

[16] This terminology is taken from [2, §4]

Proof. (i) follows immediately from the Wu formula:

$$Sq^{2i-2}(c_i) \equiv \sum_{j=0}^{i-1} c_{2i-1-j} c_j,$$

and $c_1 = 3t \equiv t \mod 2$.

(ii) Put $p_i = t_1^i + \cdots + t_8^i$ ($i \geq 0$). Then p_i's and c_i's are related to each other by the Newton formula:

$$p_n = \sum_{i=1}^{n-1} (-1)^{i-1} p_{n-i} c_i + (-1)^{n-1} n c_n.$$

In particular, considering with mod 3 coefficients, we have

$$\begin{aligned}
p_1 &\equiv c_1 \equiv 0, \\
p_2 &\equiv c_2, \\
p_4 &\equiv 2c_4 + 2c_2^2, \\
p_{10} &\equiv 2c_5^2 + c_4c_6 + c_3c_7 + c_2c_8 + 2c_3^2c_4 + 2c_2c_4^2 + 2c_2^2c_6 + c_2^3c_4 + c_2c_3c_5 + c_2^5.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{P}^1(p_2) &\equiv \mathcal{P}^1\left(\sum_i t_i^2\right) \equiv \sum_i \mathcal{P}^1(t_i^2) \equiv \sum_i 2t_i \mathcal{P}^1(t_i) \equiv \sum_i 2t_i \cdot t_i^3 \\
&\equiv \sum_i 2t_i^4 \equiv 2p_4, \\
\mathcal{P}^3(p_4) &\equiv \mathcal{P}^3\left(\sum_i t_i^4\right) \equiv \sum_i \mathcal{P}^3(t_i^4) \equiv \sum_i (2\mathcal{P}^3(t_i^2)t_i^2 + 2\mathcal{P}^2(t_i^2)\mathcal{P}^1(t_i^2)) \\
&\equiv \sum_i (2t_i^6 \cdot 2t_i^4) \equiv \sum_i t_i^{10} \equiv p_{10}.
\end{aligned}$$

Using these facts, we have easily the required results.

(iii) Similar computation yields the required results. □

Lemma 4.2. *The integral cohomology ring of E_8/T for degrees ≤ 36 is given as follows*:

$$\begin{aligned}
H^*(E_8/T; \mathbb{Z}) = \mathbb{Z}[&t_1, \ldots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \hat{\gamma}_6, \gamma_9, \gamma_{10}, \hat{\gamma}_{15}] \\
&/(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \hat{\rho}_{15}, \rho_{18}),
\end{aligned}$$

where $t_1, \ldots, t_8, t \in H^2$ are as in §2, $\gamma_i \in H^{2i}$ ($i = 3, 4, 5, 6, 9, 10, 15$) and

$$\begin{aligned}
\rho_1 &= c_1 - 3t, \\
\rho_2 &= c_2 - 4t^2, \\
\rho_3 &= c_3 - 2\gamma_3, \\
\rho_4 &= c_4 + 2t^4 - 3\gamma_4, \\
\rho_5 &= c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5, \\
\hat{\rho}_6 &= c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\hat{\gamma}_6, \\
\rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\hat{\gamma}_6) + 3t^3\gamma_5 + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8,
\end{aligned}$$

$$
\begin{aligned}
\rho_9 &= 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6 - 2\gamma_9,\\
\rho_{10} &= \gamma_5^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 - 3\gamma_{10},\\
\rho_{12} &= 15\hat{\gamma}_6^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2\gamma_3^4 + 10\gamma_3^2\hat{\gamma}_6 - 3c_8\gamma_4 - 2\gamma_4^3\\
&\quad + t(c_8\gamma_3 - 2\gamma_3^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3\gamma_4^2) + t^2(3\gamma_{10} - 25\gamma_4\hat{\gamma}_6 - c_7\gamma_3 - 16\gamma_3^2\gamma_4)\\
&\quad + t^3(25\gamma_3\hat{\gamma}_6 - 3\gamma_4\gamma_5 + 10\gamma_3^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5\gamma_4^2) + t^5(-3c_7 - 5\gamma_3\gamma_4)\\
&\quad + 4t^6\gamma_3^2 - 7t^8\gamma_4 + 4t^9\gamma_3,\\
\rho_{14} &= c_7^2 - 3c_8\hat{\gamma}_6 + 6\gamma_4\gamma_{10} - 4c_8\gamma_3^2 + 6c_7\gamma_3\gamma_4 - 6\gamma_3^2\gamma_4^2 - 12\gamma_4^2\hat{\gamma}_6 - 2\gamma_3\gamma_5\hat{\gamma}_6\\
&\quad + t(24\gamma_3\gamma_4\hat{\gamma}_6 - 8c_7\gamma_3^2 - 8c_7\hat{\gamma}_6 + 4c_8\gamma_5 - 6\gamma_3\gamma_{10} + 12\gamma_3^3\gamma_4)\\
&\quad + t^2(-2\gamma_3\gamma_4\gamma_5 + 6\gamma_4^3 + 2\gamma_3^2\hat{\gamma}_6 + 20\hat{\gamma}_6^2 - 4\gamma_3^4 - c_7\gamma_5)\\
&\quad + t^3(-12\gamma_3\gamma_4^2 + 8c_8\gamma_3 - 5c_7\gamma_4 + 3\gamma_5\hat{\gamma}_6) + t^4(3\gamma_{10} - 26\gamma_4\hat{\gamma}_6 + 6c_7\gamma_3 - 4\gamma_3^2\gamma_4)\\
&\quad + t^5(24\gamma_3\hat{\gamma}_6 + 3\gamma_4\gamma_5 + 12\gamma_3^3) + t^6(-6c_8 + 2\gamma_4^2) - 4t^7c_7 + t^8(6\hat{\gamma}_6 - 6\gamma_3^2)\\
&\quad - 6t^{10}\gamma_4 + 12t^{11}\gamma_3 - 2t^{14},\\
\hat{\rho}_{15} &= (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2(\gamma_3^2 + c_6)(\gamma_9 - c_6\gamma_3) - 2\hat{\gamma}_{15},\\
\rho_{18} &= \gamma_9^2 - 9c_8\gamma_{10} - 6\gamma_4^2\gamma_{10} - 4\gamma_3^3\gamma_9 - 10\gamma_3\hat{\gamma}_6\gamma_9 + 2\gamma_3\gamma_5\gamma_{10} - 2\gamma_3\gamma_4\gamma_5\hat{\gamma}_6\\
&\quad - 6c_7\gamma_3\gamma_4^2 + 3c_8\gamma_4\hat{\gamma}_6 + c_8\gamma_3^2\gamma_4 + 6\gamma_3^2\gamma_4^3 + 12\gamma_4^3\hat{\gamma}_6 + 2c_7^2\gamma_4 + 2c_7\gamma_3^2\gamma_5\\
&\quad - 2\gamma_3^3\gamma_4\gamma_5 + 2c_7\gamma_5\hat{\gamma}_6 + 4\gamma_3^6 - 10\hat{\gamma}_6^3 + 18\gamma_3^4\hat{\gamma}_6 + 15\gamma_3^2\hat{\gamma}_6^2 - 9c_7c_8\gamma_3\\
&\quad + t(-2\gamma_3\gamma_5\gamma_9 - 24c_7\gamma_4\hat{\gamma}_6 + 8c_8\gamma_4\gamma_5 + 4c_7\gamma_3^2\gamma_4 + 4c_7\gamma_{10} - c_8\gamma_9 + 2c_7^2\gamma_3\\
&\quad + 4c_8\gamma_3\hat{\gamma}_6 + 12\gamma_3\gamma_4\gamma_{10} - 36\gamma_3\gamma_4^2\hat{\gamma}_6 + 12\gamma_3^2\gamma_5\hat{\gamma}_6 + c_8\gamma_3^3 + 6\gamma_3^4\gamma_5 - 18\gamma_3^3\gamma_4^2)\\
&\quad + t^2(24\gamma_3^4\gamma_4 - 2c_8^2 - c_7\gamma_9 - 11\gamma_3^2\gamma_{10} + 2\gamma_3\gamma_4\gamma_9 - 2c_8\gamma_3\gamma_5 + 16c_7\gamma_3\hat{\gamma}_6 - 3c_7\gamma_4\gamma_5\\
&\quad + 75\gamma_4\hat{\gamma}_6^2 - 6\gamma_4^4 - 9c_8\gamma_4^2 + 81\gamma_3^2\gamma_4\hat{\gamma}_6 - 13\hat{\gamma}_6\gamma_{10} + 4\gamma_3\gamma_4^2\gamma_5 - c_7\gamma_3^3)\\
&\quad + t^3(-3\gamma_5\gamma_{10} - 150\gamma_3\hat{\gamma}_6^2 - 135\gamma_3^3\hat{\gamma}_6 + 6\gamma_3^2\gamma_9 - 2c_7\gamma_3\gamma_5 + 21c_7\gamma_4^2 + 15c_7c_8\\
&\quad + 3\gamma_4\gamma_5\hat{\gamma}_6 - 3\gamma_3^2\gamma_4\gamma_5 + 18\gamma_3\gamma_4^3 + 15\hat{\gamma}_6\gamma_9 + 14c_8\gamma_3\gamma_4 - 30\gamma_3^5)\\
&\quad + t^4(-13c_8\hat{\gamma}_6 + 2\gamma_4\gamma_{10} - 5c_7^2 - 33\gamma_3^2\gamma_4^2 + 3\gamma_5\gamma_9 - 28\gamma_3\gamma_5\hat{\gamma}_6 - 45\gamma_4^2\hat{\gamma}_6\\
&\quad - 41c_7\gamma_3\gamma_4 - 13\gamma_3^3\gamma_5 - 9c_8\gamma_3^2)\\
&\quad + t^5(3c_7\hat{\gamma}_6 - 6\gamma_4^2\gamma_5 + 23c_7\gamma_3^2 + 105\gamma_3\gamma_4\hat{\gamma}_6 - 6c_8\gamma_5 - 3\gamma_4\gamma_9 + 45\gamma_3^3\gamma_4)\\
&\quad + t^6(11\gamma_4^3 - 4\gamma_3\gamma_9 + 4c_7\gamma_5 + 9\gamma_3\gamma_4\gamma_5 + 12\gamma_3^4 + 66\gamma_3^2\hat{\gamma}_6 + 75\hat{\gamma}_6^2 + 2c_8\gamma_4)\\
&\quad + t^7(-33\gamma_3\gamma_4^2 + 12\gamma_3^2\gamma_5 + 15\gamma_5\hat{\gamma}_6) + t^8(-4\gamma_{10} + 21\gamma_3^2\gamma_4 - 5c_7\gamma_3 - 3\gamma_4\hat{\gamma}_6)\\
&\quad + t^9(6\gamma_9 - 42\gamma_3^3 - 99\gamma_3\hat{\gamma}_6) + t^{10}(-4c_8 - 6\gamma_4^2 - 13\gamma_3\gamma_5) + t^{11}(3c_7 + 27\gamma_3\gamma_4)\\
&\quad + t^{12}(60\hat{\gamma}_6 + 18\gamma_3^2) + 6t^{13}\gamma_5 - 9t^{14}\gamma_4 - 12t^{15}\gamma_3 + 10t^{18}.
\end{aligned}
$$

Proof. According to Toda [42, Proposition 3.2], one can give the general description of $H^*(E_8/T;\mathbb{Z})$ as follows:

$$
\begin{aligned}
H^*(E_8/T;\mathbb{Z}) &= \mathbb{Z}[t_1,\dots,t_8,t,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_9,\gamma_{10},\gamma_{15}]\\
&\qquad /(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{10},\rho_{12},\rho_{14},\rho_{15},\rho_{18},\rho_{20},\rho_{24},\rho_{30}),
\end{aligned}
$$

where $t_1,\dots,t_8,t\in H^2$ are as above and

$$\begin{aligned}
\rho_1 &= c_1 - 3t, \\
\rho_i &= \delta_i - 2\gamma_i \ (i = 3, 5, 9, 15), \\
\rho_i &= \delta_i - 3\gamma_i \ (i = 4, 10), \\
\rho_6 &= \delta_6 - 5\gamma_6.
\end{aligned}$$

Here δ_i $(i = 3, 4, 5, 6, 9, 10, 15)$ is an arbitrary element of $H^*(E_8/T; \mathbb{Z})$ satisfying

$$\begin{aligned}
&\delta_3 \equiv Sq^2(\rho_2),\ \delta_5 \equiv Sq^4(\delta_3),\ \delta_9 \equiv Sq^8(\delta_5),\ \delta_{15} \equiv Sq^{14}(\rho_8) \quad \mathrm{mod}\ 2, \\
&\delta_4 \equiv \mathcal{P}^1(\rho_2),\ \delta_{10} \equiv \mathcal{P}^3(\delta_4) \quad \mathrm{mod}\ 3, \\
&\delta_6 \equiv \mathcal{P}^1(\rho_2) \quad \mathrm{mod}\ 5.
\end{aligned}$$

Other relation ρ_j $(j = 2, 8, 12, 14, 18, 20, 24, 30)$ is determined by the maximality of the integer n_j in

(4.1) $$n_j \cdot \rho_j \equiv \iota_0^*(I_j) \quad \mathrm{mod}\ (\rho_i; i < j),$$

where $\iota_0 : E_8/T \longrightarrow BT$.

Now let us determine the generators and the relations explicitly;

(1) In view of Lemma 2.3 (i) and (4.1), we can take

$$\rho_2 = c_2 - 4t^2.$$

(2) By Lemma 4.1 (i), we have

$$\delta_3 \equiv Sq^2(\rho_2) \equiv Sq^2(c_2) \equiv c_3 + tc_2 \equiv c_3 \quad \mathrm{mod}\ (2, \rho_1, \rho_2)$$

and we can take $\delta_3 = c_3$ so that

$$\rho_3 = c_3 - 2\gamma_3.$$

(3) By Lemma 4.1 (ii), we have

$$\delta_4 \equiv \mathcal{P}^1(\rho_2) \equiv \mathcal{P}^1(c_2 - 4t^2) \equiv c_4 + c_2^2 + t^4 \equiv c_4 + 2t^4 \quad \mathrm{mod}\ (3, \rho_1, \rho_2),$$

and we can take $\delta_4 = c_4 + 2t^4$ so that

$$\rho_4 = c_4 + 2t^4 - 3\gamma_4.$$

(4) By Lemma 4.1 (i), we have

$$\begin{aligned}
\delta_5 &\equiv Sq^4(\delta_3) \equiv Sq^4(c_3) \equiv c_5 + tc_4 + c_2c_3 \equiv c_5 + tc_4 \quad \mathrm{mod}\ (2, \rho_1, \rho_2, \rho_3, \rho_4) \\
&\equiv c_5 - 3t\gamma_4 + 2t^2\gamma_3 \quad \mathrm{mod}\ (2, \rho_1, \rho_2, \rho_3, \rho_4),
\end{aligned}$$

and we can take $\delta_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3$ so that

$$\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3 - 2\gamma_5.$$

(5) By Lemma 4.1 (iii), we have

$$\begin{aligned}
\delta_6 &\equiv \mathcal{P}^1(\rho_2) \equiv \mathcal{P}^1(c_2 - 4t^2) \equiv c_6 + 2c_3^2 + 4c_2c_4 + 2c_2^3 + 2tc_5 + tc_2c_3 \\
&\quad + 4t^2c_4 + 4t^2c_2^2 + 3t^3c_3 + t^4c_2 + 2t^6 \quad \mathrm{mod}\ (5, \rho_1) \\
&\equiv c_6 + 3\gamma_3^2 + 4t\gamma_5 + t^2\gamma_4 + 3t^6 \quad \mathrm{mod}\ (5, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \\
&\equiv c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 \quad \mathrm{mod}\ (5, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5),
\end{aligned}$$

and we can take $\delta_6 = c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6$ so that

$$\hat{\rho}_6 = c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\hat{\gamma}_6.$$

(6) By Lemma 2.3 (ii), we have

$$(4.2)\quad \begin{aligned} I_8 &\equiv 2^{14}\cdot 3\cdot 5\{-18c_8 - 3c_3c_5 + 2c_4^2 + t(12c_7 - 3c_3c_4) + t^2(-6c_6 + 3c_3^2) \\ &+12t^3c_5 + 2t^4c_4 - 12t^5c_3 + 14t^8\} \mod (I_2). \end{aligned}$$

On the other hand, by the relations ρ_3, ρ_4, ρ_5, $\hat{\rho}_6$, we have

$$(4.3)\quad \begin{aligned} c_3 &= 2\gamma_3, \\ c_4 &= 3\gamma_4 - 2t^4, \\ c_5 &= 2\gamma_5 + 3t\gamma_4 - 2t^2\gamma_3, \\ c_6 &= 5\hat{\gamma}_6 + 2\gamma_3^2 + t\gamma_5 - t^2\gamma_4 + 2t^6. \end{aligned}$$

Substituting (4.3) into (4.2), we have

$$\begin{aligned} I_8 &\equiv 2^{15}\cdot 3^2\cdot 5\{-3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\hat{\gamma}_6) \\ &+ 3t^3\gamma_5 + 4t^4\gamma_4 - 6t^5\gamma_3 + t^8\} \mod (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6). \end{aligned}$$

Hence, by (4.1), we have

$$2^{15}\cdot 3^2\cdot 5\,\rho_8 \equiv \iota_0^*(I_8) \mod (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6)$$

and it follows the form of ρ_8.

(7) By Lemma 4.1 (i) , we have

$$\begin{aligned} \delta_9 &\equiv Sq^8(\delta_5) \equiv Sq^8(c_5 + tc_4) \equiv tc_8 + c_2c_7 + c_3c_6 + c_4c_5 + tc_4^2 + t^2c_7 \\ &+ t^3c_6 + t^2c_2c_5 + t^2c_3c_4 \mod (2, \rho_1) \\ &\equiv tc_8 + t^2c_7 + t^3c_6 \mod (2, \rho_1, \rho_2, \rho_3, \rho_5) \\ &\equiv tc_8 + t^2c_7 - 3t^3c_6 + 2c_6\gamma_3 \mod (2, \rho_1, \rho_2, \rho_3, \rho_5), \end{aligned}$$

and we can take $\delta_9 = 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6$ so that

$$\rho_9 = 2c_6\gamma_3 + tc_8 + t^2c_7 - 3t^3c_6 - 2\gamma_9.$$

(8) By Lemma 4.1 (ii), we have

$$\begin{aligned} \delta_{10} &\equiv \mathcal{P}^3(\delta_4) \equiv \mathcal{P}^3(c_4 + 2t^4) \equiv c_5^2 + 2c_4c_6 + 2c_3c_7 + 2c_2c_8 + c_3^2c_4 \\ &+ c_2c_4^2 + c_2^2c_6 + 2c_2c_3c_5 + 2t^{10} \mod (3, \rho_1) \\ &\equiv \gamma_5^2 + c_7\gamma_3 + 2t^2c_8 \mod (3, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6) \\ &\equiv {\gamma_5}^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 \mod (3, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6), \end{aligned}$$

and we can take $\delta_{10} = \gamma_5^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7$ so that

$$\rho_{10} = \gamma_5^2 - 2c_7\gamma_3 - t^2c_8 + 3t^3c_7 - 3\gamma_{10}.$$

(9) By (2.6), (2.5) and (2.7), we obtain

$$I_{12} \equiv 2^{18}\cdot 3^4\cdot 5\cdot 7\left\{\frac{3}{5}{c_6}^2 - c_5c_7 - c_4c_8 + \frac{1}{6}c_3c_4c_5 - \frac{2}{27}c_4^3 - \frac{1}{10}c_3^2c_6 + \frac{1}{40}c_3^4\right.$$
$$+ t\left(\frac{1}{2}c_3c_8 + \frac{7}{3}c_4c_7 - \frac{3}{5}c_5c_6 - \frac{1}{5}c_3^2c_5 + \frac{1}{6}c_3c_4^2\right)$$
$$+ t^2\left(\frac{2}{5}c_5^2 - \frac{5}{2}c_3c_7 - \frac{2}{3}c_4c_6 - \frac{1}{6}c_3^2c_4\right)$$
$$+ t^3\left(\frac{19}{10}c_3c_6 - \frac{2}{3}c_4c_5 - \frac{1}{5}c_3^3\right) + t^4\left(-\frac{1}{9}c_4^2 + \frac{19}{30}c_3c_5\right) + t^5\left(\frac{14}{3}c_7 + \frac{1}{2}c_3c_4\right)$$
$$\left. +t^6\left(-\frac{56}{15}c_6 + \frac{23}{30}c_3^2\right) - \frac{2}{15}t^7c_5 - \frac{5}{9}t^8c_4 - \frac{22}{15}t^9c_3 + \frac{154}{135}t^{12}\right\} \mod (I_2)$$
$$\equiv 2^{18}\cdot 3^4\cdot 5\cdot 7\{15\hat{\gamma}_6^2 + 2\gamma_3\gamma_4\gamma_5 - 2c_7\gamma_5 + 2\gamma_3^4 + 10\gamma_3^2\hat{\gamma}_6 - 3c_8\gamma_4 - 2\gamma_4^3$$
$$+ t(c_8\gamma_3 - 2\gamma_3^2\gamma_5 + 4c_7\gamma_4 + 6\gamma_3\gamma_4^2) + t^2(3\gamma_{10} - 25\gamma_4\hat{\gamma}_6 - c_7\gamma_3 - 16\gamma_3^2\gamma_4)$$
$$+ t^3(25\gamma_3\hat{\gamma}_6 - 3\gamma_4\gamma_5 + 10\gamma_3^3) + t^4(3c_8 + 3\gamma_3\gamma_5 + 5\gamma_4^2) + t^5(-3c_7 - 5\gamma_3\gamma_4)$$
$$+ 4t^6\gamma_3^2 - 7t^8\gamma_4 + 4t^9\gamma_3\} \mod (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \rho_{10}).$$

Hence, we have

$$2^{18}\cdot 3^4\cdot 5\cdot 7\ \rho_{12} \equiv \iota_0^*(I_{12}) \mod (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \rho_{10}).$$

From this and (4.1), the form of ρ_{12} follows. Quite similarly, we have

$$2^{20}\cdot 3^2\cdot 5^2\cdot 7\cdot 11\ \rho_{14} \equiv \iota_0^*(I_{14}) \mod (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \rho_{10}),$$
$$2^{26}\cdot 3^4\cdot 5^2\cdot 7\cdot 13\ \rho_{18} \equiv \iota_0^*(I_{18}) \mod (\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \rho_9, \rho_{10}),$$

and it follows the forms of ρ_{14} and ρ_{18}.

(10) Finally, we will determine the relation $\hat{\rho}_{15}$. Since

$$\rho_8 = -3c_8 + 3{\gamma_4}^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2{\gamma_3}^2 - 5\hat{\gamma}_6) + 3t^3\gamma_5 + 4t^4\gamma_4$$
$$- 6t^5\gamma_3 + t^8$$
$$\equiv c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8 \mod (2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6),$$

we have, by Lemma 4.1 (i),

$$\delta_{15} \equiv Sq^{14}(\rho_8)$$
$$\equiv Sq^{14}(c_8 + c_4^2 + t^2c_6 + t^4c_4 + t^8)$$
$$\equiv (c_8 + t^2c_6 + t^4c_4 + t^8)(c_7 + tc_6) \mod (2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6)$$
$$\equiv (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3)$$
$$\mod (2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \rho_9)$$

and we can take

$$\delta_{15} = (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3)$$

so that

$$\hat{\rho}_{15} = (c_8 - t^2c_6 + 2t^3\gamma_5 + 3t^4\gamma_4 - t^8)(c_7 - 3tc_6) - 2(c_6 + \gamma_3^2)(\gamma_9 - c_6\gamma_3) - 2\hat{\gamma}_{15}.$$

Consequently, we have established the lemma. □

Remark 4.3. *Since $H^*(E_8/T;\mathbb{Z})$ has a free $\mathbb{Z}$-basis consisting of Schubert classes, our generators $\gamma_3, \gamma_4, \gamma_5, \hat{\gamma}_6, \gamma_9, \gamma_{10}, \hat{\gamma}_{15}$ can be expressed as certain $\mathbb{Z}$-linear combinations of Schubert classes. The precise expression is given in* [30] (*see also* [18]).

In order to determine $\operatorname{Ker} i^*$, we need the result on the integral cohomology ring $H^*(E_7/T';\mathbb{Z})$, that was computed by the author. The result is restated as follows:

Theorem 4.4 ([36], Theorem 5.9)**.** *The integral cohomology ring of E_7/T' is given as follows*:

$$H^*(E_7/T';\mathbb{Z}) = \mathbb{Z}[t'_1, \ldots, t'_7, t', \gamma'_3, \gamma'_4, \gamma'_5, \gamma'_9] \\ /(\rho'_1, \rho'_2, \rho'_3, \rho'_4, \rho'_5, \rho'_6, \rho'_8, \rho'_9, \rho'_{10}, \rho'_{12}, \rho'_{14}, \rho'_{18}),$$

where $t'_1, \ldots, t'_7, t' \in H^2$ are as in §2, $\gamma'_i \in H^{2i}$ $(i = 3, 4, 5, 9)$ and

$$\begin{aligned}
\rho'_1 &= c'_1 - 3t', \\
\rho'_2 &= c'_2 - 4t'^2, \\
\rho'_3 &= c'_3 - 2\gamma'_3, \\
\rho'_4 &= c'_4 + 2t'^4 - 3\gamma'_4, \\
\rho'_5 &= c'_5 - 3t'\gamma'_4 + 2t'^2\gamma'_3 - 2\gamma'_5, \\
\rho'_6 &= {\gamma'_3}^2 + 2c'_6 - 2t'\gamma_5 - 3t'^2\gamma'_4 + t'^6, \\
\rho'_8 &= 3{\gamma'_4}^2 - 2\gamma'_3\gamma'_5 + t'(2c'_7 - 6\gamma'_3\gamma'_4) - 9t'^2c'_6 + 12t'^3\gamma'_5 + 15t'^4\gamma'_4 - 6t'^5\gamma'_3 - t'^8, \\
\rho'_9 &= 2c'_6\gamma'_3 + t'^2c'_7 - 3t'^3c'_6 - 2\gamma'_9, \\
\rho'_{10} &= {\gamma'_5}^2 - 2c'_7\gamma'_3 + 3t'^3c'_7, \\
\rho'_{12} &= 3c'^2_6 - 2{\gamma'_4}^3 - 2c'_7\gamma'_5 + 2\gamma'_3\gamma'_4\gamma'_5 + t'(4c'_7\gamma'_4 - 2c'_6\gamma'_5 + 6\gamma'_3{\gamma'_4}^2) \\
&\quad + t'^2(-3c'_7\gamma'_3 + 3c'_6\gamma'_4) + t'^3(-12\gamma'_4\gamma'_5 + 5c'_6\gamma'_3) + t'^4(-2\gamma'_3\gamma'_5 - 15{\gamma'_4}^2) \\
&\quad - 10t'^6c'_6 + 12t'^7\gamma'_5 + 19t'^8\gamma'_4 - 6t'^9\gamma'_3 - 2t'^{12}, \\
\rho'_{14} &= {c'_7}^2 + 6c'_7\gamma'_3\gamma'_4 - 2c'_6\gamma'_3\gamma'_5 - t'^2c'_7\gamma'_5 + t'^3(-9c'_7\gamma'_4 + 3c'_6\gamma'_5) - 6t'^4c'_7\gamma'_3 + 9t'^7c'_7, \\
\rho'_{18} &= -{\gamma'_9}^2 + 2c'_6c'_7\gamma'_5 + 6c'_7\gamma'_3{\gamma'_4}^2 - 2{c'_7}^2\gamma'_4 - 2c'_6\gamma'_3\gamma'_4\gamma'_5 + 2c'_6\gamma'_3\gamma'_9 \\
&\quad + t'(-6{c'_7}^2\gamma'_3 + 24c'_6c'_7\gamma'_4) + t'^2(-25c'_7\gamma'_4\gamma'_5 + c'_7\gamma'_9 - 18c'_6c'_7\gamma'_3) \\
&\quad + t'^3(-45c'_7{\gamma'_4}^2 + 20c'_7\gamma'_3\gamma'_5 + 3c'_6\gamma'_4\gamma'_5 - 3c'_6\gamma'_9) \\
&\quad + t'^4(11c'^2_7 + 2c'_6\gamma'_3\gamma'_5 + 48c'_7\gamma'_3\gamma'_4) + 51t'^5c'_6c'_7 - 53t'^6c'_7\gamma'_5 \\
&\quad + t'^7(-69c'_7\gamma'_4 - 3c'_6\gamma'_5) + 16t'^8c'_7\gamma'_3 + 15t'^{11}c'_7.
\end{aligned}$$

Remark 4.5. *In order to deal with the higher relations ρ'_{12}, ρ'_{14} and ρ'_{18}, the author make use of the integral cohomology ring of the homogeneous space $E_7/T^1{\cdot}Spin(12)$* ([36])*. Here we expressed the relations ρ'_{12}, ρ'_{14} and ρ'_{18} in terms of the generators $t'_1, \ldots, t'_7, t', \gamma'_3, \gamma'_4, \gamma'_5, \gamma'_9$ by computing the $W(E_7)$-invariants I'_{12}, I'_{14} and I'_{18} explicitly* (*see 2.2*).

By Lemma 4.2 and Theorem 4.4, we can find the generators of the ideal $\operatorname{Ker} i^*$:

Proposition 4.6. *For the induced homomorphism*

$$i^* : H^*(E_8/T;\mathbb{Z}) \longrightarrow H^*(C_8/T;\mathbb{Z}) \cong H^*(E_7/T';\mathbb{Z}),$$

we obtain that

$$\operatorname{Ker} i^* = (u, \tilde{\gamma}_6, \gamma_{10}, \hat{\gamma}_{15}),$$

where $\tilde{\gamma}_6 = 2\hat{\gamma}_6 + \gamma_3^2 - t^2\gamma_4 + t^6$.

Proof. By (2.2), we have

$$i^*(t_i) = t_i' \ (1 \leq i \leq 7), \ i^*(t_8) = 0, \ i^*(t) = t', \tag{4.4}$$

and therefore

$$i^*(c_n) = c_n' \ (1 \leq n \leq 7), \ i^*(c_8) = 0. \tag{4.5}$$

Then it is verified directly that

$$\begin{aligned} i^*(\gamma_i) &= \gamma_i' \ (i = 3, 4, 5, 9), \\ i^*(\hat{\gamma}_6) &= c_6' - t'\gamma_5{}' - t'^2\gamma_4{}', \\ i^*(\gamma_{10}) &= 0, \\ i^*(\hat{\gamma}_{15}) &= 0. \end{aligned} \tag{4.6}$$

Now we put

$$I = (u, \tilde{\gamma}_6, \gamma_{10}, \hat{\gamma}_{15}),$$

the ideal of $H^*(E_8/T;\mathbb{Z})$ generated by the elements in the parenthesis. Using (4.4), (4.6) and Theorem 4.4, we see that I is contained in $\operatorname{Ker} i^*$. Hence there is an induced map

$$H^*(E_8/T;\mathbb{Z})/I \longrightarrow H^*(C_8/T;\mathbb{Z}) \cong H^*(E_7/T';\mathbb{Z}).$$

Then, by Lemma 4.2, we have

$$\begin{aligned} H^*(E_8/T;\mathbb{Z})/I \cong\ & \mathbb{Z}[t_1, \ldots, t_8, t, \gamma_3, \gamma_4, \gamma_5, \hat{\gamma}_6, \gamma_9, \gamma_{10}, \hat{\gamma}_{15}] \\ & /(u, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \hat{\rho}_6, \tilde{\gamma}_6, \rho_8, \rho_9, \rho_{10}, \gamma_{10}, \rho_{12}, \rho_{14}, \hat{\rho}_{15}, \hat{\gamma}_{15}, \rho_{18}) \\ \cong\ & \mathbb{Z}[t_1, \ldots, t_7, t, \gamma_3, \gamma_4, \gamma_5, \gamma_9] \\ & /(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \tilde{\gamma}_6, \rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}, \rho_{18}) \end{aligned}$$

for degrees ≤ 36. Then it follows from Lemma 4.2, Theorem 4.4, (4.4), (4.5) and (4.6) that

$$\begin{aligned} & i^*(\rho_i) \equiv \rho_i' \ (i = 1, 2, 3, 4, 5, 8, 9, 10, 12, 14, 18), \\ & i^*(\tilde{\gamma}_6) \equiv \rho_6'. \end{aligned}$$

Therefore this map induces an isomorphism and the assertion follows. □

4.2. **Generators of $H^*(E_8/T^1{\cdot}E_7;\mathbb{Z})$.** From Proposition 4.6, we see that $H^*(E_8/C_8;\mathbb{Z})$ is generated as a ring by some four elements $\tilde{u} \in H^2$, $\tilde{v} \in H^{12}$, $\tilde{w} \in H^{20}$ and $\tilde{x} \in H^{30}$ such that

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{x}) = (u, \tilde{\gamma}_6, \gamma_{10}, \hat{\gamma}_{15}) \tag{4.7}$$

as ideals. So our next task is to describe these generators in the ring $H^*(E_8/T;\mathbb{Z})$. Hereafter we identify $H^*(E_8/C_8;\mathbb{Z})$ with the subalgebra $\operatorname{Im} p^*$ of $H^*(E_8/T;\mathbb{Z})$.

Firstly, by Lemma 2.7 and (2.15), we have

$$\begin{aligned}\tilde{J}_6 &:= \frac{1}{2^{10}\cdot 3^2\cdot 5}J_6\\ &\equiv \frac{2}{5}c_6+\frac{1}{20}c_3^2+c_5\left(-\frac{1}{5}t-\frac{1}{2}u\right)+c_4\left(\frac{1}{2}tu+\frac{1}{3}u^2\right)\\ &+c_3\left(-\frac{1}{5}t^3-\frac{1}{2}t^2u-\frac{1}{2}u^3\right)+\frac{1}{5}t^6+t^5u-\frac{1}{3}t^4u^2+t^3u^3+t^2u^4-tu^5\\ &+\frac{273}{640}u^6 \mod (I_2),\end{aligned} \tag{4.8}$$

(4.9)

$$\begin{aligned}\tilde{J}_{10} &:= \frac{1}{2^{14}\cdot 3^3\cdot 5\cdot 7}J_{10}\\ &\equiv \frac{1}{12}c_5^2-\frac{1}{3}c_3c_7+\frac{1}{2}c_3c_6u+c_4c_5\left(-\frac{1}{6}t-\frac{1}{6}u\right)+c_3c_5\left(\frac{1}{6}t^2-\frac{23}{84}u^2\right)\\ &+c_4^2\left(\frac{1}{12}t^2+\frac{1}{6}tu+\frac{1}{14}u^2\right)+c_3^2\left(\frac{1}{12}t^4+\frac{23}{84}t^2u^2-\frac{37}{96}u^4\right)\\ &+c_3c_4\left(-\frac{1}{6}t^3-\frac{1}{6}t^2u-\frac{23}{84}tu^2+\frac{1}{3}u^3\right)\\ &+c_7\left(t^3+\frac{1}{6}t^2u-\frac{17}{42}tu^2+\frac{5}{14}u^3\right)+c_6\left(-\frac{3}{2}t^3u-\frac{3}{14}t^2u^2+\frac{1}{2}tu^3-\frac{37}{84}u^4\right)\\ &+c_5\left(-\frac{1}{3}t^5+\frac{1}{6}t^4u+\frac{23}{21}t^3u^2-\frac{1}{3}t^2u^3+\frac{1}{24}tu^4+\frac{71}{336}u^5\right)\\ &+c_4\left(\frac{1}{3}t^6+\frac{1}{6}t^5u+\frac{17}{42}t^4u^2-\frac{2}{3}t^3u^3-\frac{1}{3}t^2u^4+\frac{5}{16}tu^5-\frac{131}{504}u^6\right)\\ &+c_3\left(-\frac{1}{3}t^7+\frac{1}{6}t^6u-\frac{23}{21}t^5u^2-\frac{1}{3}t^4u^3+\frac{37}{24}t^3u^4+\frac{73}{48}t^2u^5-\frac{3}{2}tu^6+\frac{239}{336}u^7\right)\\ &+\frac{1}{3}t^{10}-\frac{1}{3}t^9u+\frac{7}{6}t^8u^2+\frac{2}{3}t^7u^3-\frac{7}{8}t^6u^4-\frac{35}{8}t^5u^5+\frac{233}{72}t^4u^6+\frac{7}{24}t^3u^7\\ &-\frac{129}{56}t^2u^8+\frac{215}{168}tu^9-\frac{208601}{645120}u^{10} \mod (I_2).\end{aligned}$$

On the other hand, by Lemma 4.2, we have

$$\begin{aligned}c_3 &= 2\gamma_3,\\ c_4 &= 3\gamma_4-2t^4,\\ c_5 &= 2\gamma_5+3t\gamma_4-2t^2\gamma_3,\\ c_6 &= 5\hat{\gamma}_6+2\gamma_3^2+t\gamma_5-t^2\gamma_4+2t^6\end{aligned} \tag{4.10}$$

in $H^*(E_8/T;\mathbb{Z}) \hookrightarrow H^*(E_8/T;\mathbb{Q})$. Substituting (4.10) into (4.8), we have

$$\begin{aligned}\tilde{J}_6 &= \frac{1}{2^{10}\cdot 3^2\cdot 5}J_6\\ &= 2\hat{\gamma}_6+\gamma_3^2-u\gamma_5+\gamma_4(-t^2+u^2)-u^3\gamma_3+t^6-t^4u^2+t^3u^3\\ &+t^2u^4-tu^5+\frac{273}{640}u^6.\end{aligned} \tag{4.11}$$

Similarly, substituting (4.10) into (4.9) and using the relations ρ_8, ρ_9 and ρ_{10}, we have

$$
\begin{aligned}
\tilde{J}_{10} &= \frac{1}{2^{14}\cdot 3^3\cdot 5\cdot 7}J_{10} \\
&= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2+2u^3) \\
&\quad +\gamma_3^2\left(2t^2u^2+2tu^3+\frac{7}{24}u^4\right) + \hat{\gamma}_6\left(-5t^2u^2+5tu^3+\frac{55}{12}u^4\right) \\
&\quad +\gamma_5\left(t^4u+3t^3u^2+t^2u^3-\frac{55}{24}u^5\right) \\
&\quad +\gamma_4\left(6t^4u^2-3t^3u^3-\frac{103}{24}t^2u^4-tu^5+\frac{79}{24}u^6\right) \\
&\quad +\gamma_3\left(-6t^5u^2-2t^4u^3+4t^3u^4+6t^2u^5-4tu^6-\frac{31}{24}u^7\right) \\
&\quad +4t^7u^3+\frac{55}{24}t^6u^4-6t^5u^5-\frac{7}{24}t^4u^6+\frac{79}{24}t^3u^7+\frac{31}{24}t^2u^8-\frac{55}{24}tu^9 \\
&\quad +\frac{666919}{645120}u^{10}.
\end{aligned} \tag{4.12}
$$

Now let us determine our generators $\tilde{u}, \tilde{v}, \tilde{w}$ and $\tilde{x}$. Obviously, we can take $u = t_8$ as our generator $\tilde{u}$.

Next, since $H^*(E_8/C_8;\mathbb{Q})$ is generated by u, J_6 and J_{10} (see §3), we can put

$$\tilde{v} = \alpha\tilde{J}_6 + \beta u^6$$

for some $\alpha, \beta \in \mathbb{Q}$. On the other hand, by (4.7), we can express

$$\tilde{v} = \tilde{\gamma}_6 + f = 2\hat{\gamma}_6 + \gamma_3^2 - t^2\gamma_4 + t^6 + f$$

for some element $f \in (u) \cap H^{12}(E_8/T;\mathbb{Z})$. Then, using (4.11), we have

$$
\begin{aligned}
&2\hat{\gamma}_6 + \gamma_3^2 - t^2\gamma_4 + t^6 + f = \alpha(2\hat{\gamma}_6 + \gamma_3^2 - t^2\gamma_4 + t^6) \\
&+ \alpha(-u\gamma_5 + u^2\gamma_4 - u^3\gamma_3 - t^4u^2 + t^3u^3 + t^2u^4 - tu^5) + \left(\frac{273}{640}\alpha + \beta\right)u^6,
\end{aligned}
$$

and we can take $\alpha = 1$, $\beta = -\dfrac{273}{640}$. Thus we see that

$$
\begin{aligned}
v &= \frac{1}{2^{10}\cdot 3^2\cdot 5}J_6 - \frac{273}{640}u^6 \\
&= 2\hat{\gamma}_6 + \gamma_3^2 - u\gamma_5 + \gamma_4(-t^2+u^2) - u^3\gamma_3 + t^6 - t^4u^2 + t^3u^3 + t^2u^4 - tu^5
\end{aligned} \tag{4.13}
$$

can be chosen as our generator $\tilde{v}$.

Similarly, we can put

$$\tilde{w} = \lambda\tilde{J}_{10} + \mu u^4 v + \nu u^{10}$$

for some $\lambda, \mu, \nu \in \mathbb{Q}$. On the other hand, by (4.7), we can express

$$\tilde{w} = \gamma_{10} + g$$

for some element $g \in (u, \tilde{\gamma}_6) \cap H^{20}(E_8/T;\mathbb{Z})$. Then, using (4.12), we can take $\lambda = 1$ and hence

$$\begin{aligned}\gamma_{10} + g = {} & \gamma_{10} + u\gamma_9 - u^3 c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3)\\ & + \gamma_3^2\left\{2t^2u^2 + 2tu^3 + \left(\mu + \frac{7}{24}\right)u^4\right\}\\ & + \hat{\gamma}_6\left\{-5t^2u^2 + 5tu^3 + \left(2\mu + \frac{55}{12}\right)u^4\right\}\\ & + \gamma_5\left\{t^4u + 3t^3u^2 + t^2u^3 + \left(-\mu - \frac{55}{24}\right)u^5\right\}\\ & + \gamma_4\left\{6t^4u^2 - 3t^3u^3 + \left(-\mu - \frac{103}{24}\right)t^2u^4 - tu^5 + \left(\mu + \frac{79}{24}\right)u^6\right\}\\ & + \gamma_3\left\{-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + \left(-\mu - \frac{31}{24}\right)u^7\right\}\\ & + 4t^7u^3 + \left(\mu + \frac{55}{24}\right)t^6u^4 - 6t^5u^5 + \left(-\mu - \frac{7}{24}\right)t^4u^6 + \left(\mu + \frac{79}{24}\right)t^3u^7\\ & + \left(\mu + \frac{31}{24}\right)t^2u^8 + \left(-\mu - \frac{55}{24}\right)tu^9 + \left(\nu + \frac{666919}{645120}\right)u^{10},\end{aligned}$$

and we can take $\mu = -\dfrac{55}{24}$, $\nu = -\dfrac{666919}{645120}$. Thus we see that

$$\begin{aligned} w &= \frac{1}{2^{14}\cdot 3^3\cdot 5\cdot 7}J_{10} - \frac{55}{24}u^4 v - \frac{666919}{645120}u^{10}\\ &= \gamma_{10} + u\gamma_9 - u^3c_7 - u\gamma_4\gamma_5 + 2u^2\gamma_4^2 - 2u^2\gamma_3\gamma_5 + \gamma_3\gamma_4(-6tu^2 + 2u^3)\\ &\quad + \gamma_3^2(2t^2u^2 + 2tu^3 - 2u^4) + \hat{\gamma}_6(-5t^2u^2 + 5tu^3) + \gamma_5(t^4u + 3t^3u^2 + t^2u^3)\\ &\quad + \gamma_4(6t^4u^2 - 3t^3u^3 - 2t^2u^4 - tu^5 + u^6)\\ &\quad + \gamma_3(-6t^5u^2 - 2t^4u^3 + 4t^3u^4 + 6t^2u^5 - 4tu^6 + u^7)\\ &\quad + 4t^7u^3 - 6t^5u^5 + 2t^4u^6 + t^3u^7 - t^2u^8 \end{aligned} \tag{4.14}$$

can be chosen as our generator $\tilde{w}$.

Finally, we have to find an element x of degree 30 such that $x \equiv \hat{\gamma}_{15} \mod (u, v, w)$ in $H^*(E_8/T;\mathbb{Z})$. In order to identify the element x, we make use of the result on the mod 2 cohomology ring $H^*(E_8/T;\mathbb{Z}/2\mathbb{Z})$ (up to degrees ≤ 30) due to Kono and Ishitoya ([31, Theorem 3.10])[17]. Using their result, it can be checked directly that $u^{15} \equiv 0$ in $H^*(E_8/T;\mathbb{Z}/2\mathbb{Z})$. This means that u^{15} is divisible by 2 in the ring $H^*(E_8/T;\mathbb{Z})$. Therefore there exists an element $x \in H^{30}(E_8/T;\mathbb{Z})$ such that $2x = u^{15}$. Explicit form of x in $H^*(E_8/T;\mathbb{Z})$ is given in Appendix 6.1. Furthermore, we can check directly that

$$x \equiv \hat{\gamma}_{15} \mod (u, v, w)$$

(see also Appendix 6.1). Hence the element x can be chosen as our generator $\tilde{x}$.

[17]Note that our generators $\gamma_5, \gamma_9, \hat{\gamma}_{15}$ are slightly different from $\gamma_5, \gamma_9, \gamma'_{15}$ in [31].

4.3. **Integral cohomology ring of $E_8/T^1{\cdot}E_7$.** Using the element x, we can rewrite $\tilde{I}_{30}$ as follows:

$$
\begin{aligned}
\tilde{I}_{30} = &\ -9u^{30} - 24u^{24}v - 12u^{20}w + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\
& - 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3 \\
= &\ -36x^2 - 48u^9vx - 24u^5wx + 36u^{14}vw - 40u^{12}v^3 - 12u^{10}w^2 + 120u^8v^2w \\
& - 140u^6v^4 + 24u^4vw^2 - 40u^2v^3w - 16v^5 - 8w^3 \\
= &\ 4(-9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w \\
& - 35u^6v^4 + 6u^4vw^2 - 10u^2v^3w - 4v^5 - 2w^3).
\end{aligned}
$$

Therefore, in view of Lemma 3.1, we obtain the following main result of this paper.

Theorem 4.7. *The integral cohomology ring of $E_8/T^1{\cdot}E_7$ is given as follows*:

$$H^*(E_8/T^1{\cdot}E_7;\mathbb{Z}) = \mathbb{Z}[u,v,w,x]/(r_{15},r_{20},r_{24},r_{30}),$$

where $\deg u = 2$, $\deg v = 12$, $\deg w = 20$, $\deg x = 30$ *and*

$$
\begin{aligned}
r_{15} &= u^{15} - 2x, \\
r_{20} &= 9u^{20} + 45u^{14}v + 12u^{10}w + 60u^8v^2 + 30u^4vw + 10u^2v^3 + 3w^2, \\
r_{24} &= 11u^{24} + 60u^{18}v + 21u^{14}w + 105u^{12}v^2 + 60u^8vw + 60u^6v^3 + 9u^4w^2 \\
&\quad + 30u^2v^2w + 5v^4, \\
r_{30} &= -9x^2 - 12u^9vx - 6u^5wx + 9u^{14}vw - 10u^{12}v^3 - 3u^{10}w^2 + 30u^8v^2w \\
&\quad - 35u^6v^4 + 6u^4vw^2 - 10u^2v^3w - 4v^5 - 2w^3.
\end{aligned}
$$

Remark 4.8. *As remarked in the introduction, the integral cohomology ring $H^*(E_8/T^1{\cdot}E_7;\mathbb{Z})$ is also computed by Duan and Zhao in terms of Schubert classes* ([16, Theorem 7]). *We will verify that both results completely coincide in our forthcoming paper* [29].

5. Integral cohomology of E_8/E_7

In order to determine the integral cohomology of E_8/E_7, we consider the Gysin exact sequence associated with the following circle bundle

$$S^1 \longrightarrow E_8/E_7 \xrightarrow{\pi} E_8/T^1{\cdot}E_7, \tag{5.1}$$

where π is the natural projection. In this case, it reduces to the following short exact sequeces:

$$
\begin{aligned}
0 \longrightarrow H^{\mathrm{odd}}(E_8/E_7;\mathbb{Z}) &\longrightarrow H^*(E_8/T^1{\cdot}E_7;\mathbb{Z}) \\
&\xrightarrow{\cup u} H^*(E_8/T^1{\cdot}E_7;\mathbb{Z}) \xrightarrow{\pi^*} H^{\mathrm{even}}(E_8/E_7;\mathbb{Z}) \longrightarrow 0,
\end{aligned} \tag{5.2}
$$

where $H^{\mathrm{even}} = \oplus_{i\geq 0}H^{2i}$ and $H^{\mathrm{odd}} = \oplus_{i\geq 0}H^{2i+1}$. From the exactness of (5.2), it follows that $H^{\mathrm{even}}(E_8/E_7;\mathbb{Z})$ is isomorphic to $H^*(E_8/T^1\cdot E_7;\mathbb{Z})/(u)$. Define the elements z_i ($i = 12, 20, 30$) of $H^*(E_8/E_7;\mathbb{Z})$ as

$$z_{12} = \pi^*(v),\ z_{20} = \pi^*(w),\ z_{30} = \pi^*(x).$$

Then, by Theorem 4.7, we obtain

$$
\begin{aligned}
H^{\mathrm{even}}(E_8/E_7;\mathbb{Z}) &= \mathbb{Z}[z_{12},z_{20},z_{30}]/(2z_{30}, 3z_{20}^2, 5z_{12}^4, 4z_{12}^5 + 2z_{20}^3 + 9z_{30}^2) \\
&= \mathbb{Z}[z_{12},z_{20},z_{30}]/(2z_{30}, 3z_{20}^2, 5z_{12}^4, z_{12}^5 + z_{20}^3 + z_{30}^2).
\end{aligned}
$$

By Poincaré duality, there exist elements $z_i \in H^i(E_8/E_7;\mathbb{Z})$ ($i = 59$, 71, 79, 83, 91, 95, 103, 115) such that

$$z_{12}^3 z_{20} z_{59} = z_{12}^2 z_{20} z_{71} = z_{12}^3 z_{79} = z_{12} z_{20} z_{83} = z_{12}^2 z_{91} = z_{20} z_{95} = z_{12} z_{103} = z_{115}.$$

Then it is not hard to show that

$$\begin{aligned} z_{71} &= z_{12} z_{59}, \\ z_{79} &= z_{20} z_{59}, \\ z_{83} &= z_{12}^2 z_{59}, \\ z_{91} &= z_{12} z_{20} z_{59}, \\ z_{103} &= z_{12}^2 z_{20} z_{59}, \\ z_{115} &= z_{12}^3 z_{20} z_{59}. \end{aligned}$$

Summing up the results, we obtain the following:

Corollary 5.1. *The structure of $H^*(E_8/E_7;\mathbb{Z})$ is given by the following table*:

$H^k(E_8/E_7;\mathbb{Z})$	elements
$H^0 \cong \mathbb{Z}$	1
$H^{12} \cong \mathbb{Z}$	z_{12}
$H^{20} \cong \mathbb{Z}$	z_{20}
$H^{24} \cong \mathbb{Z}$	z_{12}^2
$H^{30} \cong \mathbb{Z}/2\mathbb{Z}$	z_{30}
$H^{32} \cong \mathbb{Z}$	$z_{12}z_{20}$
$H^{36} \cong \mathbb{Z}$	z_{12}^3
$H^{40} \cong \mathbb{Z}/3\mathbb{Z}$	z_{20}^2
$H^{42} \cong \mathbb{Z}/2\mathbb{Z}$	$z_{12}z_{30}$
$H^{44} \cong \mathbb{Z}$	$z_{12}^2 z_{20}$
$H^{48} \cong \mathbb{Z}/5\mathbb{Z}$	z_{12}^4
$H^{50} \cong \mathbb{Z}/2\mathbb{Z}$	$z_{20}z_{30}$
$H^{52} \cong \mathbb{Z}/3\mathbb{Z}$	$z_{12}z_{20}^2$
$H^{54} \cong \mathbb{Z}/2\mathbb{Z}$	$z_{12}^2 z_{30}$
$H^{56} \cong \mathbb{Z}$	$z_{12}^3 z_{20}$
$H^{59} \cong \mathbb{Z}$	z_{59}
$H^{62} \cong \mathbb{Z}/2\mathbb{Z}$	$z_{12}z_{20}z_{30}$
$H^{64} \cong \mathbb{Z}/3\mathbb{Z}$	$z_{12}^2 z_{20}^2$
$H^{66} \cong \mathbb{Z}/2$	$z_{12}^3 z_{30}$
$H^{68} \cong \mathbb{Z}/5\mathbb{Z}$	$z_{12}^4 z_{20}$
$H^{71} \cong \mathbb{Z}$	$z_{12}z_{59}$
$H^{74} \cong \mathbb{Z}/2\mathbb{Z}$	$z_{12}^2 z_{20} z_{30}$
$H^{76} \cong \mathbb{Z}/3\mathbb{Z}$	$z_{12}^3 z_{20}^2$
$H^{79} \cong \mathbb{Z}$	$z_{20}z_{59}$
$H^{83} \cong \mathbb{Z}$	$z_{12}^2 z_{59}$
$H^{86} \cong \mathbb{Z}/2\mathbb{Z}$	$z_{12}^3 z_{20} z_{30}$
$H^{91} \cong \mathbb{Z}$	$z_{12}z_{20}z_{59}$
$H^{95} \cong \mathbb{Z}$	$z_{12}^3 z_{59}$
$H^{103} \cong \mathbb{Z}$	$z_{12}^2 z_{20} z_{59}$
$H^{115} \cong \mathbb{Z}$	$z_{12}^3 z_{20} z_{59}$

6. Appendix

6.1. **Generator of degree 15.** In 4.2, we defined the element x as a rational cohomology class given by

$$x = \frac{1}{2}u^{15} \quad \text{in} \quad H^{30}(E_8/T;\mathbb{Q}).$$

We need to show that x is in fact an integral cohomology class. By Lemma 4.2, the following relations hold in $H^*(E_8/T\ \mathbb{Z})$:

$$\begin{aligned} c_1 &= 3t, \\ c_2 &= 4t^2, \\ c_3 &= 2\gamma_3, \\ c_4 &= 3\gamma_4 - 2t^4, \\ c_5 &= 2\gamma_5 + 3t\gamma_4 - 2t^2\gamma_3, \\ c_6 &= 5\hat{\gamma}_6 + 2\gamma_3^2 + t\gamma_5 - t^2\gamma_4 + 2t^6. \end{aligned} \tag{6.1}$$

Note that, by (2.11) and (6.1), the following relation holds:

$$\begin{aligned} c_8 &= uc_7 - 5u^2\hat{\gamma}_6 - 2u^2\gamma_3^2 + \gamma_5(-tu^2 + 2u^3) + \gamma_4(t^2u^2 + 3tu^3 - 3u^4) \\ &\quad + \gamma_3(-2t^2u^3 + 2u^5) - 2t^6u^2 + 2t^4u^4 - 4t^2u^6 + 3tu^7 - u^8. \end{aligned} \tag{6.2}$$

Using (6.2), we can rewrite the higher relations $\rho_8, \rho_9, \rho_{10}, \rho_{12}, \rho_{14}$ and $\hat{\rho}_{15}$. For example,

$$\begin{aligned} \rho_8 &= -3c_8 + 3\gamma_4^2 - 2\gamma_3\gamma_5 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(2\gamma_3^2 - 5\hat{\gamma}_6) + 3t^3\gamma_5 + 4t^4\gamma_4 \\ &\quad -6t^5\gamma_3 + t^8 \\ &= 3\gamma_4^2 - 2\gamma_3\gamma_5 + c_7(2t - 3u) - 6t\gamma_3\gamma_4 + \hat{\gamma}_6(-5t^2 + 15u^6) \\ &\quad +\gamma_3^2(2t^2 + 6u^2) + \gamma_5(3t^3 + 3tu^2 - 6u^3) + \gamma_4(4t^4 - 3t^2u^2 - 9tu^3 + 9u^4) \\ &\quad +\gamma_3(-6t^5 + 6t^2u^3 - 6u^5) + t^8 + 6t^6u^2 - 6t^4u^4 + 12t^2u^6 - 9tu^7 + 3u^8. \end{aligned}$$

Using the relations ρ_i $(i = 1, 2, 3, 4, 5, 8, 9, 10, 12, 14)$, $\hat{\rho}_6$, $\hat{\rho}_{15}$, we can rewrite the element $x = \frac{1}{2}u^{15}$ as follows:

$$\begin{aligned} x &= \frac{1}{2}u^{15} \\ &= \frac{1}{2}\left\{u^{15} - \hat{\rho}_{15} + u\rho_{14} - u^3\rho_{12} + (3t^4u - 3t^2u^3)\rho_{10}\right. \\ &\quad \left.+(\hat{\gamma}_6 + (t+u)\gamma_5 + u^2\gamma_4 + u^3\gamma_3 + t^6 + t^4u^2 + t^3u^3 + t^2u^4 + tu^5 + u^6)\rho_9 - 39u^7\rho_8\right\} \\ &\quad + (-tu\gamma_3 + 2t^3u^2 - 5tu^4)\rho_{10} \end{aligned}$$

$$
\begin{aligned}
&= \hat\gamma_{15} - 20\gamma_3\hat\gamma_6^2 + 3\gamma_3^2\gamma_9 - 23\gamma_3^3\hat\gamma_6 - 6\gamma_3^5 + 4\hat\gamma_6\gamma_9\\
&+ 3u\gamma_4\gamma_{10} - u\gamma_5\gamma_9 - 3u\gamma_3^2\gamma_4^2 + 3uc_7\gamma_3\gamma_4 - 6u\gamma_4^2\hat\gamma_6 + (-3t+2u)\gamma_3^3\gamma_5 + (-4t+4u)\gamma_3\gamma_5\hat\gamma_6\\
&+ (-t^2-u^2)\gamma_4\gamma_9 + (t^2+tu-u^2)c_7\gamma_3^2 + (9t^2+12tu+5u^2)\gamma_3\gamma_4\hat\gamma_6\\
&+ (5t^2+6tu+2u^2)\gamma_3^3\gamma_4 + (3t^2+4tu+u^2)c_7\hat\gamma_6\\
&+ (-6t^3-2t^2u-6tu^2+5u^3)\gamma_3^4 - u^3\gamma_3\gamma_9 + (3t^2u+u^3)\gamma_4^3 + (2t^2u+3tu^2)c_7\gamma_5\\
&+ (-45t^3+10t^2u-40tu^2)\hat\gamma_6^2 + (t^3-2t^2u+tu^2-u^3)\gamma_3\gamma_4\gamma_5\\
&+ (-33t^3+t^2u-31tu^2+13u^3)\gamma_3^2\hat\gamma_6\\
&+ (-2t^4-4t^3u-3tu^3+3u^4)c_7\gamma_4 + (-9t^4-6t^3u-18t^2u^2+5tu^3-3u^4)\gamma_5\hat\gamma_6\\
&+ (-3t^4-3t^3u-7t^2u^2+5tu^3-4u^4)\gamma_3^2\gamma_5 + (-t^4-6t^3u-t^2u^2-3tu^3)\gamma_3\gamma_4^2\\
&+ (-3t^4u-6t^3u^2+3t^2u^3+15tu^4)\gamma_{10} + (-3t^4u+t^3u^2+5t^2u^3+10tu^4-u^5)c_7\gamma_3\\
&+ (15t^5-2t^4u+3t^3u^2+14t^2u^3-16tu^4+3u^5)\gamma_3^2\gamma_4\\
&+ (39t^5-13t^4u+8t^3u^2+35t^2u^3-31tu^4-3u^5)\gamma_4\hat\gamma_6\\
&+ (t^6-t^4u^2-t^3u^3-t^2u^4-tu^5-u^6)\gamma_9\\
&+ (-13t^6+12t^5u+5t^4u^2-56t^3u^3+8t^2u^4+21tu^5+2u^6)\gamma_3\hat\gamma_6\\
&+ (6t^6+3t^5u+2t^4u^2+7t^3u^3+t^2u^4-8tu^5+3u^6)\gamma_4\gamma_5\\
&+ (-8t^6+6t^5u+2t^4u^2-22t^3u^3+6t^2u^4+8tu^5-2u^6)\gamma_3^3\\
&+ (-6t^7+t^6u-7t^4u^3+5t^3u^4+3t^2u^5+3tu^6-63u^7)\gamma_4^2\\
&+ (-t^7+2t^6u+t^5u^2-11t^4u^3+6t^3u^4+5t^2u^5+6tu^6+39u^7)\gamma_3\gamma_5\\
&+ (2t^8+6t^7u+3t^6u^2-4t^5u^3-15t^4u^4+6t^3u^5+3t^2u^6-40tu^7+59u^8)c_7\\
&+ (3t^8+t^6u^2+11t^5u^3+14t^4u^4-20t^3t^5-4t^2u^6+118tu^7+3u^8)\gamma_3\gamma_4\\
&+ (-48t^9+3t^8u-41t^7u^2+18t^6u^3+16t^5u^4-13t^4u^5-67t^3u^6+125t^2u^7\\
&- 15tu^8-291u^9)\hat\gamma_6\\
&+ (-18t^9-3t^8u-16t^7u^2+10t^6u^3-4t^5u^4-8t^4u^5-16t^3u^6-23t^2u^7-10tu^8-115u^9)\gamma_3^2\\
&+ (-6t^{10}-3t^9u-9t^8u^2+5t^7u^3-5t^6u^4-14t^4u^6-52t^3u^7+6t^2u^8-60tu^9+117u^{10})\gamma_5\\
&+ (18t^{11}-3t^{10}u+5t^9u^2+11t^8u^3-28t^7u^4+8t^6u^5+20t^5u^6-64t^4u^7-15t^3u^8\\
&+ 54t^2u^9+178tu^{10}-177u^{11})\gamma_4\\
&+ (-2t^{12}+6t^{11}u+2t^{10}u^2-20t^9u^3+11t^8u^4+22t^7u^5-8t^6u^6+83t^5u^7\\
&+ 15t^4u^8+5t^3u^9-116t^2u^{10}+tu^{11}+117u^{12})\gamma_3\\
&- 12t^{15}-t^{14}u-10t^{13}u^2+6t^{12}u^3+7t^{11}u^4-13t^{10}u^5-31t^9u^6+9t^8u^7-t^7u^8\\
&- 118t^6u^9-18t^5u^{10}+131t^4u^{11}-6t^3u^{12}-233t^2u^{13}+175tu^{14}-58u^{15},
\end{aligned}
$$

which has shown that x is in fact an integral cohomology class.

Next, we have to show that

$$x \mod (u,v,w) \equiv \hat\gamma_{15}.$$

By (4.13) and (4.14), we have

$$
\begin{aligned}
v &\equiv 2\hat\gamma_6 + \gamma_3^2 - t^2\gamma_4 + t^6 \mod (u),\\
w &\equiv \gamma_{10} \mod (u,v).
\end{aligned}
$$

Therefore, in the ring $H^*(E_8/T;\mathbb{Z})/(u,v,w)$, the following relations hold:

$$(6.3)\qquad \begin{aligned} u &= 0,\\ \gamma_3^2 &= -2\hat{\gamma}_6 + t^2\gamma_4 - t^6,\\ \gamma_{10} &= 0. \end{aligned}$$

On the other hand, we determined the ring $H^*(E_8/T;\mathbb{Z})$ up to degrees ≤ 36 (Lemma 4.2). Taking (6.3) into account, we can show directly that $x = \hat{\gamma}_{15}$ in the ring $H^*(E_8/T;\mathbb{Z})/(u,v,w)$.

6.2. **Relations of $H^*(E_8/T;\mathbb{Z})$ in degrees ≤ 18.** In Lemma 4.2, we took the element $\hat{\gamma}_6$ as one of the ring generators of $H^*(E_8/T;\mathbb{Z})$, so that the relation of degree 12 is given as follows:

$$\hat{\rho}_6 = c_6 - 2\gamma_3^2 - t\gamma_5 + t^2\gamma_4 - 2t^6 - 5\hat{\gamma}_6.$$

Since $\tilde{\gamma}_6 = 2\hat{\gamma}_6 + \gamma_3^2 - t^2\gamma_4 + t^6$ (see Proposition 4.6), we have $\hat{\gamma}_6 = -2\tilde{\gamma}_6 + c_6 - t\gamma_5 - t^2\gamma_4$, and we can replace $\hat{\gamma}_6$ with $\tilde{\gamma}_6$ as a new generator, so that the relation of degree 12 changes to

$$\tilde{\rho}_6 = \gamma_3^2 + 2c_6 - 2t\gamma_5 - 3t^2\gamma_4 + t^6 - 5\tilde{\gamma}_6.$$

Using the element $\tilde{\gamma}_6$, other relations $\rho_8, \rho_{12}, \rho_{14}, \rho_{18}$ change to the following:

$$\begin{aligned}
\rho_8 &= 3\gamma_4^2 - 2\gamma_3\gamma_5 - 3c_8 + t(2c_7 - 6\gamma_3\gamma_4) + t^2(-9c_6 + 20\tilde{\gamma}_6) + 12t^3\gamma_5 + 15t^4\gamma_4\\
&\quad - 6t^5\gamma_3 - t^8,\\
\rho_{12} &= 3c_6^2 - 2\gamma_4^3 - 2c_7\gamma_5 + 2\gamma_3\gamma_4\gamma_5 + 10\tilde{\gamma}_6^2 - 10c_6\tilde{\gamma}_6 - 3c_8\gamma_4\\
&\quad + t(4c_7\gamma_4 - 2c_6\gamma_5 + 6\gamma_3\gamma_4^2 + c_8\gamma_3) + t^2(-3c_7\gamma_3 + 3c_6\gamma_4 - 20\gamma_4\tilde{\gamma}_6)\\
&\quad + t^3(-12\gamma_4\gamma_5 + 5c_6\gamma_3) + t^4(-2\gamma_3\gamma_5 - 15\gamma_4^2 + 2c_8) + t^6(-10c_6 + 20\tilde{\gamma}_6)\\
&\quad + 12t^7\gamma_5 + 19t^8\gamma_4 - 6t^9\gamma_3 - 2t^{12},\\
\rho_{14} &= c_7^2 + 6c_7\gamma_3\gamma_4 - 2c_6\gamma_3\gamma_5 + 5c_6c_8 - 14c_8\tilde{\gamma}_6 + 6\gamma_4\gamma_{10} + 4\gamma_3\gamma_5\tilde{\gamma}_6 - 6\gamma_4^2\tilde{\gamma}_6\\
&\quad + t(-c_8\gamma_5 + 12\gamma_3\gamma_4\tilde{\gamma}_6 - 4c_7\tilde{\gamma}_6) + t^2(-c_7\gamma_5 - 9c_8\gamma_4 - 40\tilde{\gamma}_6^2 + 18c_6\tilde{\gamma}_6)\\
&\quad + t^3(-9c_7\gamma_4 + 3c_6\gamma_5 + 10c_8\gamma_3 - 24\gamma_5\tilde{\gamma}_6) + t^4(-6c_7\gamma_3 - 6\gamma_{10} - 30\gamma_4\tilde{\gamma}_6)\\
&\quad + 12t^5\gamma_3\tilde{\gamma}_6 - 5t^6c_8 + 9t^7c_7 + 2t^8\tilde{\gamma}_6,
\end{aligned}$$

$$
\begin{aligned}
\rho_{18} = {} & \gamma_9^2 - 2c_6c_7\gamma_5 - 6c_7\gamma_3\gamma_4^2 + 2c_7^2\gamma_4 + 2c_6\gamma_3\gamma_4\gamma_5 - 2c_6\gamma_3\gamma_9 - c_8\gamma_4\tilde{\gamma}_6 \\
& + 30c_6\tilde{\gamma}_6^2 + 6c_7\gamma_5\tilde{\gamma}_6 - 9c_6^2\tilde{\gamma}_6 + 6\gamma_4^3\tilde{\gamma}_6 - 6\gamma_3\gamma_4\gamma_5\tilde{\gamma}_6 - 20\tilde{\gamma}_6^3 - 6\gamma_4^2\gamma_{10} \\
& - 9c_8\gamma_{10} + 2\gamma_3\gamma_5\gamma_{10} + c_6c_8\gamma_4 - 9c_7c_8\gamma_3 \\
& + t(6c_7^2\gamma_3 - 24c_6c_7\gamma_4 + 7c_8\gamma_4\gamma_5 + 6c_6\gamma_5\tilde{\gamma}_6 - c_8\gamma_9 + 6\gamma_3\gamma_4\gamma_{10} - 3c_8\gamma_3\tilde{\gamma}_6 \\
& + 2c_6c_8\gamma_3 + 10c_7\gamma_{10} + 48c_7\gamma_4\tilde{\gamma}_6 - 18\gamma_3\gamma_4^2\tilde{\gamma}_6) \\
& + t^2(25c_7\gamma_4\gamma_5 - c_7\gamma_9 + 18c_6c_7\gamma_3 - 9c_6\gamma_4\tilde{\gamma}_6 - 9c_8\gamma_4^2 + 60\gamma_4\tilde{\gamma}_6^2 - 20\tilde{\gamma}_6\gamma_{10} \\
& + 9c_6\gamma_{10} - 2c_8^2 - 31c_7\gamma_3\tilde{\gamma}_6 - 4c_8\gamma_3\gamma_5) \\
& + t^3(45c_7\gamma_4^2 - 20c_7\gamma_3\gamma_5 - 3c_6\gamma_4\gamma_5 + 3c_6\gamma_9 - 15c_6\gamma_3\tilde{\gamma}_6 + 11c_8\gamma_3\gamma_4 + 17c_7c_8 \\
& + 36\gamma_4\gamma_5\tilde{\gamma}_6 - 12\gamma_5\gamma_{10}) \\
& + t^4(-11c_7^2 - 2c_6\gamma_3\gamma_5 - 48c_7\gamma_3\gamma_4 - 9\gamma_4\gamma_{10} - 16c_8\tilde{\gamma}_6 + 5c_6c_8 + 45\gamma_4^2\tilde{\gamma}_6 + 6\gamma_3\gamma_5\tilde{\gamma}_6) \\
& + t^5(-51c_6c_7 - 11c_8\gamma_5 + 6\gamma_3\gamma_{10} + 120c_7\tilde{\gamma}_6) \\
& + t^6(53c_7\gamma_5 - 60\tilde{\gamma}_6^2 + 30c_6\tilde{\gamma}_6 - 10c_8\gamma_4) \\
& + t^7(69c_7\gamma_4 + 3c_6\gamma_5 + c_8\gamma_3 - 36\gamma_5\tilde{\gamma}_6) \\
& + t^8(-16c_7\gamma_3 - 2\gamma_{10} - 57\gamma_4\tilde{\gamma}_6) \\
& + 18t^9\gamma_3\tilde{\gamma}_6 + 2t^{10}c_8 - 15t^{11}c_7 + 6t^{12}\tilde{\gamma}_6.
\end{aligned}
$$

Remark 6.1. *By using the element $\tilde{\gamma}_6$, it is easy to see that the relations ρ'_i $(i = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18)$ in $H^*(E_7/T'; \mathbb{Z})$ (see Theorem 4.4) correspond to the relations ρ_i $(i = 1, 2, 3, 4, 5, 8, 9, 10, 12, 14, 18)$, $\tilde{\rho}_6$ in $H^*(E_8/T; \mathbb{Z})$ under the surjective homomorphism $i^* : H^*(E_8/T; \mathbb{Z}) \longrightarrow H^*(E_7/T'; \mathbb{Z})$ (just put $\tilde{\gamma}_6 = c_8 = \gamma_{10} = 0$ in the latter relations).*

References

[1] J. F. Adams, *Lectures on Lie groups*, W. A. Benjamin, Inc., New York-Amsterdam 1969.

[2] P. Baum, *On the cohomology of homogeneous spaces*, Topology **7** (1968), 15-38.

[3] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Schubert cells and the cohomology of the spaces G/P*, London Math. Soc. Lecture Note Ser., **69**, Cambridge Univ. Press, 1982, 115-140.

[4] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115-207.

[5] A. Borel, *Kählerian coset spaces of semisimple Lie groups*, Proc. Nat. Acad. Sci. USA **40** (1954), 1147-1151.

[6] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.

[7] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251-282.

[8] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35-61.

[9] R. Bott and H. Samelson, *Application of the theory of Morse to the symmetric spaces*, Amer. J. Math. **80** (1958), 964-1029.

[10] N. Bourbaki, *Groupes et Algèbre de Lie* IV – VI, Masson, Paris, 1968.

[11] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778-782.

[12] C. Chevalley, *Sur les décomposition cellulaires des espaces G/B*, Algebraic Groups and their Generalizations: Classical Methods (W. Haboush, ed.), Proc. Sympos. Pure Math., **56**, Part 1, Amer. Math. Soc., 1994, 1-23.

[13] P. E. Chaput, L. Manivel and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, Transform. Groups **13** (2008), 47-89.

[14] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287-301.
[15] H. Duan, *Multiplicative rule of Schubert classes*, Invent. Math. **159** (2005), 407-436.
[16] H. Duan and X. Zhao, *The Chow rings of generalized Grassmannians*, arXiv:math.AG/0511332.
[17] H. Duan and X. Zhao, *Algorithm for multiplying Schubert classes*, Internat. J. Algebra Comput. **16** (2006), 1197-1210.
[18] H. Duan and X. Zhao, *The integral cohomology of a complete flag manifold G/T*, arXiv:math.AT/08012444.
[19] C. Ehresmann, *Sur la topologie de certain espaces homogènes*, Ann. of Math. **35** (1934), 396-443.
[20] W. Fulton, *Young tableaux*, London Math. Soc. Student Texts **35** (1997), Cambridge Univ. Press.
[21] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1998.
[22] A. Grothendieck, *Torsion homologique et sections rationnelles*, Séminaire Claude Chevalley, 3 (1958), exposé 5, 1-29.
[23] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, **34**, American Mathematical Society, Providence, RI, 2001.
[24] J. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge Univ. Press, Cambridge, 1990.
[25] A. Iliev and L. Manivel, *The Chow ring of the Cayley plane*, Compos. Math. **141** (2005), 146-160.
[26] K. Ishitoya, *Integral cohomology ring of the symmetric space EII*, J. Math. Kyoto Univ. **17** (1977), 375-397.
[27] K. Ishitoya and H. Toda, *On the cohomology of irreducible symmetric spaces of exceptional type*, J. Math. Kyoto Univ. **17** (1977), 225-243.
[28] V. Kac, *Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*, Invent. Math. **80** (1985), 69-79.
[29] S. Kaji and M. Nakagawa, *Multiplicative structure of the Chow rings of projective homogeneous varieties*, in preparation.
[30] S. Kaji and M. Nakagawa, *The Chow rings of the complex algebraic groups* E_6*,* E_7*, and* E_8, in preparation.
[31] A. Kono and K. Ishitoya, *Squaring operations in mod 2 cohomology of quotients of compact Lie groups by maximal tori*, Algebraic Topology (Barcelona, 1986), 192-206, Lecture Notes in Math. **1298**, Springer (1987).
[32] R. Marlin, *Anneaux de Chow des groupes algebriques* $\mathrm{SO}(n), \mathrm{Spin}(n), \mathrm{G}_2$ *et* F_4, Publications Math. d'Orsay, 95-7419 (1974).
[33] J. McCleary, *A user's guide to spectral sequences* (*Second edition*), Cambridge Studies in Advanced Mathematics **58**, Cambridge Univ. Press, Cambridge, 2001.
[34] M. L. Mehta, *Basic sets of invariant polynomials for finite reflection groups*, Comm. Alg. **16** (5) (1988), 1083-1098.
[35] T. Nagano and M. Takeuchi, *Cohomology of quaternionic Kaehler manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 57-63.
[36] M. Nakagawa, *The integral cohomology ring of E_7/T*, J. Math. Kyoto Univ. **41** (2001), 303-321.
[37] M. Nakagawa, *The integral cohomology ring of E_8/T*, in preparation.
[38] S. Nikolenko and N. Semenov, *Chow ring structure made simple*, arXiv:math.AG/0606335.
[39] S. Nikolenko and N. Semenov and K. Zainoulline, *Motivic decomposition of anisotropic varieties of type* F_4 *into generalized Rost motives*, J. K-theory **3** (2009), 85-102.
[40] P. Pragacz, *Multiplying Schubert classes*, Topics in cohomological studies of algebraic varieties, 163-174, Trends Math., Birkhäuser, Basel, 2005.
[41] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143-171.
[42] H. Toda, *On the cohomology ring of some homogeneous spaces*, J. Math. Kyoto Univ. **15** (1975), 185-199.
[43] H. Toda and T. Watanabe, *The integral cohomology ring of F_4/T and E_6/T*, J. Math. Kyoto Univ. **14** (1974), 257-286.

[44] B, Totaro, *The torsion index of E_8 and other groups*, Duke Math. J. **129** (2005), 219-248.

[45] H. C. Wang, *Closed manifolds with homogeneous complex structures*, Amer. J. Math. **76** (1954), 1-32.

[46] T. Watanabe, *The integral cohomology ring of the symmetric space $EVII$*, J. Math. Kyoto Univ. **15** (1975), 363-385.

[47] T. Watanabe, *Cohomology of a homogeneous space $E_6/T^1 \cdot SU(6)$*, Group representations: cohomology, group actions and topology (Seattle, WA, 1996), 511-518, Proc. Sympos. Pure Math., **63**, Amer. Math. Soc., Providence, RI, 1998.

Department of General Education
Kagawa National College of Technology
Takamatsu 761-8058, Japan
E-mail address: nakagawa@t.kagawa-nct.ac.jp