Multiuser Capacity in Block Fading with no Channel State Information

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Abstract

Consider M independent users, each user having his own transmit antenna, that transmit simultaneously to a receiver equipped with N antennas through a Rayleigh block-fading channel having a coherence interval of T symbols, with no channel state information available to either the transmitters or to the receiver. The total transmitted power is independent of the number of users. For a given coherence time T, we wish to identify the best multi-access strategy that maximizes the total throughput.

If perfect channel state information were available to the receiver, it is known that the total capacity increases monotonically with the number of users. If the channel state information is available to both the receiver and to all transmitters, the throughput maximizing strategy implies for $N = 1$ that only the single user who enjoys the best channel condition transmits. In the absence of channel state information one is forced to a radically different conclusion. In particular we show that if the propagation coefficients take on new independent values for every symbol (e.g., $T = 1$) then the total capacity for any $M > 1$ users is equal to the capacity for $M = 1$ user, in which case TDMA is an optimal scheme for handling multiple users. This result follows directly from a recent treatment of the single-user multiple antenna block-fading channel.

Again, motivated by the single-user results, one is lead to the following conjecture for the multiple user case: for any $T > 1$ the maximum total capacity can be achieved by no more than $M = T$ users. The conjecture is supported by establishing the asymptotic result that, for a fixed N and a constant M/T for large T, the total capacity is maximized when $M/T \to 0$, which yields a total capacity per symbol of $N \log(1 + \rho)$, where ρ is the expected SNR at the receiver. We further support the conjecture by examining the asymptotic behavior with large ρ for fixed M,T and $N \leq T$.

Index Terms: wireless communication, multiple users, Rayleigh block-fading, multi-element antenna arrays.

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1 Introduction

Multiple antenna wireless links are of ever-increasing interest to the research community because of the spectacular throughputs that are obtainable under certain conditions [4, 12]. For a single-antenna link with no channel state information available at the transmitter, fading is never beneficial. On the contrary, a multiple antenna link, when operating in a Rayleigh fading environment, has a potential throughput that increases linearly with the smaller of the number of transmit or receive antennas. Our interest is the flat-fading regime where, over the frequency support of the transmitted signals, the channel response is approximately constant.

Throughout we shall adopt a block-fading model [11], where the channel propagation matrix is constant for T symbol periods, after which it jumps to a new independent value for another T symbols, and so on. In general our goal is to maximize throughput, in the Shannon sense, implying that coding is performed over many independent T-symbol coherence intervals. We will not consider performance within a single coherence interval, where Shannon coding cannot be performed, and which requires notions such as outage capacity [11], delay-limited capacity [5], space-time autocoding [7], or multi-user diversity [2, 13].

A detailed survey of information theory for fading channels is given in [2]. For our purposes it is convenient to subdivide the multiple-antenna problem area as shown in Table 1, with a limited number of references provided for the sub-areas. Usually channel state information is acquired

	CSI at transmitter & receiver	CSI at receiver	no CSI
single user	[12]	[4, 12]	[1, 10, 16]
multi-user	[8]	[12]	

Table 1: Wireless links require special treatment, depending on the number of users, and the amount of channel state information (CSI) that is available.

by the transmission of a set of known training signals, with the required symbol duration of the training signals proportional to the number of transmit antennas. The overhead that is represented by training can be considerable. In fact, where one is free to choose the number of antennas to maximize the net throughput for a fixed coherence interval, half of the coherence interval is used for training [9]. Training, however, is not always inefficient. See [16, 6], which demonstrate that training may preserve basic “degrees of freedom” in the system. There certainly is no fundamental mandate to perform training, so the assumption of no CSI is the most natural one to make.

For a single user, the assumption of no CSI at the receiver versus the assumption that the receiver has CSI changes the problem drastically. With CSI at the receiver only, the capacity attaining signals are independent among the transmit antennas, and capacity increases linearly with the smaller of the number of transmit and receive antennas. Without CSI, the capacity attaining signals are temporally orthogonal among the transmit antennas within the coherence interval, and there is no point in making the number of transmit antennas M greater than the symbol duration of the coherence interval T. When CSI is available to both the transmitter and the receiver, the single user can further improve performance by water-pouring optimization, which also introduces signal correlation among the antennas.

In contrast to the single user, multiple users must transmit independent signals. When CSI is available to the receiver alone, the multiple users can collectively duplicate the throughput of a single user, whose optimal signals are independent among the transmit antennas. With CSI available to both transmitter and receiver (and for the case of a single receive antenna, $N = 1$), maximum throughput is achieved by a strategy called "controlled TDMA" [2], where only the user who possesses the strongest propagation coefficient transmits during each coherence interval. This specific result does not necessarily hold for $N > 1$. A similar problem, but with two users and $N > 1$ receive antennas is considered in [15], where it is shown that the capacity region can be characterized as the solution to a convex optimization problem, which determines also the optimal two-user water pouring strategy.

There are some interesting parallels between multiple antenna links that operate in Rayleigh fading, and coded direct-sequence CDMA links [13]. When CSI is available either to the receiver, or to the transmitters and receiver, the channel coefficients play the role of signature sequences of length N (processing gain), where M stands for the number of users.

The focus of this note is the case where no CSI is available to either the transmitters or to the receiver. Here we conjecture that, from the standpoint of maximizing throughput, there is no point in making the number of users M greater than the symbol-duration of the coherence interval T, and we establish the validity of this conjecture for three special cases.

2 Signal Model

We use a block-fading model, with coherence interval T [11, 2, 10] where M independent users simultaneously transmit to a single receiver equipped with N antennas in a flat-fading environment, where each user has sole access to one of M transmit antennas, and where nobody has any channel state information. The fading is described by a $M \times N$ complex-valued propagation matrix H, which remains constant for T symbol periods, after which it jumps to a new independent value for another T symbol, and so on. During a coherence interval the M users collectively transmit a $T \times M$ complex matrix S, whose columns are statistically independent, and the receiver records a $T \times N$ complex matrix X,

$$X = \sqrt{\frac{\rho}{M}} SH + W \ , \tag{2.1}$$

where W is a $T \times N$ vector of additive receiver noise, whose components are independent, zero-mean complex Gaussian with unit variance $(\mathrm{CN}(0,1))$. The components of H are assumed independent and zero-mean, with unit-variance. The independence of the columns of S is necessary if the users are to act with no cooperation among themselves. [1] We enforce an expected power constraint

$$\operatorname{tr} \mathrm{E}\left\{SS^\dagger\right\} = T \cdot M \ . \tag{2.2}$$

This constraint, when combined with the normalization $1/\sqrt{M}$ in (2.1), implies that the total transmitted power remains constant as the number of users changes, and that ρ represents the expected SNR at each receiver antenna.

We wish to choose the joint probability density of the components of S, subject to the independence of its M columns, and subject to the power constraint (2.2), to maximize the mutual information with no CSI available to anyone (receiver and transmitters),

$$I(X;S) \triangleq \mathrm{E}\left\{\log\left(\frac{p(X|S)}{p(X)}\right)\right\} \ . \tag{2.3}$$

This maximization yields the total throughput or capacity.

[1] In general, conditional independence is required, when conditioning is made on an auxiliary simple random variable [3] resembling a time-varying strategy. In the setting considered here this aspect is not effective.

It is convenient to assume Rayleigh fading, where the independent components of H are distributed as $\mathrm{CN}(0,1)$, although this assumption can be relaxed for some of our asymptotic results. For Rayleigh fading the conditional density takes the form

$$p(X \mid S) = \frac{\exp\left(-X^\dagger \Lambda^{-1} X\right)}{\pi^{TN} \det^N \Lambda}\,, \tag{2.4}$$

where $\Lambda = I_T + (\rho/M)\, SS^\dagger$ and I_T denotes the $T \times T$ identity matrix.

3 Upper Bound on Capacity; Capacity for T = 1

Neither the transmitter nor the receiver has CSI and we wish to maximize the mutual information (2.3), a difficult problem due to the constraint that the columns of S be statistically independent. If we relaxed this constraint and permitted dependent columns we would obtain an upper bound on the multiuser capacity, which is the cooperative multiple-user capacity equaling the capacity for a single user who has access to M transmit antennas. This problem was studied for the case $T = M = 1$ by [1] and for general T, N and M by [10], with the following results:

1. The capacity for $M > T$ is equal to the capacity for $M = T$;
2. In choosing $p(S)$ to maximize mutual information, the transmitted matrix can, without penalty, be constrained to have the factorization

$$S = \Phi V\,, \tag{3.1}$$

   where Φ and V are statistically independent, Φ is an isotropically random $T \times T$ unitary matrix (i.e., the columns are orthonormal, and the joint probability density of the elements of Φ is unchanged if Φ is premultiplied by any deterministic unitary matrix), and V is real, nonnegative, $T \times M$ diagonal.

Note that for the capacity-attaining signal, the columns of S are *not* independent.

In contrast, if perfect CSI were available to the receiver [12] the components of S would be independent $\mathrm{CN}(0,1)$, and the capacity would increase monotonically with M. If perfect CSI were available to the receiver, this single-user capacity could be achieved by M independent users [12].

If CSI were available to both receiver and transmitter, and if $N = 1$, only the user who enjoys the best channel condition should transmit in each block [8]. This strategy, referred to as controlled random TDMA in [2], does not necessarily scale with N for an arbitrary propagation matrix H. This can be deduced by examining H matrices that satisfy the Welch bound, that is $\frac{1}{M}H^\dagger H = I_N$, and $\frac{1}{N}HH^\dagger$ has diagonals equal to one. In this case it can be shown [14], via the analogy to coded DS-CDMA previously mentioned, that capacity equal to $N\log(1+\rho)$ is achieved only if all $M \geq N$ users transmit i.i.d. equal-power signals.

We conclude that the absence of CSI drastically changes the problem, and that the total M-user capacity is generally less than the single-user/M-antenna capacity.

For the special case where $T = 1$ (arbitrarily fast fading) a single user having M antennas can attain the same performance with a single antenna, so the total capacity for M users is equal to the capacity for one user. This establishes the optimality of TDMA for $T = 1$ and unavailable CSI. This conclusion extends also to all fading matrices H which are statistically invariant under a unitary transformation.

For the general case $T > 1$ an obvious conjecture [2], in light of the results cited from [10], is that the total capacity for any $M > T$ is equal to the total capacity for $M \leq T$. At present we are unable to prove this conjecture, but in the following sections we make some headway by studying the case where T and M grow big simultaneously, or ρ grows asymptotically large.

4 Asymptotic Results for Large $\mathbf{T}, \mathbf{M}$

We begin this section by deriving upper and lower bounds on the mutual information (2.3). Throughout, we only assume that the components of H are independent, zero-mean, and unit variance. Again we require that the columns of S be independent, and we observe the power constraint (2.2).

We expand the mutual information as follows,

$$\begin{aligned} I(X;S) &= \mathrm{E}\left\{\log\left[\frac{p(X|S)\cdot p(X|S,H)}{p(X)\cdot p(X|S,H)}\right]\right\} \\ &= \mathrm{E}\left\{\log\left[\frac{p(X|S,H)}{p(X)}\right]\right\} - \mathrm{E}\left\{\log\left[\frac{p(X|S,H)}{p(X|S)}\right]\right\} \end{aligned}$$

$$= \quad I(X;S,H) - I(X;H|S) \; . \tag{4.1}$$

We obtain an upper bound on mutual information by dropping the second of the two terms above,

$$I(X;S) \leq I(X;S,H) \tag{4.2}$$

We obtain a lower bound on the first term in (4.1) as follows,

$$\begin{aligned} I(X;S,H) &= \mathrm{E}\left\{\log\left[\frac{p(X|S,H)\cdot p(X|H)}{p(X)\cdot p(X|H)}\right]\right\} \\ &= \mathrm{E}\left\{\log\left[\frac{p(X|S,H)}{p(X|H)}\right]\right\} + \mathrm{E}\left\{\log\left[\frac{p(X|H)}{p(X)}\right]\right\} \\ &= I(X;S|H) + I(X;H) \\ &\geq I(X;S|H) \; . \end{aligned} \tag{4.3}$$

The combination of (4.1), (4.2), and (4.3) gives

$$I(X;S|H) - I(X;H|S) \leq I(X;S) \leq I(X;S,H). \tag{4.4}$$

We now maximize the upper bound in (4.4). The mutual information $I(X;S,H)$ corresponds to an additive Gaussian noise channel, $X = (\rho/M)^{1/2}Z + W$, where $Z = SH$, and where the transmitter has complete control of Z. Recall that the elements of H are independent, zero-mean, and unit-variance, implying that, conditioned on S, the columns of Z are independent, each with covariance matrix equal to $SS^\dagger$. We maximize $I(X;S,H)$ by making Z zero-mean complex Gaussian, with independent columns whose covariance matrices are equal to $\mathrm{E}\left\{SS^\dagger\right\}$, which yields

$$I(X;S,H) \leq N \, \log\det\left(I_T + \frac{\rho}{M}\mathrm{E}\left\{SS^\dagger\right\}\right) . \tag{4.5}$$

Subject to the power constraint (2.2), this is maximized for $\mathrm{E}\left\{SS^\dagger\right\} = M \cdot I_T$, which gives

$$I(X;S,H) \leq TN\log(1+\rho) \; . \tag{4.6}$$

The upper bound in (4.4), combined with (4.6) gives an upper bound on mutual information that is valid for any density $p(S)$ that satisfies the power constraint (2.2),

$$I(X;S) \leq TN\log(1+\rho). \tag{4.7}$$

We choose S, within (4.4) to be zero-mean complex Gaussian, with independent components having unit variance, which serves to maximize $I(X;S|H)$ [12],

$$I(X;S_G|H) = T \cdot \mathrm{E}\left\{\log\det\left(I_N + \frac{\rho}{M}H^\dagger H\right)\right\}\ , \tag{4.8}$$

where the subscripted S_G indicates that S has the Gaussian density. The combination of (4.4), (4.7), and (4.8) yields

$$T \cdot \mathrm{E}\left\{\log\det\left(I_N + \frac{\rho}{M}H^\dagger H\right)\right\} - I(X;H|S_G) \leq I(X;S_G) \leq TN\log(1+\rho). \tag{4.9}$$

Next we obtain a lower bound on the second term in (4.9), $-I(X;H|S_G)$, by choosing $p(H)$ to maximize $I(X;H|S_G)$. In so doing, we constrain H to be independent of S_G, and to have elements that are independent, zero-mean, unit-variance. The mutual information $I(X;H|S_G)$ corresponds to the fictitious case of a user sending the signal H through a random propagation matrix S_G, where S_G is known at the receiver. Therefore the mutual information is maximized by making the elements of H complex Gaussian [12], which gives

$$\begin{aligned} I(X;H|S_G) &\leq N \cdot \mathrm{E}\left\{\log\det\left(I_T + \frac{\rho}{M}S_G S_G^\dagger\right)\right\} \\ &= N \cdot \mathrm{E}\left\{\log\det\left(I_M + \frac{\rho}{M}S_G^\dagger S_G\right)\right\}, \end{aligned} \tag{4.10}$$

where we have used a common matrix identity to obtain the final expression. We can further bound $I(X;H|S_G)$ by using Jensen's inequality to bring the expectation inside of the log-determinant,

$$\begin{aligned} N \cdot \mathrm{E}\left\{\log\det\left(I_M + \frac{\rho}{M}S_G^\dagger S_G\right)\right\} &\leq N \cdot \log\det\left(I_M + \frac{\rho}{M}\mathrm{E}\left\{S_G^\dagger S_G\right\}\right) \\ &= N \cdot \log\det\left(I_M + \frac{\rho T}{M}I_M\right) \\ &= MN\log\left(1 + \frac{\rho T}{M}\right). \end{aligned} \tag{4.11}$$

The combination of (4.9), (4.10), and (4.11) yields

$$\mathrm{E}\left\{\log\det\left(I_N + \frac{\rho}{M}H^\dagger H\right)\right\} - \frac{MN}{T}\log\left(1 + \frac{\rho T}{M}\right) \leq \frac{I(X;S_G)}{T} \leq N\log(1+\rho). \tag{4.12}$$

We simultaneously let T and M become big while maintaining a constant ratio $\beta = T/M$. This implies that $H^\dagger H/M \to I_N$, so

$$N\log(1+\rho) - \frac{N}{\beta}\log\left(1+\rho\beta\right) \le \frac{I(X;S_G)}{T} \le N\log(1+\rho). \tag{4.13}$$

Finally we let the ratio grow big, $\beta = T/M \to \infty$, which yields the asymptotic result

$$\frac{I(X;S_G)}{T} = N\log(1+\rho)\ . \tag{4.14}$$

The expression $N\log(1+\rho)$ is equal to the capacity for a single user having an unlimited number of transmit antennas, where the receiver has perfect knowledge of the propagation matrix [12]. We have shown that this same capacity can collectively be attained by M independent users, where no CSI is available to anyone, in the limit as T and M become large, with $M \ll T$. We conclude that the capacity could not be increased by having $M > T$ users, in agreement with our original conjecture.

5 Asymptotics with Large ρ

To support further our conjecture we now focus on the case where T, M and N are fixed, while the signal-to-noise ratio $\rho \to \infty$. This asymptotically high SNR region is appropriate for multiple users operating in an extremely high data rate regime. Again, resorting to an upper bound by allowing the full cooperation at the transmitter sites we invoke recent results by Zheng & Tse [16] characterizing the asymptotic capacity of the single user M-transmit, N-receiver systems with coherence time T.

The capacity behaviour [16] is

$$\frac{C(M,N,T;\rho)}{\log\rho} \underset{\rho\to\infty}{\longrightarrow} TM^*\left(1-\frac{M^*}{T}\right)\ , \tag{5.1}$$

where

$$M^* = \min\left\{M, N, \left\lfloor\frac{T}{2}\right\rfloor\right\}\ . \tag{5.2}$$

Here $C(M,N,T;\rho)$ stands for the capacity over the coherence interval T (not normalized) with M transmit and N receive antennas, and $\text{SNR} = \rho$. With $N = 1$, this result supports our conjecture, as immaterial of what the coherence time T is (fixed though), having a single operating transmitter yields the optimal possible asymptotics with respect to ρ.

Clearly, we cannot infer from [16] the optimal strategy for $N > 1$. For $N < \left\lfloor \frac{T}{2} \right\rfloor$, the fact that the single-user capacity does not require more than N transmitters doesn't imply necessarily the same conclusion in the multiple user case. This is because, in the single user case, capacity achieving signals are in general correlated for different transmitting antennas, and this cannot be the case for multiple users.

To investigate the behavior for $1 < N < T$, we resort again to independent Gaussian signals, and we explicitly assume that the components of H are independent Gaussian. We denote this by a subscript G. We find

$$I_G(X; S) \geq I_G(X\,;\,S|H) - I_G(X\,;\,H|S)\ . \tag{5.3}$$

Notice that

$$I_G(X\,;\,S|H) = T \cdot \mathcal{C}_{\mathrm{SI}}(M, N; \rho)\ , \tag{5.4}$$

where $\mathcal{C}_{\mathrm{SI}}(M, N; \rho)$ stands for the single user normalized capacity per unit time with M transmitters and N receiving antennas and signal-to-noise ratio ρ, with CSI available to the receiver [12]. The symmetry of the signal model (2.1) yields,

$$I_G(X\,;\,H|S) = N \cdot \mathcal{C}_{\mathrm{SI}}(M, T\,;\,\rho)\ . \tag{5.5}$$

Symmetry considerations [12] imply that

$$\mathcal{C}_{\mathrm{SI}}(a, b; b\rho) = \mathcal{C}_{\mathrm{SI}}(b, a; a\rho)\ . \tag{5.6}$$

Now, it is known [12] that

$$\frac{\mathcal{C}_{\mathrm{SI}}(M, N\,;\,\rho)}{\log \rho} \underset{\rho \to \infty}{\longrightarrow} \min(M, N)\ , \tag{5.7}$$

which when combined with (5.3) yields

$$\frac{I_G(X\,;\,S)}{\log \rho} \underset{\rho \to \infty}{\longrightarrow} \geq T\,\min(M, N) - N\,\min(M, T)\ . \tag{5.8}$$

Consider now the case of fixed N and T and $N \leq T$. Clearly, the asymptotic expression is optimized by letting $M = N$, which yields,

$$\frac{I_G(X\,;\,S)}{\log \rho} \underset{\rho \to \infty}{\longrightarrow} \geq TN \left(1 - \frac{N}{T}\right)\ . \tag{5.9}$$

Note that for $2N \leq T$, this yields the optimal performance as deduced by (5.1), (5.2). For $\left\lfloor \frac{T}{2} \right\rfloor < N \leq T$, there is a gap between the behavior of (5.9) (where $M = N$) and (5.1) which is optimized for $M = \left\lfloor \frac{T}{2} \right\rfloor < N$. We conjecture that (5.9) reflects the actual asymptotic behavior in this case and the penalty with respect to (5.1) is attributed to the independent signaling at each transmitting antenna.

6 Conclusions

The absence of CSI drastically alters the behavior of a wireless link for independent multiusers. We conjecture that, from the standpoint of maximizing throughput, the number of users should not exceed the symbol duration of the coherence interval. We have established the correctness of the conjecture for three special cases: 1) a coherence interval comprising one symbol, 2) a coherence interval of unlimited duration, and 3) an unlimited SNR.

As mentioned in the introduction, there is an interesting analogy between multiple user, multiple antenna links and coded DS-CDMA with random signatures. In the regime considered here, the signatures are chosen in an absolutely random fashion for every T-symbol coherence interval, without their values being known by any of the receivers, whether legitimate or not. It is not surprising that our results also conform with the autocoding capacity [7], where, for the case of unknown CSI the number of actively operating users at each time instant is small compared with T.

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