

Association schemes related to universally optimal configurations, Kerdock codes and extremal Euclidean line-sets

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Abstract

H. Cohn et. al. proposed an association scheme of 64 points in \mathbb{R}^{14} which is conjectured to be a universally optimal code. We show that this scheme has a generalization in terms of Kerdock codes, as well as in terms of maximal real mutually unbiased bases. These schemes also related to extremal line-sets in Euclidean spaces and Barnes-Wall lattices. D. de Caen and E. R. van Dam constructed two infinite series of formally dual 3-class association schemes. We explain this formal duality by constructing two dual abelian schemes related to quaternary linear Kerdock and Preparata codes.

Keywords: universally optimal configurations, association schemes, dual schemes, Kerdock codes, Preparata codes, mutually unbiased bases, Barnes-Wall lattices.

1 Introduction

In [8] Henry Cohn and others defined and studied universally optimal configurations in Euclidean spaces, spherical point configurations that minimize broad class of functions like potential energy functions. They also obtained experimental results and conjectured that a some three class association scheme on 64 points determines universally optimal configuration in \mathbb{R}^{14} . This scheme is uniquely determined by their parameters [2] and has automorphism group $4^3 : (2 \times L_3(2))$, where $2 \times L_3(2)$ is the stabilizer of a point. It has the following first and second eigenmatrices:

$$P = Q = \begin{pmatrix} 1 & 14 & 42 & 7 \\ 1 & -6 & 6 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -2 & -6 & 7 \end{pmatrix}.$$

The scheme generates a configuration of 64 vectors on a sphere of squared radius 7 in \mathbb{R}^{14} . These vectors generate integral lattice with automorphism group $2^{14} : (2^3 : L_3(2))$. The lattice

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can be obtained by construction A from binary shortened projective [14,4,7] code. Theta series of the lattice is equal to $\theta(q) = 1 + 28q^4 + 1024q^7 + 2156q^8 + \dots$. So the 64 vectors are not minimal vectors of the lattice.

We show that the scheme has a generalization in terms of binary and quaternary Kerdock and Preparata codes, as well as in terms of maximal real mutually unbiased bases (MUB). Starting from doubly shortened binary (shortened quaternary) Kerdock code of length N (resp. of length $N/2$), where $N = 2^{m+1}$ with odd m , one can construct a 3-class association scheme of size $N^2/4$, which leads to a spherical code in \mathbb{R}^{N-2} of size $N^2/4$. As binary Kerdock code we understand a binary code of length N obtained from a Kerdock set [18], so Kerdock code is inside of a second order Reed-Muller code (see for details Section 3), and it has nonzero distances $2^m \pm 2^{(m-1)/2}$, 2^m , 2^{m+1} . Furthermore, we define Kerdock-like code as a binary code of length N with nonzero distances $(N \pm \sqrt{N})/2$, $N/2$ and N (in particular, Kerdock codes are Kerdock-like codes). It is an open question whether Kerdock-like code is actually a Kerdock code. We do not even know whether N must have the form 2^{m+1} .

Any collection of maximal mutually unbiased bases in \mathbb{R}^N is in one-to-one correspondence with Kerdock-like codes. In particular, any Kerdock code determines maximal MUB. We show that in fact any maximal MUB in \mathbb{R}^N (equivalently, any Kerdock-like code) determines a 3-class association scheme in \mathbb{R}^{N-2} of size $N^2/4$ with the same parameters as schemes obtained from Kerdock codes (Theorem 5).

D. de Caen and E. R. van Dam [6] constructed two infinite series of formally dual 3-class association schemes, related to Kerdock sets. We explain this formal duality by constructing two dual abelian schemes related to quaternary linear Kerdock and Preparata codes (Theorem 1). The situation is similar to one, when the formal duality between binary nonlinear Kerdock and Preparata codes was explained by duality between quaternary linear Kerdock and Preparata codes [17].

We also note that maximal real mutually unbiased bases determine a 4-class association scheme of size $N^2 + 2N$ in \mathbb{R}^N , as well as a 3-class association scheme of size $N^2/2$ in \mathbb{R}^{N-1} (which corresponds to the association scheme obtained from a shortened Kerdock-like code).

R. L. Griess Jr. showed [15] that the 64 point code and other tricosine codes can also be constructed using minimal vectors of Barnes-Wall lattice. We give later explanation of this phenomenon in terms of extremal line-sets and Kerdock codes.

It seems that the schemes on shortened Kerdock codes also might be candidates for being universally optimal (or optimal) configurations in $\mathbb{R}^{2^{m+1}-2}$. We note that in \mathbb{R}^{14} for θ -code, $\theta = \frac{1}{7}$, Levenshtein's bound is 69.6 (corresponding scheme has 64 points). Similarly, in \mathbb{R}^{62} for θ -code, $\theta = \frac{3}{31}$, Levenshtein's bound is 1081 (corresponding scheme has 1024 points).

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2 Quaternary Kerdock and Preparata codes and dual abelian association schemes

In this section we construct two dual abelian association schemes in terms of quaternary linear Kerdock and Preparata codes.

By quaternary linear Kerdock and Preparata codes we mean the class of codes determined in [7]. Recall that \mathbb{Z}_4 -linear Kerdock code K and \mathbb{Z}_4 -linear Preparata code P are linear codes over \mathbb{Z}_4 of length $q = 2^m = N/2$, $m \geq 3$, m odd:

$$0 \subset K \subseteq P \subset \mathbb{Z}_4^q.$$

They are dual codes: $K^\perp = P$. Moreover, $K = P$ for $m = 3$ and $K \neq P$ for $m > 3$. The image under the Gray map of the quaternary Kerdock (resp. Preparata) code is binary nonlinear Kerdock (resp. "Preparata") code. For $m = 3$ the Gray image of $K = P$ is the famous binary nonlinear Nordstrom-Robinson code of length 16. Consider shortened Kerdock and punctured Preparata codes:

$$0 \subset K_{\text{short}} \subseteq P_{\text{punct}} \subset \mathbb{Z}_4^{q-1} = A.$$

For a code C , the punctured code comes from deleting the coordinate at position i , and the shortened code from deleting the 0 at i from the words of the subcode of C having 0's at i . Since the automorphism group of Kerdock code acts transitively on coordinates we can consider shortening and puncturing at any fixed (same) position. Note that $K_{\text{short}}^\perp = P_{\text{punct}}$. Therefore, one has nondegenerate bilinear pairing

$$(K_{\text{short}}, A/P_{\text{punct}}) \rightarrow \mathbb{Z}_4,$$

which gives us duality between K_{short} and A/P_{punct} . We have $|K_{\text{short}}| = |A/P_{\text{punct}}| = q^2 = 4^m$. We can consider A/P_{punct} as character group of K_{short} or vice versa. Therefore, an abelian association scheme on K_{short} defines dual abelian scheme on A/P_{punct} (cosets of P_{punct}).

Shortened \mathbb{Z}_4 -Kerdock code [7, 17] is a code of length $2^m - 1$, m odd. It has $4^m = N^2/4$ codewords and nonzero codewords have Lee weights $2^m + 2^{(m-1)/2}$, $2^m - 2^{(m-1)/2}$, and 2^m . We are going to show that the following relations on the shortened Kerdock code will determine an abelian 3 class association scheme:

$$(x, y) \in \begin{cases} R_0, & \text{if } x - y \text{ has weight } 0, \\ R_1, & \text{if } x - y \text{ has weight } 2^m + 2^{(m-1)/2}, \\ R_2, & \text{if } x - y \text{ has weight } 2^m - 2^{(m-1)/2}, \\ R_3, & \text{if } x - y \text{ has weight } 2^m. \end{cases} \quad (1)$$

Cosets of punctured \mathbb{Z}_4 -Preparata code $C = P_{\text{punct}}$ have Lee weights 0, 1 and 2. Furthermore, for cosets $a + C$ we can choose $a = (0, \dots, 0, \pm 1, 0, \dots, 0)$, $a = (0, \dots, 0, +1, \dots, -1, 0, \dots, 0)$ or $a = (0, \dots, 0, 2, 0, \dots, 0)$ (see Lemma 2). The following relations

$$(x, y) \in \begin{cases} R'_0, & \text{if } x - y = C, \\ R'_1, & \text{if } x - y = (0, \dots, 0, \pm 1, 0, \dots, 0) + C, \\ R'_2, & \text{if } x - y = (0, \dots, 0, +1, \dots, -1, 0, \dots, 0) + C, \\ R'_3, & \text{if } x - y = (0, \dots, 0, 2, 0, \dots, 0) + C. \end{cases} \quad (2)$$

on A/C will define a three class association scheme which is dual to the previous scheme.

Theorem 1 *The relations (1) on codewords of shortened \mathbb{Z}_4 -Kerdock code define a three class (abelian) association scheme, with the first and the second eigenmatrices given by:*

$$P = \begin{pmatrix} 1 & \frac{(N-2\sqrt{N})(N-2)}{8} & \frac{(N+2\sqrt{N})(N-2)}{8} & \frac{N}{2} - 1 \\ 1 & -\frac{\sqrt{N}(N-4)}{8} & \frac{\sqrt{N}(N-4)}{8} & -1 \\ 1 & \frac{\sqrt{N}}{2} & -\frac{\sqrt{N}}{2} & -1 \\ 1 & -\frac{N-2\sqrt{N}}{4} & -\frac{N+2\sqrt{N}}{4} & \frac{N}{2} - 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & N - 2 & \frac{(N-2)(N-4)}{4} & \frac{N}{2} - 1 \\ 1 & -\sqrt{N} - 2 & \sqrt{N} + 2 & -1 \\ 1 & \sqrt{N} - 2 & -\sqrt{N} + 2 & -1 \\ 1 & -2 & -\frac{N}{2} + 2 & \frac{N}{2} - 1 \end{pmatrix}.$$

The relations (2) on cosets of punctured \mathbb{Z}_4 -Preparata code define an association scheme which is dual to the former scheme, so the scheme has the following first and second eigenmatrices: $P' = Q$, $Q' = P$.

If we take the particular class of Kerdock and Preparata codes considered in [17] then the scheme has automorphism group $4^m : Aut(K_{\text{short}})$, where $Aut(K_{\text{short}}) \cong 2 \times L_3(2)$ for $m = 3$ and $Aut(K_{\text{short}}) \cong 2 \times (\mathbb{F}_{2^m}^* : Aut(\mathbb{F}_{2^m}))$ for $m > 3$. We also note that the abelian group K_{short} is isomorphic to the Galois ring $GR(4, m) = \mathbb{Z}_4[\xi]$, $\xi^{q-1} = 1$, $q = 2^m$. Isomorphism is given by map $\gamma \mapsto (Tr(\gamma\xi^0), Tr(\gamma\xi^1), \dots, Tr(\gamma\xi^{q-2}))$, where $\gamma \in GR(4, m)$ (see for details [17]).

At first we study the structure of cosets of punctured \mathbb{Z}_4 -Preparata code $C = P_{\text{punct}}$.

Lemma 2 a) There exists a partition $A/C = V_0 \cup V_1 \cup V_2 \cup V_3$ into four sets, where V_0, V_1, V_2 and V_3 are cosets of the form C , $(0, \dots, 0, \pm 1, 0, \dots, 0) + C$, $(0, \dots, 0, +1, 0, \dots, 0, -1, 0, \dots, 0) + C$ and $(0, \dots, 2, \dots, 0) + C$, respectively.

b) The previous statement remains true if the set V_2 is enumerated by elements of the form $(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0) + C$ and $(0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0) + C$.

c) Finally, V_2 can be enumerated by elements of the form $(0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0) + C$.

Proof. a) First we note that $|V_0| = 1$, $|V_1| = 2(q-1)$, $|V_2| = (q-1)(q-2)$, $|V_3| = q-1$, so $|V_0| + |V_1| + |V_2| + |V_3| = q^2 = |A/C|$. It remains for us to prove that elements of all V_i are different. Minimum Lee weight of \mathbb{Z}_4 -Preparata code P is 6, therefore minimum Lee weight of punctured code C is either 4 (if there is an element $(2, \pm 1, \pm 1, \pm 1, \pm 1, 0, \dots, 0) \in P$) or 5. If the elements of V_i are not different, then C contains elements of the form $(1, 1, -1, -1, 0, \dots, 0)$ or $(2, 1, -1, 0, \dots, 0)$, which means $(2, 1, 1, -1, -1, 0, \dots, 0) \in P$ or $(2, 2, 1, -1, 0, \dots, 0) \in P$. These cases are not possible, since codewords of P are zero sum vectors $((1, \dots, 1) \in K$ by definition and $P = K^\perp$) and the codeword $(0, 0, 2, 2, 0, \dots, 0)$ of weight 4 does not belong to P .

Parts b) and c) can be proved similarly. \square

Proof of Theorem 1. First we prove that the relations (2) on cosets of punctured \mathbb{Z}_4 -Preparata code define an association scheme. Consider the following elements in $\mathbb{C}[A/C]$ (the group ring of the group A/C over the field of complex numbers):

$$D_i = \sum_{v \in V_i} Z^v, \quad 0 \leq i \leq 3.$$

We will show that the subalgebra $\langle D_0, D_1, D_2, D_3 \rangle$, generated by elements D_0, D_1, D_2 and D_3 , is Schurian.

The following equalities are obtained just from counting elements in D_i :

$$\begin{aligned} D_1 \cdot (D_0 + D_1 + D_2 + D_3) &= 2(q-1)(D_0 + D_1 + D_2 + D_3), \\ D_2 \cdot (D_0 + D_1 + D_2 + D_3) &= (q-1)(q-2)(D_0 + D_1 + D_2 + D_3), \\ D_3 \cdot (D_0 + D_1 + D_2 + D_3) &= (q-1)(D_0 + D_1 + D_2 + D_3), \end{aligned}$$

Further, we have

$$(D_0 + D_3)^2 = (D_0 + D_3)(D_0 + D_3) = q(D_0 + D_3),$$

which implies

$$D_3^2 = (q-1)D_0 + (q-2)D_3.$$

Lemma 2 b) implies that

$$D_1^2 = 2(q-1)D_0 + 4D_2 + 2D_3.$$

Finally, by lemma 2 c) we have

$$D_3D_1 = D_1 + 2D_2.$$

Matrices of multiplications by D_1, D_2, D_3 with respect to basis D_0, D_1, D_2, D_3 are given by

$$\rho_1 = \begin{pmatrix} 0 & 2q-2 & 0 & 0 \\ 1 & 0 & 2(q-2) & 1 \\ 0 & 4 & 2(q-4) & 2 \\ 0 & 2 & 2(q-2) & 0 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 0 & 0 & (q-1)(q-2) & 0 \\ 0 & 2(q-2) & (q-4)(q-2) & q-2 \\ 1 & 2(q-4) & q^2-6q+12 & q-3 \\ 0 & 2(q-2) & (q-3)(q-2) & 0 \end{pmatrix},$$

$$\rho_3 = \begin{pmatrix} 0 & 0 & 0 & q-1 \\ 0 & 1 & q-2 & 0 \\ 0 & 2 & q-3 & 0 \\ 1 & 0 & 0 & q-2 \end{pmatrix}.$$

It is easy to see that vectors $v_1 = (1, 2q-2, (q-1)(q-2), q-1)$, $v_2 = (1, -\sqrt{2q}-2, \sqrt{2q}+2, -1)$, $v_3 = (1, \sqrt{2q}-2, -\sqrt{2q}+2, -1)$, $v_4 = (1, -2, -q+2, q-1)$ are common left eigenvectors for matrices $\rho_i = {}^t B_i$. They are strokes of the matrix

$$P' = Q = \begin{pmatrix} 1 & 2q-2 & (q-1)(q-2) & q-1 \\ 1 & -\sqrt{2q}-2 & \sqrt{2q}+2 & -1 \\ 1 & \sqrt{2q}-2 & -\sqrt{2q}+2 & -1 \\ 1 & -2 & -q+2 & q-1 \end{pmatrix}.$$

Now we are going to prove that relations (1) define an association scheme dual to scheme (2). The latter abelian scheme determines a dual scheme on K_{short} , with partition $K_{\text{short}} = V'_0 \cup V'_1 \cup V'_2 \cup V'_3$. We prove that this partition corresponds to sets of codewords of Lee weights $0, 2^m + 2^{(m-1)/2}, 2^m - 2^{(m-1)/2}$, and 2^m respectively.

Indeed, according to [11, section 4.7.1], for any element $u \in V'_j$ we should have

$$Q_{jk} = \sum_{v \in V_k} i^{-(v,u)},$$

where $i = \sqrt{-1}$. Such equations determine elements in V'_j uniquely. For example, let us take codeword $u \in K_{\text{short}}$ of Lee weight $2^m + 2^{(m-1)/2}$ (other cases can be considered analogously). Then codeword u has form $(2^a, 1^b, (-1)^c, 0^{q-1-a-b-c})$, $2a + b + c = 2^m + 2^{(m-1)/2}$. Since $2u = (0^a, 2^b, 2^c, 0^{q-1-a-b-c}) \in K_{\text{short}}$, we have $b + c = 2^{m-1}$ and $a = (q + \sqrt{2q})/4$.

For $k = 1$ we have $V_1 = \{(0, \dots, 0, \pm 1, 0, \dots, 0) + C\}$, $|V_1| = 2(q-1)$, and

$$\sum_{v \in V_1} i^{-(v,u)} = \frac{q + \sqrt{2q}}{4} i^2 + b i^{-1} + c i + \left(\frac{q - \sqrt{2q}}{4} - 1 \right) +$$

$$\frac{q + \sqrt{2q}}{4} i^2 + b i + c i^{-1} + \left(\frac{q - \sqrt{2q}}{4} - 1 \right) = -\sqrt{2q} - 2.$$

Therefore, $u \in V'_1$.

Finally we show that one can use a shortening (puncturing) at any position. It is enough to show that the automorphism group of Kerdock code is transitive on coordinates. It just follows from the definition of \mathbb{Z}_4 -Kerdock code [7]. Let V be a m -dimensional vector space over \mathbb{F}_2 and R be a binary symmetric $m \times m$ matrix. The map $T_R : V \rightarrow \mathbb{Z}_4$ is given by

$$T_R(v) = \sum_{i=1}^m R_{ii} \widehat{v}_i^2 + 2 \sum_{i < j} R_{ij} \widehat{v}_i \widehat{v}_j,$$

where the entries of the matrix R are identified with the elements 0, 1 of \mathbb{Z}_4 and $\widehat{v} \in \mathbb{Z}_4^m$ is congruent to $v \in V$ modulo 2 (such map is called \mathbb{Z}_4 -valued quadratic form on V). It is easy to see that

$$T_R(u + v) = T_R(u) + T_R(v) + 2\widehat{u}R\widehat{v}^T. \quad (3)$$

Let S be a set of 2^m binary symmetric $m \times m$ matrices such that the difference of any two is non-singular. Then the associated \mathbb{Z}_4 -Kerdock code is defined as the set

$$\{(T_R(v) + 2\widehat{s} \cdot \widehat{v} + \varepsilon)_v \mid R \in S, s \in V, \varepsilon \in \mathbb{Z}_4\}$$

Now (3) implies that the automorphism group of a Kerdock code is transitive on coordinates, since each translation $v \mapsto v + u$ of V is an automorphism of Kerdock code.

Theorem is proved. \square

3 Binary Kerdock codes and association schemes

In this section we discuss binary Kerdock codes and related association schemes.

Recall the definition of binary nonlinear Kerdock codes [7]. Let V be a $(m + 1)$ -dimensional vector space over \mathbb{F}_2 , m odd. Any polynomial function f on V can be considered as codeword of length 2^{m+1} , evaluating f on all vectors of V : $(f(v_1), \dots, f(v_{2^{m+1}}))$. The set of all polynomials of degree at most k is called k th order Reed-Muller code $RM(k, m + 1)$. The first order Reed-Muller code $RM(1, m + 1)$ consists of affine functions.

The second order Reed-Muller code $RM(2, m + 1)$ consists of elements $Q + f$, where $f \in RM(1, m + 1)$ and Q is a quadratic form. A quadratic form is a map $Q : V \mapsto \mathbb{F}_2$ such that $Q(0) = 0$ and

$$B_Q(x, y) = Q(x + y) + Q(x) + Q(y)$$

is bilinear. For quadratic form Q and affine function f the element $Q + f \in RM(2, m + 1)$ has $2^m \pm 2^{(m-1+r)/2}$ zeroes, where r is the dimension of the radical of the bilinear form B_Q , associated with Q .

A quadratic form is called non-singular if its associated bilinear form is non-singular. Non-singular form Q has a type $\chi(Q) = \pm 1$, where Q has precisely $2^m + \chi(Q)2^{(m-1)/2}$ zeroes. The projective quadrics associated to non-singular quadratic forms with $\chi(Q) = +1$ are called hyperbolic, those with $\chi(Q) = -1$ elliptic.

A Kerdock set is a set of 2^m alternating bilinear forms (or, equivalently, binary skew-symmetric $(m + 1) \times (m + 1)$ matrices) such that the difference of any two is non-singular. Let S be a Kerdock set. Then the binary Kerdock code $\mathcal{K}(S)$ is a set of elements $Q + f$, where $f \in RM(1, m + 1)$ and Q is a quadratic form such that its associated bilinear form B_Q belongs to S . So Kerdock code \mathcal{K} lives in $RM(2, m + 1)$ and contains $RM(1, m + 1)$:

$$RM(1, m + 1) \subset \mathcal{K} \subset RM(2, m + 1).$$

Actually the quadratic forms, generating a Kerdock code, are chosen so that the minimum distance between any of two of the cosets $Q + RM(1, m + 1)$ and $Q' + RM(1, m + 1)$ is as large as possible, namely $2^m - 2^{(m-1)/2}$, which occurs if and only if $Q + Q'$ is non-singular; and Kerdock set provides the maximum possible number of such cosets.

In [7] it was shown that any binary Kerdock code can be obtained from a \mathbb{Z}_4 -Kerdock code with the aid of Gray map. If this code is \mathbb{Z}_4 -linear then its shortened code defines an association scheme, as we showed in the previous section. Actually the structure of an association scheme can be defined for any doubly shortened binary Kerdock code and, therefore, for any shortened \mathbb{Z}_4 -Kerdock code.

Proposition 3 *Let C be a shortened in two (arbitrary) positions binary Kerdock code. Then distinct codewords in C have only distances $2^m + 2^{(m-1)/2}$, $2^m - 2^{(m-1)/2}$, 2^m and these values determine three class association scheme on $q^2 = N^2/4$ points with the first and second eigenmatrices P and Q .*

Proof. It follows from Theorem 8 in [6]. There it was constructed five class association schemes, and by fusion ($R_1 \cup R_2$, $R_3 \cup R_4$, R_5) one can get a three class schemes with required parameters. We need to show that vertices of that scheme are actually codewords of doubly shortened Kerdock code. Indeed, the vertex set consists of all ordered pairs (B, Q) , where Q is a quadratic form, such that its associated bilinear form B belongs to a fixed Kerdock set S , and $Q(v) = 0$ for a fixed vector in V . Actually we can remove B from the description of the vertex set, since it can be determined uniquely from Q . Therefore the vertex set consists of quadratic forms Q , which are codewords from the Kerdock code, defined from Kerdock set S . Moreover, we have $Q(0) = Q(v) = 0$, so these codewords determine doubly shortened (in 0 and v) code.

Further, we note that if $Q \not\equiv Q' \pmod{RM(1, m + 1)}$ then distance

$$d(Q, Q') = \#\text{nonzeros}(Q + Q') = 2^{m+1} - (2^m + \chi(Q + Q')2^{(m-1)/2}) = 2^m - \chi(Q + Q')2^{(m-1)/2},$$

so

$$d(Q, Q') = 2^m \pm 2^{(m-1)/2} \Leftrightarrow \chi(Q + Q') = \mp 1.$$

If $Q \equiv Q' \pmod{RM(1, m + 1)}$ then $d(Q, Q') = 2^m$.

Finally, we note that the automorphism group of a Kerdock code is transitive on coordinates, since each translation $u \mapsto u + a$ of V is an automorphism of Kerdock code (it leaves invariant each coset $Q + RM(1, m + 1)$). Using this automorphism and taking $v = b - a$ we can send the pair $(0, v)$ to any fixed pair (a, b) . \square

One might ask what happens if we consider a shortened binary Kerdock code.

Proposition 4 *Let C be a shortened binary Kerdock code. Then distinct codewords in C have only distances $2^m + 2^{(m-1)/2}$, $2^m - 2^{(m-1)/2}$, 2^m and these values determine three class association scheme on $2q^2 = N^2/2$ points with the following first and second eigenmatrices:*

$$P = \begin{pmatrix} 1 & \frac{(N-\sqrt{N})(N-2)}{4} & \frac{(N+\sqrt{N})(N-2)}{4} & N-1 \\ 1 & -\frac{\sqrt{N}(N-2)}{4} & \frac{\sqrt{N}(N-2)}{4} & -1 \\ 1 & \frac{\sqrt{N}}{2} & -\frac{\sqrt{N}}{2} & -1 \\ 1 & -\frac{N-\sqrt{N}}{2} & -\frac{N+\sqrt{N}}{2} & N-1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & N-1 & \frac{(N-2)(N-1)}{2} & \frac{N}{2}-1 \\ 1 & -\sqrt{N}-1 & \sqrt{N}+1 & -1 \\ 1 & \sqrt{N}-1 & -\sqrt{N}+1 & -1 \\ 1 & -1 & -\frac{N}{2}+1 & \frac{N}{2}-1 \end{pmatrix}.$$

Proof. See Delsarte's thesis [12], Example 2 on page 82. \square

In his thesis [12, Example 2 on page 82] Delsarte considers a binary code Y of length $N - 1$ with three nonzero distances $2^m \pm 2^{(m-1)/2}$, 2^m . He shows that $|Y| \leq N^2/2$, so shortened Kerdock code is optimal in the sense that it is a code with maximal possible number of codewords among binary codes of length $N - 1$ with mentioned three nonzero distances. We repeated his reasoning for a doubly shortened Kerdock code. Let us take a binary code Y of length $N - 2$ with three nonzero distances $2^m \pm 2^{(m-1)/2}$, 2^m . Calculations show that

$$\alpha(z) = 4N^{-2}|Y|(\Phi_0(z) + \frac{N+2}{2N-2}\Phi_1(z) + \frac{3}{N-1}\Phi_2(z) + \frac{3}{2N-2}\Phi_3(z)),$$

where notations are taken from [12]. Then theorem 5.23(ii) from [12] implies that $|Y| \leq N^2/4$. It means that doubly shortened Kerdock code is a code with maximal possible number of codewords among binary codes of length $N - 2$ with three nonzero distances $2^m \pm 2^{(m-1)/2}$, 2^m .

4 Mutually unbiased bases and association schemes

In the previous sections, in particular in Theorem 1 and Proposition 3, we have constructed 3-class association schemes Y in \mathbb{R}^{N-2} of size $|Y| = N^2/4$, with $N = 2^{m+1}$ and m odd, from Kerdock codes. These association schemes are generalizations of the association scheme of 64 points in \mathbb{R}^{14} which is a candidate of universal optimal code considered by Cohn and Kumar [9]. In this section we will show that from such Kerdock codes we can obtain the specific line systems in real Euclidean space \mathbb{R}^N , or equivalently, maximal real MUB (mutually unbiased bases) in real Euclidean space \mathbb{R}^N . (The reader is referred to [4] for the details on MUB.) It is an interesting question whether any such line system in real Euclidean space \mathbb{R}^N can be obtained from a Kerdock code, but this question is still open. We will show also that from any maximal real MUB, or equivalently from any such line system in real Euclidean space \mathbb{R}^N , we obtain an association scheme which has the same parameters as those in the association scheme obtained in Theorem 1. So, this result (Theorem 5 below) is a generalization of Proposition 3. Our proof is based on the method due to Delsarte-Goethals-Seidel [14], and does not depend on the specific structures of the codes.

Two orthonormal bases \mathcal{B} and \mathcal{B}' in \mathbb{R}^N are called mutually unbiased if $|(x, y)| = 1/\sqrt{N}$ for any $x \in \mathcal{B}$ and $y \in \mathcal{B}'$. It is known [7, 13] that there can be at most $N/2 + 1$ mutually unbiased bases in dimension N , and constructions reaching this upper bound are known only for values $N = 2^{m+1}$. Let us define Kerdock-like codes as binary codes of length N with N^2 codewords and with nonzero distances $(N \pm \sqrt{N})/2$, $N/2$ and N (in particular, Kerdock codes are Kerdock-like codes). Any collection of $N/2 + 1$ mutually unbiased bases is in one-to-one correspondence with Kerdock-like codes. Indeed, let $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{N/2}$ be mutually unbiased bases, \mathcal{B}_0 be the standard orthonormal basis. Consider the set L of all vectors $x \in \mathbb{R}^N$ such that x or $-x$ belongs to one of bases $\mathcal{B}_1, \dots, \mathcal{B}_{N/2}$. Then $|L| = N^2$. The elements of L have form $(\pm 1/\sqrt{N}, \dots, \pm 1/\sqrt{N})$, since \mathcal{B}_0 and \mathcal{B}_i ($i > 0$) are mutually unbiased. One can convert L to a binary code C by changing $1/\sqrt{N}$ and $-1/\sqrt{N}$ to 0 and 1, respectively. Then it is easy to see that C is a Kerdock-like code (the conditions $(x, y) = \pm 1/\sqrt{N}$, 0, 1 for $x, y \in L$ mean exactly that distances $d(x', y') = (N \mp \sqrt{N})/2$, $N/2$, N for corresponding images $x', y' \in C$).

In the construction of association schemes we used doubly shortened binary Kerdock codes. Now we do similar procedure for mutually unbiased bases: take vectors from L with $1/\sqrt{N}$ in two fixed coordinates and then delete these coordinates. We will get a configuration of $N^2/4$ vectors in \mathbb{R}^{N-2} of equal length, such that cosines of angles between distinct vectors are equal to $\frac{-\sqrt{N-2}}{N-2}$, $\frac{-2}{N-2}$ and $\frac{\sqrt{N-2}}{N-2}$.

Theorem 5 Let M be a maximal mutually unbiased bases in \mathbb{R}^N and $X = M \cup (-M)$. For $u, v \in X$ such that $(u, v) = 0$, we put $Y' := \{x \in X \mid (u, x) = (v, x) = \frac{1}{\sqrt{N}}\}$, and let Y be the orthogonal projection of vectors Y' to $\langle u, v \rangle^\perp$ rescaled to make a spherical code. Then Y is an association scheme with the first and second eigenmatrices given by:

$$P = \begin{pmatrix} 1 & \frac{(N-2\sqrt{N})(N-2)}{8} & \frac{(N+2\sqrt{N})(N-2)}{8} & \frac{N}{2} - 1 \\ 1 & -\frac{\sqrt{N}(N-4)}{8} & \frac{\sqrt{N}(N-4)}{8} & -1 \\ 1 & \frac{\sqrt{N}}{2} & -\frac{\sqrt{N}}{2} & -1 \\ 1 & -\frac{N-2\sqrt{N}}{4} & -\frac{N+2\sqrt{N}}{4} & \frac{N}{2} - 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & N-2 & \frac{(N-2)(N-4)}{4} & \frac{N}{2} - 1 \\ 1 & -\sqrt{N}-2 & \sqrt{N}+2 & -1 \\ 1 & \sqrt{N}-2 & -\sqrt{N}+2 & -1 \\ 1 & -2 & -\frac{N}{2}+2 & \frac{N}{2} - 1 \end{pmatrix}.$$

Proof. We put $\alpha = \frac{-\sqrt{N}-2}{N-2}$, $\beta = \frac{\sqrt{N}-2}{N-2}$, $\gamma = \frac{-2}{N-2}$. Since $A(Y) = \{\alpha, \beta, \gamma\}$, Y is a 3-distance set. The annihilator polynomial $F(x) := \prod_{\alpha \in A(Y)} \frac{x-\alpha}{1-\alpha}$ has the Gegenbauer polynomial expansion

$$F(x) = \frac{4}{N^2}Q_0(x) + \frac{2(N^2+6)(N-2)}{N^3(N-1)}Q_1(x) + \frac{(N-2)^3(N+3)}{N^3(N-1)}Q_2(x) + \frac{6(N-2)(N-3)}{N^2(N-1)}Q_3(x).$$

As $|Y| = N^2/4$, Theorem 6.5 of [14] implies that Y is a spherical 3-design.

By Lemma 7.3 of [14], for $0 \leq i, j \leq 2$, $i+j \neq 4$ and $z := (\xi, \eta)$, the intersection numbers $p_{\alpha, \beta}(\xi, \eta)$ satisfy the linear equation

$$\sum_{x, y \in A(Y)} x^i y^j p_{x, y}(\xi, \eta) = \frac{N^2}{4} F_{i, j}(z) - z^i - z^j + \delta_{1, z},$$

where $t^i = \sum_{k=0}^i f_{i, k} Q_k(t)$ and $F_{i, j}(t) = \sum_{k=0}^{\min(i, j)} f_{i, k} f_{j, k} Q_k(t)$.

Now let $z = (\xi, \eta)$ be fixed. Then we get following equation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \alpha & \beta & \gamma & \alpha & \beta \\ \alpha^2 & \beta^2 & \gamma^2 & \alpha^2 & \beta^2 & \gamma^2 & \alpha^2 & \beta^2 \\ \alpha & \alpha & \alpha & \beta & \beta & \beta & \gamma & \gamma \\ \alpha^2 & \alpha\beta & \alpha\gamma & \alpha\beta & \beta^2 & \beta\gamma & \alpha\gamma & \beta\gamma \\ \alpha^3 & \alpha\beta^2 & \alpha\gamma^2 & \alpha^2\beta & \beta^3 & \beta\gamma^2 & \alpha^2\gamma & \beta^2\gamma \\ \alpha^2 & \alpha^2 & \alpha^2 & \beta^2 & \beta^2 & \beta^2 & \gamma^2 & \gamma^2 \\ \alpha^3 & \alpha^2\beta & \alpha^2\gamma & \alpha\beta^2 & \beta^3 & \beta^2\gamma & \alpha\gamma^2 & \beta\gamma^2 \end{pmatrix} \begin{pmatrix} p_{\alpha, \alpha}(\xi, \eta) \\ p_{\beta, \alpha}(\xi, \eta) \\ p_{\gamma, \alpha}(\xi, \eta) \\ p_{\alpha, \beta}(\xi, \eta) \\ p_{\beta, \beta}(\xi, \eta) \\ p_{\gamma, \beta}(\xi, \eta) \\ p_{\alpha, \gamma}(\xi, \eta) \\ p_{\beta, \gamma}(\xi, \eta) \end{pmatrix} =$$

$$\begin{pmatrix} F_{0,0}(z) - z^0 - z^0 + \delta_{1,z} - \gamma^0 p_{\gamma, \gamma}(\xi, \eta) \\ F_{1,0}(z) - z^1 - z^0 + \delta_{1,z} - \gamma^1 p_{\gamma, \gamma}(\xi, \eta) \\ F_{2,0}(z) - z^2 - z^0 + \delta_{1,z} - \gamma^2 p_{\gamma, \gamma}(\xi, \eta) \\ F_{0,1}(z) - z^0 - z^1 + \delta_{1,z} - \gamma^1 p_{\gamma, \gamma}(\xi, \eta) \\ F_{1,1}(z) - z^1 - z^1 + \delta_{1,z} - \gamma^2 p_{\gamma, \gamma}(\xi, \eta) \\ F_{2,1}(z) - z^2 - z^1 + \delta_{1,z} - \gamma^3 p_{\gamma, \gamma}(\xi, \eta) \\ F_{0,2}(z) - z^0 - z^2 + \delta_{1,z} - \gamma^2 p_{\gamma, \gamma}(\xi, \eta) \\ F_{1,2}(z) - z^1 - z^2 + \delta_{1,z} - \gamma^3 p_{\gamma, \gamma}(\xi, \eta) \end{pmatrix},$$

where

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \alpha & \beta & \gamma & \alpha & \beta \\ \alpha^2 & \beta^2 & \gamma^2 & \alpha^2 & \beta^2 & \gamma^2 & \alpha^2 & \beta^2 \\ \alpha & \alpha & \alpha & \beta & \beta & \beta & \gamma & \gamma \\ \alpha^2 & \alpha\beta & \alpha\gamma & \alpha\beta & \beta^2 & \beta\gamma & \alpha\gamma & \beta\gamma \\ \alpha^3 & \alpha\beta^2 & \alpha\gamma^2 & \alpha^2\beta & \beta^3 & \beta\gamma^2 & \alpha^2\gamma & \beta^2\gamma \\ \alpha^2 & \alpha^2 & \alpha^2 & \beta^2 & \beta^2 & \beta^2 & \gamma^2 & \gamma^2 \\ \alpha^3 & \alpha^2\beta & \alpha^2\gamma & \alpha\beta^2 & \beta^3 & \beta^2\gamma & \alpha\gamma^2 & \beta\gamma^2 \end{pmatrix} = (\alpha - \beta)^6(\alpha - \gamma)^4(\beta - \gamma)^4 \neq 0.$$

In case of $(\xi, \eta) = 1$ we have $p_{\gamma, \gamma}(\xi, \eta) = N/2 - 1$. In case of $(\xi, \eta) = \alpha$ or β , we have $p_{\gamma, \gamma}(\xi, \eta) = 0$. Finally, in case of $(\xi, \eta) = \gamma$ we have $p_{\gamma, \gamma}(\xi, \eta) = N/2 - 2$.

Therefore, intersection numbers $p_{x,y}(\xi, \eta)$ are determined uniquely by $p_{\gamma, \gamma}(\xi, \eta)$. Hence Y is an association scheme. This completes the proof of Theorem 5. \square

Remark. In Theorem 7.4 of [14] it is mentioned that if $t \geq 2s - 3$, then for any fixed $z = \tilde{\gamma} = (\xi, \eta)$, the intersection numbers $p_{\tilde{\alpha}, \tilde{\beta}}(\xi, \eta)$ are uniquely determined by $p_{\tilde{\gamma}, \tilde{\gamma}}(\xi, \eta)$. Our claim is that intersection numbers $p_{\tilde{\alpha}, \tilde{\beta}}(\xi, \eta)$ are uniquely determined by $p_{\gamma, \gamma}(\xi, \eta)$ with a suitable γ which is not necessarily equal to $\tilde{\gamma}$. So, our argument is slightly general than in Theorem 7.4 of [14].

Here we mention some more useful information. We can easily calculate the intersection matrices $B_i = (p_{i,j}^k)_{0 \leq j \leq 3, 0 \leq k \leq 3}$ and Krein parameter matrices $B_i^* = (q_{i,j}^k)_{0 \leq j \leq 3, 0 \leq k \leq 3}$. Namely, they are given as follows:

$$\begin{aligned} B_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(N-2\sqrt{N})(N-2)}{8} & \frac{(N+2\sqrt{N})(N-7\sqrt{N}+12)}{16} & \frac{(N-2\sqrt{N})(N-\sqrt{N}-4)}{16} & \frac{(N-2\sqrt{N})(N-2\sqrt{N}-4)}{16} \\ 0 & \frac{(N+2\sqrt{N})(N-\sqrt{N}-4)}{16} & \frac{(N-2\sqrt{N})(N+\sqrt{N}-4)}{16} & \frac{N(N-4)}{16} \\ 0 & \frac{N-2\sqrt{N}-4}{4} & \frac{N-2\sqrt{N}}{4} & 0 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{(N+2\sqrt{N})(N-\sqrt{N}-4)}{16} & \frac{(N-2\sqrt{N})(N+\sqrt{N}-4)}{16} & \frac{N(N-4)}{16} \\ \frac{(N+2\sqrt{N})(N-2)}{8} & \frac{(N+2\sqrt{N})(N+\sqrt{N}-4)}{16} & \frac{(N-2\sqrt{N})(N+7\sqrt{N}+12)}{16} & \frac{(N+2\sqrt{N})(N+2\sqrt{N}-4)}{16} \\ 0 & \frac{N+2\sqrt{N}}{4} & \frac{N+2\sqrt{N}-4}{4} & 0 \end{pmatrix}, \\ B_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{N-2\sqrt{N}-4}{4} & \frac{N-2\sqrt{N}}{4} & 0 \\ 0 & \frac{N+2\sqrt{N}}{4} & \frac{N+2\sqrt{N}-4}{4} & 0 \\ \frac{N}{2} - 1 & 0 & 0 & \frac{N}{2} - 2 \end{pmatrix}; \\ B_1^* &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ N-2 & 0 & 4 & 2 \\ 0 & N-4 & N-8 & N-4 \\ 0 & 1 & 2 & 0 \end{pmatrix}, \\ B_2^* &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & N-4 & N-8 & N-4 \\ \frac{(N-2)(N-4)}{4} & \frac{(N-4)(N-8)}{4} & \frac{N^2-12N+48}{4} & \frac{(N-4)(N-6)}{4} \\ 0 & \frac{N}{2} - 2 & \frac{N}{2} - 3 & 0 \end{pmatrix}, \\ B_3^* &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & \frac{N}{2} - 2 & \frac{N}{2} - 3 & 0 \\ \frac{N}{2} - 1 & 0 & 0 & \frac{N}{2} - 2 \end{pmatrix}. \end{aligned}$$

The original association scheme X of size $N^2 + 2N$ in \mathbb{R}^N (attached to a real MUB) is a class 4 association scheme with the following parameters. Note that this is a 5-spherical design and of degree $s = 4$. It is interesting to note that they are Q -polynomial association schemes (and not P -polynomial association scheme for $N \geq 4$). The reader is referred to Bannai-Bannai [3] for more details, where this fact was first noticed. These association schemes are possible candidates of universally optimal codes in the sense of Cohn-Kumar [9].

$$\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{N^2}{2} & \frac{(N+\sqrt{N})(N-2)}{4} & \frac{N^2}{4} & \frac{(N-\sqrt{N})(N-2)}{4} & 0 \\ 0 & N-1 & 0 & N-1 & 0 \\ 0 & \frac{(N-\sqrt{N})(N-2)}{4} & \frac{N^2}{4} & \frac{(N+\sqrt{N})(N-2)}{4} & \frac{N^2}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
B_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & N-1 & 0 & N-1 & 0 \\ 2(N-1) & 0 & 2(N-2) & 0 & 2(N-1) \\ 0 & N-1 & 0 & N-1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\
B_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{(N-\sqrt{N})(N-2)}{4} & \frac{N^2}{4} & \frac{(N+\sqrt{N})(N-2)}{4} & \frac{N^2}{2} \\ 0 & N-1 & 0 & N-1 & 0 \\ \frac{N^2}{2} & \frac{(N+\sqrt{N})(N-2)}{4} & \frac{N^2}{4} & \frac{(N-\sqrt{N})(N-2)}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\
B_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \\
B_1^* &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ N & 0 & \frac{2N}{N+2} & 0 & 0 \\ 0 & N-1 & 0 & N-1 & 0 \\ 0 & 0 & \frac{N^2}{N+2} & 0 & N \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
B_2^* &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & N-1 & 0 & N-1 & 0 \\ \frac{(N+2)(N-1)}{2} & 0 & \frac{(N+2)(N-2)}{2} & 0 & \frac{(N+2)(N-1)}{2} \\ 0 & \frac{N(N-1)}{2} & 0 & \frac{N(N-1)}{2} & 0 \\ 0 & 0 & \frac{N}{2} & 0 & 0 \end{pmatrix}, \\
B_3^* &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{N^2}{N+2} & 0 & N \\ 0 & \frac{N(N-1)}{2} & 0 & \frac{N(N-1)}{2} & 0 \\ \frac{N^2}{2} & 0 & \frac{N^3}{2(N+2)} & 0 & \frac{N(N-2)}{2} \\ 0 & \frac{N}{2} & 0 & \frac{N}{2} - 1 & 0 \end{pmatrix}, \\
B_4^* &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{N}{2} & 0 & 0 \\ 0 & \frac{N}{2} & 0 & \frac{N}{2} - 1 & 0 \\ \frac{N}{2} & 0 & 0 & 0 & \frac{N}{2} - 1 \end{pmatrix};
\end{aligned}$$

$$P = \begin{pmatrix} 1 & \frac{N^2}{2} & 2(N-1) & \frac{N^2}{2} & 1 \\ 1 & \frac{N^{\frac{3}{2}}}{2} & 0 & -\frac{N^{\frac{3}{2}}}{2} & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -\sqrt{N} & 0 & \sqrt{N} & -1 \\ 1 & -N & 2(N-1) & -N & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & N & \frac{(N-1)(N+2)}{2} & \frac{N^2}{2} & \frac{N}{2} \\ 1 & \sqrt{N} & 0 & -\sqrt{N} & -1 \\ 1 & 0 & -\frac{N}{2} - 1 & 0 & \frac{N}{2} \\ 1 & -\sqrt{N} & 0 & \sqrt{N} & -1 \\ 1 & -N & \frac{(N-1)(N+2)}{2} & -\frac{N^2}{2} & \frac{N}{2} \end{pmatrix}.$$

The intermediate association scheme Z between X and Y , and of size $N^2/2$ in \mathbb{R}^{N-1} , where Z is obtained from vectors $Z' = \{x \in X \mid (x, u) = 1/\sqrt{N}\}$ by orthogonal projection to $\langle u \rangle^\perp$ for any fixed $u \in X$ and rescaling to make a spherical code, is a class 3 association scheme. This is a spherical 3-design and of degree $s = 3$ with the following parameters. It is interesting to note that they are Q -polynomial association schemes (and not P -polynomial association scheme for $N \geq 4$.) It seems that this example was already recognized in the list of W. Martin's homepage (of such association schemes) as those coming from the linked symmetric designs. These association schemes are also possible candidates of universally optimal codes in the sense of Cohn-Kumar [9]. Here we describe the parameters of the association scheme Z .

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{(N-\sqrt{N})(N-2)}{4} & \frac{(N-3\sqrt{N})(N-4)}{8} & \frac{(N-\sqrt{N})(N-4)}{8} & \frac{(N-2\sqrt{N})(N-2)}{8} \\ 0 & \frac{(N+\sqrt{N})(N-4)}{8} & \frac{(N-\sqrt{N})(N-4)}{8} & \frac{N(N-2)}{8} \\ 0 & \frac{(\sqrt{N}-2)(\sqrt{N}+1)}{2} & \frac{N-\sqrt{N}}{2} & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{(N+\sqrt{N})(N-2)}{4} & \frac{(N+\sqrt{N})(N-4)}{8} & \frac{(N-\sqrt{N})(N-4)}{8} & \frac{N(N-2)}{8} \\ 0 & \frac{N+\sqrt{N}}{2} & \frac{(\sqrt{N}-1)(\sqrt{N}+2)}{2} & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \frac{(\sqrt{N}-2)(\sqrt{N}+1)}{2} & \frac{N-\sqrt{N}}{2} & 0 \\ 0 & \frac{N+\sqrt{N}}{2} & \frac{(\sqrt{N}-1)(\sqrt{N}+2)}{2} & 0 \\ N-1 & 0 & 0 & N-2 \end{pmatrix};$$

$$B_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ N-1 & 0 & 2 & 0 \\ 0 & N-2 & N-4 & N-1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & N-2 & N-4 & N-1 \\ \frac{(N-2)(N-1)}{2} & \frac{(N-4)(N-2)}{2} & \frac{N^2-6N+12}{2} & \frac{(N-4)(N-1)}{2} \\ 0 & \frac{N}{2} - 1 & \frac{N}{2} - 2 & 0 \end{pmatrix},$$

$$B_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{N}{2} - 1 & \frac{N}{2} - 2 & 0 \\ \frac{N}{2} - 1 & 0 & 0 & \frac{N}{2} - 2 \end{pmatrix}.$$

The first and the second eigenmatrices are the same as in Proposition 4.

Remarks. (1) We have association schemes X, Z, Y of sizes $N^2+2N, N^2/2, N^2/4$ (respectively) in $\mathbb{R}^N, \mathbb{R}^{N-1}, \mathbb{R}^{N-2}$ (respectively), where N must be an even power of 2. Currently all of them are possible candidates of universally optimal codes in the sense of Cohn-Kumar, at least for $N \geq 16$. It is shown in Cohn et. al. [10] that X in \mathbb{R}^4 (i.e., for $N = 4$) is not universally optimal. (It is an open question whether it is optimal or not.) Z for \mathbb{R}^3 (i.e., for $N = 4$) is not universally optimal nor optimal. On the other hand, Y for \mathbb{R}^2 (i.e., for $N = 4$) is universally optimal. Although it is a wild guess without firm ground, we think Y may be most likely to be universally optimal among X, Z , and Y .

(2) It is an interesting open question whether these association schemes are uniquely determined by the parameters. The uniqueness of Y for \mathbb{R}^{14} (i.e., for $N = 16$) was obtained in [2]. On the other hand, the uniqueness of X for \mathbb{R}^{16} (i.e., for $N = 16$) was proved by Akio Nakamura [22] in his masters degree thesis of Kyushu University in 1997 (it follows also from the uniqueness of the Nordstrom-Robinson code). We note that the uniqueness of Z for \mathbb{R}^{15} (i.e., for $N = 16$) is also obtained. The claim is essentially obtained in [21], it can be proved also by method of [22]. So, it would be interesting what will happen in particular for $N = 64$ for X, Z , and Y . The result of Kantor [19] implies that if $N = 2^{m+1}$ with odd m , and if m is not a prime, then there are non-isomorphic line systems, and so there are non-isomorphic association schemes with the same parameters, i.e., the uniqueness is break down.

(3) Quite recently it was shown [5] that the problem of constructing of s pairwise mutually unbiased bases in \mathbb{K}^n ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) is equivalent to the problem of constructing of s Cartan subalgebras of $sl_n(\mathbb{K})$ that are pairwise orthogonal with respect to Killing form and are closed under the adjoint operation. In particular, a complete collection of mutually unbiased bases in \mathbb{C}^n is equivalent to an orthogonal decomposition of Lie algebra $sl_n(\mathbb{C})$, closed under the adjoint operation. So there is a link to the well-developed theory [20] of orthogonal decompositions of Lie algebras.

5 Extremal line-sets and Barnes-Wall lattices

In this section we discuss connections between mutually unbiased bases, extremal line-sets in \mathbb{R}^N with prescribed angles and minimum vectors of Barnes-Wall lattices.

Fix any positive integer $N > 1$. Let M be a set of unit vectors in \mathbb{R}^N , such that $|(a, b)| \in \{0, 1/\sqrt{N}\}$ for all $a \neq b$ in M (so, in particular, $M \cap (-M) = \emptyset$). Then $|M| \leq N(N+2)/2$, and if $|M|$ reaches this upper bound, then M is a set of $N/2 + 1$ mutually unbiased bases [7, Proposition 3.12]. Constructions of such extremal line-sets are known only for $N = 2^{m+1}$, m odd [7].

Our final observation is that known constructions of extremal line-sets (or, equivalently, maximal sets of mutually unbiased bases) are connected to the minimum vectors of Barnes-Wall lattices. We show that vectors of known maximal real MUB after suitable rescaling will become minimal vectors of a Barnes-Wall lattice. Therefore, vectors of association schemes X, Y, Z can be obtained from a set of minimal vectors of the Barnes-Wall lattices.

First we recall the construction from [7]. Label the standard basis of \mathbb{R}^N as e_v , with $v \in$

$V = \mathbb{Z}_2^{m+1}$. For $b \in V$, define the permutation matrix $X(b)$ and diagonal matrix $Y(b)$ as follows:

$$X(b) : e_v \mapsto e_{v+b} \quad \text{and} \quad Y(b) := \text{diag}[(-1)^{b \cdot v}].$$

The groups $X(V) := \{X(b) \mid b \in V\}$ and $Y(V) := \{Y(b) \mid b \in V\}$ are contained in $O(\mathbb{R}^N)$ and are isomorphic to the additive group V . Let $E := \langle X(V), Y(V) \rangle$. Then $E = 2_+^{1+2(m+1)}$ is an extraspecial 2-group of order $2^{1+2(m+1)}$ and $\bar{E} = E/Z(E)$ is elementary abelian group of order $2^{2(m+1)}$. We identify the center $Z(E)$ of E with \mathbb{Z}_2 and consider the map $Q : \bar{E} \rightarrow \mathbb{Z}_2$ defined by $Q(\bar{e}) = e^2$ for any $\bar{e} \in \bar{E}$ and any preimage e of \bar{e} in E . Then Q is a non-singular quadratic form on \bar{E} . So \bar{E} is an $\Omega^+(2m+2, 2)$ -space. The action of E on \mathbb{R}^N can be extended to the action of the group $2_+^{1+2(m+1)}\Omega^+(2m+2, 2)$.

The space \bar{E} contains $(2^{m+1} - 1)(2^m + 1)$ singular points. An orthogonal spread of \bar{E} is a family Σ of $2^m + 1$ totally singular $(m+1)$ -spaces such that every singular point of \bar{E} belongs to exactly one member of Σ .

Let A be a subgroup of E such that its image \bar{A} in \bar{E} is totally singular $(m+1)$ -space of \bar{E} . Then the set $\mathcal{F}(A)$ of A -irreducible subspaces of \mathbb{R}^N is an orthogonal frame: a set of 2^{m+1} pairwise orthogonal lines through the origin. For an orthogonal spread Σ of the $\Omega^+(2m+2, 2)$ -space \bar{E} we let

$$\mathcal{F}(\Sigma) := \bigcup_{\bar{A} \in \Sigma} \mathcal{F}(A).$$

Then $\mathcal{F}(\Sigma)$ consists of $2^{m+1}(2^m + 1)$ lines of \mathbb{R}^N such that, if u_1 and u_2 are unit vectors in different members of $\mathcal{F}(\Sigma)$, then $|(u_1, u_2)| = 0$ or $2^{-(m+1)/2}$. Therefore, $\mathcal{F}(A)$ determines orthonormal basis and $\mathcal{F}(\Sigma)$ determines a set of $N/2 + 1$ mutually unbiased bases. These line-sets $\mathcal{F}(\Sigma)$ are extremal in the sense that $|\mathcal{F}(\Sigma)|$ meets an upper bound obtained in [13] for line-sets in \mathbb{R}^N with prescribed angles.

The binary Kerdock code $\mathcal{K}(\Sigma)$ can be recovered [7] from $\mathcal{F}(\Sigma)$:

$$\mathcal{K}(\Sigma) = \{(c_v)_v \in \mathbb{Z}_2^N \mid \langle ((-1)^{c_v})_v \rangle \in \mathcal{F}(\Sigma)\}.$$

Take unit vectors from line-set $\mathcal{F}(\Sigma)$, rescale them to vectors of norm \sqrt{N} , then these vectors will be minimum vectors of a Barnes-Wall lattice. Indeed, we note that for odd m the minimum norm of Barnes-Wall lattice is \sqrt{N} , the automorphism group is $G = 2^{1+2(m+1)}\Omega^+(2m+2, 2)$, G acts transitively [15] on the set of minimum vectors, the vector $c = N^{-1/4} \sum_{v \in V} e_v$ is a minimum vector, and any even lattice of rank N invariant under the group G is similar to a Barnes-Wall lattice [15]. So all the minimum vectors of Barnes-Wall lattice are obtained from c by action of the group G . On the other hand, in notations of [7] we have $c = N^{1/4} e_b^*$ for $b = 0$, and any unit vector of $\mathcal{F}(\Sigma)$ is obtained from e_b^* by action of some element of G .

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