

The Limit of the Boltzmann Equation to the Euler Equations for Riemann Problems

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Abstract

The convergence of the Boltzmann equation to the compressible Euler equations when the Knudsen number tends to zero has been a long standing open problem in the kinetic theory. In the setting of Riemann solution that contains the generic superposition of shock, rarefaction wave and contact discontinuity to the Euler equations, we succeed in justifying this limit by introducing hyperbolic waves with different solution backgrounds to capture the extra masses carried by the hyperbolic approximation of the rarefaction wave and the diffusion approximation of contact discontinuity.

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1 Introduction

As the fundamental equation in statistical mechanics, the Boltzmann equation takes the form of

$$f_t + \xi \cdot \nabla_X f = \frac{1}{\varepsilon} Q(f, f),$$

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where $f(t, X, \xi)$ is the density distribution function of particles at time t with location X and velocity ξ . In this equation, the physical parameter $\varepsilon > 0$ called Knudsen number is proportional to the mean free path of the interacting particles.

It was known since its derivation that the Boltzmann equation is closely related to the systems of fluid dynamics, in particular, the systems of Euler and Navier-Stokes equations. In fact, the first derivation of the fluid dynamical components and systems from the kinetic equations can be traced back to the dates of Maxwell and Boltzmann. Their early derivations rest on some arguments as how the various terms in a kinetic equation balance each other. These balance arguments seem arbitrary to some extent. For this, Hilbert proposed a systematic expansion in 1912, and Enskog and Chapman independently proposed another expansion in 1916 and 1917 respectively.

Either the Hilbert expansion or Chapman-Enskog expansion yields the compressible Euler equations in the leading order with respect to the Knudsen number ε , and the compressible Navier-Stokes equations, Burnett equations in the subsequent orders. To justify these formal approximations in rigorous mathematics, that is, hydrodynamic limits, has been proved to be extremely challenging and most remains open, in part because the basic well-posedness and regularity questions are still mostly unsolved for these fluid equations.

The justification of the fluid limits of the Boltzmann equation is also related to the Hilbert's sixth problem, "Mathematical treatment of the axioms of physics", in which it says that "The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua. Further, the mathematician has the duty to test exactly in each instance whether the new axioms are compatible with the previous ones."

The goal of this paper is to justify the limiting process of the Boltzmann equation to the system of the compressible Euler equations in the setting of Riemann solutions. The Riemann problem was first formulated and studied by Riemann in 1860s when he studied the one space dimensional isentropic gas dynamics with initial data being two constant states. The solution to this problem turns out to be fundamental in the theory of hyperbolic conservation laws because it not only captures the local and global behavior of solutions, but also fully represents the effect of the nonlinearity in the structure of the solutions. It is now well known that for the system of Euler equations, there are three basic wave patterns, that is, shock wave, rarefaction wave and contact discontinuity. These three types of waves have essential differences, that is, shock is compressive, rarefaction is expansive, and contact discontinuity has some diffusive structure. Therefore, how to study the hydrodynamic limit of Boltzmann equation for the full Riemann solution that consists of the superposition of these three typical waves is still very challenging in mathematics.

In this paper, by introducing two types of hyperbolic waves that carry the extra masses in the hyperbolic approximation of the rarefaction wave profile and the diffusive approximation of the contact discontinuity, we succeed in proving rigorously that there exists a family of solutions to the Boltzmann equation that converges to a Maxwellian determined by a Riemann solution consisting of three basic wave patterns when the Knudsen number tends to zero. Furthermore, a convergence rate is obtained in term of the Knudsen number.

By coping with the essential properties of individual wave pattern, the hydrodynamic limit for a single wave was justified in the previous works separately. More precisely, by using the compressibility of shock wave profile, Yu [41] showed that when the solution of the Euler equations (1.2) contains only non-interacting shocks, there exists a sequence of solutions to the Boltzmann equation that converge to a local Maxwellian defined by the solution of the Euler equations (1.2) uniformly away from the shock in any fixed time interval. In this work, a generalized Hilbert expansion was introduced, and the analytic technique of matching the inner and outer expansions developed by Goodman-Xin [16] for conservation laws was used. On the other hand, by using the time decay properties of the rarefaction wave, similar problem was studied by Xin-Zeng [40]. Moreover, by using the diffusive structure in the contact discontinuity as for the Navier-Stokes equations, the hydrodynamic limit to the contact discontinuity was proved by Huang-Wang-Yang in [19].

However, up to now, how to deal with the general Riemann solution that consists of all three basic

waves is still a challenging open problem. This is mainly due to the difficulty in handling the wave interactions and also unifying the different approaches in the analysis used for each single wave pattern. In order to overcome these difficulties in justifying the limit, our main idea in this paper is to introduce two families of hyperbolic waves, called hyperbolic wave I and II, that capture the propagation of the extra mass created by the approximate hyperbolic rarefaction wave profile in the viscous setting and the diffusion approximation of contact discontinuity.

We now briefly explain why the two families of hyperbolic waves we introduced are essential for the proof. As in the previous works on the rarefaction wave in the setting of either Navier-Stokes equations or the Boltzmann equation, the approximate rarefaction wave is constructed as a hyperbolic wave profile. Therefore, we need to precisely capture the error in the second order of the approximation for the Boltzmann equation in term of the Knudsen number, that is, in the Navier-Stokes level. And this reduces to study the propagation of the extra mass induced by the viscosity and heat conductivity. For this, we introduce the hyperbolic wave I as a solution to the linearized system around the approximate rarefaction wave profile with source terms given by the viscosity and heat conductivity induced by the rarefaction wave profile to recover the viscous terms. We can show that the hyperbolic wave I decays like the first-order derivative of the rarefaction wave profile so that the decay properties given in Lemma 2.2 are good enough to carry out the analysis.

The main difficulty comes from the approximation of the contact discontinuity. First of all, such an approximation, that is, 2-viscous contact wave, behaves like a diffusion wave profile as for the Navier-Stokes equations. Due to the lack of sufficient decay in ε and the non-conservative error terms when taking the anti-derivative of the perturbation, we need to remove the leading error terms and non-conservative terms in such approximation before taking the anti-derivative. The hyperbolic wave II is constructed to remove these error terms due to the viscous contact wave approximation. Note that the construction of the hyperbolic wave II can not be done simply around the 2-viscous contact wave as the hyperbolic wave I for the rarefaction wave profile. Otherwise, the wave interaction terms thus induced will lead to insufficiently decay in ε due to 2-viscous contact wave and it seems that these terms are essential in wave interactions. Instead, it is constructed around the superposition of the approximate 1-rarefaction wave, the hyperbolic wave I, the 2-viscous contact wave and the 3-shock profile as a whole. Thus some wave interaction terms can be absorbed in the hyperbolic wave II and the other wave interaction terms can be handled by some subtle and careful calculations. Due to the non-conservative terms and insufficient decay rates of ε of error terms induced by the 2-contact wave, we can not use the anti-derivative technique to analyze the hyperbolic wave II. Since the derivative of 3-shock profile is negative and tends to infinity as the Knudsen number $\varepsilon \rightarrow 0+$, we have to impose the condition at the time $t = T$ (see (2.74) below) for the linearized hyperbolic system of hyperbolic wave II so that the monotonicity of 3-shock wave is fully utilized. This idea is different from the previous stability analysis on the shock profile which is based on the anti-derivative technique.

With the help of these two hyperbolic waves and the corresponding new estimates, we can justify the limiting process from the Boltzmann equation to compressible Euler equations for the generic Riemann problems by elaborate analysis after a hyperbolic scaling.

We now formulate the problem. Consider the Boltzmann equation with slab symmetry

$$f_t + \xi_1 f_x = \frac{1}{\varepsilon} Q(f, f), \quad (1.1)$$

where $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ and $x \in \mathbf{R}^1$. Here, the collision operator takes the form of

$$Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\mathbf{R}^3} \int_{\mathbf{S}_+^2} \left(f(\xi') g(\xi'_*) + f(\xi'_*) g(\xi') - f(\xi) g(\xi_*) - f(\xi_*) g(\xi) \right) B(|\xi - \xi_*|, \hat{\theta}) d\xi_* d\Omega,$$

where ξ', ξ'_* are the velocities after an elastic collision of two particles with velocities ξ, ξ_* before the collision. Here, $\hat{\theta}$ is the angle between the relative velocity $\xi - \xi_*$ and the unit vector Ω in $\mathbf{S}_+^2 = \{\Omega \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0\}$. The conservations of momentum and energy yield the following relations between the velocities before and after collision:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \Omega] \Omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega] \Omega.$$

We will concentrate on the hard sphere model where the cross-section takes the form of

$$B(|\xi - \xi_*|, \hat{\theta}) = |(\xi - \xi_*, \Omega)| = |\xi - \xi_*| \cos \hat{\theta}.$$

On the other hand, it is noted that the analysis can be applied to at least hard potential if we can assume Lemma 2.3 on the shock wave profile holds true.

As we mentioned earlier, formally, when the Knudsen number ε tends to zero, the limit of the Boltzmann equation (1.1) is the system of compressible Euler equations that consists of conservations of mass, momentum and energy:

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = 0, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = 0, \quad i = 2, 3, \\ [\rho(e + \frac{|u|^2}{2})]_t + [\rho u_1(E + \frac{|u|^2}{2}) + p u_1]_x = 0, \end{cases} \quad (1.2)$$

where

$$\begin{cases} \rho(t, x) = \int_{\mathbf{R}^3} \varphi_0(\xi) f(t, x, \xi) d\xi, \\ \rho u_i(t, x) = \int_{\mathbf{R}^3} \varphi_i(\xi) f(t, x, \xi) d\xi, \quad i = 1, 2, 3, \\ \rho(e + \frac{|u|^2}{2})(t, x) = \int_{\mathbf{R}^3} \varphi_4(\xi) f(t, x, \xi) d\xi. \end{cases} \quad (1.3)$$

Here, ρ is the density, $u = (u_1, u_2, u_3)$ is the macroscopic velocity, e is the internal energy, and $p = R\rho\theta$ with R being the gas constant is the pressure. Note that the temperature θ is related to the internal energy by $e = \frac{3}{2}R\theta$, and $\varphi_i(\xi) (i = 0, 1, 2, 3, 4)$ are the collision invariants given by

$$\varphi_0(\xi) = 1, \quad \varphi_i(\xi) = \xi_i \quad (i = 1, 2, 3), \quad \varphi_4(\xi) = \frac{1}{2}|\xi|^2, \quad (1.4)$$

that satisfy

$$\int_{\mathbf{R}^3} \varphi_i(\xi) Q(g_1, g_2) d\xi = 0, \quad \text{for } i = 0, 1, 2, 3, 4.$$

Instead of using either Hilbert expansion or Chapman-Enskog expansion, we will apply the macro-micro decomposition introduced in [28]. For a solution $f(t, x, \xi)$ of (1.1), set

$$f(t, x, \xi) = \mathbf{M}(t, x, \xi) + \mathbf{G}(t, x, \xi),$$

where the local Maxwellian $\mathbf{M}(t, x, \xi) = \mathbf{M}_{[\rho, u, \theta]}(\xi)$ represents the macroscopic component of the solution defined by the five conserved quantities, i.e., the mass density $\rho(t, x)$, the momentum $\rho u(t, x)$, and the total energy $\rho(e + \frac{1}{2}|u|^2)(t, x)$ given in (1.3), through

$$\mathbf{M} = \mathbf{M}_{[\rho, u, \theta]}(t, x, \xi) = \frac{\rho(t, x)}{\sqrt{(2\pi R\theta(t, x))^3}} e^{-\frac{|\xi - u(t, x)|^2}{2R\theta(t, x)}}. \quad (1.5)$$

And $\mathbf{G}(t, x, \xi)$ represents the microscopic component.

From now on, the inner product of g_1 and g_2 in $L^2_\xi(\mathbf{R}^3)$ with respect to a given Maxwellian $\tilde{\mathbf{M}}$ is denoted by:

$$\langle g_1, g_2 \rangle_{\tilde{\mathbf{M}}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{\mathbf{M}}} g_1(\xi) g_2(\xi) d\xi. \quad (1.6)$$

If $\tilde{\mathbf{M}}$ is the local Maxwellian \mathbf{M} defined in (1.5), the macroscopic space is spanned by the following five pairwise orthogonal base,

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}} \mathbf{M}, \\ \chi_i(\xi) \equiv \frac{\xi_i - u_i}{\sqrt{R\theta\rho}} \mathbf{M} \quad \text{for } i = 1, 2, 3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}} \left(\frac{|\xi - u|^2}{R\theta} - 3 \right) \mathbf{M}, \\ \langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad i, j = 0, 1, 2, 3, 4. \end{cases} \quad (1.7)$$

For brevity, if $\tilde{\mathbf{M}}$ is the local Maxwellian \mathbf{M} , we will simply use $\langle \cdot, \cdot \rangle$ to denote $\langle \cdot, \cdot \rangle_{\mathbf{M}}$. By using the above base, the macroscopic projection \mathbf{P}_0 and microscopic projection \mathbf{P}_1 can be defined as

$$\mathbf{P}_0 g = \sum_{j=0}^4 \langle g, \chi_j \rangle \chi_j, \quad \mathbf{P}_1 g = g - \mathbf{P}_0 g.$$

Note that a function $g(\xi)$ is called microscopic if

$$\int g(\xi) \varphi_i(\xi) d\xi = 0, \quad i = 0, 1, 2, 3, 4,$$

where again $\varphi_i(\xi)$ represents the collision invariants.

Notice that the solution $f(t, x, \xi)$ to the Boltzmann equation (1.1) satisfies

$$\mathbf{P}_0 f = \mathbf{M}, \quad \mathbf{P}_1 f = \mathbf{G},$$

and the Boltzmann equation (1.1) becomes

$$(\mathbf{M} + \mathbf{G})_t + \xi_1 (\mathbf{M} + \mathbf{G})_x = \frac{1}{\varepsilon} [2Q(\mathbf{M}, \mathbf{G}) + Q(\mathbf{G}, \mathbf{G})]. \quad (1.8)$$

By integrating the product of the equation (1.8) and the collision invariants $\varphi_i(\xi)$ ($i = 0, 1, 2, 3, 4$) with respect to ξ over \mathbf{R}^3 , one has the following system for the fluid variables (ρ, u, θ) :

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = - \int \xi_1^2 \mathbf{G}_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = - \int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ [\rho(e + \frac{|u|^2}{2})]_t + [\rho u_1(e + \frac{|u|^2}{2}) + p u_1]_x = - \int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi. \end{cases} \quad (1.9)$$

Note that the above fluid-type system is not self-contained and one more equation for the microscopic component \mathbf{G} is needed and it can be obtained by applying the projection operator \mathbf{P}_1 to (1.8):

$$\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} [\mathbf{L}_{\mathbf{M}} \mathbf{G} + Q(\mathbf{G}, \mathbf{G})]. \quad (1.10)$$

Here $\mathbf{L}_{\mathbf{M}}$ is the linearized collision operator of $Q(f, f)$ with respect to the local Maxwellian \mathbf{M} given by

$$\mathbf{L}_{\mathbf{M}} g = 2Q(\mathbf{M}, g) = Q(\mathbf{M}, g) + Q(g, \mathbf{M}).$$

Note that the null space \mathfrak{N} of $\mathbf{L}_{\mathbf{M}}$ is spanned by the macroscopic variables:

$$\chi_j(\xi), \quad j = 0, 1, 2, 3, 4.$$

Furthermore, there exists a positive constant $\tilde{\sigma} > 0$ such that for any function $g(\xi) \in \mathfrak{N}^\perp$, cf. [17],

$$\langle g, \mathbf{L}_{\mathbf{M}} g \rangle \leq -\tilde{\sigma} \langle \nu(|\xi|) g, g \rangle,$$

where $\nu(|\xi|) = O(1)(1 + |\xi|)$ is the collision frequency for the hard sphere model.

Consequently, the linearized collision operator $\mathbf{L}_{\mathbf{M}}$ is a dissipative operator on $L^2(\mathbf{R}^3)$, and its inverse $\mathbf{L}_{\mathbf{M}}^{-1}$ is a bounded operator on \mathfrak{N}^\perp . It follows from (1.10) that

$$\mathbf{G} = \varepsilon \mathbf{L}_{\mathbf{M}}^{-1} [\mathbf{P}_1(\xi_1 \mathbf{M}_x)] + \Pi, \quad (1.11)$$

with

$$\Pi = \mathbf{L}_{\mathbf{M}}^{-1} [\varepsilon (\mathbf{G}_t + \mathbf{P}_1(\xi_1 \mathbf{G}_x)) - Q(\mathbf{G}, \mathbf{G})]. \quad (1.12)$$

Plugging (1.11) into (1.9) gives

$$\left\{ \begin{array}{l} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4\varepsilon}{3}(\mu(\theta)u_{1x})_x - \int \xi_1^2 \Pi_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = \varepsilon(\mu(\theta)u_{ix})_x - \int \xi_1 \xi_i \Pi_x d\xi, \quad i = 2, 3, \\ [\rho(\theta + \frac{|u|^2}{2})]_t + [\rho u_1(\theta + \frac{|u|^2}{2}) + p u_1]_x = \varepsilon(\kappa(\theta)\theta_x)_x + \frac{4\varepsilon}{3}(\mu(\theta)u_1 u_{1x})_x \\ \quad + \varepsilon \sum_{i=2}^3 (\mu(\theta)u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_x d\xi, \end{array} \right. \quad (1.13)$$

where the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\kappa(\theta) > 0$ are smooth functions of the temperature θ . Here, we normalize the gas constant R to be $\frac{2}{3}$ so that $e = \theta$ and $p = \frac{2}{3}\rho\theta$.

Since the problem considered in this paper is one dimensional in the space variable $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to rewrite the equation (1.1) and the system (1.2) in the *Lagrangian* coordinates. For this, set the coordinate transformation:

$$(t, x) \Rightarrow \left(t, \int_{(0,0)}^{(t,x)} \rho(\tau, y) dy - (\rho u_1)(\tau, y) d\tau \right), \quad (1.14)$$

where $\int_A^B f dy + g d\tau$ represents a line integraton from point A to point B on $\mathbf{R}^+ \times \mathbf{R}$. Here, the value of the integration is unique because of the conservation of mass.

We will still denote the *Lagrangian* coordinates by (t, x) for the simplicity of notations. Then (1.1) and (1.2) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = \frac{1}{\varepsilon} Q(f, f), \quad (1.15)$$

and

$$\left\{ \begin{array}{l} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ u_{it} = 0, \quad i = 2, 3, \\ (\theta + \frac{|u|^2}{2})_t + (p u_1)_x = 0. \end{array} \right. \quad (1.16)$$

Moreover, (1.9)-(1.13) take the form of

$$\left\{ \begin{array}{l} v_t - u_{1x} = 0, \\ u_{1t} + p_x = - \int \xi_1^2 \mathbf{G}_x d\xi, \\ u_{it} = - \int \xi_1 \xi_i \mathbf{G}_x d\xi, \quad i = 2, 3, \\ (\theta + \frac{|u|^2}{2})_t + (p u_1)_x = - \int \frac{1}{2} \xi_1 |\xi|^2 \mathbf{G}_x d\xi, \end{array} \right. \quad (1.17)$$

$$\mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x) + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x) = \frac{1}{\varepsilon} (\mathbf{L}_M \mathbf{G} + Q(\mathbf{G}, \mathbf{G})), \quad (1.18)$$

with

$$\mathbf{G} = \varepsilon \mathbf{L}_M^{-1} \left(\frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_x) \right) + \Pi_1, \quad (1.19)$$

$$\Pi_1 = \mathbf{L}_M^{-1} \left[\varepsilon \left(\mathbf{G}_t - \frac{u_1}{v} \mathbf{G}_x + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x) \right) - Q(\mathbf{G}, \mathbf{G}) \right], \quad (1.20)$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4\varepsilon}{3} \left(\frac{\mu(\theta)}{v} u_{1x} \right)_x - \int \xi_1^2 \Pi_{1x} d\xi, \\ u_{it} = \varepsilon \left(\frac{\mu(\theta)}{v} u_{ix} \right)_x - \int \xi_1 \xi_i \Pi_{1x} d\xi, \quad i = 2, 3, \\ \left(\theta + \frac{|u|^2}{2} \right)_t + (pu_1)_x = \varepsilon \left(\frac{\kappa(\theta)}{v} \theta_x \right)_x + \frac{4\varepsilon}{3} \left(\frac{\mu(\theta)}{v} u_1 u_{1x} \right)_x \\ \quad + \varepsilon \sum_{i=2}^3 \left(\frac{\mu(\theta)}{v} u_i u_{ix} \right)_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_{1x} d\xi. \end{cases} \quad (1.21)$$

The Riemann problem for the Euler system (1.16) is an initial value problem with initial data

$$(v, u, \theta)(t = 0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases}$$

where, $u = (u_1, u_2, u_3)$, $u_{\pm} = (u_{1\pm}, 0, 0)$ and $v_{\pm} > 0, u_{1\pm}, \theta_{\pm} > 0$ are constants. It is known that the generic solution to the Riemann problem consists of three waves that propagates at different speeds, that is, shock, rarefaction wave and contact discontinuity, cf. [10, 27]. We denote this solution by $(\tilde{V}, \tilde{U}, \tilde{\Theta})(t, x)$. Note that $\tilde{U} = (\tilde{U}_1, 0, 0)$.

Given the right end state (v_+, u_{1+}, θ_+) , the following wave curves for the left end state (v, u_1, θ) in the phase space are defined with $v > 0$ and $\theta > 0$ for the Euler equations (1.16).

- Contact discontinuity curve:

$$CD(v_+, u_{1+}, \theta_+) = \{(v, u_1, \theta) | u_1 = u_{1+}, p = p_+, v \neq v_+\}. \quad (1.22)$$

- i -Rarefaction wave curve ($i = 1, 3$):

$$R_i(v_+, u_{1+}, \theta_+) := \left\{ (v, u_1, \theta) \left| v < v_+, u_1 = u_{1+} - \int_{v_+}^v \lambda_i(\eta, s_+) d\eta, s(v, \theta) = s_+ \right. \right\}, \quad (1.23)$$

where $s_+ = s(v_+, \theta_+)$ and $\lambda_i = \lambda_i(v, s)$ is the i -th characteristic speed of (1.16).

- i -Shock wave curve ($i = 1, 3$):

$$S_i(v_+, u_{1+}, \theta_+) := \left\{ (v, u_1, \theta) \left| \begin{array}{l} -s_i(v_+ - v) - (u_{1+} - u_1) = 0, \\ -s_i(u_{1+} - u_1) + (p_+ - p) = 0, \\ -s_i(E_+ - E) + (p_+ u_{1+} - pu_1) = 0, \end{array} \right. \text{ and } \lambda_{i+} < s_i < \lambda_{i-} \right\}, \quad (1.24)$$

where $E = \theta + \frac{|u|^2}{2}, p = \frac{2\theta}{3v}, E_+ = \theta_+ + \frac{|u_+|^2}{2}, p_+ = \frac{2\theta_+}{3v_+}$, $\lambda_{i\pm} = \lambda_i(v_{\pm}, \theta_{\pm})$ and s_i is the i -shock speed.

For definiteness, we consider the case when the solution to the Riemann problem is a superposition of a 1-rarefaction and a 3-shock wave with a contact discontinuity in between, that is, $(v_-, u_{1-}, \theta_-) \in R_1\text{-}CD\text{-}S_3(v_+, u_{1+}, \theta_+)$. Then there exist uniquely two intermediate states (v_*, u_{1*}, θ_*) and (v^*, u_1^*, θ^*) such that $(v_-, u_{1-}, \theta_-) \in R_1(v_*, u_{1*}, \theta_*)$, $(v_*, u_{1*}, \theta_*) \in CD(v^*, u_1^*, \theta^*)$ and $(v^*, u_1^*, \theta^*) \in S_3(v_+, u_{1+}, \theta_+)$.

Hence, the wave pattern $(\tilde{V}, \tilde{U}, \tilde{\mathcal{E}})(t, x)$ can be written as

$$\begin{pmatrix} \tilde{V} \\ \tilde{U}_1 \\ \tilde{\mathcal{E}} \end{pmatrix} (t, x) = \begin{pmatrix} v^{r_1} + v^{cd} + v^{s_3} \\ u_1^{r_1} + u_1^{cd} + u_1^{s_3} \\ E^{r_1} + E^{cd} + E^{s_3} \end{pmatrix} (t, x) - \begin{pmatrix} v_* + v^* \\ u_{1*} + u_1^* \\ E_* + E^* \end{pmatrix}, \quad \tilde{U}_i = 0, (i = 2, 3), \quad (1.25)$$

where $(v^{r_1}, u_1^{r_1}, \theta^{r_1})(t, x)$ is the 1-rarefaction wave defined in (1.23) with the right state (v_+, u_{1+}, θ_+) given by (v_*, u_{1*}, θ_*) , $(v^{cd}, u_1^{cd}, \theta^{cd})(t, x)$ is the contact discontinuity defined in (1.22) with the states (v_-, u_{1-}, θ_-) and (v_+, u_{1+}, θ_+) given by (v_*, u_{1*}, θ_*) and (v^*, u_1^*, θ^*) respectively, and $(v^{s_3}, u_1^{s_3}, \theta^{s_3})(t, x)$ is the 3-shock wave defined in (1.24) with the left state (v_-, u_{1-}, θ_-) given by (v^*, u_1^*, θ^*) .

Consequently, we can define

$$\tilde{\Theta}(t, x) = (\tilde{\mathcal{E}}(t, x) - \frac{\tilde{U}(t, x)^2}{2}). \quad (1.26)$$

Due to the singularity of the rarefaction wave at $t = 0$, in this paper, we consider the problem in the time interval $[h, T]$ for any small fixed $h > 0$ up to any arbitrarily fixed time $T > 0$. To investigate the interaction between the waves and the initial layer is another interesting topic that will not be discussed here. With the above preparation, the main result can be stated as follows.

Theorem 1.1 *Let $(\tilde{V}, \tilde{U}, \tilde{\Theta})(t, x)$ be a Riemann solution to the Euler equations which is a superposition of a 1-rarefaction wave, a 2-contact discontinuity and a 3-shock wave, and $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)|$ be the wave strength. There exist a small positive constant δ_0 , and a global Maxwellian $\mathbf{M}_\star = \mathbf{M}_{[v_\star, u_\star, \theta_\star]}$ such that if the wave strength satisfies $\delta \leq \delta_0$, then in any time interval $[h, T]$ with $0 < h < T$, there exists a positive constant $\varepsilon_0 = \varepsilon_0(\delta, h, T)$, such that if the Knudsen number $\varepsilon \leq \varepsilon_0$, then the Boltzmann equation admits a family of smooth solutions $f^{\varepsilon, h}(t, x, \xi)$ satisfying*

$$\sup_{(t, x) \in \Sigma_{h, T}} \|f^{\varepsilon, h}(t, x, \xi) - \mathbf{M}_{[\tilde{V}, \tilde{U}, \tilde{\Theta}]}(t, x, \xi)\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_\star})}} \leq C_{h, T} \varepsilon^{\frac{1}{5}} |\ln \varepsilon|,$$

where $\Sigma_{h, T} = \{(t, x) | h \leq t \leq T, |x| \geq h, |x - s_3 t| \geq h\}$, the norm $\|\cdot\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_\star})}}$ is $\|\cdot\|_{L_\xi^2(\mathbf{R}^3)}$ and the positive constant $C_{h, T}$ depends on h and T but is independent of ε . Consequently, when $\varepsilon \rightarrow 0+$ and then $h \rightarrow 0+, T \rightarrow +\infty$, we have

$$\|f^{\varepsilon, h}(\xi) - \mathbf{M}_{[\tilde{V}, \tilde{U}, \tilde{\Theta}]}(\xi)\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_\star})}}(t, x) \rightarrow 0, \quad \text{a.e. in } \mathbf{R}^+ \times \mathbf{R}.$$

Remark 1 *Theorem 1.1 shows that away from the initial time $t = 0$, the contact discontinuity at $x = 0$ and the shock discontinuity at $x = s_3 t$, for small total wave strength $\delta \leq \delta_0$ and Knudsen number $\varepsilon \leq \varepsilon_0$, there exists a family of smooth solutions $f^{\varepsilon, h}(t, x, \xi)$ of the Boltzmann equation which tends to the Maxwellian $\mathbf{M}_{[\tilde{V}, \tilde{U}, \tilde{\Theta}]}(t, x, \xi)$ with $(\tilde{V}, \tilde{U}, \tilde{\Theta})(t, x)$ being the Riemann solution to the Euler equations as a superposition of a 1-rarefaction wave, a 2-contact discontinuity and a 3-shock wave when $\varepsilon \rightarrow 0$ with a convergence rate $\varepsilon^{\frac{1}{5}} |\ln \varepsilon|$. Note that this superposition of waves is the most generic case for the Riemann problem. Similar results hold for any other superpositions of waves by using the same analysis.*

Remark 2 *The proof of the above theorem crucially depends on the introduction of two kinds of hyperbolic waves. The hyperbolic wave I was constructed by Huang-Wang-Yang [21] for the compressible Navier-Stokes equations to recover the viscous terms to the inviscid approximation of rarefaction wave pattern where the rarefaction wave structure plays an important role in the construction.*

The hyperbolic wave II is constructed to remove the error terms due to the viscous contact wave approximation. Note that the construction of the hyperbolic wave II can not be done simply around the contact wave approximation as the hyperbolic wave I for the rarefaction wave. Otherwise, the wave interaction terms thus induced will lead to insufficiently decay in term of the Knudsen number. Instead, it is constructed around the superposition of the approximate 1-rarefaction wave, the hyperbolic wave I, the 2-viscous contact wave and the 3-shock profile as a whole. Moreover, it also takes care of the non-conservative terms in the previous reduced system so that energy estimates can be taken for anti-derivative of the perturbation.

Remark 3 *Note that the analysis can also be applied to the vanishing viscosity limit of the one dimensional compressible Navier-Stokes equations. In fact, the vanishing viscosity limit of the one dimensional compressible Navier-Stokes equations in some sense can be viewed as a special case of hydrodynamic limit of Boltzmann equation to the Euler equations by neglecting the microscopic effect.*

Remark 4 *If the total wave strength $\delta = |(v_+ - v_-, u_+ - u_-, \theta_+ - \theta_-)| \leq \delta_0$, then from the wave curves defined in (1.22), (1.23) and (1.24), we know that $\delta^{R_1}, \delta^{CD}, \delta^{S_3} \leq C\delta_0$ where $\delta^{R_1}, \delta^{CD}, \delta^{S_3}$ are the wave strengths of rarefaction wave, contact discontinuity and shock wave, respectively.*

Let us now review some previous works on the hydrodynamic limits to the Boltzmann equation. For the case when the Euler equations have smooth solutions, the vanishing Knudsen number limit of the Boltzmann equation has been studied even in the case with an initial layer, cf. Caffisch [6], Lachowicz [26], Nishida [33] and Ukai-Asona [37] etc. However, as well-known, solutions of the Euler equations in general develop singularities, such as shock waves and contact discontinuities. Therefore, how to verify

the hydrodynamic limit from the Boltzmann equation to the Euler equations with basic wave patterns becomes a natural problem in the process to the general setting. In this direction, with slab symmetry, as mentioned earlier, there were studies on each individual wave pattern. For superposition of different types of waves, to our knowledge, there is only one result given in [20] about the superposition of two rarefaction waves and one contact discontinuity.

On the other hand, for the incompressible equations, there are works, such as those by Bardos-Golse-Levermore, Bardos-Levermore-Ukai-Yang, Bardos-Ukai, Golse-Saint Raymond, Levermore-Masmoudi and Sone which studied direct derivations of the incompressible Navier-Stokes equations in the long time scaling, about which more is known, cf. [2, 4, 3, 15, 25, 35, 36] and the references therein. In particular, Golse and Saint-Raymond showed that the limits of suitably rescaled sequences of the DiPerna-Lions renormalized solutions to the Boltzmann equation are the Leray solutions to the incompressible Navier-Stokes equations. However, even in this aspect, the uniqueness and regularity of the solution are still big issues. Since we will concentrate on the compressible Euler limit in this paper, we will not go into details about the incompressible limits.

Furthermore, the Boltzmann equation provides more information than the classical fluid dynamical systems so that it describes some phenomena which can not be modeled by using the classical systems, such as Euler and Navier-Stokes equations. This kind of interesting phenomena, such as the thermal creep flow in a rarefied gas was known since the time of Maxwell. Some mathematical formulations and numerical computations on the basis of kinetic equations were studied since 1960s, cf. the works by Sone [35, 36]. However, the justification of this kind of fluid dynamics is almost open with rigorous mathematical theory.

Finally, we briefly outline the proof of the theorem. Firstly, we define the individual wave profile. Then we introduce the first family of hyperbolic wave by linearizing around the approximate rarefaction profile and by adding the viscosity and heat conductivity terms induced by the profile. Then we define the first approximation of the superposition of this hyperbolic wave together with the three basic wave patterns so that it takes care of the hyperbolicity of the rarefaction wave in the viscous setting.

Based on this, we linearize the fluid system around this profile and consider the propagation of the extra error due to the contact discontinuity approximation and then define a second set of hyperbolic wave. By adding these two sets of hyperbolic waves to the superposition of the three basic wave profiles, we will perform the energy estimate on the Boltzmann equation with suitable initial data through the macro-micro decomposition. Precisely, for the macroscopic component, we will consider the anti-derivative of the perturbation after applying a hyperbolic scaling. By using the dissipation in the fluid-type system and the linearized Boltzmann operator on the microscopic component, we can close the energy estimate through a suitable chosen a priori assumption. Then the statements in the theorem follow.

The rest of the paper will be arranged as follows. In Section 2, we will construct the approximate solutions to the Boltzmann equation corresponding to the basic wave patterns to the Euler system. Then we obtain the detailed information on the difference between the Riemann solution to Euler system and the approximate solution to the Boltzmann equation by the construction. In Section 3, we will construct a family of solutions to the Boltzmann equation around the approximate solution by using energy method to close the a priori estimate. Since the proofs of two Propositions 3.1 and 3.2 about the lower and higher order energy estimates respectively are very technical and long, we put them to the Appendices.

Notations: Throughout this paper, the positive generic constants which are independent of ε, T, h are denoted by $c, C, C_i (i = 1, 2, 3, \dots)$, while $C_{h,T}$ represents a generic positive constant depending on h and T but independent of ε . And we will use $\|\cdot\|$ to denote the standard $L_2(\mathbf{R}; dy)$ norm, and $\|\cdot\|_{H^i}$ ($i = 1, 2, 3, \dots$) to denote the Sobolev $H^i(\mathbf{R}; dy)$ norm. Sometimes, we also use $O(1)$ to denote a uniform bounded constant which is independent of ε, T, h .

2 Approximate Wave Patterns

In this section, we will construct the approximate wave profile that consists of three basic wave patterns and two hyperbolic waves. For this, we will firstly recall the construction of the approximate rarefaction wave for the Boltzmann equation. Then we will introduce the hyperbolic waves I to correct the error terms coming from the hyperbolic approximation. Then we will construct the viscous contact wave

to Boltzmann equation and study the non-conservative error terms. The viscous shock profile to the Boltzmann equation will then be recalled. With the above wave patterns, we will introduce the hyperbolic wave II to take care of the error terms due to the viscous contact wave by avoiding the interaction between the viscous contact wave with the wave patterns defined earlier.

2.1 Rarefaction Wave

For the rarefaction wave, since there is no exact rarefaction wave profile for either the Navier-Stokes equations or the Boltzmann equation, the following approximate rarefaction wave profile satisfying the Euler equations was introduced in [32, 39]. For the completeness of the presentation, we include its definition and the properties obtained in the above two papers as follows.

If $(v_-, u_{1-}, \theta_-) \in R_1(v_+, u_{1+}, \theta_+)$, then there exists a 1-rarefaction wave $(v^{r_1}, u_1^{r_1}, E^{r_1})(x/t)$ which is a global solution to the following Riemann problem

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = 0, \\ E_t + (pu_1)_x = 0, \\ (v, u_1, \theta)(t=0, x) = \begin{cases} (v_-, u_{1-}, \theta_-), & x < 0, \\ (v_+, u_{1+}, \theta_+), & x > 0. \end{cases} \end{cases} \quad (2.1)$$

Consider the following inviscid Burgers equation with Riemann data

$$\begin{cases} w_t + ww_x = 0, \\ w(t=0, x) = \begin{cases} w_-, & x < 0, \\ w_+, & x > 0. \end{cases} \end{cases} \quad (2.2)$$

If $w_- < w_+$, then the above Riemann problem admits a rarefaction wave solution

$$w^r(t, x) = w^r\left(\frac{x}{t}\right) = \begin{cases} w_-, & \frac{x}{t} \leq w_-, \\ \frac{x}{t}, & w_- \leq \frac{x}{t} \leq w_+, \\ w_+, & \frac{x}{t} \geq w_+. \end{cases} \quad (2.3)$$

As in [39], the approximate rarefaction wave $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ to the problem (2.1) can be constructed by the solution of the Burgers equation

$$\begin{cases} w_t + ww_x = 0, \\ w(0, x) = w_\sigma(x) = w\left(\frac{x}{\sigma}\right) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} \tanh \frac{x}{\sigma}, \end{cases} \quad (2.4)$$

where $\sigma > 0$ is a small parameter to be determined later to be $\varepsilon^{\frac{1}{5}}$. Note that the solution $w_\sigma^r(t, x)$ of the problem (2.4) is given by

$$w_\sigma^r(t, x) = w_\sigma(x_0(t, x)), \quad x = x_0(t, x) + w_\sigma(x_0(t, x))t.$$

The smooth approximate rarefaction wave profile denoted by $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ can be defined by

$$\begin{cases} S^{R_1}(t, x) = s(V^{R_1}(t, x), \Theta^{R_1}(t, x)) = s_+, \\ w_\pm = \lambda_{1\pm} := \lambda_1(v_\pm, \theta_\pm), \\ w_\sigma^r(t, x) = \lambda_1(V^{R_1}(t, x), s_+), \\ U_1^{R_1}(t, x) = u_{1+} - \int_{v_+}^{V^{R_1}(t, x)} \lambda_1(v, s_+) dv, \\ U_i^{R_1}(t, x) \equiv 0, \quad i = 2, 3. \end{cases} \quad (2.5)$$

Note that $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ defined above satisfies

$$\begin{cases} V_t^{R_1} - U_{1x}^{R_1} = 0, \\ U_{1t}^{R_1} + P_x^{R_1} = 0, \\ U_{it}^{R_1} = 0, \quad i = 2, 3, \\ \mathcal{E}_t^{R_1} + (P^{R_1} U_1^{R_1})_x = 0, \end{cases} \quad (2.6)$$

where $P^{R_1} = p(V^{R_1}, \Theta^{R_1}) = \frac{2\Theta^{R_1}}{3V^{R_1}}$ and $\mathcal{E}^{R_1} = \Theta^{R_1} + \frac{|U^{R_1}|^2}{2}$. The properties of the rarefaction wave profile can be summarized as follows.

Lemma 2.1 ([39]) *The approximate rarefaction waves $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ constructed in (2.5) have the following properties:*

(1) $U_{1x}^{R_1}(t, x) > 0$ for $x \in \mathbf{R}$, $t > 0$;

(2) For any $1 \leq p \leq +\infty$, the following estimates holds,

$$\begin{aligned} \|(V^{R_1}, U_1^{R_1}, \Theta^{R_1})_x\|_{L^p(dx)} &\leq C \min \{ \delta^{R_1} \sigma^{-1+1/p}, (\delta^{R_1})^{1/p} t^{-1+1/p} \}, \\ \|(V^{R_1}, U_1^{R_1}, \Theta^{R_1})_{xx}\|_{L^p(dx)} &\leq C \min \{ \delta^{R_1} \sigma^{-2+1/p}, \sigma^{-1+1/p} t^{-1} \}, \end{aligned}$$

where the positive constant C depends only on p and the wave strength;

(3) If $x \geq \lambda_{1+}^{R_1} t$, then

$$\begin{aligned} |(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x) - (v_+, u_+, \theta_+)| &\leq C e^{-\frac{2|x-\lambda_{1+}t|}{\sigma}}, \\ |\partial_x^k (V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)| &\leq \frac{C}{\sigma^k} e^{-\frac{2|x-\lambda_{1+}t|}{\sigma}}, \quad k = 1, 2; \end{aligned}$$

(4) There exist positive constants C and σ_0 such that for $\sigma \in (0, \sigma_0)$ and $t > 0$,

$$\sup_{x \in \mathbf{R}} |(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})(t, x) - (v^{r_1}, u^{r_1}, E^{r_1})\left(\frac{x}{t}\right)| \leq \frac{C}{t} [\sigma \ln(1+t) + \sigma |\ln \sigma|].$$

2.2 Hyperbolic Wave I

Since the whole wave profile consists of a shock wave whose rate of change in the shock region is of the order of ε^{-1} , we have to consider the anti-derivative of the perturbation in order to cope with the correct sign as in the stability analysis. From (2.6), we know that the approximate rarefaction wave $(V^{R_1}, U^{R_1}, \Theta^{R_1})(t, x)$ satisfies the compressible Euler equations exactly without viscous terms. Thus if we carry out the energy estimates to the anti-derivative variables, the error terms due to the viscous terms from the approximate rarefaction wave are not good enough to get the desired estimates. In order to overcome this difficulty, we introduce the hyperbolic wave I to recover these viscous terms.

This hyperbolic wave denoted by $(d_1, d_2, d_3)(t, x)$ can be defined as follows. Consider a linear system

$$\begin{cases} d_{1t} - d_{2x} = 0, \\ d_{2t} + (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3)_x = \frac{4}{3} \varepsilon \left(\frac{\mu(\Theta^{R_1}) U_{1x}^{R_1}}{V^{R_1}} \right)_x, \\ d_{3t} + [(pu_1)_v^{R_1} d_1 + (pu_1)_{u_1}^{R_1} d_2 + (pu_1)_E^{R_1} d_3]_x = \varepsilon \left(\frac{\kappa(\Theta^{R_1}) \Theta_x^{R_1}}{V^{R_1}} \right)_x + \frac{4}{3} \varepsilon \left(\frac{\mu(\Theta^{R_1}) U_1^{R_1} U_{1x}^{R_1}}{V^{R_1}} \right)_x, \end{cases} \quad (2.7)$$

where $p = \frac{R\theta}{v} = p(v, u, E) = \frac{2E-u^2}{3v}$ and $p_v^{R_1} = p_v(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1})$ etc. Note that the left hand side of the above system is the linearization of the Euler equation around the rarefaction wave approximation. We want to solve this linear hyperbolic system (2.7) on the time interval $[h, T]$. Firstly, we diagonalize the above system by rewriting it as

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_t + \left[A^{R_1} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \right]_x = \begin{pmatrix} 0 \\ H_1^{R_1} \\ H_2^{R_1} \end{pmatrix}, \quad (2.8)$$

where $H_1^{R_1} = \varepsilon \left(\frac{\mu(\Theta^{R_1}) U_{1x}^{R_1}}{V^{R_1}} \right)_x$, $H_2^{R_1} = \varepsilon \left(\frac{\kappa(\Theta^{R_1}) \Theta_x^{R_1}}{V^{R_1}} \right)_x + \varepsilon \left(\frac{\mu(\Theta^{R_1}) U_1^{R_1} U_{1x}^{R_1}}{V^{R_1}} \right)_x$. Here, the matrix

$$A^{R_1} = \begin{pmatrix} 0 & -1 & 0 \\ p_v^{R_1} & p_{u_1}^{R_1} & p_E^{R_1} \\ (pu_1)_v^{R_1} & (pu_1)_{u_1}^{R_1} & (pu_1)_E^{R_1} \end{pmatrix}$$

has three distinct eigenvalues $\lambda_1^{R_1} := \lambda_1(V^{R_1}, s_\pm) < 0 \equiv \lambda_2^{R_1} < \lambda_3(V^{R_1}, s_\pm) := \lambda_3^{R_1}$ and the corresponding left and right eigenvectors denoted $l_j^{R_1}, r_j^{R_1}$ ($j = 1, 2, 3$) respectively, satisfy

$$L^{R_1} A^{R_1} R^{R_1} = \text{diag}(\lambda_1^{R_1}, 0, \lambda_3^{R_1}) \equiv \Lambda^{R_1}, \quad L^{R_1} R^{R_1} = \text{Id.},$$

Here $L^{R_1} = (l_1^{R_1}, l_2^{R_1}, l_3^{R_1})^t$, $R^{R_1} = (r_1^{R_1}, r_2^{R_1}, r_3^{R_1})$ with $l_i^{R_1} = l_i(V^{R_1}, U_1^{R_1}, s_+)$ and $r_i^{R_1} = r_i(V^{R_1}, U_1^{R_1}, s_+)$ ($i = 1, 2, 3$) and Id. is the 3×3 identity matrix. Now we set

$$(D_1, D_2, D_3)^t = L^{R_1}(d_1, d_2, d_3)^t. \quad (2.9)$$

Then

$$(d_1, d_2, d_3)^t = R^{R_1}(D_1, D_2, D_3)^t, \quad (2.10)$$

and (D_1, D_2, D_3) satisfies the system

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}_t + \left[\Lambda^{R_1} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} \right]_x = L^{R_1} \begin{pmatrix} 0 \\ H_1^{R_1} \\ H_2^{R_1} \end{pmatrix} + L_t^{R_1} R^{R_1} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + L_x^{R_1} R^{R_1} \Lambda^{R_1} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}. \quad (2.11)$$

Due to the fact that the 1-Riemann invariant is constant along the approximate rarefaction wave curve, we have

$$L_t^{R_1} = -\lambda_1^{R_1} L_x^{R_1}.$$

Substituting the above equation into (2.11), we obtain the diagonalized system

$$\begin{cases} D_{1t} + (\lambda_1^{R_1} D_1)_x = b_{12}^{R_1} H_1^{R_1} + b_{13}^{R_1} H_2^{R_1} + a_{12}^{R_1} V_x^{R_1} D_2 + a_{13}^{R_1} V_x^{R_1} D_3, \\ D_{2t} = b_{22}^{R_1} H_1^{R_1} + b_{23}^{R_1} H_2^{R_1} + a_{22}^{R_1} V_x^{R_1} D_2 + a_{23}^{R_1} V_x^{R_1} D_3, \\ D_{3t} + (\lambda_3^{R_1} D_3)_x = b_{32}^{R_1} H_1^{R_1} + b_{33}^{R_1} H_2^{R_1} + a_{32}^{R_1} V_x^{R_1} D_2 + a_{33}^{R_1} V_x^{R_1} D_3, \end{cases} \quad (2.12)$$

where $a_{ij}^{R_1}, b_{ij}^{R_1}$ are some given functions of $V^{R_1}, U_1^{R_1}$ and $S^{R_1} = s_\pm$. Note that in the diagonalized system (2.12), the equations of D_2, D_3 are decoupled from D_1 because of the property of the rarefaction wave.

Now we impose the following boundary condition to the above linear hyperbolic system (2.12) in the domain $(t, x) \in [h, T] \times \mathbf{R}$:

$$D_1(t = h, x) = 0, \quad D_2(t = T, x) = D_3(t = T, x) = 0. \quad (2.13)$$

With this boundary condition, we can solve the linear diagonalized hyperbolic system (2.12) under the conditions (2.13). Moreover, we have the following estimates on the solution.

Lemma 2.2 *There exists a positive constant $C_{h,T}$ independent of ε such that*

(1)

$$\left\| \frac{\partial^k}{\partial x^k} d_i(t, \cdot) \right\|_{L^2(dx)}^2 \leq C_{h,T} \frac{\varepsilon^2}{\sigma^{2k+1}}, \quad i = 1, 2, 3, \quad k = 0, 1, 2, 3.$$

(2) *If $x > \lambda_1 + t$, then we have*

$$|d_i(x, t)| \leq C_{h,T} \frac{1}{\sigma} e^{-\frac{|x - \lambda_1 + t|}{\sigma}}, \quad |d_{ix}(x, t)| \leq C_{h,T} \frac{1}{\sigma^2} e^{-\frac{|x - \lambda_1 + t|}{\sigma}}, \quad i = 1, 2, 3.$$

The proof of Lemma 2.2 can be done similarly as in [21] for the compressible Navier-Stokes equations.

2.3 Viscous Contact Wave

In this subsection, we construct the contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ for the Boltzmann equation motivated by [23]. Consider the Euler system (1.16) with a Riemann initial data

$$(v, u, \theta)(t = 0, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases} \quad (2.14)$$

where $u_{\pm} = (u_{1\pm}, 0, 0)$ and $v_{\pm} > 0, \theta_{\pm} > 0, u_{1\pm}$ are given constants. It is known (cf. [34]) that the Riemann problem (1.16), (2.14) admits a contact discontinuity

$$(v^{cd}, u^{cd}, \theta^{cd})(t, x) = \begin{cases} (v_-, u_-, \theta_-), & x < 0, \\ (v_+, u_+, \theta_+), & x > 0, \end{cases} \quad (2.15)$$

provided that $(v_-, u_{1-}, \theta_-) \in CD(v_+, u_{1+}, \theta_+)$, that is,

$$u_{1+} = u_{1-}, \quad p_- := \frac{2\theta_-}{3v_-} = p_+ := \frac{2\theta_+}{3v_+}. \quad (2.16)$$

Then for the Navier-Stokes equations, by the energy equation (1.21)₄ and the mass equation (1.21)₁ with $v \approx \frac{2\theta}{3p_+}$, cf. [20], we can obtain the following nonlinear diffusion equation

$$\theta_t = \varepsilon(a(\theta)\theta_x)_x, \quad a(\theta) = \frac{9p_+\kappa(\theta)}{10\theta}. \quad (2.17)$$

From [1] and [11], we know that the nonlinear diffusion equation (2.17) admits a unique self-similar solution $\hat{\Theta}(\eta)$, $\eta = \frac{x}{\sqrt{\varepsilon(1+t)}}$ satisfying the boundary conditions $\hat{\Theta}(\pm\infty, t) = \theta_{\pm}$. Let $\delta^{CD} = |\theta_+ - \theta_-|$, then $\hat{\Theta}(t, x)$ has the property that

$$\hat{\Theta}_x(t, x) = \frac{O(1)\delta^{CD}}{\sqrt{\varepsilon(1+t)}} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad (2.18)$$

with some positive constant c depending only on θ_{\pm} .

Correspondingly, we can define the Navier-Stokes profile by

$$\begin{aligned} \hat{V} &= \frac{2}{3p_+}\hat{\Theta}, \\ \hat{U}_1 &= u_{1+} + \frac{2\varepsilon a(\hat{\Theta})}{3p_+}\hat{\Theta}_x, \quad \hat{U}_i = 0, i = 2, 3. \end{aligned} \quad (2.19)$$

For the Boltzmann equation, if we still use the above Navier-Stokes profile $(\hat{V}, \hat{U}, \hat{\Theta})$, we can not get any decay with respect to the Knudsen number ε due to the non-fluid component. Hence, we construct a Boltzmann contact wave as follows. Set

$$\mathbf{G}^{CD}(t, x, \xi) = \frac{3\varepsilon}{2v\theta} \mathbf{L}_M^{-1} \left\{ \mathbf{P}_1 \left[\xi_1 \left(\frac{|\xi - u|^2}{2\theta} \Theta_x^{CD} + \xi \cdot U_x^{CD} \right) \mathbf{M} \right] \right\}, \quad (2.20)$$

and

$$\Pi_{11}^{CD} = \mathbf{L}_M^{-1} \left[\varepsilon \left(-\frac{u_1}{v} \mathbf{G}_x^{CD} + \frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{G}_x^{CD}) \right) - Q(\mathbf{G}^{CD}, \mathbf{G}^{CD}) \right], \quad (2.21)$$

where $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ is the viscous contact wave for the Boltzmann equation to be constructed later.

Note that for the Boltzmann equation, the leading terms in the energy equation (1.21)₄ can be written as

$$\theta_t + p_+ u_{1x} = \varepsilon \left(\frac{\kappa(\theta)\theta_x}{v} \right)_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_{11x}^{CD} d\xi + u_{1+} \int \frac{1}{2} \xi_1^2 \Pi_{11x}^{CD} d\xi. \quad (2.22)$$

By the definition of Π_{11}^{CD} in (2.21), we have

$$- \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_{11}^{CD} d\xi + u_{1+} \int \frac{1}{2} \xi_1^2 \Pi_{11}^{CD} d\xi = \Delta_{11} + \Delta_{12}, \quad (2.23)$$

where

$$\Delta_{11} = \varepsilon^2 \left[g_{11} \theta_x \Theta_x^{CD} + g_{12} v_x \Theta_x^{CD} + g_{13} (\Theta_x^{CD})^2 + g_{14} \Theta_{xx}^{CD} \right], \quad (2.24)$$

with $g_{1i} = g_{1i}(v, u, \theta)$, ($i = 1, 2, 3, 4$) being smooth functions of (v, u, θ) , and

$$\Delta_{12} = O(1) \varepsilon^2 \left[(|v_x| + |u_x| + |\theta_x| + |\Theta_x^{CD}| + |U_x^{CD}|) |U_x^{CD}| + |u_x| |\Theta_x^{CD}| + |U_{xx}^{CD}| \right]. \quad (2.25)$$

Thus, by choosing the leading term and dropping the higher order term Δ_{12x} in (2.22), we have

$$\theta_t = \varepsilon(a(\theta)\theta_x)_x + \frac{3}{5} \Delta_{11x}, \quad (2.26)$$

where $a(\theta)$ is defined in (2.17) and Δ_{11} is defined in (2.24). To represent the microscopic effect on the wave profile, we want to define Θ^{CD} to be close to $\hat{\Theta}(\frac{x}{\sqrt{\varepsilon(1+t)}}) + \hat{\Theta}^{nf}(t, x)$ with $\hat{\Theta}$ being determined by (2.17), (2.18) and $\hat{\Theta}^{nf}$ represents the part of the nonlinear diffusion wave coming from the non-fluid component not appearing in the Navier-Stokes level. Moreover, the term $\hat{\Theta}^{nf}$ decays faster than $\hat{\Theta}$ so that it can be viewed as the perturbation around the Navier-Stokes profile $\hat{\Theta}$. To construct $\hat{\Theta}^{nf}$, we linearize the equation (2.26) around the Navier-Stokes profile $\hat{\Theta}$ and drop all the higher order terms. This leads to a linear diffusion equation for $\hat{\Theta}^{nf}$

$$\hat{\Theta}_t^{nf} = \varepsilon(a(\hat{\Theta})\hat{\Theta}_x^{nf})_x + \varepsilon(a'(\hat{\Theta})\hat{\Theta}_x)\hat{\Theta}_x^{nf} + \frac{3}{5} \tilde{\Delta}_{11x}, \quad (2.27)$$

where $\tilde{\Delta}_{11} = \varepsilon^2(\tilde{g}_{11} + \frac{2}{3p_+}\tilde{g}_{12} + \tilde{g}_{13})(\hat{\Theta}_x)^2 + \varepsilon^2\tilde{g}_{14}\hat{\Theta}_{xx}$ with $\tilde{g}_{1i} = \tilde{g}_{1i}(\hat{V}, \hat{U}, \hat{\Theta})$ ($i = 1, 2, 3, 4$). Integrating (2.27) with respect to x yields that

$$\Xi_{1t} = \varepsilon a(\hat{\Theta})\Xi_{1xx} + \varepsilon a'(\hat{\Theta})\hat{\Theta}_x\Xi_{1x} + \frac{3}{5} \tilde{\Delta}_{11}, \quad (2.28)$$

where

$$\Xi_1(t, x) = \int_{-\infty}^x \hat{\Theta}^{nf}(t, x) dx. \quad (2.29)$$

Note that $\tilde{\Delta}_{11}$ takes the form of $\frac{\varepsilon}{1+t} A^1(\frac{x}{\sqrt{\varepsilon(1+t)}})$ and satisfies that

$$|\tilde{\Delta}_{11}| = O(\delta^{CD}) \varepsilon(1+t)^{-1} e^{-\frac{x^2}{4a(\theta_{\pm})\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty.$$

We can check that there exists a self-similar solution $\Xi_1(\frac{x}{\sqrt{\varepsilon(1+t)}})$ for (2.27) with the boundary conditions $\Xi_1(-\infty) = 0, \Xi_1(+\infty) = \Xi_{1+}$. Here Ξ_{1+} can be any given constant satisfying $|\Xi_{1+}| < \delta^{CD}$. It is worthy to point out that even though the function $\Xi_1(t, x)$ depends on the constant Ξ_{1+} , $\hat{\Theta}^{nf}(t, x) = \Xi_{1x}(t, x) \rightarrow 0$ as $x \rightarrow \pm\infty$. That is, the choice of the constant Ξ_{1+} has no influence on the ansatz as long as $|\Xi_{1+}| < \delta^{CD}$. From now on, we fix Ξ_{1+} so that the function $\Xi_1(t, x)$ is uniquely determined and its derivative $\Xi_{1x} = \hat{\Theta}^{nf}$ has the property

$$|\hat{\Theta}^{nf}| = |\Xi_{1x}| = O(\delta^{CD}) \varepsilon^{\frac{1}{2}}(1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4a(\theta_{\pm})\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty. \quad (2.30)$$

Then we apply the similar procedure to construct the second and the third components of the velocity of the contact wave denoted by U_i^{CD} ($i = 2, 3$) as follows. The leading part of the equation for u_i in (1.21) _{i} ($i = 2, 3$) is

$$u_{it} = \varepsilon \left(\frac{3p_+ \mu(\theta)}{2\theta} u_{ix} \right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi. \quad (2.31)$$

Firstly, we have

$$- \int \xi_1 \xi_i \Pi_{11}^{CD} d\xi = \Delta_{i1} + \Delta_{i2}, \quad (2.32)$$

where

$$\Delta_{i1} = \varepsilon^2 \left[g_{i1} \theta_x \Theta_x^{CD} + g_{i2} v_x \Theta_x^{CD} + g_{i3} (\Theta_x^{CD})^2 + g_{i4} \Theta_{xx}^{CD} \right], \quad (2.33)$$

with g_{ij} , ($i = 2, 3$, $j = 1, 2, 3, 4$) being the smooth functions of (v, u, θ) and

$$\Delta_{i2} = O(1)\varepsilon^2 \left[(|v_x| + |u_x| + |\theta_x| + |\Theta_x^{CD}| + |U_x^{CD}|) |U_x^{CD}| + |u_x| |\Theta_x^{CD}| + |U_{xx}^{CD}| \right]. \quad (2.34)$$

Thus we expect the viscous contact wave U_i^{CD} ($i = 2, 3$) to satisfy the following linear equation

$$U_{it}^{CD} = \varepsilon \left(\frac{3p_+ \mu(\hat{\Theta})}{2\hat{\Theta}} U_{ix}^{CD} \right)_x + \tilde{\Delta}_{i1x}, \quad i = 2, 3, \quad (2.35)$$

where where $\tilde{\Delta}_{i1} = \varepsilon^2 (\tilde{g}_{i1} + \frac{2}{3p_+} \tilde{g}_{i2} + \tilde{g}_{i3}) (\hat{\Theta}_x)^2 + \varepsilon^2 \tilde{g}_{i4} \hat{\Theta}_{xx}$ with $\tilde{g}_{ij} = \tilde{g}_{ij}(\hat{V}, \hat{U}, \hat{\Theta})$ ($i = 2, 3$, $j = 1, 2, 3, 4$). Integrating (2.35) with respect to x yields that

$$\Xi_{it} = \varepsilon \frac{3p_+ \mu(\hat{\Theta})}{2\hat{\Theta}} \Xi_{ixx} + \tilde{\Delta}_{i1}, \quad (2.36)$$

where

$$\Xi_i(t, x) = \int_{-\infty}^x U_i^{CD}(t, x) dx. \quad (2.37)$$

Note that $\tilde{\Delta}_{i1}$ takes the form $\frac{\varepsilon}{1+t} A^i \left(\frac{x}{\sqrt{\varepsilon(1+t)}} \right)$, $i = 2, 3$ and satisfies that

$$|\tilde{\Delta}_{i1}| = O(\delta^{CD}) \varepsilon (1+t)^{-1} e^{-\frac{x^2}{4a(\theta_{\pm})\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty,$$

We can check that there exists a self-similar solution $\Xi_i \left(\frac{x}{\sqrt{\varepsilon(1+t)}} \right)$ for (2.35) with the boundary conditions $\Xi_i(-\infty) = 0$, $\Xi_i(+\infty) = \Xi_{i+}$, ($i = 2, 3$). Again, here Ξ_{i+} can be any given constant satisfying $|\Xi_{i+}| < \delta^{CD}$. As we explained before, the choice of the constant Ξ_{i+} has no influence on the ansatz as long as $|\Xi_{1+}| < \delta^{CD}$. We fix Ξ_{i+} so that the function $\Xi_i(t, x)$ is uniquely determined and the derivative $\Xi_{ix} = U_i^{CD}$ has the property

$$|U_i^{CD}| = |\Xi_{ix}| = O(\delta^{CD}) \varepsilon^{\frac{1}{2}} (1+t)^{-\frac{1}{2}} e^{-\frac{x^2}{4b(\theta_{\pm})\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty, \quad (2.38)$$

with $b(\theta_{\pm}) = \max\{a(\theta_{\pm}), \frac{3p_+ \mu(\theta_{\pm})}{2\theta_{\pm}}\}$.

In summary, the viscous contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ can be defined by

$$\begin{aligned} V^{CD} &= \frac{2}{3p_+} (\hat{\Theta} + \hat{\Theta}^{nf}), \\ U_1^{CD} &= u_{1+} + \frac{2}{3p_+} \left[\varepsilon a(\hat{\Theta}) \hat{\Theta}_x + \varepsilon (a(\hat{\Theta}) \hat{\Theta}^{nf})_x + \frac{3}{5} \tilde{\Delta}_{11} \right], \\ U_i^{CD} &= \Xi_{ix}, \quad (i = 2, 3), \\ \Theta^{CD} &= \hat{\Theta} + \hat{\Theta}^{nf} + H, \end{aligned} \quad (2.39)$$

where

$$H = -\frac{\varepsilon V^{CD}}{p_+} \left[a(\hat{\Theta}) \hat{\Theta}_t + (a(\hat{\Theta}) \hat{\Theta}^{nf})_t \right] + \frac{4\varepsilon^2 \mu(\hat{\Theta})}{3p_+} \left[(a(\hat{\Theta}) \hat{\Theta}_x)_x + (a(\hat{\Theta}) \hat{\Theta}^{nf})_{xx} \right] - \frac{3}{2} V^{CD} \int \xi_1^2 \tilde{\Pi}_{11} d\xi, \quad (2.40)$$

is chosen such that the momentum equation that the viscous contact wave satisfies has an error term with sufficient decay in ε . Without H , the error term in the momentum equation decay like $\varepsilon^{\frac{1}{2}}$. In order to get ε order decay in the error term, we should introduce the higher order approximate term H in the definition of Θ^{CD} in the contact wave. Here, $\int \xi_1^2 \tilde{\Pi}_{11} d\xi$ in (2.40) is the corresponding function defined in (2.21) by replacing both the variables (v, u, θ) and $(V^{CD}, U^{CD}, \Theta^{CD})$ by $(\hat{V}, \hat{U}, \hat{\Theta})$, and it satisfies

$$\int \xi_1^2 \tilde{\Pi}_{11} d\xi = O(1)\varepsilon^2 |(\hat{\Theta}_x^2, \hat{\Theta}_{xx})|. \quad (2.41)$$

Hence, by (2.18), (2.30) and (2.41), we have

$$H = O(\delta^{CD})\varepsilon(1+t)^{-2}e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty. \quad (2.42)$$

Now the contact wave $(V^{CD}, U^{CD}, \Theta^{CD})(t, x)$ defined in (2.39) satisfies the following system

$$\begin{cases} V_t^{CD} - U_{1x}^{CD} = 0, \\ U_{1t}^{CD} + P_x^{CD} = \frac{4\varepsilon}{3}\left(\frac{\mu(\Theta^{CD})}{V^{CD}}U_{1x}^{CD}\right)_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi + Q_1^{CD}, \\ U_{it}^{CD} = \varepsilon\left(\frac{\mu(\Theta^{CD})}{V^{CD}}U_{ix}^{CD}\right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi + Q_i^{CD}, \quad i = 2, 3, \\ \mathcal{E}_t^{CD} + (P^{CD}U_1^{CD})_x = \varepsilon\left(\frac{\kappa(\Theta^{CD})}{V^{CD}}\Theta_x^{CD}\right)_x + \frac{4\varepsilon}{3}\left(\frac{\mu(\Theta^{CD})U_1^{CD}U_{1x}^{CD}}{V^{CD}}\right)_x \\ + \sum_{i=2}^3 \varepsilon\left(\frac{\mu(\Theta^{CD})U_i^{CD}U_{ix}^{CD}}{V^{CD}}\right)_x - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi + Q_4^{CD}, \end{cases} \quad (2.43)$$

where $P^{CD} = \frac{2\Theta^{CD}}{3V^{CD}}$, $\mathcal{E}^{CD} = \Theta^{CD} + \frac{|U^{CD}|^2}{2}$, $-\int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi$ ($i = 1, 2, 3$) and $-\int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi$ are the corresponding functions defined in (2.21) by replacing the variables (v, u, θ) by $(V^{CD}, U^{CD}, \Theta^{CD})$, respectively. Moreover,

$$\begin{aligned} Q_1^{CD} &= \frac{2\varepsilon}{5p_+} \tilde{\Delta}_{11t} - \frac{8\varepsilon^2}{15p_+} \left(\frac{\mu(\hat{\Theta}) \tilde{\Delta}_{11x}}{V^{CD}}\right)_x - \frac{4\varepsilon}{3} \left(\frac{\mu(\Theta^{CD}) - \mu(\hat{\Theta})}{V^{CD}} U_{1x}^{CD}\right)_x + \int \xi_1^2 (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi \\ &= O(1)\delta^{CD} \varepsilon(1+t)^{-2} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty, \end{aligned} \quad (2.44)$$

$$\begin{aligned} Q_i^{CD} &= -\varepsilon \left(\frac{\mu(\Theta^{CD}) - \mu(\hat{\Theta})}{V^{CD}} U_{ix}^{CD}\right)_x + \int \xi_1 \xi_i (\Pi_{i1}^{CD} - \tilde{\Pi}_{i1})_x d\xi + \tilde{\Delta}_{i2x} \\ &= O(1)\delta^{CD} \varepsilon(1+t)^{-2} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty, \quad i = 2, 3, \end{aligned} \quad (2.45)$$

and

$$\begin{aligned} Q_4^{CD} &= -\frac{5\varepsilon}{3} \left[\left(a(\Theta^{CD}) - a(\hat{\Theta}) - a'(\hat{\Theta})(\hat{\Theta}^{CD} - \hat{\Theta}) \right) (\hat{\Theta}_x + \hat{\Theta}_x^{nf}) + a(\Theta^{CD}) H_x \right. \\ &\quad \left. + a'(\hat{\Theta})(\hat{\Theta}^{CD} - \hat{\Theta}) \hat{\Theta}_x^{nf} \right]_x + \frac{3p_+ \varepsilon}{2} \left(\frac{\kappa(\Theta^{CD}) \Theta_x^{CD} H}{\Theta^{CD}(\Theta^{CD} - H)} \right)_x + H_t \\ &\quad + \frac{2U_{1x}^{CD} H}{3V^{CD}} - \frac{4\varepsilon \mu(\Theta^{CD})}{3V^{CD}} (U_{1x}^{CD})^2 - \sum_{i=2}^3 \frac{\varepsilon \mu(\Theta^{CD})}{V^{CD}} (U_{ix}^{CD})^2 \\ &\quad + \int \xi_1 \frac{|\xi|^2}{2} (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi + (U_1^{CD} - u_{1+}) \int \xi_1^2 (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi \\ &\quad + \sum_{i=2}^3 U_i^{CD} \int \xi_1 \xi_i (\Pi_{11}^{CD} - \tilde{\Pi}_{11})_x d\xi + \sum_{i=1}^3 U_i^{CD} Q_i^{CD} + \tilde{\Delta}_{12x} \\ &= O(1)\delta^{CD} \varepsilon(1+t)^{-2} e^{-\frac{cx^2}{\varepsilon(1+t)}}, \quad \text{as } x \rightarrow \pm\infty, \end{aligned} \quad (2.46)$$

with some positive constant $c > 0$ depending only on θ_\pm and $\tilde{\Delta}_{i2x}$ ($i = 1, 2, 3$) being the corresponding functions defined in (2.25) and (2.34) by replacing both (v, u, θ) and $(V^{CD}, U^{CD}, \Theta^{CD})$ by $(\hat{V}, \hat{U}, \hat{\Theta})$.

Note that from (2.18), we have

$$|(V^{CD}, U^{CD}, \Theta^{CD})(t, x) - (v^{cd}, u^{cd}, \theta^{cd})(t, x)| = O(1)\delta^{CD} e^{-\frac{cx^2}{2\varepsilon(1+t)}}. \quad (2.47)$$

2.4 Shock Profile

In this subsection, we will firstly recall the shock profile $F^{S_3}(x - \bar{s}_3 t, \xi)$ of the Boltzmann equation (1.1) in Eulerian coordinates with its existence and properties given in the papers by Caffisch-Nicolaenko [7] and Liu-Yu [30], [31]. And then we will state the corresponding properties in the Lagrangian coordinates used in this paper.

First of all, $F^{S_3}(x - \bar{s}_3 t, \xi)$ satisfies

$$\begin{cases} -\bar{s}_3(F^{S_3})' + \xi_1(F^{S_3})' = \frac{1}{\varepsilon}Q(F^{S_3}, F^{S_3}), \\ F^{S_3}(\pm\infty, \xi) = \mathbf{M}_\pm(\xi) := \mathbf{M}_{[\rho_\pm, u_\pm, \theta_\pm]}(\xi), \end{cases} \quad (2.48)$$

where $' = \frac{d}{d\vartheta}$, $\vartheta = x - \bar{s}_3 t$, $u_\pm = (u_{1\pm}, 0, 0)$ and $(\rho_\pm, u_\pm, \theta_\pm)$ satisfy Rankine-Hugoniot condition

$$\begin{cases} -\bar{s}_3(\rho_+ - \rho_-) + (\rho_+ u_{1+} - \rho_- u_{1-}) = 0, \\ -\bar{s}_3(\rho_+ u_{1+} - \rho_- u_{1-}) + (\rho_+ u_{1+}^2 + p_+ - \rho_- u_{1-}^2 - p_-) = 0, \\ -\bar{s}_3(\rho_+ E_+ - \rho_- E_-) + (\rho_+ u_{1+} E_+ + p_+ u_{1+} - \rho_- u_{1-} E_- - p_- u_{1-}) = 0, \end{cases} \quad (2.49)$$

and Lax entropy condition

$$\lambda_{3+}^E < \bar{s}_3 < \lambda_{3-}^E, \quad (2.50)$$

with \bar{s}_3 being 3-shock wave speed and $\lambda_3^E = u_1 + \frac{\sqrt{10\theta}}{3}$ being the third characteristic eigenvalue of the Euler equations in the Eulerian coordinate and $\lambda_{3\pm}^E = u_{1\pm} + \frac{\sqrt{10\theta_\pm}}{3}$.

By the macro-micro decomposition around the local Maxwellian \mathbf{M}^{S_3} , set

$$F^{S_3}(x, t, \xi) = \mathbf{M}^{S_3}(x, t, \xi) + \mathbf{G}^{S_3}(x, t, \xi),$$

where

$$\mathbf{M}^{S_3}(x, t, \xi) = \mathbf{M}_{[\rho^{S_3}, u^{S_3}, \theta^{S_3}]}(x, t, \xi) = \frac{\rho^{S_3}(x, t)}{\sqrt{(2\pi R\theta^{S_3}(x, t))^3}} e^{-\frac{|\xi - u^{S_3}(x, t)|^2}{2R\theta^{S_3}(x, t)}},$$

with

$$\begin{pmatrix} \rho^{S_3} \\ \rho^{S_3} u_i^{S_3} \\ \rho^{S_3} (\theta^{S_3} + \frac{|u^{S_3}|^2}{2}) \end{pmatrix} = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ \xi_i \\ \frac{|\xi|^2}{2} \end{pmatrix} F^{S_3}(x, t, \xi) d\xi, \quad i = 1, 2, 3. \quad (2.51)$$

With respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbf{M}^{S_3}}$ defined in (1.6), we can now define the macroscopic projection $\mathbf{P}_0^{S_3}$ and microscopic projection $\mathbf{P}_1^{S_3}$ by

$$\mathbf{P}_0^{S_3} g = \sum_{j=0}^4 \langle g, \chi_j^{S_3} \rangle_{\mathbf{M}^{S_3}} \chi_j^{S_3}, \quad \mathbf{P}_1^{S_3} g = g - \mathbf{P}_0^{S_3} g, \quad (2.52)$$

where $\chi_j^{S_3}$ ($j = 0, 1, 2, 3, 4$) are the corresponding pairwise orthogonal base defined in (1.7) by replacing $(\rho, u, \theta, \mathbf{M})$ by $(\rho^{S_3}, u^{S_3}, \theta^{S_3}, \mathbf{M}^{S_3})$.

Under the above macro-micro decomposition, the solution $F^{S_3} = F^{S_3}(x - \bar{s}_3 t, \xi)$ satisfies

$$\mathbf{P}_0^{S_3} F^{S_3} = \mathbf{M}^{S_3}, \quad \mathbf{P}_1^{S_3} F^{S_3} = \mathbf{G}^{S_3},$$

and the Boltzmann equation (2.48) becomes

$$(\mathbf{M}^{S_3} + \mathbf{G}^{S_3})_t + \xi_1(\mathbf{M}^{S_3} + \mathbf{G}^{S_3})_x = \frac{1}{\varepsilon} [2Q(\mathbf{M}^{S_3}, \mathbf{G}^{S_3}) + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})]. \quad (2.53)$$

Correspondingly, we have the following fluid-type system for the fluid components of shock profile:

$$\begin{cases} \rho_t^{S_3} + (\rho^{S_3} u_1^{S_3})_x = 0, \\ (\rho^{S_3} u_1^{S_3})_t + [\rho^{S_3} (u_1^{S_3})^2 + p^{S_3}]_x = \frac{4\varepsilon}{3} (\mu(\theta^{S_3}) u_{1x}^{S_3})_x - \int \xi_1^2 \Pi_x^{S_3} d\xi, \\ (\rho^{S_3} u_i^{S_3})_t + (\rho^{S_3} u_1^{S_3} u_i^{S_3})_x = \varepsilon (\mu(\theta^{S_3}) u_{ix}^{S_3})_x - \int \xi_1 \xi_i \Pi_x^{S_3} d\xi, \quad i = 2, 3, \\ [\rho^{S_3} (\theta^{S_3} + \frac{|u^{S_3}|^2}{2})]_t + [\rho^{S_3} u_1^{S_3} (\theta^{S_3} + \frac{|u^{S_3}|^2}{2}) + p^{S_3} u_1^{S_3}]_x = \varepsilon (\kappa(\theta^{S_3}) \theta_x^{S_3})_x \\ + \frac{4\varepsilon}{3} (\mu(\theta^{S_3}) u_1^{S_3} u_{1x}^{S_3})_x + \varepsilon \sum_{i=2}^3 (\mu(\theta^{S_3}) u_i^{S_3} u_{ix}^{S_3})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Pi_x^{S_3} d\xi. \end{cases} \quad (2.54)$$

In fact, from the invariance of the equation (2.48) by changing ξ_i with $-\xi_i$ and the fact that $u_{i\pm} = 0$, we have $u_i^{S_3} = \int \xi_1 \xi_i \Pi_x^{S_3} d\xi \equiv 0$ for $i = 2, 3$.

And the equation for the non-fluid component \mathbf{G}^{S_3} is

$$\mathbf{G}_t^{S_3} + \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3}) + \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3}) = \frac{1}{\varepsilon} [\mathbf{L}_{\mathbf{M}^{S_3}} \mathbf{G}^{S_3} + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})],$$

Here $\mathbf{L}_{\mathbf{M}^{S_3}}$ is the linearized collision operator of $Q(F^{S_3}, F^{S_3})$ with respect to the local Maxwellian \mathbf{M}^{S_3} :

$$\mathbf{L}_{\mathbf{M}^{S_3}} g = 2Q(\mathbf{M}^{S_3}, g) = Q(\mathbf{M}^{S_3}, g) + Q(g, \mathbf{M}^{S_3}).$$

Thus

$$\mathbf{G}^{S_3} = \varepsilon \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} [\mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3})] + \Pi^{S_3},$$

$$\Pi^{S_3} = \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} [\varepsilon(\mathbf{G}_t^{S_3} + \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3})) - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})].$$

Now we recall the properties of the shock profile $F^{S_3}(x - \bar{s}_3 t, \xi)$ that are given or can be induced by Liu-Yu in Theorem 6.8, [31].

Lemma 2.3 ([31]) *If the shock wave strength δ^{S_3} is small enough, then the Boltzmann equation (1.1) admits a 3-shock profile solution $F^{S_3}(x - \bar{s}_3 t, \xi)$ uniquely up to a shift satisfying the following properties:*

- (1) *The shock profile converges to its far fields exponentially fast with an exponent proportional to the magnitude of the shock wave strength, that is*

$$\begin{cases} |(\rho^{S_3} - \rho_{\pm}, u_{1\pm}^{S_3} - u_{1\pm}, \theta^{S_3} - \theta_{\pm})| \leq C \delta^{S_3} e^{-c_{\pm} \frac{\delta^{S_3} |\vartheta|}{\varepsilon}}, & \text{as } \vartheta \rightarrow \pm\infty, \\ \left(\int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C (\delta^{S_3})^2 e^{-c_{\pm} \frac{\delta^{S_3} |\vartheta|}{\varepsilon}}, & \text{as } \vartheta \rightarrow \pm\infty, \end{cases}$$

with δ^{S_3} being the 3-shock strength and \mathbf{M}_0 being the global Maxwellian which is close to the shock profile with its precise definition given in Theorem 6.8, [31].

- (2) *Compressibility of 3-shock profile:*

$$(\lambda_3^E)_{\vartheta} < 0, \quad \lambda_3^E = u_1^{S_3} + \frac{\sqrt{10\theta^{S_3}}}{3}.$$

- (3) *The following properties hold:*

$$\rho_{\vartheta}^{S_3} \sim u_{1\vartheta}^{S_3} \sim \theta_{\vartheta}^{S_3} \sim (\lambda_3^E)_{\vartheta} \sim \frac{1}{\varepsilon} \left(\int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}},$$

where $A \sim B$ denotes the equivalence of the quantities A and B , and

$$\begin{cases} u_i^{S_3} \equiv 0, \quad \int \xi_1 \xi_i \Pi^{S_3} d\xi \equiv 0, \quad i = 2, 3, \\ |\partial_{\vartheta}^k (\rho^{S_3}, u_{1\vartheta}^{S_3}, \theta^{S_3})| \leq C \frac{(\delta^{S_3})^{k-1}}{\varepsilon^{k-1}} |(\rho_{\vartheta}^{S_3}, u_{1\vartheta}^{S_3}, \theta_{\vartheta}^{S_3})|, \quad k \geq 2, \\ \left(\int \frac{\nu(|\xi|) |\partial_{\vartheta}^k \mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C \frac{(\delta^{S_3})^k}{\varepsilon^k} \left(\int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}}, \quad k \geq 1, \\ \left| \int \xi_1 \varphi_i(\xi) \Pi_{\vartheta}^{S_3} d\xi \right| \leq C \delta^{S_3} |u_{1\vartheta}^{S_3}|, \quad i = 1, 2, 3, 4, \end{cases}$$

where $\varphi_i(\xi)$ ($i = 1, 2, 3, 4$) are the collision invariants defined in (1.4).

Now we rewrite this shock profile in Lagrangian coordinate by using the transformation (1.14) and use (\tilde{t}, \tilde{x}) for the Lagrangian coordinate to distinguish it from the Eulerian coordinate (t, x) at this moment. Then the shock profile in Lagrangian coordinate can be written as $\tilde{F}^{S_3}(\tilde{x} - s_3\tilde{t}, \xi)$ with s_3 determined by the 3-shock wave curve given in (1.24). First, from the Rankine-Hugoniot condition in Eulerian and Lagrangian coordinates, we have

$$-\bar{s}_3(\rho_+ - \rho_-) + (\rho_+ u_{1+} - \rho_- u_{1-}) = 0,$$

and

$$-s_3(v_+ - v_-) - (u_{1+} - u_{1-}) = 0,$$

with $v_{\pm} = \frac{1}{\rho_{\pm}}$, respectively. Thus we have the following relation between \bar{s}_3 and s_3

$$s_3 = \rho_{\pm}(\bar{s}_3 - u_{1\pm}). \quad (2.55)$$

On the other hand, we have from (2.54)₁ that

$$\rho^{S_3}(x - s_3t)[\bar{s}_3 - u_1^{S_3}(x - s_3t)] = \text{const.} = \rho_{\pm}(\bar{s}_3 - u_{1\pm}). \quad (2.56)$$

Note that

$$\begin{aligned} \tilde{x} - s_3\tilde{t} &= -\int_0^t \rho^{S_3} u_1^{S_3}(-\bar{s}_3\tau) d\tau + \int_0^x \rho^{S_3}(y - \bar{s}_3\tau) dy - \int_0^t \rho_{\pm}(\bar{s}_3 - u_{1\pm}) d\tau \\ &= -\int_0^t \rho^{S_3} u_1^{S_3}(-\bar{s}_3\tau) d\tau + \int_0^x \rho^{S_3}(y - \bar{s}_3\tau) dy - \int_0^t \rho^{S_3}(-s_3\tau)[\bar{s}_3 - u_1^{S_3}(-s_3\tau)] d\tau \\ &= \int_0^x \rho^{S_3}(y - \bar{s}_3\tau) dy - \bar{s}_3 \int_0^t \rho^{S_3}(-s_3\tau) d\tau = \int_0^{x - \bar{s}_3 t} \rho^{S_3}(\eta) d\eta, \end{aligned}$$

where in the second equality we have used (2.56).

This shows that under the Lagrangian transformation (1.14), the shock profile $F^{S_3}(x - \bar{s}_3t, \xi)$ in Eulerian coordinate can be exactly transformed to the shock profile $\tilde{F}^{S_3}(\tilde{x} - s_3\tilde{t}, \xi)$ in Lagrangian coordinate. Moreover, we have

$$\tilde{F}_{\eta}^{S_3}(\eta, \xi) = \rho^{S_3} F_{\vartheta}^{S_3}(\vartheta, \xi)$$

with $\eta = \tilde{x} - s_3\tilde{t}$.

For simplicity of the notations, from now on, we use (t, x) to denote the Lagrangian coordinate and $F^{S_3}(\eta, \xi)$ with $\eta = x - s_3t$ to denote the 3-shock profile of Boltzmann equation in Lagrangian coordinate. And in the Lagrangian coordinate, we have the following Lemma.

Lemma 2.4 *Assume that $(v_-, u_-, \theta_-) \in S_3(v_+, u_+, \theta_+)$, then there exists a unique shock profile $F^{S_3}(\eta, \xi)$ with $\eta = x - s_3t$ up to a shift, to the Boltzmann equation (1.15) in Lagrangian coordinate. Moreover, there are positive constants c_{\pm} and C such that for $\eta \in \mathbf{R}$,*

$$\left\{ \begin{array}{l} s_3 V_{\eta}^{S_3} = -U_{1\eta}^{S_3} > 0, \\ U_i^{S_3} \equiv 0, \quad \int \xi_1 \xi_i \Pi_1^{S_3} d\xi \equiv 0, \quad i = 2, 3, \\ (|V^{S_3} - v_{\pm}|, |U_1^{S_3} - u_{1\pm}|, |\Theta^{S_3} - \theta_{\pm}|) \leq C \delta^{S_3} e^{-\frac{c_{\pm} \delta^{S_3} |\eta|}{\varepsilon}}, \quad \text{as } \eta \rightarrow \pm\infty, \\ \left(\int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} \leq C (\delta^{S_3})^2 e^{-c_{\pm} \frac{\delta^{S_3} |\eta|}{\varepsilon}}, \quad \text{as } \eta \rightarrow \pm\infty. \end{array} \right.$$

Furthermore, we have

$$V_{\eta}^{S_3} \sim U_{1\eta}^{S_3} \sim \Theta_{\eta}^{S_3} \sim \frac{1}{\varepsilon} \left(\int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}},$$

and

$$\begin{aligned} |\partial_{\eta}^k (V^{S_3}, U_1^{S_3}, \Theta^{S_3})| &\leq C \frac{(\delta^{S_3})^{k-1}}{\varepsilon^{k-1}} |(V_{\eta}^{S_3}, U_{1\eta}^{S_3}, \Theta_{\eta}^{S_3})|, \quad k \geq 2, \\ \left(\int \frac{\nu(|\xi|) |\partial_{\eta}^k \mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}} &\leq C \frac{(\delta^{S_3})^k}{\varepsilon^k} \left(\int \frac{\nu(|\xi|) |\mathbf{G}^{S_3}|^2}{\mathbf{M}_0} d\xi \right)^{\frac{1}{2}}, \quad k \geq 1, \end{aligned}$$

and

$$\left| \int \xi_1 \varphi_i(\xi) \Pi_{1\eta}^{S_3} d\xi \right| \leq C \delta^{S_3} |U_{1\eta}^{S_3}|, \quad i = 1, 2, 3, 4,$$

with $\varphi_i(\xi)$ being the collision invariants.

Furthermore, we have

$$\begin{cases} V_t^{S_3} - U_{1x}^{S_3} = 0, \\ U_{1t}^{S_3} + P_x^{S_3} = \frac{4}{3} \varepsilon \left(\frac{\mu(\Theta^{S_3}) U_{1x}^{S_3}}{V^{S_3}} \right)_x - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi, \\ U_{it}^{S_3} = \varepsilon \left(\frac{\mu(\Theta^{S_3}) U_{ix}^{S_3}}{V^{S_3}} \right)_x - \int \xi_1 \xi_i \Pi_{1x}^{S_3} d\xi, \quad i = 2, 3, \\ \mathcal{E}_t^{S_3} + (P^{S_3} U_1^{S_3})_x = \varepsilon \left(\frac{\kappa(\Theta^{S_3}) \Theta_x^{S_3}}{V^{S_3}} \right)_x + \frac{4}{3} \varepsilon \left(\frac{\mu(\Theta^{S_3}) U_1^{S_3} U_{1x}^{S_3}}{V^{S_3}} \right)_x \\ + \varepsilon \sum_{i=2}^3 \left(\frac{\mu(\Theta^{S_3}) U_i^{S_3} U_{ix}^{S_3}}{V^{S_3}} \right)_x - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi, \end{cases} \quad (2.57)$$

where $\mathcal{E}^{S_3} = \Theta^{S_3} + \frac{|U^{S_3}|^2}{2}$ and $(v_\pm, u_\pm, \theta_\pm)$ satisfy Rankine-Hugoniot condition and Lax entropy condition and s_3 is 3-shock wave speed.

Correspondingly, we have the following equation for the non-fluid part of 3-shock profile.

$$\mathbf{G}_t^{S_3} - \frac{U_1^{S_3}}{V^{S_3}} \mathbf{G}_x^{S_3} + \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3}) + \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3}) = \frac{1}{\varepsilon} [\mathbf{L}_{\mathbf{M}^{S_3}} \mathbf{G}^{S_3} + Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})].$$

Here, $\mathbf{L}_{\mathbf{M}^{S_3}}$ is the linearized collision operator of $Q(F^{S_3}, F^{S_3})$ with respect to the local Maxwellian \mathbf{M}^{S_3} :

$$\mathbf{L}_{\mathbf{M}^{S_3}} g = 2Q(\mathbf{M}^{S_3}, g) = Q(\mathbf{M}^{S_3}, g) + Q(g, \mathbf{M}^{S_3}).$$

Thus

$$\begin{aligned} \mathbf{G}^{S_3} &= \varepsilon \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} \left[\frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_x^{S_3}) \right] + \Pi_1^{S_3}, \\ \Pi_1^{S_3} &= \mathbf{L}_{\mathbf{M}^{S_3}}^{-1} \left[\varepsilon \left(\mathbf{G}_t^{S_3} - \frac{U_1^{S_3}}{V^{S_3}} \mathbf{G}_x^{S_3} + \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_x^{S_3}) \right) - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3}) \right]. \end{aligned} \quad (2.58)$$

2.5 Hyperbolic Wave II

The purpose of this subsection is to construct the second hyperbolic wave. Up to now, we can define the following approximate composite wave profile $(\bar{V}, \bar{U}, \bar{\mathcal{E}})(t, x)$

$$\begin{aligned} \begin{pmatrix} \bar{V} \\ \bar{U}_1 \\ \bar{\mathcal{E}} \end{pmatrix} (t, x) &= \begin{pmatrix} V^{R_1} + d_1 + V^{CD} + V^{S_3} \\ U_1^{R_1} + d_2 + U_1^{CD} + U_1^{S_3} \\ \mathcal{E}^{R_1} + d_3 + \mathcal{E}^{CD} + \mathcal{E}^{S_3} \end{pmatrix} (t, x) - \begin{pmatrix} v_* + v^* \\ u_{1*} + u_{1*}^* \\ E_* + E^* \end{pmatrix}, \\ \bar{U}_i &= U_i^{CD}, \quad i = 2, 3, \end{aligned} \quad (2.59)$$

where $\bar{\mathcal{E}} = \bar{\Theta} + \frac{|\bar{U}|^2}{2}$, $(V^{R_1}, U_1^{R_1}, \mathcal{E}^{R_1})(t, x)$ is the 1-rarefaction wave defined in (2.5) with the right state (v_+, u_{1+}, E_+) replaced by (v_*, u_{1*}, E_*) , $(V^{CD}, U_1^{CD}, \mathcal{E}^{CD})(t, x)$ is the viscous contact wave defined in (2.39) with the states (v_-, u_{1-}, E_-) and (v_+, u_{1+}, E_+) replaced by (v_*, u_{1*}, E_*) and (v^*, u_{1*}^*, E^*) respectively, and $(V^{S_3}, U_1^{S_3}, \mathcal{E}^{S_3})(t, x)$ is the fluid part of 3-shock profile of Boltzmann equation defined in (2.57) with the left state (v_-, u_{1-}, E_-) replaced by (v^*, u_{1*}^*, E^*) .

Moreover, we can check that this profile satisfies

$$\left\{ \begin{array}{l} \bar{V}_t - \bar{U}_{1x} = 0, \\ \bar{U}_{1t} + \bar{P}_x = \frac{4}{3}\varepsilon\left(\frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}}\right)_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi + \bar{Q}_{1x} + Q_1^{CD}, \\ \bar{U}_{it} = \varepsilon\left(\frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}}\right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi - \int \xi_1 \xi_i \Pi_{1x}^{S_3} d\xi + \bar{Q}_{ix} + Q_i^{CD}, \quad i = 2, 3, \\ \bar{\mathcal{E}}_t + (\bar{P}\bar{U}_1)_x = \varepsilon\left(\frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{\bar{V}}\right)_x + \frac{4}{3}\varepsilon\left(\frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1x}}{\bar{V}}\right)_x + \sum_{i=2}^3 \varepsilon\left(\frac{\mu(\bar{\Theta})\bar{U}_i\bar{U}_{ix}}{\bar{V}}\right)_x \\ \quad - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi + \bar{Q}_{4x} + Q_4^{CD}, \end{array} \right. \quad (2.60)$$

where $\bar{P} = p(\bar{V}, \bar{\Theta})$, Q_i^{CD} ($i = 1, 2, 3, 4$) are defined in (2.44), (2.45) and (2.46), respectively, and

$$\begin{aligned} \bar{Q}_1 &= \left[\bar{P} - (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3) - P^{R_1} - P^{CD} - P^{S_3} + p_* + p^* \right] \\ &\quad - \frac{4}{3}\varepsilon \left[\frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}} - \frac{\mu(\Theta^{R_1})U_{1x}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1x}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1x}^{S_3}}{V^{S_3}} \right], \\ \bar{Q}_i &= -\varepsilon \left[\frac{\mu(\bar{\Theta})}{\bar{V}} - \frac{\mu(\Theta^{CD})}{V^{CD}} \right] U_{ix}^{CD}, \quad i = 2, 3, \end{aligned}$$

$$\begin{aligned} \bar{Q}_4 &= \left[\bar{P}\bar{U}_1 - ((pu_1)_v^{R_1} d_1 + (pu_1)_{u_1}^{R_1} d_2 + (pu_1)_E^{R_1} d_3) - P^{R_1} U_1^{R_1} - P^{CD} U_1^{CD} - P^{S_3} U_1^{S_3} \right. \\ &\quad \left. + p_* u_{1*} + p^* u_1^* \right] - \varepsilon \left[\frac{\kappa(\bar{\Theta})\bar{\Theta}_x}{\bar{V}} - \frac{\kappa(\Theta^{R_1})\Theta_x^{R_1}}{V^{R_1}} - \frac{\kappa(\Theta^{CD})\Theta_x^{CD}}{V^{CD}} - \frac{\kappa(\Theta^{S_3})\Theta_x^{S_3}}{V^{S_3}} \right] \\ &\quad - \frac{4}{3}\varepsilon \left[\frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1x}}{\bar{V}} - \frac{\mu(\Theta^{R_1})U_1^{R_1}U_{1x}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_1^{CD}U_{1x}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_1^{S_3}U_{1x}^{S_3}}{V^{S_3}} \right] \\ &\quad - \sum_{i=2}^3 \varepsilon \left[\frac{\mu(\bar{\Theta})}{\bar{V}} - \frac{\mu(\Theta^{CD})}{V^{CD}} \right] U_i^{CD} U_{ix}^{CD}. \end{aligned}$$

From (2.60), we have that

$$\begin{aligned} \bar{\Theta} &= \Theta^{R_1} + d_3 + \Theta^{CD} + \Theta^{S_3} - (\theta_* + \theta^*) \\ &\quad - \frac{1}{2} [|\bar{U}|^2 - |U^{R_1}|^2 - |U^{CD}|^2 - |U^{S_3}|^2 + u_{1*}^2 + (u_1^*)^2]. \end{aligned} \quad (2.61)$$

Direct computation yields

$$\begin{aligned} \bar{Q}_1 &= O(1) \left[|(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, U_1^{R_1} - u_{1*}, d_1, d_2, d_3)| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \right. \\ &\quad \left. \Theta^{CD} - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| \right. \\ &\quad \left. + |(V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| |(V^{CD} - v^*, U_1^{CD} - u_{1*}, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*)| \right. \\ &\quad \left. + \varepsilon |U_{1x}^{R_1}| |(V^{CD} - v_*, \Theta^{CD} - \theta_*, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \right. \\ &\quad \left. + \varepsilon |U_{1x}^{CD}| |(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \right. \\ &\quad \left. + \varepsilon |U_{1x}^{S_3}| |(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{CD} - v^*, \Theta^{CD} - \theta^*)| \right] \\ &\quad + O(1) \left[|(d_1, d_2, d_3)|^2 + \varepsilon |d_{2x}| + \varepsilon |U_{1x}^{R_1}| |(d_1, d_2, d_3)| \right] \\ &:= \bar{Q}_{11} + \bar{Q}_{12}, \end{aligned} \quad (2.62)$$

$$\bar{Q}_i = O(1) \varepsilon \left[|(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, U_1^{R_1} - u_{1*}, d_1, d_2, d_3, V^{S_3} - v^*, \Theta^{S_3} - \theta^*, U_1^{S_3} - u_1^*)| |U_{ix}^{CD}| \right], \quad i = 2, 3, \quad (2.63)$$

and

$$\begin{aligned}
\bar{Q}_4 &= O(1) \left[|(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, d_1, d_2, d_3)| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \right. \\
&\quad \left. \Theta^{CD} - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| \right. \\
&\quad \left. + |(V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| |(V^{CD} - v^*, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*)| \right. \\
&\quad \left. + \varepsilon |(U_{1x}^{R_1}, \Theta_x^{R_1})| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \Theta^{CD} - \theta_*, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \right. \\
&\quad \left. + \varepsilon |(U_{1x}^{CD}, \Theta_x^{CD})| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{S_3} - v^*, \Theta^{S_3} - \theta^*)| \right. \\
&\quad \left. + \varepsilon |(U_{1x}^{S_3}, \Theta_x^{S_3})| |(V^{R_1} - v_*, \Theta^{R_1} - \theta_*, d_1, d_2, d_3, V^{CD} - v^*, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*)| \right] \\
&\quad + O(1) \left[|(d_1, d_2, d_3)|^2 + \varepsilon |(d_{2x}, d_{3x})| + \varepsilon |(U_{1x}^{R_1}, \Theta_x^{R_1})| |(d_1, d_2, d_3)| \right] \\
&:= \bar{Q}_{41} + \bar{Q}_{42}.
\end{aligned} \tag{2.64}$$

Here, \bar{Q}_{11} , \bar{Q}_i ($i = 2, 3$) and \bar{Q}_{41} represent the interaction of waves in different families, \bar{Q}_{12} and \bar{Q}_{42} represent the error terms coming from the approximate rarefaction wave and the hyperbolic wave I.

Firstly, we estimate the interaction terms \bar{Q}_{11} , \bar{Q}_i ($i = 2, 3$) and \bar{Q}_{41} by dividing the whole domain $\Omega = \{(t, x) | (t, x) \in [h, T] \times \mathbf{R}\}$ into three regions:

$$\begin{aligned}
\Omega_{R_1} &= \{(t, x) \in \Omega | 2x \leq \lambda_{1*}t\}, \\
\Omega_{CD} &= \{(t, x) \in \Omega | \lambda_{1*}t < 2x < s_3t\}, \\
\Omega_{S_3} &= \{(t, x) \in \Omega | 2x \geq s_3t\},
\end{aligned}$$

where $\lambda_{1*} = \lambda_1(v_*, \theta_*)$ and s_3 is the 3-shock speed.

From Lemma 2.1, we have the following estimates in each region:

- In Ω_{R_1} ,

$$\begin{aligned}
|V^{S_3} - v^*| &= O(1) \delta^{S_3} e^{-\frac{c\delta^{S_3}|x-s_3t|}{\varepsilon}} = O(1) e^{-\frac{C|x|}{\varepsilon}} e^{-\frac{C_h}{\varepsilon}}, \\
|(V^{CD} - v_*, V^{CD} - v^*)| &= O(1) \delta^{CD} e^{-\frac{C|x||\lambda_{1*}|t}{8\varepsilon(1+t)}} e^{-\frac{C(\lambda_{1*}t)^2}{8\varepsilon(1+t)}} = O(1) e^{-\frac{C_h|x|}{\varepsilon}} e^{-\frac{C_h}{\varepsilon}};
\end{aligned}$$

- In Ω_{CD} ,

$$\begin{aligned}
|(V^{R_1} - v_*, d_1, d_2, d_3)| &= O(1) e^{-\frac{2|x-\lambda_{1*}t|}{\sigma}} = O(1) e^{-\frac{C_h|x|}{\sigma}} e^{-\frac{C_h}{\sigma}}, \\
|V^{S_3} - v^*| &= O(1) e^{-\frac{c\delta^{S_3}|x-s_3t|}{\varepsilon}} = O(1) e^{-\frac{C_h|x|}{\varepsilon}} e^{-\frac{C_h}{\varepsilon}};
\end{aligned}$$

- In Ω_{S_3} ,

$$\begin{aligned}
|(V^{R_1} - v_*, d_1, d_2, d_3)| &= O(1) e^{-\frac{2|x-\lambda_{1*}t|}{\sigma}} = O(1) e^{-\frac{2|x|}{\sigma}} e^{-\frac{C_h}{\sigma}}, \\
|(V^{CD} - v_*, V^{CD} - v^*)| &= O(1) \delta^{CD} e^{-\frac{C|x||s_3t|}{8\varepsilon(1+t)}} e^{-\frac{C(s_3t)^2}{8\varepsilon(1+t)}} = O(1) e^{-\frac{C_h|x|}{\varepsilon}} e^{-\frac{C_h}{\varepsilon}}.
\end{aligned}$$

Note that we just give the pointwise estimates of V component and d_i ($i = 1, 2, 3$) in each region, similar estimates hold also for the U_1 and Θ components. In summary, we have

$$|(\bar{Q}_{11}, \bar{Q}_2, \bar{Q}_3, \bar{Q}_{41})| = C_{h,T} e^{-\frac{C_h|x|}{\sigma}} e^{-\frac{C_h}{\sigma}}, \tag{2.65}$$

with $\sigma = \varepsilon^{\frac{1}{5}}$ and for some positive constants $C_{h,T}$ and C_h independent of ε .

In order to remove the non-conservative error terms Q_i^{CD} , ($i = 1, 2, 3, 4$) coming from the definition of the viscous contact wave, we now introduce the following hyperbolic wave $\vec{b} \triangleq (b_1, b_{21}, b_{22}, b_{23}, b_3)$:

$$\begin{cases}
b_{1t} - b_{21x} = 0, \\
b_{21t} + [\bar{P}_v b_1 + \bar{P}_{u_1} b_{21} + \bar{P}_{u_2} b_{22} + \bar{P}_{u_3} b_{23} + \bar{P}_E b_3]_x = -Q_1^{CD}, \\
b_{22t} = -Q_2^{CD}, \\
b_{23t} = -Q_3^{CD}, \\
b_{3t} + [(\bar{P}\bar{U}_1)_v b_1 + (\bar{P}\bar{U}_1)_{u_1} b_{21} + (\bar{P}\bar{U}_1)_{u_2} b_{22} + (\bar{P}\bar{U}_1)_{u_3} b_{23} + (\bar{P}\bar{U}_1)_E b_3]_x = -Q_4^{CD},
\end{cases} \tag{2.66}$$

where $P = \frac{2\Theta}{3V} = P(V, U, \mathcal{E})$ and $\bar{P}_v = P_v(\bar{V}, \bar{U}, \bar{\mathcal{E}})$, etc. For later use, we denote $b_2 = (b_{21}, b_{22}, b_{23})$. Now we want to solve this linear hyperbolic system (2.66) on the interval $[h, T]$. Firstly, we diagonalize the above system. Rewrite the system (2.66) as

$$\begin{pmatrix} b_1 \\ b_{21} \\ b_{22} \\ b_{23} \\ b_3 \end{pmatrix}_t + \left[\bar{A}(\bar{V}, \bar{U}, \bar{\mathcal{E}}) \begin{pmatrix} b_1 \\ b_{21} \\ b_{22} \\ b_{23} \\ b_3 \end{pmatrix} \right]_x = \begin{pmatrix} 0 \\ -Q_1^{CD} \\ -Q_2^{CD} \\ -Q_3^{CD} \\ -Q_4^{CD} \end{pmatrix}, \quad (2.67)$$

where the matrix

$$\bar{A}(\bar{V}, \bar{U}, \bar{\mathcal{E}}) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ \bar{P}_v & \bar{P}_{u_1} & \bar{P}_{u_2} & \bar{P}_{u_3} & \bar{P}_E \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (\bar{P}\bar{U}_1)_v & (\bar{P}\bar{U}_1)_{u_1} & (\bar{P}\bar{U}_1)_{u_2} & (\bar{P}\bar{U}_1)_{u_3} & (\bar{P}\bar{U}_1)_E \end{pmatrix}$$

has three distinct eigenvalues $\bar{\lambda}_1 := \lambda_1(\bar{V}, \bar{P}) < 0 = \bar{\lambda}_2 < \lambda_3(\bar{V}, \bar{P}) := \bar{\lambda}_3$, (here $\bar{\lambda}_2$ being 3-repeated eigenvalues) with the corresponding left and right eigenvectors denoted by

$$\bar{l}_1, \bar{l}_{21}, \bar{l}_{22}, \bar{l}_{23}, \bar{l}_3; \quad \bar{r}_1, \bar{r}_{21}, \bar{r}_{22}, \bar{r}_{23}, \bar{r}_3.$$

It holds that

$$\begin{aligned} \bar{L}\bar{A}\bar{R} &= \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_2, \bar{\lambda}_2, \bar{\lambda}_3) \equiv \bar{\Lambda}, \\ \bar{L}\bar{R} &= \text{Id}. \end{aligned}$$

Here $\bar{L} = (\bar{l}_1, \bar{l}_{21}, \bar{l}_{22}, \bar{l}_{23}, \bar{l}_3)^t$, $\bar{R} = (\bar{r}_1, \bar{r}_{21}, \bar{r}_{22}, \bar{r}_{23}, \bar{r}_3)$ with $\bar{L} = \bar{L}(\bar{V}, \bar{U}, \bar{\mathcal{E}})$ and $\bar{R} = \bar{R}(\bar{V}, \bar{U}, \bar{\mathcal{E}})$ ($i = 1, 2, 3$) and Id. is the 5×5 identity matrix. Specially, we can choose

$$\bar{l}_{21} = (\bar{P}, -\bar{U}_1, 0, 0, 1), \quad \bar{l}_{22} = (0, 0, 1, 0, 0), \quad \bar{l}_{23} = (0, 0, 0, 1, 0). \quad (2.68)$$

Set

$$\vec{B} \triangleq (B_1, B_{21}, B_{22}, B_{23}, B_3)^t = \bar{L} \cdot (b_1, b_{21}, b_{22}, b_{23}, b_3), \quad (2.69)$$

then

$$(b_1, b_{21}, b_{22}, b_{23}, b_3)^t = \bar{R} \cdot (B_1, B_{21}, B_{22}, B_{23}, B_3)^t, \quad (2.70)$$

and \vec{B} satisfies the system

$$\begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix}_t + \left[\bar{\Lambda} \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix} \right]_x = \bar{L} \begin{pmatrix} 0 \\ -Q_1^{CD} \\ -Q_2^{CD} \\ -Q_3^{CD} \\ -Q_4^{CD} \end{pmatrix} + \bar{L}_t \bar{R} \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix} + \bar{L}_x \bar{R} \bar{\Lambda} \begin{pmatrix} B_1 \\ B_{21} \\ B_{22} \\ B_{23} \\ B_3 \end{pmatrix}. \quad (2.71)$$

For simplicity, denote

$$\vec{Q}^{CD} = (0 - Q_1^{CD}, -Q_2^{CD}, -Q_3^{CD}, -Q_4^{CD})^t. \quad (2.72)$$

So we obtain a diagonalized system

$$\begin{cases} B_{1t} + (\bar{\lambda}_1 B_1)_x = \bar{l}_1 \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{1t} + \bar{\lambda}_i \bar{l}_{1x}) \cdot \bar{r}_i B_i + \bar{l}_{1t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}, \\ B_{21t} = \bar{l}_{21} \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{21t} + \bar{\lambda}_i \bar{l}_{21x}) \cdot \bar{r}_i B_i + \bar{l}_{21t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}, \\ B_{22t} = \bar{l}_{22} \cdot \vec{Q}^{CD}, \\ B_{23t} = \bar{l}_{23} \cdot \vec{Q}^{CD}, \\ B_{3t} + (\bar{\lambda}_3 B_3)_x = \bar{l}_3 \cdot \vec{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{3t} + \bar{\lambda}_i \bar{l}_{3x}) \cdot \bar{r}_i B_i + \bar{l}_{3t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}. \end{cases} \quad (2.73)$$

Now we impose the following boundary condition to the linear hyperbolic system (2.73) on the domain $(t, x) \in [h, T] \times \mathbf{R}$:

$$(B_1, B_{21}, B_{22}, B_{23}, B_3)(t = T, x) = 0. \quad (2.74)$$

We can solve the linear diagonalized hyperbolic system (2.73) under the condition (2.74) to have the following lemma.

Lemma 2.5 *There exists a positive constant δ_0 such that if the wave strength $\delta \leq \delta_0$, then there exists a positive constant $C_{h,T}$ which is independent of ε , such that*

$$\begin{aligned} & \left\| \frac{\partial^k}{\partial x^k} (b_1, b_{21}, b_{22}, b_{23}, b_3)(t, \cdot) \right\|_{L^2(dx)}^2 + \int_h^T \left\| \sqrt{|U_{1x}^{S_3}|} \frac{\partial^k}{\partial x^k} (b_1, b_{21}, b_{22}, b_{23}, b_3)(t, \cdot) \right\|_{L^2(dx)}^2 dt \\ & \leq C_{h,T} \varepsilon^{\frac{5}{2}-2k}, \quad k = 0, 1, 2, 3. \end{aligned} \quad (2.75)$$

Proof: From the wave curves defined in (1.22)-(1.24), we know $\delta^{CD} + \delta^{S_3} \leq C\delta_0$. Let $N = \frac{1}{\delta_0}$ and $\delta \leq \delta_0 \ll 1$, then we have

$$0 < c_0 \leq \left(\frac{\hat{V}}{v^*} \right)^{\pm N}, \left(\frac{V^{S_3}}{v^*} \right)^{\pm N} \leq C_0 < \infty,$$

where \hat{V} is defined in (2.19) and c_0, C_0 are independent of δ_0 .

Without loss of generality, we assume that $v^* = 1$ and $\hat{V}_x > 0$. If $v^* \neq 1$, then we just replace $\hat{V}^{\pm N}$ and $(V^{S_3})^{\pm N}$ by $\left(\frac{\hat{V}}{v^*} \right)^{\pm N}$ and $\left(\frac{V^{S_3}}{v^*} \right)^{\pm N}$, respectively.

Firstly, multiplying the equation (2.73)₁ by $[(\hat{V})^{-N} + (V^{S_3})^{-N}]B_1$, we obtain

$$\begin{aligned} & \left(\frac{1}{2} [\hat{V}^{-N} + (V^{S_3})^{-N}] B_1^2 \right)_t + \frac{1}{2} N \left(\hat{V}^{-N-1} \hat{V}_t + (V^{S_3})^{-N-1} V_t^{S_3} \right) B_1^2 \\ & \quad + (\bar{\lambda}_1(\bar{V}, \bar{P}) B_1)_x [\hat{V}^{-N} + (V^{S_3})^{-N}] B_1 \\ & = [\hat{V}^{-N} + (V^{S_3})^{-N}] B_1 \left(\bar{l}_1 \cdot \bar{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{1t} + \bar{\lambda}_i \bar{l}_{1x}) \cdot \bar{r}_i B_i + \bar{l}_{1t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} \right). \end{aligned} \quad (2.76)$$

Now the second and third terms on the left hand side of (2.76) can be estimated by

$$\begin{aligned} & \frac{1}{2} N \left(\hat{V}^{-N-1} \hat{V}_t + (V^{S_3})^{-N-1} V_t^{S_3} \right) B_1^2 + (\bar{\lambda}_1(\bar{V}, \bar{P}) B_1)_x [\hat{V}^{-N} + (V^{S_3})^{-N}] B_1 \\ & \leq C_{h,T} B_1^2 + C(\hat{V}_x + |U_{1x}^{S_3}|) B_1^2 - \frac{1}{2} N |\bar{\lambda}_1(\bar{V}, \bar{P})| \hat{V}^{-N-1} \hat{V}_x B_1^2 \\ & \quad - \frac{1}{2} N \left(1 + \frac{|\bar{\lambda}_1(\bar{V}, \bar{P})|}{s_3} \right) (V^{S_3})^{-N-1} |U_{1x}^{S_3}| B_1^2 + (\dots)_x. \end{aligned} \quad (2.77)$$

Then we estimate the right hand side of (2.76) term by term. On one hand,

$$|[\hat{V}^{-N} + (V^{S_3})^{-N}] B_1 \bar{l}_1 \cdot \bar{Q}^{CD}| \leq C B_1^2 + C |\bar{Q}^{CD}|^2. \quad (2.78)$$

On the other hand, we have

$$|[\hat{V}^{-N} + (V^{S_3})^{-N}] (\bar{l}_{1t} + \bar{\lambda}_1 \bar{l}_{1x}) \cdot \bar{r}_1 B_1^2| \leq C_{h,T} B_1^2 + C \hat{V}_x B_1^2 + C |U_{1x}^{S_3}| B_1^2, \quad (2.79)$$

$$\begin{aligned} & |[\hat{V}^{-N} + (V^{S_3})^{-N}] B_1 \bar{l}_{1t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}| \\ & \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| (B_1^2 + |B_{21}|^2 + |B_{22}|^2 + |B_{23}|^2), \end{aligned} \quad (2.80)$$

and

$$\begin{aligned}
& |[\hat{V}^{-N} + (V^{S_3})^{-N}](\bar{l}_{1t} + \bar{\lambda}_3 \bar{l}_{1x}) \cdot \bar{r}_3 B_1 B_3| \\
& \leq C_{h,T}(B_1^2 + B_3^2) + C\hat{V}_x(B_1^2 + B_3^2) + C|\lambda_3^{S_3} - s_3| |U_{1x}^{S_3}|(B_1^2 + B_3^2) \\
& \quad + C_{h,T} \left[e^{-\frac{c_h}{\sigma}} e^{-\frac{c_h |x|}{\sigma}} + |(d_{1x}, d_{2x}, d_{3x})| \right] (B_1^2 + B_3^2) \\
& \leq C_{h,T}(B_1^2 + B_3^2) + C\hat{V}_x(B_1^2 + B_3^2) + C\delta |U_{1x}^{S_3}|(B_1^2 + B_3^2). \tag{2.81}
\end{aligned}$$

Substituting (2.77)–(2.81) into (2.76) and choosing N large enough give

$$\begin{aligned}
& \left(\frac{1}{2} [\hat{V}^{-N} + (V^{S_3})^{-N}] B_1^2 \right)_t - \frac{1}{4} N |\bar{\lambda}_1(\bar{V}, \bar{P})| \hat{V}^{-N-1} |\hat{V}_x| B_1^2 - \frac{1}{4} N \left(1 + \frac{|\bar{\lambda}_1(\bar{V}, \bar{P})|}{s_3} \right) (V^{S_3})^{-N-1} |U_{1x}^{S_3}| B_1^2 \\
& \geq -C_{h,T} |\bar{B}|^2 - C_{h,T} |\bar{Q}^{CD}|^2 - C |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 - C |\hat{V}_x| B_3^2 - C\delta |U_{1x}^{S_3}| B_3^2. \tag{2.82}
\end{aligned}$$

By multiplying the equation (2.73) $_{j+1}$ by $(V^{S_3})^{-N} B_{2j}$ ($j = 1, 2, 3$), and taking the summation of the equations together, we obtain

$$\begin{aligned}
& \left(\frac{1}{2} (V^{S_3})^{-N} \sum_{j=1}^3 |B_{2j}|^2 \right)_t - \frac{1}{2} N (V^{S_3})^{-N-1} |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 \\
& = (V^{S_3})^{-N} \left(\sum_{j=1}^3 B_{2j} \bar{l}_{2j} \cdot \bar{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{21t} + \bar{\lambda}_i \bar{l}_{21x}) \cdot \bar{r}_i B_i B_{21} + \bar{l}_{21t} \sum_{j=1}^3 \bar{r}_{2j} B_{2j} B_{21} \right). \tag{2.83}
\end{aligned}$$

It is easy to check that

$$|(V^{S_3})^{-N} \sum_{j=1}^3 B_{2j} \bar{l}_{2j} \cdot \bar{Q}^{CD}| \leq C \sum_{j=1}^3 |B_{2j}|^2 + C |\bar{Q}^{CD}|^2. \tag{2.84}$$

From the construction of viscous contact wave and (2.68), it holds that $l_{21x}^{CD} = O(1)\delta^{CD}$. Then the terms on the right hand side of (2.83) can be estimated by

$$\begin{aligned}
& |(V^{S_3})^{-N} (\bar{l}_{21t} + \bar{\lambda}_1 \bar{l}_{21x}) \cdot \bar{r}_1 B_1 B_{21}| \\
& \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| (B_1^2 + |B_{21}|^2) + C_{h,T} \left[e^{-\frac{c_h}{\sigma}} e^{-\frac{c_h |x|}{\sigma}} + |(d_{1x}, d_{2x}, d_{3x})| \right] (B_1^2 + |B_{21}|^2) \\
& \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| (B_1^2 + |B_{21}|^2). \tag{2.85}
\end{aligned}$$

Similar to (2.81) and (2.85), we have

$$|(V^{S_3})^{-N} (\bar{l}_{21t} + \bar{\lambda}_3 \bar{l}_{21x}) \cdot \bar{r}_3 B_3 B_{21}| \leq C_{h,T} |\bar{B}|^2 + C\delta |U_{1x}^{S_3}| (B_3^2 + |B_{21}|^2), \tag{2.86}$$

and

$$|(V^{S_3})^{-N} \bar{l}_{21t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} B_{21}| \leq C_{h,T} |\bar{B}|^2 + C |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2. \tag{2.87}$$

Substituting (2.84)–(2.87) into (2.83) and choosing N large enough give

$$\begin{aligned}
& \left(\frac{1}{2} (V^{S_3})^{-N} \sum_{j=1}^3 |B_{2j}|^2 \right)_t - \frac{1}{4} N (V^{S_3})^{-N-1} |U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 \\
& \geq -C_{h,T} |\bar{B}|^2 - C |\bar{Q}^{CD}|^2 - C\delta |U_{1x}^{S_3}| B_3^2 - C |U_{1x}^{S_3}| (B_1^2 + B_{21}^2). \tag{2.88}
\end{aligned}$$

Multiplying (2.73) $_5$ by $\hat{V}^N B_3$ yields

$$\begin{aligned}
& \left(\frac{1}{2} \hat{V}^N B_3^2 \right)_t + \frac{1}{2} \bar{\lambda}_{3x}(\bar{V}, \bar{P}) \hat{V}^N B_3^2 - \frac{\bar{\lambda}_3}{2} N \hat{V}^{N-1} |\hat{V}_x| B_3^2 \\
& = \frac{1}{2} N \hat{V}^{N-1} \hat{V}_t B_3^2 + \hat{V}^N B_3 \left(\bar{l}_3 \cdot \bar{Q}^{CD} + \sum_{i=1,3} (\bar{l}_{3t} + \bar{\lambda}_i \bar{l}_{3x}) \cdot \bar{r}_i B_i + \bar{l}_{3t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j} \right). \tag{2.89}
\end{aligned}$$

The wave interaction estimations imply

$$\bar{\lambda}_{3x} = \lambda_{3x}^{R_1} + \lambda_{3x}^{CD} + \lambda_{3x}^{S_3} + (\bar{\lambda}_{3x} - \lambda_{3x}^{R_1} - \lambda_{3x}^{CD} - \lambda_{3x}^{S_3}) \leq \lambda_{3x}^{S_3} + C|\hat{V}_x| + C_{h,T}, \quad (2.90)$$

so that

$$\frac{1}{2}\bar{\lambda}_{3x}\hat{V}^N B_3^2 \leq \frac{1}{2}\lambda_{3x}^{S_3}\hat{V}^N B_3^2 + C|\hat{V}_x|\hat{V}^N B_3^2 + C_{h,T}B_3^2. \quad (2.91)$$

The other terms on the right hand side of (2.89) can be estimated similarly to (2.81)-(2.85). Then we have

$$\left| \frac{1}{2}N\hat{V}^{N-1}\hat{V}_t B_3^2 + \hat{V}^N B_3 \bar{l}_3 \cdot \vec{Q}^{CD} \right| \leq CB_3^2 + C|\vec{Q}^{CD}|^2, \quad (2.92)$$

$$\begin{aligned} & |\hat{V}^N(\bar{l}_{3t} + \bar{\lambda}_1 \bar{l}_{3x}) \cdot \bar{r}_1 B_1 B_3| \\ & \leq C_{h,T}(B_1^2 + B_3^2) + \beta|U_{1x}^{S_3}|\hat{V}^N B_3^2 + C_\beta|U_{1x}^{S_3}|B_1^2 + C|\hat{V}_x|\hat{V}^N(B_1^2 + B_3^2), \end{aligned} \quad (2.93)$$

$$|\hat{V}^N(\bar{l}_{3t} + \bar{\lambda}_3 \bar{l}_{3x}) \cdot \bar{r}_3 B_3^2| \leq C_{h,T}\hat{V}^N B_3^2 + C\delta|U_{1x}^{S_3}|\hat{V}^N B_3^2 + C|\hat{V}_x|\hat{V}^N B_3^2, \quad (2.94)$$

and

$$|\hat{V}^N B_3 \bar{l}_{3t} \cdot \sum_{j=1}^3 \bar{r}_{2j} B_{2j}| \leq C_{h,T}|\vec{B}|^2 + \beta|U_{1x}^{S_3}|\hat{V}^N B_3^2 + C_\beta|U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2. \quad (2.95)$$

By choosing β and δ_0 small enough, substituting (2.91)–(2.94) into (2.89) gives

$$\begin{aligned} & \left(\frac{1}{2}\hat{V}^N B_3^2 \right)_t - \frac{|\lambda_{3x}^{S_3}|}{4}\hat{V}^N B_3^2 - \frac{\bar{\lambda}_3}{4}N\hat{V}^{N-1}|\hat{V}_x|B_3^2 \\ & \geq -C_{h,T}|\vec{B}|^2 - C|\vec{Q}^{CD}|^2 - C|\hat{V}_x|\hat{V}^N B_1^2 - C_\beta|U_{1x}^{S_3}|(B_1^2 + \sum_{j=1}^3 |B_{2j}|^2). \end{aligned} \quad (2.96)$$

By combining (2.82),(2.88) and (2.96) and noticing that $\delta_0 \ll 1$ and $N \gg 1$, we can get

$$\begin{aligned} & \left(\frac{1}{2}[\hat{V}^{-N} + (V^{S_3})^{-N}]B_1^2 + \frac{1}{2}(V^{S_3})^{-N} \sum_{j=1}^3 |B_{2j}|^2 + \frac{1}{2}\hat{V}^N B_3^2 \right)_t \\ & - \frac{1}{4}N|\bar{\lambda}_1|\hat{V}^{-N-1}|\hat{V}_x|B_1^2 - \frac{1}{4}N(1 + \frac{|\bar{\lambda}_1|}{s_3})\hat{V}^{-N-1}|U_{1x}^{S_3}|B_1^2 \\ & - \frac{1}{4}N(V^{S_3})^{-N-1}|U_{1x}^{S_3}| \sum_{j=1}^3 |B_{2j}|^2 - \frac{|\lambda_{3x}^{S_3}|}{4}\hat{V}^N B_3^2 - \frac{\bar{\lambda}_3}{4}N\hat{V}^{N-1}|\hat{V}_x|B_3^2 \\ & \geq -C_{h,T}|\vec{B}|^2 - C_{h,T}|Q^{CD}|^2. \end{aligned} \quad (2.97)$$

Integrating (2.97) over $[t, T] \times \mathbf{R}$ with $t \in (h, T)$ then yields

$$\begin{aligned} & \int_{\mathbf{R}} |\vec{B}|^2 dx + \int_t^T \int_{\mathbf{R}} \left[|\hat{V}_x|(B_1^2 + B_3^2) + |U_x^{S_3}||\vec{B}|^2 \right] dx dt \\ & \leq C_{h,T} \int_{\mathbf{R}} |\vec{B}|^2 + C_{h,T} \varepsilon^{\frac{5}{2}}, \quad \forall t \in [h, T]. \end{aligned} \quad (2.98)$$

Applying Gronwall inequality to (2.98) gives

$$\int_{\mathbf{R}} |\vec{B}|^2 dx + \int_t^T \int_{\mathbf{R}} \left[|\hat{V}_x|(B_1^2 + B_3^2) + |U_x^{S_3}||\vec{B}|^2 \right] dx dt \leq C_{h,T} \varepsilon^{\frac{5}{2}}, \quad \forall t \in [h, T]. \quad (2.99)$$

This completes the proof for the case when $k = 0$ in Lemma 2.5. The case $k = 1, 2, 3$ can be proved similarly to the differentiated system, and we omit the details for brevity. \square

2.6 Superposition of Waves

With the above preparation, finally, the approximate superposition wave $(V, U, \mathcal{E})(t, x)$ can be defined by

$$\begin{pmatrix} V \\ U_i \\ \mathcal{E} \end{pmatrix} (t, x) = \begin{pmatrix} \bar{V} + b_1 \\ \bar{U}_i + b_{2i} \\ \bar{\mathcal{E}} + b_3 \end{pmatrix} (t, x), \quad i = 1, 2, 3, \quad (2.100)$$

where $\mathcal{E} = \Theta + \frac{|U|^2}{2}$.

Thus, we have

$$\Theta = \bar{\Theta} - \sum_{i=1}^3 \bar{U}_i b_{2i} + b_3 - \frac{|b_2|^2}{2}, \quad (2.101)$$

where $b_2 = (b_{21}, b_{22}, b_{23})^t$ and $|b_2|^2 = \sum_{i=1}^3 b_{2i}^2$.

From the construction of the contact wave and Lemma 2.1 and by noting that $\sigma = \varepsilon^{\frac{1}{5}}$, we have the following relation between the approximate wave pattern $(V, U, \mathcal{E}, \Theta)(t, x)$ of the Boltzmann equation and the inviscid superposition wave pattern $(\bar{V}, \bar{U}, \bar{\mathcal{E}}, \bar{\Theta})(t, x)$ to the Euler equations

$$\begin{aligned} & |(V, U, \mathcal{E}, \Theta)(t, x) - (\bar{V}, \bar{U}, \bar{\mathcal{E}}, \bar{\Theta})(t, x)| \\ & \leq C \left[|(V^{R_1}, U^{R_1}, \mathcal{E}^{R_1}, \Theta^{R_1})(t, x) - (v^{r_1}, u^{r_1}, E^{r_1}, \theta^{r_1})(t, x)| + |(d_1, d_2, d_3)(t, x)| \right. \\ & \quad + |(V^{CD}, U^{CD}, \mathcal{E}^{CD}, \Theta^{CD})(t, x) - (v^{cd}, u^{cd}, E^{cd}, \theta^{cd})(t, x)| \\ & \quad \left. + |(V^{S_3}, U^{S_3}, \mathcal{E}^{S_3}, \Theta^{S_3})(t, x) - (v^{s_3}, u^{s_3}, E^{s_3}, \theta^{s_3})(t, x)| + |(b_1, b_{21}, b_{22}, b_{23}, b_3)(t, x)| \right] \\ & \leq C_{h,T} \left[\frac{1}{t} (\sigma \ln(1+t) + \sigma |\ln \sigma|) + \frac{\varepsilon}{\sigma} + \delta^{CD} e^{-\frac{cx^2}{\varepsilon(1+t)}} + \delta^{S_3} e^{-c \frac{\delta^{S_3} |x-s_3 t|}{\varepsilon}} + \varepsilon^{\frac{3}{4}} \right] \\ & \leq C_{h,T} \left[\varepsilon^{\frac{1}{5}} |\ln \varepsilon| + \delta^{CD} e^{-\frac{cx^2}{\varepsilon(1+t)}} + \delta^{S_3} e^{-c \frac{\delta^{S_3} |x-s_3 t|}{\varepsilon}} \right]. \end{aligned} \quad (2.102)$$

Moreover, the approximate wave pattern $(V, U, \mathcal{E}, \Theta)(t, x)$ satisfies

$$\begin{cases} V_t - U_{1x} = 0, \\ U_{1t} + P_x = \frac{4}{3} \varepsilon \left(\frac{\mu(\Theta) U_{1x}}{V} \right)_x - \int \xi_1^2 \Pi_{11x}^{CD} d\xi - \int \xi_1^2 \Pi_{1x}^{S_3} d\xi + \bar{Q}_{1x} + Q_{1x}, \\ U_{it} = \varepsilon \left(\frac{\mu(\Theta) U_{ix}}{V} \right)_x - \int \xi_1 \xi_i \Pi_{11x}^{CD} d\xi - \int \xi_1 \xi_i \Pi_{1x}^{S_3} d\xi + \bar{Q}_{ix} + Q_{ix}, \quad i = 2, 3, \\ \mathcal{E}_t + (PU_1)_x = \varepsilon \left(\frac{\kappa(\Theta) \Theta_x}{V} \right)_x + \frac{4}{3} \varepsilon \left(\frac{\mu(\Theta) U_1 U_{1x}}{V} \right)_x + \sum_{i=2}^3 \varepsilon \left(\frac{\mu(\Theta) U_i U_{ix}}{V} \right)_x \\ \quad - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11x}^{CD} d\xi - \int \xi_1 \frac{|\xi|^2}{2} \Pi_{1x}^{S_3} d\xi + \bar{Q}_{4x} + Q_{4x}, \end{cases} \quad (2.103)$$

where $P = p(V, \Theta)$ and

$$\begin{aligned} Q_1 &= \left[P - \bar{P} - (\bar{P}_v b_1 + \bar{P}_u \cdot b_2 + \bar{P}_E b_3) \right] - \frac{4}{3} \varepsilon \left[\frac{\mu(\Theta) U_{1x}}{V} - \frac{\mu(\bar{\Theta}) \bar{U}_{1x}}{V} \right], \\ &:= Q_{11} + Q_{12}, \\ Q_i &= -\varepsilon \left[\frac{\mu(\Theta) U_{ix}}{V} - \frac{\mu(\bar{\Theta}) \bar{U}_{ix}}{V} \right], \quad i = 2, 3, \\ Q_4 &= \left[PU_1 - \bar{P} \bar{U}_1 - ((\bar{P} \bar{U}_1)_v b_1 + (\bar{P} \bar{U}_1)_u \cdot b_2 + (\bar{P} \bar{U}_1)_E b_3) \right] \\ &\quad - \varepsilon \left[\left(\frac{\kappa(\Theta) \Theta_x}{V} - \frac{\kappa(\bar{\Theta}) \bar{\Theta}_x}{V} \right) + \frac{4}{3} \left(\frac{\mu(\Theta) U_1 U_{1x}}{V} - \frac{\mu(\bar{\Theta}) \bar{U}_1 \bar{U}_{1x}}{V} \right) \right. \\ &\quad \left. + \sum_{i=2}^3 \left(\frac{\mu(\Theta) U_i U_{ix}}{V} - \frac{\mu(\bar{\Theta}) \bar{U}_i \bar{U}_{ix}}{V} \right) \right] \\ &:= Q_{41} + Q_{42}. \end{aligned} \quad (2.104)$$

Straightforward calculation shows that

$$(Q_{11}, Q_{41}) = O(1) |\bar{b}|^2. \quad (2.105)$$

3 Proof of Main Result

With the above preparation, we will give the proof of the main theorem as follows. For this, we will first reformulate the problem in the following subsection. The energy estimates will then be given in the second subsection.

3.1 Reformulation of the Problem

We now reformulate the system by introducing a scaling for the independent variables. Set

$$y = \frac{x}{\varepsilon}, \quad \tau = \frac{t}{\varepsilon}. \quad (3.1)$$

In the following, we will also use the notations $(v, u, \theta)(\tau, y)$, $\mathbf{G}(\tau, y, \xi)$, $\Pi_1(\tau, y, \xi)$ and $(V, U, \Theta)(\tau, y)$, etc., in the scaled independent variables. Set the perturbation around the superposition wave $(V, U, \Theta)(\tau, y)$ by

$$\begin{aligned} (\phi, \psi, \omega, \zeta)(\tau, y) &= (v - V, u - U, E - \mathcal{E}, \theta - \Theta)(\tau, y), \\ \tilde{\mathbf{G}}(\tau, y, \xi) &= \mathbf{G}(\tau, y, \xi) - \mathbf{G}^{S_3}(\tau, y, \xi), \\ \tilde{f}(\tau, y, \xi) &= f(\tau, y, \xi) - F^{S_3}(\tau, y, \xi). \end{aligned} \quad (3.2)$$

Under this scaling, the hydrodynamic limit problem is reduced to a time asymptotic stability problem for the Boltzmann equation.

In particular, we can choose the initial value as

$$(\phi, \psi, \omega)(\tau = \frac{h}{\varepsilon}, y) = (0, 0, 0), \quad \tilde{\mathbf{G}}(\tau = \frac{h}{\varepsilon}, y, \xi) = 0. \quad (3.3)$$

Introduce the anti-derivative variables

$$(\Phi, \Psi, \bar{W})(\tau, y) = \int_{-\infty}^y (\phi, \psi, \omega)(\tau, y') dy'.$$

Then $(\Phi, \Psi, \bar{W})(\tau, y)$ satisfies that

$$\left\{ \begin{array}{l} \Phi_\tau - \Psi_{1y} = 0, \\ \Psi_{1\tau} + (p - P) = \frac{4}{3} \left(\frac{\mu(\theta)u_{1y}}{v} - \frac{\mu(\Theta)U_{1y}}{V} \right) - \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \bar{Q}_1 - Q_1, \\ \Psi_{i\tau} = \left(\frac{\mu(\theta)u_{iy}}{v} - \frac{\mu(\Theta)U_{iy}}{V} \right) - \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \bar{Q}_i - Q_i, \quad i = 2, 3, \\ \bar{W}_\tau + (pu_1 - PU_1) = \left(\frac{\kappa(\theta)\theta_y}{v} - \frac{\kappa(\Theta)\Theta_y}{V} \right) + \frac{4}{3} \left(\frac{\mu(\theta)u_1 u_{1y}}{v} - \frac{\mu(\Theta)U_1 U_{1y}}{V} \right) \\ \quad + \sum_{i=2}^3 \left(\frac{\mu(\theta)u_i u_{iy}}{v} - \frac{\mu(\Theta)U_i U_{iy}}{V} \right) - \int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \bar{Q}_4 - Q_4. \end{array} \right. \quad (3.4)$$

To precisely capture the dissipation of heat conduction, we introduce another variable related to the absolute temperature

$$W = \bar{W} - U \cdot \Psi = \bar{W} - \sum_{i=1}^3 U_i \Psi_i,$$

then

$$\zeta = W_y - \left(\frac{|\Psi_y|^2}{2} - U_y \cdot \Psi \right). \quad (3.5)$$

Linearizing the system (3.4) around the approximate wave pattern $(V, U, \Theta)(\tau, y)$ implies that

$$\left\{ \begin{array}{l} \Phi_\tau - \Psi_{1y} = 0, \\ \Psi_{1\tau} - \frac{Z}{V}\Phi_y + \frac{2}{3V}W_y + \frac{2}{3V}U_y \cdot \Psi - \frac{4}{3}\frac{\mu'(\Theta)}{V}(W_y + U_y \cdot \Psi)U_{1y} = \frac{4}{3}\frac{\mu(\Theta)}{V}\Psi_{1yy} \\ \quad - \int \xi_1^2(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi + N_1 - \bar{Q}_1 - Q_1, \\ \Psi_{i\tau} + \frac{\mu(\Theta)U_{iy}}{V^2}\Phi_y - \frac{\mu'(\Theta)}{V}(W_y + U_y \cdot \Psi)U_{iy} = \frac{\mu(\Theta)}{V}\Psi_{iyy} \\ \quad - \int \xi_1 \xi_i(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi + N_i - \bar{Q}_i - Q_i, i = 2, 3, \\ W_\tau + Z\Psi_{1y} - \sum_{i=2}^3 \frac{\mu(\Theta)U_{iy}}{V}\Psi_{iy} + U_\tau \cdot \Psi - \frac{\kappa(\Theta)}{V}(U_y \cdot \Psi)_y + \frac{\kappa(\Theta)}{V^2}\Theta_y\Phi_y \\ - \frac{\kappa'(\Theta)}{V}(W_y + U_y \cdot \Psi)\Theta_y = \frac{\kappa(\Theta)}{V}W_{yy} - \int \xi_1 \frac{|\xi|^2}{2}(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi \\ + \sum_{i=1}^3 U_i \int \xi_1 \xi_i(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})d\xi + N_4 - \bar{Q}_4 + \sum_{i=1}^3 U_i \bar{Q}_i - Q_4 + \sum_{i=1}^3 U_i Q_i, \end{array} \right. \quad (3.6)$$

where

$$Z = P - \frac{4}{3}\frac{\mu(\Theta)U_{1y}}{V}, \quad (3.7)$$

$$\begin{aligned} N_1 &= \frac{p-P}{V}\Phi_y + \frac{1}{3V}|\Psi_y|^2 + \frac{4}{3}\left(\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V}\right)\Psi_{1yy} \\ &\quad + \frac{4}{3}U_{1y}\left[\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2}\Phi_y - \frac{\mu'(\Theta)}{V}\left(\zeta + \frac{|\Psi_y|^2}{2}\right)\right] \\ &= O(1)\left[|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{1yy}|^2\right], \end{aligned} \quad (3.8)$$

$$\begin{aligned} N_i &= \left(\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V}\right)\Psi_{iyy} + U_{iy}\left[\frac{\mu(\theta)}{v} - \frac{\mu(\Theta)}{V} + \frac{\mu(\Theta)}{V^2}\Phi_y - \frac{\mu'(\Theta)}{V}\left(\zeta + \frac{|\Psi_y|^2}{2}\right)\right] \\ &= O(1)\left[|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{iyy}|^2\right], \quad i = 2, 3, \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} N_4 &= -(p-P)\Psi_{1y} - \frac{\kappa(\Theta)}{V}\Psi_y \cdot \Psi_{yy} + \left(\frac{\kappa(\theta)}{v} - \frac{\kappa(\Theta)}{V}\right)\zeta_y \\ &\quad + \frac{4}{3}\left(\frac{\mu(\theta)u_{1y}}{v} - \frac{\mu(\Theta)U_{1y}}{V}\right)\Psi_{1y} + \sum_{i=2}^3 \left(\frac{\mu(\theta)u_{iy}}{v} - \frac{\mu(\Theta)U_{iy}}{V}\right)\Psi_{iy} \\ &\quad + \Theta_y\left[\frac{\kappa(\theta)}{v} - \frac{\kappa(\Theta)}{V} + \frac{\kappa(\Theta)}{V^2}\Phi_y - \frac{\kappa'(\Theta)}{V}\left(\zeta + \frac{|\Psi_y|^2}{2}\right)\right] \\ &= O(1)\left[|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{yy}|^2 + |\zeta_y|^2\right]. \end{aligned} \quad (3.10)$$

We now derive the equation for the non-fluid component $\tilde{\mathbf{G}}(\tau, y, \xi)$ in the scaled independent variables. From (1.18), we have

$$\begin{aligned} \tilde{\mathbf{G}}_\tau - \mathbf{L}_M \tilde{\mathbf{G}} &= \frac{u_1}{v}\tilde{\mathbf{G}}_y - \frac{1}{v}\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) - \left[\frac{1}{v}\mathbf{P}_1(\xi_1 \mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_y^{S_3})\right] \\ &\quad + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_1, \end{aligned} \quad (3.11)$$

where

$$J_1 = (\mathbf{L}_M - \mathbf{L}_{M^{S_3}})\mathbf{G}^{S_3} + \left(\frac{u}{v} - \frac{U_1^{S_3}}{V^{S_3}}\right)\mathbf{G}_y^{S_3} - \left[\frac{1}{v}\mathbf{P}_1(\xi_1 \mathbf{G}_y^{S_3}) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1 \mathbf{G}_y^{S_3})\right]. \quad (3.12)$$

Let

$$\mathbf{G}^{R_1}(\tau, y, \xi) = \frac{3}{2v\theta}\mathbf{L}_M^{-1}\{\mathbf{P}_1[\xi_1\left(\frac{|\xi - u|^2}{2\theta}\Theta_y^{R_1} + \xi \cdot U_y^{R_1}\right)\mathbf{M}]\}, \quad (3.13)$$

and

$$\tilde{\mathbf{G}}_1(\tau, y, \xi) = \tilde{\mathbf{G}}(\tau, y, \xi) - \mathbf{G}^{R_1}(\tau, y, \xi) - \mathbf{G}^{CD}(\tau, y, \xi), \quad (3.14)$$

where $\mathbf{G}^{CD}(\tau, y, \xi)$ is defined in (2.20).

Then $\mathbf{G}_1(\tau, y, \xi)$ satisfies

$$\tilde{\mathbf{G}}_{1\tau} - \mathbf{L}_M \tilde{\mathbf{G}}_1 = \frac{u_1}{v} \tilde{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_1 + J_2 - \mathbf{G}_\tau^{R_1} - \mathbf{G}_\tau^{CD}. \quad (3.15)$$

with

$$\begin{aligned} J_2 &= - \left[\frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) - \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_y^{S_3}) \right. \\ &\quad \left. - \frac{3}{2v\theta} \mathbf{P}_1 \left(\xi_1 \left[\frac{|\xi - u|^2}{2\theta} (\Theta_y^{R_1} + \Theta_y^{CD}) + \xi \cdot (U_y^{R_1} + U_y^{CD}) \right] \mathbf{M} \right) \right] \\ &= - \left[\frac{3}{2v\theta} \mathbf{P}_1 \left(\xi_1 \left[\frac{|\xi - u|^2}{2\theta} (\theta - \Theta^{R_1} - \Theta^{CD})_y + \xi \cdot (u - U^{R_1} - U^{CD})_y \right] \mathbf{M} \right) \right. \\ &\quad \left. - \frac{3}{2V^{S_3} \Theta^{S_3}} \mathbf{P}_1^{S_3} \left(\xi_1 \left[\frac{|\xi - U^{S_3}|^2}{2\Theta^{S_3}} \Theta_y^{S_3} + \xi \cdot U_y^{S_3} \right] \mathbf{M}^{S_3} \right) \right]. \end{aligned} \quad (3.16)$$

Notice that in (3.14) and (3.15), \mathbf{G}^{R_1} and \mathbf{G}^{CD} are subtracted from $\tilde{\mathbf{G}}$ when carrying out the lower order energy estimates because $\int_{\frac{h}{\varepsilon}}^{\frac{T}{\varepsilon}} \|(\Theta_y^{R_1}, U_{1y}^{R_1})\|_{L^2(dy)}^2 d\tau$ is uniformly bounded with respect to ε , while $\int_{\frac{h}{\varepsilon}}^{\frac{T}{\varepsilon}} \|\Theta_y^{CD}\|_{L^2(dy)}^2 d\tau$ is only of the order of $\varepsilon^{-\frac{1}{2}}$. Both do not give any decay with respect to Knudsen number ε in the above integrals.

From (1.15) and the scaling transformation (3.1), we have

$$f_\tau - \frac{u_1}{v} f_y + \frac{\xi_1}{v} f_y = Q(f, f). \quad (3.17)$$

Thus, we have the equation for \tilde{f} defined in (3.2)

$$\tilde{f}_\tau - \frac{u_1}{v} \tilde{f}_y + \frac{\xi_1}{v} \tilde{f}_y = \mathbf{L}_M \tilde{\mathbf{G}} + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_F, \quad (3.18)$$

with

$$J_F = \left(\frac{u_1}{v} - \frac{U_1^{S_3}}{V^{S_3}} \right) F_y^{S_3} - \left(\frac{1}{v} - \frac{1}{V^{S_3}} \right) \xi_1 F_y^{S_3} + 2Q(\mathbf{M} - \mathbf{M}^{S_3}, \mathbf{G}^{S_3}) + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}). \quad (3.19)$$

The estimation on the fluid and non-fluid components governed by the above equations will be given in the next subsection. In the following, we will state the main estimate we want to obtain and also give the a priori estimate.

Note that to prove the main theorem in this paper, it is sufficient to prove the following theorem on the Boltzmann equation (3.17) in the scaled independent variables based on the construction of the approximate wave pattern.

Theorem 3.1 *There exist a small positive constants δ_1 and a global Maxwellian $\mathbf{M}_\star = \mathbf{M}_{[v_\star, u_\star, \theta_\star]}$ such that if the wave strength δ satisfies $\delta \leq \delta_1$, then on the time interval $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$ for any $0 < h < T$, there is a positive constant $\varepsilon_1(\delta, h, T)$. If the Knudsen number $\varepsilon \leq \varepsilon_1$, then the problem (3.17) admits a family of smooth solution $f^{\varepsilon, h}(\tau, y, \xi)$ satisfying*

$$\sup_{\tau \in [\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]} \sup_{y \in \mathbf{R}} \|f^{\varepsilon, h}(\tau, y, \xi) - \mathbf{M}_{[V, U, \Theta]}(\tau, y, \xi)\|_{L_\xi^2(\frac{1}{\sqrt{\mathbf{M}_\star})}} \leq C\varepsilon^{\frac{1}{5}}. \quad (3.20)$$

Consider the reformulated system (3.6) and (3.15). Since the local existence of solution to (3.6) and (3.15) is known, cf. [18] and [38], to prove the existence on the time interval $[\frac{h}{\varepsilon}, \frac{T}{\varepsilon}]$, we only need to close the following a priori estimate by the continuity argument. Set

$$\begin{aligned} \mathcal{N}(\tau) &= \sup_{\frac{h}{\varepsilon} \leq \tau' \leq \tau} \left\{ \|(\Phi, \Psi, W)(\tau', \cdot)\|^2 + \|(\phi, \psi, \zeta)(\tau', \cdot)\|_1^2 + \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy \right. \\ &\quad \left. + \sum_{|\alpha'|=1} \int \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy + \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_\star} d\xi dy \right\} \leq \chi^2 = \varepsilon^{\frac{1}{10}}, \quad \forall \tau \in [\frac{h}{\varepsilon}, \frac{T}{\varepsilon}], \end{aligned} \quad (3.21)$$

where $\partial^\alpha, \partial^{\alpha'}$ denote the derivatives with respect to y and τ , and \mathbf{M}_\star is a global Maxwellian to be chosen.

Remark 5 In the paper, we simply choose the initial data for the Boltzmann equation (3.17) as

$$f^{\varepsilon,h}(\tau = \frac{h}{\varepsilon}, y, \xi) = \mathbf{M}_{[V,U,\Theta]}(\frac{h}{\varepsilon}, y, \xi) + \mathbf{G}^{S_3}(\frac{h}{\varepsilon}, y, \xi), \quad (3.22)$$

so that

$$\mathcal{N}(\tau)|_{\tau=\frac{h}{\varepsilon}} \leq C_{h,T} \varepsilon^{\frac{1}{2}}. \quad (3.23)$$

In this case, the functional measuring the perturbation $\mathcal{N}(\tau)$ is smaller at the initial time $\tau = \frac{h}{\varepsilon}$ than the estimate given in Theorem 3.1 in the whole time interval when ε is small.

Note that the a priori assumption (3.21) implies that

$$\|(\Phi, \Psi, W)\|_{L^\infty}^2 + \|(\phi, \psi, \zeta, \omega)\|_{L^\infty}^2 \leq C\chi^2, \quad (3.24)$$

and

$$\begin{aligned} \left\| \int \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi \right\|_{L^y_\infty} &\leq C \left(\int \int \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\tilde{\mathbf{G}}_{1y}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\ &\leq C \left(\int \int \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\tilde{\mathbf{G}}_y|^2 + |\mathbf{G}_y^{R_1}|^2 + |\mathbf{G}_y^{CD}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\ &\leq C\chi \left[\chi + \|(v_y, u_y, \theta_y)(\Theta_y^{R_1}, U_y^{R_1}, \Theta_y^{CD}, U_y^{CD})\|_{L^2(dy)} \right. \\ &\quad \left. + \|(\Theta_{yy}^{R_1}, U_{yy}^{R_1}, \Theta_{yy}^{CD}, U_{yy}^{CD})\|_{L^2(dy)} \right] \leq C_{h,T}\chi^2. \end{aligned} \quad (3.25)$$

Furthermore, for $|\alpha'| = 1$,

$$\begin{aligned} \left\| \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi \right\|_{L^y_\infty} &\leq C \left(\int \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}_y|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\ &\leq C \left(\int \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \cdot \left(\int \int \frac{|\partial^{\alpha'} \tilde{f}_y|^2 + |\partial^{\alpha'} (\mathbf{M} - \mathbf{M}^{S_3})_y|^2}{\mathbf{M}_\star} d\xi dy \right)^{\frac{1}{2}} \\ &\leq C\chi \left[\chi + \sum_{|\alpha|=2} \|\partial^\alpha (v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})\| \right. \\ &\quad \left. + \|(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})(\sum_{|\alpha|=2} \partial^\alpha + \sum_{|\alpha'|=1} \partial^{\alpha'}) (V^{S_3}, U^{S_3}, \Theta^{S_3})\| \right. \\ &\quad \left. + \sum_{|\alpha'|=1} \|\partial^{\alpha'} (v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3}) \cdot \partial^{\alpha'} (V^{S_3}, U^{S_3}, \Theta^{S_3})\| \right. \\ &\quad \left. + \sum_{|\alpha'|=1} \|\partial^{\alpha'} (v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})\|_{L^4(dy)}^2 \right] \leq C_{h,T}\chi^2. \end{aligned} \quad (3.26)$$

From (1.17) and (2.103), we have

$$\left\{ \begin{array}{l} \phi_\tau - \psi_{1y} = 0, \\ \psi_{1\tau} + (p - P)_y = -\frac{4}{3} \left[\left(\frac{\mu(\Theta)}{V} U_{1y} \right)_y - \left(\frac{\mu(\Theta^{S_3})}{V^{S_3}} U_{1y}^{S_3} \right)_y \right] + \int \xi_1^2 \Pi_{11y}^{CD} d\xi \\ \quad - \bar{Q}_{1y} - Q_{1y} - \int \xi_1^2 \tilde{\mathbf{G}}_y d\xi, \\ \psi_{i\tau} = -\left(\frac{\mu(\Theta)}{V} U_{iy} \right)_y + \int \xi_1 \xi_i \Pi_{11y}^{CD} d\xi - \bar{Q}_{iy} - Q_{iy} - \int \xi_1 \xi_i \tilde{\mathbf{G}}_y d\xi, \quad i = 2, 3, \\ \omega_\tau + (pu_1 - PU_1)_y = -\left[\left(\frac{\kappa(\Theta)}{V} \Theta_y \right)_y - \left(\frac{\kappa(\Theta^{S_3})}{V^{S_3}} \Theta_y^{S_3} \right)_y \right] - \frac{4}{3} \left[\left(\frac{\mu(\Theta)}{V} U_1 U_{1y} \right)_y \right. \\ \quad \left. - \left(\frac{\mu(\Theta^{S_3})}{V^{S_3}} U_1^{S_3} U_{1y}^{S_3} \right)_y \right] - \sum_{i=2}^3 \left(\frac{\mu(\Theta)}{V} U_i U_{iy} \right)_y + \int \xi_1 \frac{|\xi|^2}{2} \Pi_{11y}^{CD} d\xi \\ \quad - \bar{Q}_{4y} - Q_{4y} - \frac{1}{2} \int \xi_1 |\xi|^2 \tilde{\mathbf{G}}_y d\xi. \end{array} \right. \quad (3.27)$$

And from (2.62), we get

$$\begin{aligned}
\bar{Q}_{1y} &= \varepsilon[\bar{P} - P^{R_1} - P^{CD} - P^{S_3} - (p_v^{R_1} d_1 + p_{u_1}^{R_1} d_2 + p_E^{R_1} d_3)]_x \\
&\quad + \varepsilon^2 \left(\frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}} - \frac{\mu(\Theta^{R_1})U_{1x}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1x}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1x}^{S_3}}{V^{S_3}} \right)_x \\
&= O(1)\varepsilon \left[|(V_x^{R_1}, U_x^{R_1}, \mathcal{E}_x^{R_1}, d_{ix})(|d_i|^2, V^{S_3} - v_*, U^{S_3} - u_*, \mathcal{E}^{S_3} - E_*)| + |d_i d_{ix}| \right] \\
&\quad + O(1)\varepsilon^2 \left[|(d_{2xx}, d_{1x}d_{2x}, V_x^{R_1}d_{2x}, U_x^{R_1}d_{1x})| + |(U_{xx}^{R_1}, U_x^{R_1}V_x^{R_1})(d_1, V^{S_3} - v_*)| \right. \\
&\quad \left. + |(U_{xx}^{S_3}, U_x^{S_3}V_x^{S_3})(V^{R_1} - v_*, d_1)| + |(U_x^{S_3}, V_x^{S_3})(U_x^{R_1}, V_x^{R_1}, d_{1x}, d_{2x})| \right] \\
&:= \bar{Q}_{13} + \bar{Q}_{14},
\end{aligned} \tag{3.28}$$

where \bar{Q}_{13} represents the wave interaction satisfying

$$\begin{aligned}
\bar{Q}_{13} &= O(1)\varepsilon \left[|(V_x^{R_1}, U_x^{R_1}, \mathcal{E}_x^{R_1}, d_{ix})(V^{S_3} - v_*, U^{S_3} - u_*, \mathcal{E}^{S_3} - E_*)| \right] \\
&\quad + O(1)\varepsilon^2 \left[|(U_{xx}^{R_1}, U_x^{R_1}V_x^{R_1})(d_1, V^{S_3} - v_*)| + |(U_{xx}^{S_3}, U_x^{S_3}V_x^{S_3})(V^{R_1} - v_*, d_1)| \right. \\
&\quad \left. + |(U_x^{S_3}, V_x^{S_3})(U_x^{R_1}, V_x^{R_1}, d_{1x}, d_{2x})| \right] \leq C_{h,T} e^{-\frac{c|x|}{\sigma}} e^{-\frac{c\hbar}{\sigma}},
\end{aligned} \tag{3.29}$$

and \bar{Q}_{14} represents the terms related to the hyperbolic waves d_i ($i = 1, 2, 3$) satisfying

$$\begin{aligned}
\bar{Q}_{14} &= O(1)\varepsilon \left[|(V_x^{R_1}, U_x^{R_1}, \mathcal{E}_x^{R_1}, d_{ix})||d_i|^2 + |d_i d_{ix}| \right] \\
&\quad + O(1)\varepsilon^2 \left[|(d_{2xx}, d_{1x}d_{2x}, V_x^{R_1}d_{2x}, U_x^{R_1}d_{1x})| \right].
\end{aligned} \tag{3.30}$$

Then we have

$$\int_{\frac{\hbar}{\varepsilon}}^{\tau} \int |\bar{Q}_{1y}|^2 dy d\tau \leq C \int_{\frac{\hbar}{\varepsilon}}^{\tau} \int (|\bar{Q}_{13}|^2 + |\bar{Q}_{14}|^2) dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}. \tag{3.31}$$

Similar estimates hold for \bar{Q}_{iy} ($i = 2, 3, 4$).

By (2.104), straightforward calculation gives

$$\begin{aligned}
Q_{1y} &= \varepsilon[P - \bar{P} - (\bar{p}_v b_1 + \bar{p}_u \cdot b_2 + \bar{p}_E b_3)]_x - \frac{4}{3}\varepsilon^2 \left(\frac{\mu(\Theta)U_{1x}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1x}}{\bar{V}} \right)_x \\
&= O(1)\varepsilon \sum_{i=1}^3 \left[|b_i|^2 |(\bar{V}_x, \bar{U}_{1x}, \bar{\mathcal{E}}_x)| + |b_i| |b_{ix}| \right] \\
&\quad + O(1)\varepsilon^2 \sum_{i=1}^3 \left[|b_{ixx}| + |b_{ix}|^2 + |b_i| |(\bar{U}_{xx}, \bar{V}_x \bar{U}_x, \bar{V}_x \bar{\Theta}_x)| + |b_{ix}| |(\bar{V}_x, \bar{U}_x, \bar{\Theta}_x)| \right].
\end{aligned} \tag{3.32}$$

Hence,

$$\int_{\frac{\hbar}{\varepsilon}}^{\tau} \int |Q_{1y}|^2 dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}. \tag{3.33}$$

Similar estimates hold for Q_{iy} ($i = 2, 3, 4$).

Thus from the system (3.27), we have

$$\|(\phi_\tau, \psi_\tau, \omega_\tau)\|^2 \leq C_{h,T} \chi^2, \tag{3.34}$$

and

$$\|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 \leq C \|(\phi_\tau, \psi_\tau, \omega_\tau, U_\tau \cdot \psi)\|^2 \leq C_{h,T} \chi^2. \tag{3.35}$$

Now we want to obtain the estimates on $\|\partial^\alpha(\phi, \psi, \zeta)\|^2$ for $|\alpha| = 2$. For brevity, we only calculate $|\partial_{yy}(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})|$, and the others can be estimated similarly. From (1.3) and (2.51), we have

$$\begin{aligned}
&|\partial_{yy}(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})| \\
&= O(1)|(V_{yy}^{S_3}, U_{yy}^{S_3}, \Theta_{yy}^{S_3}, (V_y^{S_3})^2, (U_y^{S_3})^2, (\Theta_y^{S_3})^2)| \cdot |(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})| \\
&\quad + O(1)|(V_y^{S_3}, U_y^{S_3}, \Theta_y^{S_3})| \cdot |(v_y - V_y^{S_3}, u_y - U_y^{S_3}, \theta_y - \Theta_y^{S_3})| \\
&\quad + O(1)|(v_y - V_y^{S_3}, u_y - U_y^{S_3}, \theta_y - \Theta_y^{S_3})|^2 + O(1) \sum_{i=0}^4 \int |\varphi_i(\xi)| \cdot |\partial_{yy} \tilde{f}| d\xi.
\end{aligned} \tag{3.36}$$

Therefore, we have

$$\begin{aligned} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 &\leq C \left[\|\partial^\alpha(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})\|^2 + \|\partial^\alpha(V^{R_1}, U^{R_1}, \Theta^{R_1})\|^2 \right. \\ &\quad + \|\partial^\alpha(V^{CD}, U^{CD}, \Theta^{CD})\|^2 + \|\partial^\alpha(d_1, d_2, d_3)\|^2 \\ &\quad \left. + \|\partial^\alpha(b_1, b_{21}, b_{22}, b_{23}, b_3)\|^2 \right] + C \int \int \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_\star} d\xi dy \leq C_{h,T} \chi^2. \end{aligned} \quad (3.37)$$

Finally, by noticing the fact that $f = \mathbf{M} + \mathbf{G}$ and $F^{S_3} = \mathbf{M}^{S_3} + \mathbf{G}^{S_3}$, (3.37), with $|\alpha| = 2$ yields,

$$\int \int \frac{|\partial^\alpha \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy \leq C \int \int \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_\star} d\xi dy + C \int \int \frac{|\partial^\alpha(\mathbf{M} - \mathbf{M}^{S_3})|^2}{\mathbf{M}_\star} d\xi dy \leq C_{h,T} \chi^2, \quad (3.38)$$

where in the last inequality we have used a similar argument used for (3.26).

Before closing the a priori estimate (3.21), we list some basic lemmas based on the celebrated H-theorem for later use. The first lemma is from [14].

Lemma 3.2 *There exists a positive constant C such that*

$$\int \frac{\nu(|\xi|)^{-1} Q(f, g)^2}{\tilde{\mathbf{M}}} d\xi \leq C \left\{ \int \frac{\nu(|\xi|) f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{g^2}{\tilde{\mathbf{M}}} d\xi + \int \frac{f^2}{\tilde{\mathbf{M}}} d\xi \cdot \int \frac{\nu(|\xi|) g^2}{\tilde{\mathbf{M}}} d\xi \right\},$$

where $\tilde{\mathbf{M}}$ can be any Maxwellian so that the above integrals are well-defined.

Based on Lemma 3.2, the following three lemmas are taken from [29]. And their proofs are straightforward by using Cauchy inequality.

Lemma 3.3 *If $\theta/2 < \theta_\star < \theta$, then there exist two positive constants $\tilde{\sigma} = \tilde{\sigma}(v, u, \theta; v_\star, u_\star, \theta_\star)$ and $\eta_0 = \eta_0(v, u, \theta; v_\star, u_\star, \theta_\star)$ such that if $|v - v_\star| + |u - u_\star| + |\theta - \theta_\star| < \eta_0$, we have for $g(\xi) \in \mathfrak{N}^\perp$,*

$$- \int \frac{g \mathbf{L}_\mathbf{M} g}{\mathbf{M}_\star} d\xi \geq \tilde{\sigma} \int \frac{\nu(|\xi|) g^2}{\mathbf{M}_\star} d\xi.$$

Lemma 3.4 *Under the assumptions in Lemma 3.3, we have for each $g(\xi) \in \mathfrak{N}^\perp$,*

$$\int \frac{\nu(|\xi|)}{\mathbf{M}} |\mathbf{L}_\mathbf{M}^{-1} g|^2 d\xi \leq \tilde{\sigma}^{-2} \int \frac{\nu(|\xi|)^{-1} g^2}{\mathbf{M}} d\xi, \quad \text{and} \quad \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_\mathbf{M}^{-1} g|^2 d\xi \leq \tilde{\sigma}^{-2} \int \frac{\nu(|\xi|)^{-1} g^2}{\mathbf{M}_\star} d\xi.$$

Lemma 3.5 *Under the conditions in Lemma 3.3, for any positive constants k and λ , it holds that*

$$\left| \int \frac{g_1 \mathbf{P}_1(|\xi|^k g_2)}{\mathbf{M}_\star} d\xi - \int \frac{g_1 |\xi|^k g_2}{\mathbf{M}_\star} d\xi \right| \leq C_{k,\lambda} \int \frac{\lambda |g_1|^2 + \lambda^{-1} |g_2|^2}{\mathbf{M}_\star} d\xi,$$

where the constant $C_{k,\lambda}$ depends on k and λ .

3.2 Energy Estimates

To close the a priori estimate (3.21) and to prove Theorem 3.1, we need the following energy estimates given in Propositions 3.1 and Proposition 3.2. First, the lower order estimates to the system (3.6) and (3.15) are given in the following Proposition.

Proposition 3.1 *Under the assumptions of Theorem 3.1, there exist positive constants C and $C_{h,T}$ independent of ε such that*

$$\begin{aligned} &\sup_{\frac{h}{\varepsilon} \leq \tau_1 \leq \tau} \left[\|(\Phi, \Psi, W, \Phi_y)(\tau_1, \cdot)\|^2 + \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau_1, y, \xi) d\xi dy \right] \\ &\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \left[\|\sqrt{|U_{1y}^{S_3}}|(\Psi, W)\|^2 + \|(\Phi_y, \Psi_y, W_y, \zeta, \Psi_\tau, W_\tau)\|^2 \right] d\tau + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\ &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau \\ &\quad + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{2}{5}}. \end{aligned}$$

To derive the estimate on the higher order derivatives, applying ∂_y to the system (3.39), gives

$$\left\{ \begin{array}{l} \phi_{y\tau} - \psi_{1yy} = 0, \\ \psi_{1y\tau} - \frac{Z}{V}\phi_{yy} + \frac{2}{3V}\zeta_{yy} + H_5 = \left(\frac{4}{3}\frac{\mu(\Theta)}{V}\psi_{1y}\right)_{yy} \\ \quad - \int \xi_1^2(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_{yy}d\xi + N_{5y} - \bar{Q}_{1yy} - Q_{1yy}, \\ \psi_{iy\tau} + H_{i+4} = \left(\frac{\mu(\Theta)}{V}\psi_{iy}\right)_{yy} - \int \xi_1\xi_i(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_{yy}d\xi \\ \quad + (N_{i+4})_y - \bar{Q}_{iyy} - Q_{iyy}, i = 2, 3, \\ \zeta_{y\tau} + Z\psi_{1yy} + H_8 = \left(\frac{\kappa(\Theta)}{V}\zeta_y\right)_{yy} - \int \xi_1\frac{|\xi|^2}{2}(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_{yy}d\xi \\ \quad + \sum_{i=1}^3 \left[\psi_i \int \xi_1\xi_i\Pi_{1y}d\xi\right]_y + \sum_{i=1}^3 \left[U_i \int \xi_1\xi_i(\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi\right]_y \\ \quad + N_{8y} - \bar{Q}_{4yy} - Q_{4yy} + \sum_{i=1}^3 \left[U_i(\bar{Q}_{iy} + Q_{iy})\right]_y, \end{array} \right. \quad (3.46)$$

where

$$H_5 = -\frac{4}{3}\frac{\mu'(\Theta)}{V}U_{1y}\zeta_{yy} - 2\left(\frac{Z}{V}\right)_y\phi_y - \left(\frac{Z}{V}\right)_{yy}\phi + \left(\frac{4}{3V}\right)_y\zeta_y \\ + \left(\frac{2}{3V}\right)_{yy}\zeta - \left(\frac{8\mu'(\Theta)}{3V}U_{1y}\right)_y\zeta_y - \left(\frac{4\mu'(\Theta)}{3V}U_{1y}\right)_{yy}\zeta, \quad (3.47)$$

$$H_{i+4} = \frac{\mu(\Theta)U_{iy}}{V^2}\phi_{yy} - \frac{\mu'(\Theta)}{V}U_{iy}\zeta_{yy} + 2\left(\frac{\mu(\Theta)U_{iy}}{V^2}\right)_y\phi_y + \left(\frac{\mu(\Theta)U_{iy}}{V^2}\right)_{yy}\phi \\ - 2\left(\frac{\mu'(\Theta)}{V}U_{iy}\right)_y\zeta_y - \left(\frac{\mu'(\Theta)}{V}U_{iy}\right)_{yy}\zeta, \quad i = 2, 3, \quad (3.48)$$

and

$$H_8 = Z_y\psi_{1y} - \left[\frac{U_{1y}}{V}(Z\phi - \frac{2}{3}\zeta + \frac{4\mu'(\Theta)U_{1y}}{3}\zeta + \frac{4\mu(\Theta)}{3}\psi_{1y})\right]_y + \left(\frac{\kappa(\Theta)}{V^2}\Theta_y\phi\right)_{yy} \\ - \sum_{i=2}^3 \left[\frac{U_{iy}}{V}\left(-\frac{\mu(\Theta)U_{iy}}{V}\phi + \mu'(\Theta)U_{iy}\zeta + 2\mu(\Theta)\psi_{iy}\right)\right]_y - \left(\frac{\kappa'(\Theta)}{V}\Theta_y\zeta\right)_{yy}. \quad (3.49)$$

By using the above two systems and the equation for the non-fluid component, we can reach the following proposition for the higher order energy estimates.

Proposition 3.2 *Under the assumptions of Theorem 3.1, there exist positive constants C and $C_{h,T}$ independent of ε such that*

$$\sup_{\frac{h}{\varepsilon} \leq \tau_1 \leq \tau} \left[\|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)(\tau_1, \cdot)\|^2 + \sum_{|\alpha'|=1} \int \int \frac{|\partial^{\alpha'} \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star}(\tau_1, y, \xi) d\xi dy + \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star}(\tau_1, y, \xi) d\xi dy \right] \\ + \int_{\frac{h}{\varepsilon}}^{\tau} \sum_{1 \leq |\alpha| \leq 2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 d\tau + \sum_{1 \leq |\alpha| \leq 2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ \leq C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.$$

Again, the proof of Proposition 3.2 will be given in the Appendices.

By combining the above lower and higher order estimates given in Propositions 3.1 and 3.2 and choosing the wave strength δ , the bound on the a priori estimate χ and the Knudsen number ε to be

suitably small, we obtain

$$\begin{aligned} & \mathcal{N}(\tau) + \int_{\frac{h}{\varepsilon}}^{\tau} \left[\sum_{0 \leq |\alpha| \leq 2} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 + \|\sqrt{|U_{1y}^{S_3}|}(\Psi, W)\|^2 \right] d\tau + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_*} d\xi dy d\tau \\ & + \sum_{1 \leq |\alpha| \leq 2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\partial^\alpha \tilde{\mathbf{G}}|^2}{\mathbf{M}_*}(\tau, y, \xi) d\xi dy d\tau \leq C_{h,T} \varepsilon^{\frac{2}{3}}. \end{aligned}$$

Therefore, we close the a priori assumption (3.21) and then complete the proof of Theorem 3.1.

4 Appendices

As mentioned before, since the proofs of Propositions 3.1 and 3.2 are technical and long, we put them in the following three subsections.

4.1 Proof of Proposition 3.1

Proof of Proposition 3.1. Firstly, from the fact that

$$v_- < V^{R_1} < v_*, \quad V^{CD} \in (\min\{v_*, v^*\}, \max\{v_*, v^*\}) \quad \text{and} \quad v^* < V^{S_3} < v_+,$$

we have

$$v_- - \delta^{CD} - \|(d_1, b_1)\|_{L^\infty} \leq V \leq v_+ + \delta^{CD} + \|(d_1, b_1)\|_{L^\infty}.$$

Thus

$$\frac{v_-}{2} \leq V \leq 2v_+, \quad \text{if } \varepsilon \ll 1 \quad \text{and} \quad \delta^{CD} \ll 1. \quad (4.1.1)$$

From (2.57)₁ and (2.57)₂, we can obtain

$$Z^{S_3} := P^{S_3} - \frac{4}{3} \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}}{V^{S_3}} = (a_1 - s_3^2 V^{S_3}) - \int \xi_1^2 \Pi_1^{S_3} d\xi, \quad (4.1.2)$$

where $a_1 = p_+ + s_3^2 v_+ = p^* + s_3^2 v^*$.

Since

$$0 < p_+ \leq (a_1 - s_3^2 V^{S_3}) \leq p^*, \quad \text{and} \quad \left| \int \xi_1^2 \Pi_1^{S_3} d\xi \right| \leq C \delta^{S_3}, \quad (4.1.3)$$

we have

$$p_+ - C \delta^{S_3} \leq Z^{S_3} \leq p^* + C \delta^{S_3},$$

and then

$$\frac{p_+}{2} \leq Z^{S_3} \leq 2p^*, \quad \text{if } \delta^{S_3} \ll 1. \quad (4.1.4)$$

Now we estimate Z defined in (3.7) as follows.

$$\begin{aligned} Z &= Z^{S_3} + (P^{R_1} - p_*) - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} + (P^{CD} - p^*) - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} \\ &+ (P - P^{R_1} - P^{CD} - P^{S_3} + p_* + p^*) \\ &- \frac{4}{3} \left[\frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}}{V^{S_3}} \right] \\ &:= Z^{S_3} + (P^{R_1} - p_*) - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} + (P^{CD} - p^*) - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} + Q_5, \end{aligned} \quad (4.1.5)$$

where Q_5 is related to the hyperbolic waves and the wave interaction terms given by

$$\begin{aligned} Q_5 &= O(1) \sum_{i=1}^3 |(d_i, b_i)| + O(1)|(d_{2y}, b_{2y})| + O(1)e^{-\frac{c_h}{\sigma}} \\ &= O(1) \left(\frac{\varepsilon}{\sigma} + \varepsilon^{\frac{3}{4}} + \frac{\varepsilon^2}{\sigma^2} + e^{-\frac{c_h}{\sigma}} \right), \end{aligned} \quad (4.1.6)$$

with $\sigma = \varepsilon^{\frac{1}{5}}$.

From the properties of the approximate rarefaction wave, we have

$$0 \leq P^{R_1} - p_* \leq p_- - p_*, \quad \text{and} \quad \frac{4}{3} \frac{\mu(\Theta^{R_1}) U_{1y}^{R_1}}{V^{R_1}} \leq C_{h,T} \varepsilon. \quad (4.1.7)$$

And the properties of the viscous contact wave imply that

$$P^{CD} - p^* \leq C_{h,T} \varepsilon, \quad \text{and} \quad \frac{4}{3} \frac{\mu(\Theta^{CD}) U_{1y}^{CD}}{V^{CD}} \leq C_{h,T} \varepsilon^{\frac{3}{2}}. \quad (4.1.8)$$

Substituting (4.1.2), (4.1.6), (4.1.7) and (4.1.8) into (4.1.5), we know that there exist positive constants c and C such that

$$0 < c \leq Z \leq C, \quad (4.1.9)$$

provided $\varepsilon \ll 1$ and the wave strength $\delta \ll 1$.

Step 1. Estimation on $\|(\Phi, \Psi, W)(\tau, \cdot)\|^2$.

By multiplying (3.6)₁ by Φ , (3.6)₂ by $\frac{V}{Z} \Psi_1$, (3.6)₃ by $V^{S_3} \Psi_i$, (3.6)₄ by $\frac{2}{3Z^2} W$ respectively and adding all of them together, we have

$$\begin{aligned} I_1(\Phi, \Psi, W)_\tau + I_2(\Psi, W) + I_3(\Psi_y, W_y) &= I_4(\Psi, W, \Phi_y, \Psi_y, W_y) \\ &+ \frac{V \Psi_1}{Z} (N_1 - \bar{Q}_1 - Q_1) + V^{S_3} \sum_{i=2}^3 (N_i - \bar{Q}_i - Q_i) \Psi_i \\ &+ \frac{2W}{3Z^2} (N_4 - \bar{Q}_4 + \sum_{i=1}^3 U_i \bar{Q}_i - Q_4 + \sum_{i=1}^3 U_i Q_i) + K_1 + (\dots)_y, \end{aligned} \quad (4.1.10)$$

where

$$\begin{aligned} I_1(\Phi, \Psi, W) &= \frac{\Phi^2}{2} + \frac{V}{2Z} \Psi_1^2 + \frac{V^{S_3}}{2} \sum_{i=2}^3 \Psi_i^2 + \frac{W^2}{3Z^2}, \\ I_2(\Psi, W) &= \left[\frac{2}{3Z} U_{1y} - \frac{1}{2} \left(\frac{V}{Z} \right)_\tau \right] \Psi_1^2 - V_\tau^{S_3} \sum_{i=2}^3 \frac{\Psi_i^2}{2} - \left(\frac{1}{3Z^2} \right)_\tau W^2, \\ I_3(\Psi_y, W_y) &= \frac{4\mu(\Theta)}{3Z} \Psi_{1y}^2 + \frac{V^{S_3} \mu(\Theta)}{V} \sum_{i=2}^3 \Psi_{iy}^2 + \frac{2\kappa(\Theta)}{3Z^2 V} W_y^2, \\ I_4(\Psi, W, \Phi_y, \Psi_y, W_y) &= -\frac{2}{3Z^2} W \Psi_1 (U_{1\tau} + Z_y) - \frac{2W}{3Z^2} \sum_{i=2}^3 U_{i\tau} \Psi_i \\ &- \left[\frac{2\Psi_1}{3Z} + \frac{\mu(\Theta) V^{S_3}}{V^2} \Phi_y - \frac{\mu'(\Theta) V^{S_3}}{V} (W_y + U_y \cdot \Psi) \right] \sum_{i=2}^3 U_{iy} \Psi_i \\ &- \frac{4}{3} \left(\frac{\mu(\Theta)}{Z} \right)_y \Psi_1 \Psi_{1y} - \sum_{i=2}^3 \left(\frac{V^{S_3} \mu(\Theta)}{V} \right)_y \Psi_i \Psi_{iy} - \frac{2}{3} \left(\frac{\kappa(\Theta)}{V Z^2} \right)_y W W_y \\ &+ \frac{4}{3} \frac{\mu'(\Theta)}{Z} U_{1y} \Psi_1 (W_y + U_y \cdot \Psi) - \frac{2\kappa(\Theta)}{3V^2 Z^2} \Theta_y \Phi_y W + \frac{2\kappa(\Theta)}{3V Z^2} W (U_y \cdot \Psi)_y \\ &+ \frac{2\kappa'(\Theta)}{3V Z^2} \Theta_y W (W_y + U_y \cdot \Psi) - \sum_{i=2}^3 \frac{2\mu(\Theta) U_{iy}}{3Z^2 V} W \Psi_{iy} := \sum_{i=1}^{11} I_4^i, \end{aligned} \quad (4.1.11)$$

and K_1 denotes the non-fluid parts given by

$$\begin{aligned} K_1 &= -\frac{V}{Z} \Psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi - \sum_{i=2}^3 V^{S_3} \Psi_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \\ &+ \frac{2W}{3Z^2} \left[-\int \frac{1}{2} \xi_1 |\xi|^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi + \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right]. \end{aligned} \quad (4.1.13)$$

From (4.1.9), we have

$$c(\Phi, \Psi, W)^2 \leq I_1 \leq C(\Phi, \Psi, W)^2, \quad (4.1.14)$$

for some positive constants c and C .

Now we estimate I_i , ($i = 2, 3, 4$) term by term. Note that

$$Z_\tau^{S_3} = -s_3^2 U_{1y}^{S_3} + s_3 \int \xi_1^2 \Pi_{1y}^{S_3} d\xi, \quad (4.1.15)$$

and

$$Z_\tau = -s_3^2 U_{1y}^{S_3} + s_3 \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + Z_\tau^{CD} + Z_\tau^{R_1} + Q_6, \quad (4.1.16)$$

where $Z^{R_1} = P^{R_1} - \frac{4}{3} \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}}$, $Z^{CD} = P^{CD} - \frac{4}{3} \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}}$ and

$$\begin{aligned} Q_6 &= (P - P^{R_1} - P^{CD} - P^{S_3})_\tau \\ &\quad - \frac{4}{3} \left[\frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1y}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}}{V^{S_3}} \right]_\tau \\ &= \frac{2}{3} \left(\frac{\Theta_\tau}{V} - \frac{\Theta_\tau^{R_1}}{V^{R_1}} - \frac{\Theta_\tau^{CD}}{V^{CD}} - \frac{\Theta_\tau^{S_3}}{V^{S_3}} \right) - \frac{2}{3} \left(\frac{\Theta V_\tau}{V^2} - \frac{\Theta^{R_1} V_\tau^{R_1}}{(V^{R_1})^2} - \frac{\Theta^{CD} V_\tau^{CD}}{(V^{CD})^2} - \frac{\Theta^{S_3} V_\tau^{S_3}}{(V^{S_3})^2} \right) \\ &\quad - \frac{4}{3} \left(\frac{\mu(\Theta)U_{1y\tau}}{V} - \frac{\mu(\Theta^{R_1})U_{1y\tau}^{R_1}}{V^{R_1}} - \frac{\mu(\Theta^{CD})U_{1y\tau}^{CD}}{V^{CD}} - \frac{\mu(\Theta^{S_3})U_{1y\tau}^{S_3}}{V^{S_3}} \right) \\ &\quad + \frac{4}{3} \left[\frac{\mu(\Theta)U_{1y}V_\tau}{V^2} - \frac{\mu(\Theta^{R_1})U_{1y}^{R_1}V_\tau^{R_1}}{(V^{R_1})^2} - \frac{\mu(\Theta^{CD})U_{1y}^{CD}V_\tau^{CD}}{(V^{CD})^2} - \frac{\mu(\Theta^{S_3})U_{1y}^{S_3}V_\tau^{S_3}}{(V^{S_3})^2} \right] \\ &\quad - \frac{4}{3} \left[\frac{\mu'(\Theta)U_{1y}\Theta_\tau}{V} - \frac{\mu'(\Theta^{R_1})U_{1y}^{R_1}\Theta_\tau^{R_1}}{V^{R_1}} - \frac{\mu'(\Theta^{CD})U_{1y}^{CD}\Theta_\tau^{CD}}{V^{CD}} - \frac{\mu'(\Theta^{S_3})U_{1y}^{S_3}\Theta_\tau^{S_3}}{V^{S_3}} \right] := \sum_{i=1}^5 Q_{6i}. \end{aligned} \quad (4.1.17)$$

Then we get for the first term of I_2 that

$$\begin{aligned} \frac{2}{3Z} U_{1y} - \frac{1}{2} \left(\frac{V}{Z} \right)_\tau &= \frac{1}{6Z} U_{1y} + \frac{V}{2Z^2} Z_\tau \\ &= \frac{U_{1y}^{S_3}}{6Z^2} (Z^{S_3} - 3s_3^2 V^{S_3}) + \frac{U_{1y}^{S_3}}{6Z^2} [(Z - Z^{S_3}) - 3s_3^2 (V - V^{S_3})] + \frac{s_3 V}{2Z^2} \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \\ &\quad + \frac{1}{6Z} (U_{1y}^{R_1} + U_{1y}^{CD} + d_{2y} + b_{21y}) + \frac{V}{2Z^2} (Z_\tau^{CD} + Z_\tau^{R_1} + \sum_{i=1}^5 Q_{6i}) \\ &= \frac{|U_{1y}^{S_3}|}{6Z^2} [-(a_1 - s_3^2 V^{S_3}) + 3s_3^2 V^{S_3} + \int \xi_1^2 \Pi_{1y}^{S_3} d\xi] + \frac{s_3 V}{2Z^2} \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + Q_7, \end{aligned} \quad (4.1.18)$$

where

$$\begin{aligned} Q_7 &= \frac{U_{1y}^{S_3}}{6Z^2} [(Z - Z^{S_3}) - 3s_3^2 (V - V^{S_3})] + \frac{1}{6Z} (U_{1y}^{R_1} + U_{1y}^{CD} + d_{2y} + b_{21y}) \\ &\quad + \frac{V}{2Z^2} (Z_\tau^{CD} + Z_\tau^{R_1} + \sum_{i=1}^5 Q_{6i}). \end{aligned} \quad (4.1.19)$$

Firstly, straightforward calculation gives

$$\begin{aligned} -(a_1 - s_3^2 V^{S_3}) + 3s_3^2 V^{S_3} &\geq 4p_* - C\delta^{S_3} \geq 3p_*, \quad \text{if } \delta^{S_3} \ll 1, \\ \left| \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \right| &\leq C\delta^{S_3}, \quad \text{and } \left| \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \right| \leq C\delta^{S_3} |U_{1y}^{S_3}|. \end{aligned}$$

And (4.1.18) implies that

$$\left[\frac{2}{3Z} U_{1y} - \frac{1}{2} \left(\frac{V}{Z} \right)_\tau \right] \Psi_1^2 \geq C^{-1} |U_{1y}^{S_3}| \Psi_1^2 + Q_7 \Psi_1^2, \quad (4.1.20)$$

provided $\delta \ll 1$.

By the definition of the approximate wave pattern defined in (2.100) and (2.101), we have

$$\begin{aligned}
Q_{61} &= O(1)\varepsilon \left[|(\Theta_t^{R_1}, U_{1t}^{R_1})|(V^{CD} - v_*, U_1^{CD} - u_{1*}, V^{S_3} - v^*, U_1^{S_3} - u_1^*)| + |(\Theta_t^{R_1} d_1, U_{1t}^{R_1} d_2)| \right. \\
&\quad + |(\Theta_t^{CD}, U_{1t}^{CD})|(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, V^{S_3} - v^*, U_1^{S_3} - u_1^*, d_1, d_2)| \\
&\quad + |(\Theta_t^{S_3}, U_{1t}^{S_3})|(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, V^{CD} - v^*, U_1^{CD} - u_1^*, d_1, d_2)| \\
&\quad \left. + |(d_{2t}, d_{3t})| + |b_1| |(\Theta_t^{R_1}, \Theta_t^{CD}, \Theta_t^{S_3})| \right] + \frac{2}{3V}(b_3 - \bar{U} \cdot b_2 - \frac{|b_2|^2}{2})_\tau \\
&\leq C_{h,T} \varepsilon + C|(b_2, b_{2\tau}, b_{3\tau})|.
\end{aligned} \tag{4.1.21}$$

On the other hand, direct calculation gives

$$\begin{aligned}
Q_{63} &= O(1)\varepsilon^2 \left[|U_{1xt}^{R_1}| |(V^{CD} - v_*, U_1^{CD} - u_{1*}, \Theta^{CD} - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*)| \right. \\
&\quad + |U_{1xt}^{CD}| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, V^{S_3} - v^*, U_1^{S_3} - u_1^*, \Theta^{S_3} - \theta^*, d_1, d_2, d_3)| \\
&\quad + |U_{1xt}^{S_3}| |(V^{R_1} - v_*, U_1^{R_1} - u_{1*}, \Theta^{R_1} - \theta_*, V^{CD} - v^*, U_1^{CD} - u_1^*, \Theta^{CD} - \theta^*, d_1, d_2, d_3)| \\
&\quad \left. + |U_{1xt}^{R_1}| |(d_1, d_2, d_3)| + |(b_1, b_2, b_3)| |(U_{1xt}^{R_1}, U_{1xt}^{CD}, U_{1xt}^{S_3})| + |d_{2xt}| \right] - \frac{4\mu(\Theta)}{3V} b_{21y\tau} \\
&\leq C_{h,T} \varepsilon + C|b_{21y\tau}|.
\end{aligned} \tag{4.1.22}$$

Similar estimates hold for Q_{62} , Q_{64} and Q_{65} .

Moreover, we have

$$|Z_\tau^{R_1}| = |(P^{R_1} - \frac{4}{3} \frac{\mu(\Theta^{R_1}) U_{1y}^{R_1}}{V^{R_1}})_\tau| \leq C_{h,T} \varepsilon, \tag{4.1.23}$$

and

$$|Z_\tau^{CD}| = |(P^{CD} - \frac{4}{3} \frac{\mu(\Theta^{CD}) U_{1y}^{CD}}{V^{CD}})_\tau| \leq C_{h,T} \varepsilon^2. \tag{4.1.24}$$

Now from (4.1.5), (4.1.19) and (4.1.21)-(4.1.24), we have

$$\begin{aligned}
\int_{\frac{h}{\varepsilon}}^\tau \int |Q_7| \Psi_1^2 dy d\tau &\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int |(b, b_y, b_\tau, b_{y\tau})| |\Psi_1|^2 dy d\tau \\
&\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \|(b, b_y, b_\tau, b_{y\tau})\| \|\Psi_1\| \|\Psi_1\|_{L^\infty} d\tau \\
&\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau + C_{h,T} \varepsilon^{\frac{3}{4}} \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_1\|^{\frac{3}{2}} \|\Psi_{1y}\|^{\frac{1}{2}} d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_{1y}\|^2 d\tau + C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau.
\end{aligned} \tag{4.1.25}$$

Substituting (4.1.25) into (4.1.20) gives that

$$\begin{aligned}
&\int_{\frac{h}{\varepsilon}}^\tau \int \left[\frac{2}{3Z} U_{1y} - \frac{1}{2} \left(\frac{V}{Z} \right)_\tau \right] \Psi_1^2 dy d\tau \\
&\geq C^{-1} \int_{\frac{h}{\varepsilon}}^\tau \int |U_{1y}^{S_3}| \Psi_1^2 dy d\tau - \beta \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_{1y}\|^2 d\tau - C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int \Psi_1^2 dy d\tau,
\end{aligned} \tag{4.1.26}$$

provided $\delta^{S_3} \ll 1$.

On the other hand, we have

$$-V_\tau^{S_3} \sum_{i=2}^3 \frac{\Psi_i^2}{2} = \sum_{i=2}^3 |U_{1y}^{S_3}| \frac{\Psi_i^2}{2}.$$

Similar to (4.1.26), we have

$$\begin{aligned}
&\int_{\frac{h}{\varepsilon}}^\tau \int \left[- \left(\frac{1}{3Z^2} \right)_\tau W^2 \right] dy d\tau = \int_{\frac{h}{\varepsilon}}^\tau \int \frac{2}{3Z^3} Z_\tau W^2 dy d\tau \\
&= \int_{\frac{h}{\varepsilon}}^\tau \int \frac{2}{3Z^3} \left(-s_3^2 U_{1y}^{S_3} + s_3 \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + Z_\tau^{CD} + Z_\tau^{R_1} + Q_6 \right) W^2 dy d\tau \\
&\geq C^{-1} \int_{\frac{h}{\varepsilon}}^\tau \int |U_{1y}^{S_3}| W^2 dy d\tau - \beta \int_{\frac{h}{\varepsilon}}^\tau \|W_y\|^2 d\tau - C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \int W^2 dy d\tau.
\end{aligned}$$

Thus we complete the estimation for I_2 as

$$\begin{aligned} \int_{\frac{h}{\varepsilon}}^{\tau} \int I_2(\Psi, W) dy d\tau &\geq C^{-1} \int_{\frac{h}{\varepsilon}}^{\tau} \int |U_{1y}^{S_3}| |(\Psi, W)|^2 dy d\tau \\ &\quad - \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_{1y}, W_y)\|^2 d\tau - C_{h,T,\beta} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\Psi, W)|^2 dy d\tau. \end{aligned} \quad (4.1.27)$$

Direct computation yields that

$$I_3(\Psi_y, W_y) \geq C^{-1} |(\Psi_y, W_y)|^2. \quad (4.1.28)$$

Now we estimate I_4 in (4.1.12). Note that

$$U_{1\tau} + Z_y = - \int \xi_1^2 \Pi_{11y}^{CD} d\xi - \int \xi_1^2 \Pi_{1y}^{S_3} d\xi + \bar{Q}_{1y} + Q_{1y}.$$

Thus

$$\begin{aligned} I_4^1 &= \frac{2}{3Z^2} W \Psi_1 \left[\int \xi_1^2 \Pi_{11y}^{CD} d\xi + \int \xi_1^2 \Pi_{1y}^{S_3} d\xi - \bar{Q}_{1y} - Q_{1y} \right] \\ &= \frac{2}{3Z^2} W \Psi_1 \left(\int \xi_1^2 \Pi_{11y}^{CD} d\xi + \int \xi_1^2 \Pi_{1y}^{S_3} d\xi \right) + \left(\frac{2W\Psi_1}{3Z^2} \right)_y (\bar{Q}_1 + Q_1) + (\cdots)_y \\ &\leq \beta |(\Psi_{1y}, W_y)|^2 + C_\beta \delta^{S_3} |U_{1y}^{S_3}| |(\Psi_1, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi_1, W)|^2 + (\cdots)_y, \end{aligned}$$

where from now on, β is a small positive constant to be determined and C_β is some positive constant depending on β but independent of h, T and ε , while $C_{h,T,\beta}$ depends on h, T, β but independent of ε .

Similarly, we have

$$\begin{aligned} I_4^2 &= \sum_{i=2}^3 \frac{2W\Psi_i}{3Z^2} \left[- \left(\frac{\mu(\Theta)U_{iy}}{V} \right)_y + \int \xi_1 \xi_i \Pi_{11y}^{CD} d\xi + \int \xi_1 \xi_i \Pi_{1y}^{S_3} d\xi - \bar{Q}_{iy} - Q_{iy} \right] \\ &= \sum_{i=2}^3 \frac{2W\Psi_i}{3Z^2} \left(\int \xi_1 \xi_i \Pi_{11y}^{CD} d\xi + \int \xi_1 \xi_i \Pi_{1y}^{S_3} d\xi \right) + \sum_{i=2}^3 \left(\frac{2W\Psi_i}{3Z^2} \right)_y \left(\frac{\mu(\Theta)U_{iy}}{V} + \bar{Q}_i + Q_i \right) + (\cdots)_y \\ &\leq \beta \sum_{i=2}^3 |(\Psi_{iy}, W_y)|^2 + C_\beta \delta^{S_3} \sum_{i=2}^3 |U_{1y}^{S_3}| |(\Psi_i, W)|^2 + C_{h,T,\beta} \varepsilon \sum_{i=2}^3 |(\Psi_i, W)|^2 + (\cdots)_y. \end{aligned}$$

Next, we can show that

$$\begin{aligned} I_4^3 &= \sum_{i=2}^3 U_i \left(\frac{2\Psi_1\Psi_i}{3Z} \right)_y - \left[\frac{\mu(\Theta)V^{S_3}}{V^2} \Phi_y - \frac{\mu'(\Theta)V^{S_3}}{V} (W_y + U_y \cdot \Psi) \right] \sum_{i=2}^3 U_{iy} \Psi_i + (\cdots)_y \\ &\leq \beta |(\Phi_y, \Psi_y, W_y)|^2 + C_\beta \delta^{S_3} |U_{1y}^{S_3}| |(\Psi, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi, W)|^2 + (\cdots)_y, \end{aligned} \quad (4.1.29)$$

where we have used the fact that $U_i = \bar{U}_i + b_{i+1} = U_i^{CD} + b_{i+1} = O(1)\varepsilon^{\frac{1}{2}}$, ($i = 2, 3$) and that $U_{iy} = U_{iy}^{CD} + (b_{i+1})_y = O(1)\varepsilon^{\frac{3}{4}}$, $i = 2, 3$.

By Hölder inequality, we have

$$\sum_{i=4}^{11} I_4^i \leq \beta |(\Phi_y, \Psi_y, W_y)|^2 + C_\beta \delta^{S_3} |U_{1y}^{S_3}| |(\Psi, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi, W)|^2. \quad (4.1.30)$$

Hence, we have

$$\begin{aligned} I_4(\Psi, W, \Phi_y, \Psi_y, W_y) &\leq \beta |(\Phi_y, \Psi_y, W_y)|^2 + (C_\beta \delta^{S_3} + C_{h,T} \varepsilon) |U_{1y}^{S_3}| |(\Psi, W)|^2 + C_{h,T,\beta} \varepsilon |(\Psi, W)|^2 + (\cdots)_y. \end{aligned} \quad (4.1.31)$$

Now we estimate the nonlinear terms $\frac{V}{Z} N_1 \Psi_1$ by

$$\begin{aligned} \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} N_1 \Psi_1 dy d\tau &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left[|\Phi_y|^2 + |\Psi_y|^2 + |\zeta|^2 + |\Psi_{1yy}|^2 \right] dy d\tau \\ &\leq C \chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta, \Psi_{1yy})\|^2 d\tau. \end{aligned} \quad (4.1.32)$$

Similarly, we have

$$\int_{\frac{h}{\varepsilon}}^{\tau} \int \sum_{i=2,3} V^{S_3} N_i \Psi_i + \frac{2W}{3Z^2} N_4 dy d\tau \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta, \Psi_{yy}, \zeta_y)\|^2 d\tau.$$

We now turn to estimate the terms $-\frac{V}{Z}\Psi_1\bar{Q}_1$, $-\sum_{i=2}^3 V^{S_3}\bar{Q}_i\Psi_i$ and $-\frac{2W}{3Z^2}(\bar{Q}_4 - \sum_{i=1}^3 U_i\bar{Q}_i)$. From (2.62) and (2.64), we have

$$\begin{aligned} & |-\frac{V}{Z}\Psi_1\bar{Q}_1|, |-\sum_{i=2}^3 V^{S_3}\bar{Q}_i\Psi_i|, |-\frac{2W}{3Z^2}(\bar{Q}_4 - \sum_{i=1}^3 U_i\bar{Q}_i)| \\ & \leq C|(\Psi, W)|(|\bar{Q}_{11}, \bar{Q}_2, \bar{Q}_3, \bar{Q}_{41}|) + C|(\Psi, W)|(|\bar{Q}_{12}, \bar{Q}_{42}|). \end{aligned} \quad (4.1.33)$$

And from (2.65), we have

$$\begin{aligned} & \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)|(|\bar{Q}_{11}, \bar{Q}_2, \bar{Q}_3, \bar{Q}_{41}|) dy d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)| e^{-\frac{c|x|}{\sigma}} e^{-\frac{C_h}{\sigma}} dy d\tau \\ & \leq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C_{h,T} e^{-\frac{C_h}{2\sigma}}. \end{aligned} \quad (4.1.34)$$

On the other hand, from (2.62)-(2.64), Lemma 2.3 and noting that $\sigma = \varepsilon^{\frac{1}{5}}$, we have

$$\begin{aligned} & \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)|(|\bar{Q}_{12}, \bar{Q}_{42}|) dy d\tau \\ & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi, W)| \left[\sum_{i=1}^3 |d_i|^2 + \varepsilon|(d_{2x}, d_{3x})| + \varepsilon|(U_{1x}^{R_1}, \Theta_x^{R_1})|(d_1, d_2, d_3) \right] dy d\tau \\ & \leq C_{h,T} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|_{L^2(dy)} \sum_{i=1}^3 \left[\|d_i\|_{L^\infty} \|d_i\|_{L^2(dy)} + \varepsilon \|d_{ix}\|_{L^2(dy)} + \varepsilon \|d_i\|_{L^2(dy)} \right] d\tau \\ & \leq C_{h,T} \left(\frac{\varepsilon}{\sigma}\right)^{\frac{3}{2}} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|_{L^2(dy)} d\tau \leq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{2}{5}}. \end{aligned} \quad (4.1.35)$$

Now we estimate the terms $-\frac{V}{Z}\Psi_1Q_1$, $-\sum_{i=2}^3 V^{S_3}Q_i\Psi_i$ and $-\frac{2W}{3Z^2}(Q_4 - \sum_{i=1}^3 U_iQ_i)$. For this, from (2.104) and (2.105), we have

$$\begin{aligned} & \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[-\frac{V}{Z}\Psi_1Q_{11} - \frac{2W}{3Z^2}(Q_{41} - U_1Q_{11}) \right] dy d\tau \\ & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi_1, W)|(|Q_{11}, Q_{41}|) dy d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi_1, W)| \sum_{i=1}^3 |b_i|^2 dy d\tau \\ & \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)} \sum_{i=1}^3 \|b_i\|_{L^\infty} \|b_i\|_{L^2(dy)} d\tau \\ & \leq C_{h,T} \varepsilon^{\frac{3}{2}} \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\| d\tau \leq \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|^2 d\tau + C_{h,T} \varepsilon. \end{aligned} \quad (4.1.36)$$

On the other hand, from Lemma 2.5 with the properties of the hyperbolic wave II, we can show that

$$\begin{aligned} & \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[-\frac{V}{Z}\Psi_1Q_{12} - \sum_{i=2}^3 V^{S_3}Q_i\Psi_i - \frac{2W}{3Z^2}(Q_{42} - U_1Q_{12} - \sum_{i=2}^3 U_iQ_i) \right] dy d\tau \\ & = \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[\left(\frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left(\frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1y}}{\bar{V}} \right) \right. \\ & \quad + \sum_{i=2}^3 \left(V^{S_3}\Psi_i - \frac{2U_iW}{3Z^2} \right) \left(\frac{\mu(\Theta)U_{iy}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{iy}}{\bar{V}} \right) \\ & \quad + \frac{2W}{3Z^2} \left(\frac{\mu(\Theta)\Theta_y}{V} - \frac{\mu(\bar{\Theta})\bar{\Theta}_y}{\bar{V}} \right) + \frac{8W}{9Z^2} \left(\frac{\mu(\Theta)U_1U_{1y}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_1\bar{U}_{1y}}{\bar{V}} \right) \\ & \quad \left. + \frac{2W}{3Z^2} \sum_{i=2}^3 \left(\frac{\mu(\Theta)U_iU_{iy}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_i\bar{U}_{iy}}{\bar{V}} \right) \right] dy d\tau := \sum_{i=1}^5 I_{5i}. \end{aligned} \quad (4.1.37)$$

For brevity, we only compute I_{51} in the following because I_{5i} ($i = 2, 3, 4, 5$) can be estimated similarly.

$$\begin{aligned}
I_{51} &= \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left(\frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left(\frac{\mu(\Theta)U_{1y}}{V} - \frac{\mu(\bar{\Theta})\bar{U}_{1y}}{\bar{V}} \right) dy d\tau \\
&= \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left(\frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left[\frac{\mu(\Theta)}{V} b_{2y} + \left(\frac{\mu(\Theta)}{V} - \frac{\mu(\bar{\Theta})}{\bar{V}} \right) \bar{U}_{1y} \right] dy d\tau \\
&= - \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left[\left(\frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \frac{\mu(\Theta)}{V} \right]_y b_2 dy d\tau \\
&\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} \left(\frac{4V\Psi_1}{3Z} - \frac{8U_1W}{9Z^2} \right) \left(\frac{\mu(\Theta)}{V} - \frac{\mu(\bar{\Theta})}{\bar{V}} \right) \bar{U}_{1y} dy d\tau := I_{51}^1 + I_{51}^2.
\end{aligned} \tag{4.1.38}$$

We can further have

$$\begin{aligned}
I_{51}^1 &= O(1) \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |b_2| \left[|(\Psi_{1y}, W_y)| + |(V_y, U_{1y}, \Theta_y, Z_y)| |(\Psi_1, W)| \right] dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \left[\|(\Psi_{1y}, W_y)\|_{L^2(dy)}^2 + \|\sqrt{|U_{1y}^{S_3}|}(\Psi_1, W)\|_{L^2(dy)}^2 \right] d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)}^2 d\tau + C_{\beta} \int_{\frac{h}{\varepsilon}}^{\tau} \|(b_1, b_2, b_3)\|_{L^2(dy)}^2 d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \left[\|(\Psi_{1y}, W_y)\|_{L^2(dy)}^2 + \|\sqrt{|U_{1y}^{S_3}|}(\Psi_1, W)\|_{L^2(dy)}^2 \right] d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)}^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.39}$$

Similarly, we have

$$\begin{aligned}
I_{51}^2 &= O(1) \int_{\frac{h}{\varepsilon}}^{\tau} \int_{\mathbf{R}} |(\Psi_1, W)| |\bar{U}_{1y}| |(b_1, b_2, b_3)| dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|\sqrt{|U_{1y}^{S_3}|}(\Psi_1, W)\|_{L^2(dy)}^2 d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_1, W)\|_{L^2(dy)}^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.40}$$

By integrating (4.1.10) with respect to y and τ , then combining all the above estimates, and choosing $\beta, \delta^{S_3}, \varepsilon$ and χ small enough, we have

$$\begin{aligned}
&\|(\Phi, \Psi, W)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \left[\|\sqrt{|U_{1y}^{S_3}|}(\Psi, W)\|^2 + \|(\Psi_y, W_y)\|^2 \right] d\tau \\
&\leq C(\beta + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|\Phi_y\|^2 d\tau + C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\zeta, \Psi_{yy}, \zeta_y)\|^2 d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int K_1 dy d\tau + C_{h,T,\beta} \varepsilon^{\frac{2}{5}}.
\end{aligned} \tag{4.1.41}$$

Now we turn to estimate the microscopic term $\int_{\frac{h}{\varepsilon}}^{\tau} \int K_1 dy d\tau$ in (4.1.41). Here, we will only estimate

$K_{11} =: - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi dy d\tau$ because the other terms in $\int_{\frac{h}{\varepsilon}}^{\tau} \int K_1 dy d\tau$ can be estimated similarly. Let \mathbf{M}_{\star} be a global Maxwellian with the state $(v_{\star}, u_{\star}, \theta_{\star})$ satisfying $\frac{1}{2}\theta < \theta_{\star} < \theta$ and $|v - v_{\star}| + |u - u_{\star}| + |\theta - \theta_{\star}| \leq \eta_0$ so that Lemma 3.3 holds, and Lemmas 2.3 and 2.4 hold with \mathbf{M}_0 being replaced by \mathbf{M}_{\star} . Note that the above choice of the global Maxwellian \mathbf{M}_{\star} can be obtained if the total wave strength δ is small enough. By the definition of Π_1, Π_{11}^{CD} and $\Pi_1^{S_3}$ given in (1.20), (2.21) and (2.58) respectively, we have

$$\begin{aligned}
\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3} &= \mathbf{L}_{\mathbf{M}}^{-1} \left[\tilde{\mathbf{G}}_{\tau} - \frac{u_1}{v} \tilde{\mathbf{G}}_{1y} + \frac{1}{v} \mathbf{P}_1 (\xi_1 \tilde{\mathbf{G}}_{1y}) - Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1) \right] \\
&\quad - \mathbf{L}_{\mathbf{M}}^{-1} [2Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})] + J_3 + J_4 + J_5.
\end{aligned} \tag{4.1.42}$$

Here

$$\bar{\mathbf{G}}^{CD}(\tau, y, \xi) = \frac{3}{2V^{CD}\Theta^{CD}}\mathbf{L}_{\mathbf{M}^{CD}}^{-1}\{\mathbf{P}_1^{CD}[\xi_1(\frac{|\xi - U^{CD}|^2}{2\Theta^{CD}}\Theta_y^{CD} + \xi \cdot U_y^{CD})\mathbf{M}^{CD}]\}, \quad (4.1.43)$$

$$\begin{aligned} J_3 = & (\mathbf{L}_{\mathbf{M}}^{-1} - \mathbf{L}_{\mathbf{M}^{S_3}}^{-1})[\mathbf{G}_\tau^{S_3} - Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3})] - (\frac{u_1}{v}\mathbf{L}_{\mathbf{M}}^{-1} - \frac{U_1^{S_3}}{V^{S_3}}\mathbf{L}_{\mathbf{M}^{S_3}}^{-1})\mathbf{G}_y^{S_3} \\ & + (\frac{1}{v}\mathbf{L}_{\mathbf{M}}^{-1}\mathbf{P}_1 - \frac{1}{V^{S_3}}\mathbf{L}_{\mathbf{M}^S}^{-1}\mathbf{P}_1^{S_3})(\xi_1\mathbf{G}_y^{S_3}) - \mathbf{L}_{\mathbf{M}}^{-1}[2Q(\mathbf{G}^{S_3}, \mathbf{G}^{R_1} + \mathbf{G}^{CD})], \end{aligned} \quad (4.1.44)$$

$$\begin{aligned} J_4 = & -(\frac{u_1}{v}\mathbf{L}_{\mathbf{M}}^{-1}\mathbf{G}_y^{CD} - \frac{U_1^{CD}}{V^{CD}}\mathbf{L}_{\mathbf{M}^{CD}}^{-1}\bar{\mathbf{G}}_y^{CD}) - [\mathbf{L}_{\mathbf{M}}^{-1}Q(\mathbf{G}^{CD}, \mathbf{G}^{CD}) - \mathbf{L}_{\mathbf{M}^{CD}}^{-1}Q(\bar{\mathbf{G}}^{CD}, \bar{\mathbf{G}}^{CD})] \\ & + [\frac{1}{v}\mathbf{L}_{\mathbf{M}}^{-1}\mathbf{P}_1(\xi_1\mathbf{G}_y^{CD}) - \frac{1}{V^{CD}}\mathbf{L}_{\mathbf{M}^{CD}}^{-1}\mathbf{P}_1^{CD}(\xi_1\bar{\mathbf{G}}_y^{CD})] - \mathbf{L}_{\mathbf{M}}^{-1}[2Q(\mathbf{G}^{R_1}, \mathbf{G}^{CD})], \end{aligned} \quad (4.1.45)$$

and

$$J_5 = \mathbf{L}_{\mathbf{M}}^{-1}[-\frac{u_1}{v}\mathbf{G}_y^{R_1} + \frac{1}{v}\mathbf{L}_{\mathbf{M}}^{-1}\mathbf{P}_1(\xi_1\mathbf{G}_y^{R_1}) - Q(\mathbf{G}^{R_1}, \mathbf{G}^{R_1})]. \quad (4.1.46)$$

By (4.1.42), we have

$$\begin{aligned} K_{11} = & -\int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\tilde{\mathbf{G}}_\tau) d\xi dy d\tau + \int_{\frac{h}{\varepsilon}}^\tau \int \frac{u_1 V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\tilde{\mathbf{G}}_{1y}) d\xi dy d\tau \\ & - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}[\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})] d\xi dy d\tau + \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V \Psi_1}{Z} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}[Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)] d\xi dy d\tau \\ & + \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}[2Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})] d\xi dy d\tau - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V \Psi_1}{Z} \int \xi_1^2 J_3 d\xi dy d\tau \\ & - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V \Psi_1}{Z} \int \xi_1^2 J_4 d\xi dy d\tau - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V \Psi_1}{Z} \int \xi_1^2 J_5 d\xi dy d\tau := \sum_{i=1}^8 K_{11}^i. \end{aligned} \quad (4.1.47)$$

For the integral K_{11}^1 , we have

$$\begin{aligned} K_{11}^1 = & -\int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\tilde{\mathbf{G}}_{1\tau}) d\xi dy d\tau \\ & - \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\mathbf{G}_\tau^{R_1} + \mathbf{G}_\tau^{CD}) d\xi dy d\tau =: K_{11}^{11} + K_{11}^{12}. \end{aligned} \quad (4.1.48)$$

Note that the linearized operator $\mathbf{L}_{\mathbf{M}}^{-1}$ satisfies, for any $h \in \mathfrak{N}^\perp$,

$$(\mathbf{L}_{\mathbf{M}}^{-1}g)_\varsigma = \mathbf{L}_{\mathbf{M}}^{-1}(g_\varsigma) - 2\mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}g, M_\varsigma)\}, \quad \text{for } \varsigma = \tau, y. \quad (4.1.49)$$

Then we have

$$\begin{aligned} K_{11}^{11} = & -\int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z}\Psi_1 \int \xi_1^2 (\mathbf{L}_{\mathbf{M}}^{-1}\tilde{\mathbf{G}}_1)_\tau d\xi dy d\tau - 2 \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}\tilde{\mathbf{G}}_1, M_\tau)\} d\xi dy d\tau \\ = & -\int \int \left[\frac{V}{Z}\Psi_1 \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\tilde{\mathbf{G}}_1) \right](\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^\tau \int \left(\frac{V}{Z}\Psi_1 \right)_\tau \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\tilde{\mathbf{G}}_1) d\xi dy d\tau \\ & - 2 \int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z}\Psi_1 \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}\tilde{\mathbf{G}}_1, M_\tau)\} d\xi dy d\tau. \end{aligned} \quad (4.1.50)$$

The Hölder inequality and Lemma 3.4 yield

$$\left| \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}(\tilde{\mathbf{G}}_1) d\xi \right|^2 \leq C \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi. \quad (4.1.51)$$

Moreover, from Lemmas 3.2-3.4, we have

$$\begin{aligned} \int \xi_1^2 \mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}\tilde{\mathbf{G}}_1, M_\tau)\} d\xi & \leq C \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_{\mathbf{M}}^{-1}\{Q(\mathbf{L}_{\mathbf{M}}^{-1}\tilde{\mathbf{G}}_1, M_\tau)\}|^2 d\xi \right)^{\frac{1}{2}} \\ & \leq C \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\mathbf{L}_{\mathbf{M}}^{-1}\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} \cdot \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |M_\tau|^2 d\xi \right)^{\frac{1}{2}} \\ & \leq C|(v_\tau, u_\tau, \theta_\tau)| \left(\int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}}. \end{aligned} \quad (4.1.52)$$

Combining (4.1.50)-(4.1.52) gives

$$\begin{aligned}
K_{11}^{11} &\leq \beta \left[\|\Psi(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_{1\tau}, \sqrt{|U_{1y}^{S_3}|}\Psi_1)\|^2 d\tau \right] \\
&\quad + C_\beta \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi_1\|^2 d\tau + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 d\tau.
\end{aligned} \tag{4.1.53}$$

On the other hand, by (3.13) and (2.20), we have

$$\begin{aligned}
K_{11}^{12} &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V}{Z} \Psi_1 \int \xi_1^2 \mathbf{L}_M^{-1}(\mathbf{G}_\tau^{R_1} + \mathbf{G}_\tau^{CD}) d\xi dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left[|(\Theta_{y\tau}^{R_1}, U_{1y\tau}^{R_1}, \Theta_{y\tau}^{CD}, U_{y\tau}^{CD})| + |(\Theta_y^{R_1}, U_{1y}^{R_1}, \Theta_y^{CD}, U_y^{CD})| \right] \|(v_\tau, u_\tau, \theta_\tau)\| dy d\tau \\
&\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi_1\|^2 d\tau + C(C_{h,T} \varepsilon + \delta) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}},
\end{aligned} \tag{4.1.54}$$

which, together with (4.1.53), imply

$$\begin{aligned}
K_{11}^1 &\leq \beta \left[\|\Psi(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_{1\tau}, \sqrt{|U_{1y}^{S_3}|}\Psi_1)\|^2 d\tau \right] + C_{h,T} \varepsilon^{\frac{1}{2}} \\
&\quad + C_\beta \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi_1\|^2 d\tau + C[\delta + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi)] \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2 d\tau.
\end{aligned} \tag{4.1.55}$$

We now turn to K_{11}^2 . By (4.1.49), we have

$$\begin{aligned}
K_{11}^2 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \left(\frac{u_1 V \Psi_1}{v Z} \right)_y \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_1) d\xi dy d\tau + 2 \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{u_1 V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_M^{-1} \{ Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_1, \mathbf{M}_y) \} d\xi dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left[|\Psi_{1y}| + |\Psi_1| |(V_y, Z_y)| + |\Psi_1| |(v_y, u_y, \theta_y)| \right] \left(\int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_{1y}, \sqrt{|U_{1y}^{S_3}|}\Psi_1)\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi_1\|^2 d\tau + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau.
\end{aligned} \tag{4.1.56}$$

To estimate K_{11}^3 , notice that

$$\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y}) = [\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_1)]_y + \sum_{j=0}^4 \langle \xi_1 \tilde{\mathbf{G}}_1, \chi_j \rangle \mathbf{P}_1(\chi_{jy}). \tag{4.1.57}$$

Then from (4.1.49) and (4.1.57), we have

$$\begin{aligned}
K_{11}^3 &= \int_{\frac{h}{\varepsilon}}^{\tau} \int \left(\frac{V \Psi_1}{v Z} \right)_y \int \xi_1^2 \mathbf{L}_M^{-1}[\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_1)] d\xi dy d\tau \\
&\quad - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_M^{-1} \left[\sum_{j=0}^4 \langle \xi_1 \tilde{\mathbf{G}}_1, \chi_j \rangle \mathbf{P}_1(\chi_{jy}) \right] d\xi dy d\tau \\
&\quad - 2 \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V \Psi_1}{v Z} \int \xi_1^2 \mathbf{L}_M^{-1} \{ Q(\mathbf{L}_M^{-1}[\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_1)], \mathbf{M}_y) \} d\xi dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \int (|U_{1y}^{S_3}| |\Psi_1|^2 + |\Psi_{1y}|^2) dy d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C_{h,T}(\varepsilon^{\frac{1}{4}} + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\phi_y, \psi_y, \zeta_y)|^2 dy d\tau.
\end{aligned} \tag{4.1.58}$$

On the other hand,

$$\begin{aligned}
K_{11}^4 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\mathbf{L}_M^{-1} \{Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)\}|^2 d\xi \right)^{\frac{1}{2}} dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau,
\end{aligned} \tag{4.1.59}$$

and

$$\begin{aligned}
K_{11}^5 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi \right)^{\frac{1}{2}} \left[\left(\int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\mathbf{G}^{S_3}|^2 d\xi \right)^{\frac{1}{2}} + |(\Theta_y^{R_1}, \Theta_y^{CD}, U_{1y}^{R_1}, U_y^{CD})| \right] dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \int |U_{1y}^{S_3}| |\Psi_1|^2 dy d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau.
\end{aligned} \tag{4.1.60}$$

By (4.1.44), we have

$$\begin{aligned}
K_{11}^6 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \left\{ (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{S_3}}^{-1}) \mathbf{G}_\tau^{S_3} - (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{S_3}}^{-1}) Q(\mathbf{G}^{S_3}, \mathbf{G}^{S_3}) \right. \\
&\quad - \left(\frac{u_1}{v} \mathbf{L}_M^{-1} - \frac{U_1^{S_3}}{V^{S_3}} \mathbf{L}_{M^{S_3}}^{-1} \right) \mathbf{G}_y^{S_3} + \left(\frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1 - \frac{1}{V^{S_3}} \mathbf{L}_{M^S}^{-1} \mathbf{P}_1^{S_3} \right) (\xi_1 \mathbf{G}_y^{S_3}) \\
&\quad \left. - \mathbf{L}_M^{-1} [2Q(\mathbf{G}^{S_3}, \mathbf{G}^{R_1} + \mathbf{G}^{CD})] \right\} d\xi dy d\tau := \sum_{i=1}^5 K_{11}^{6i}.
\end{aligned} \tag{4.1.61}$$

For brevity, we will only consider the term K_{11}^{61} because similar estimates hold for the terms K_{11}^{6i} ($i = 2, 3, 4, 5$). Direct calculation yields

$$\begin{aligned}
K_{11}^{61} &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \mathbf{L}_M^{-1} [2Q(\mathbf{M}^{S_3} - \mathbf{M}, \mathbf{L}_{M^{S_3}}^{-1}(\mathbf{G}_\tau^{S_3}))] d\xi \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left(\int \frac{\nu(|\xi|) |\mathbf{M}^{S_3} - \mathbf{M}|^2}{\mathbf{M}_*} d\xi \right)^{\frac{1}{2}} \left(\int \frac{\nu(|\xi|) |\mathbf{L}_{M^{S_3}}^{-1}(\mathbf{G}_\tau^{S_3})|^2}{\mathbf{M}_*} d\xi \right)^{\frac{1}{2}} dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| |(v - V^{S_3}, u - U^{S_3}, \theta - \Theta^{S_3})| \left(\int \frac{\nu^{-1}(|\xi|) |\mathbf{G}_\tau^{S_3}|^2}{\mathbf{M}_*} d\xi \right)^{\frac{1}{2}} dy d\tau \\
&\leq C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \int |U_{1y}^{S_3}| |\Psi_1|^2 dy d\tau + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\Phi_y, \Psi_y, \zeta)|^2 dy d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.62}$$

By (4.1.45), we have

$$\begin{aligned}
K_{11}^7 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \left\{ - \left(\frac{u_1}{v} \mathbf{L}_M^{-1} \mathbf{G}_y^{CD} - \frac{U_1^{CD}}{V^{CD}} \mathbf{L}_{M^{CD}}^{-1} \bar{\mathbf{G}}_y^{CD} \right) \right. \\
&\quad - \left[\mathbf{L}_M^{-1} Q(\mathbf{G}^{CD}, \mathbf{G}^{CD}) - \mathbf{L}_{M^{CD}}^{-1} Q(\bar{\mathbf{G}}^{CD}, \bar{\mathbf{G}}^{CD}) \right] \\
&\quad \left. + \left[\frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1 (\xi_1 \mathbf{G}_y^{CD}) - \frac{1}{V^{CD}} \mathbf{L}_{M^{CD}}^{-1} \mathbf{P}_1^{CD} (\xi_1 \bar{\mathbf{G}}_y^{CD}) \right] - \mathbf{L}_M^{-1} [2Q(\mathbf{G}^{R_1}, \mathbf{G}^{CD})] \right\} d\xi dy d\tau := \sum_{i=1}^4 K_{11}^{7i}.
\end{aligned}$$

For illustration, we only consider K_{11}^{71} as follows.

$$\begin{aligned}
K_{11}^{71} &= \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \left[\left(\frac{u_1}{v} - \frac{U_1^{CD}}{V^{CD}} \right) \mathbf{L}_M^{-1} \mathbf{G}_y^{CD} + \frac{U_1^{CD}}{V^{CD}} (\mathbf{L}_M^{-1} - \mathbf{L}_{M^{CD}}^{-1}) \mathbf{G}_y^{CD} \right. \\
&\quad \left. + \frac{U_1^{CD}}{V^{CD}} \mathbf{L}_{M^{CD}}^{-1} (\mathbf{G}_y^{CD} - \bar{\mathbf{G}}_y^{CD}) \right] d\xi dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| |(v - V^{CD}, u - U^{CD}, \theta - \Theta^{CD})| \cdot \left[|(\Theta_{yy}^{CD}, U_{yy}^{CD})| + |(\Theta_y^{CD}, U_y^{CD})| |(v_y, u_y, \theta_y)| \right] dy d\tau \\
&\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.63}$$

Then, by (4.1.46), we have

$$\begin{aligned}
K_{11}^8 &= - \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{V\Psi_1}{Z} \int \xi_1^2 \mathbf{L}_M^{-1} \left[-\frac{u_1}{v} \mathbf{G}_y^{R_1} + \frac{1}{v} \mathbf{L}_M^{-1} \mathbf{P}_1(\xi_1 \mathbf{G}_y^{R_1}) + Q(\mathbf{G}^{R_1}, \mathbf{G}^{R_1}) \right] d\xi dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1| \left[|(v_y, u_y, \theta_y)| |(\Theta_y^{R_1}, U_y^{R_1})| + |(\Theta_y^{R_1}, U_y^{R_1})|^2 + |(\Theta_{yy}^{R_1}, U_{yy}^{R_1})| \right] dy d\tau \\
&\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \int |\Psi_1|^2 dy d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.64}$$

By collecting all the above estimates, we have

$$\begin{aligned}
K_{11} &\leq \beta \left[\|\Psi(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_{1\tau}, \sqrt{|U_{1y}^{S_3}} \Psi_1)\|^2 d\tau \right] + C_\beta \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy \\
&\quad + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C(\chi + \beta + \delta) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau \\
&\quad + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi_1\|^2 d\tau + C[\delta + C_{h,T}\chi] \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.65}$$

Therefore, we have

$$\begin{aligned}
&\|(\Phi, \Psi, W)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \left[\|\sqrt{|U_{1y}^{S_3}}(\Psi, W)\|^2 + \|(\Psi_y, W_y)\|^2 \right] d\tau \\
&\leq C \int \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi, W)\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_\tau, W_\tau)\|^2 d\tau \\
&\quad + C(\chi + \beta + \delta) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \zeta)\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\
&\quad + C(\delta + C_{h,T}\chi) \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{2}{5}},
\end{aligned} \tag{4.1.66}$$

where we have used the smallness of $\delta, \beta, \varepsilon$ and χ .

Step 2. Estimation on $\|\Phi_y(\tau, \cdot)\|^2$.

Note that the dissipation term does not contain the term $\|\Phi_y\|^2$. To complete the lower order energy estimate, we have to estimate Φ_y . From (3.6)₂, we have

$$\begin{aligned}
\frac{4}{3} \frac{\mu(\Theta)}{V} \Phi_{y\tau} - \Psi_{1\tau} + \frac{Z}{V} \Phi_y &= \frac{2}{3V} W_y + \frac{2}{3V} U_y \cdot \Psi \\
&\quad - \frac{4}{3} \frac{\mu'(\Theta)}{V} U_{1y} (W_y + U_y \cdot \Psi) - N_1 + \bar{Q}_1 + Q_1 + \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi.
\end{aligned} \tag{4.1.67}$$

Multiplying (4.1.67) by Φ_y yields

$$\begin{aligned}
\left(\frac{2\mu(\Theta)}{3V} \Phi_y^2 - \Phi_y \Psi_1 \right)_\tau + \frac{Z}{V} \Phi_y^2 &= \left(\frac{2\mu(\Theta)}{3V} \right)_\tau \Phi_y^2 + \Psi_{1y}^2 + \left[\frac{2}{3V} U_y \cdot \Psi + \frac{2}{3V} W_y \right. \\
&\quad \left. - \frac{4}{3} \frac{\mu'(\Theta)}{V} U_{1y} (W_y + U_y \cdot \Psi) - N_1 + \bar{Q}_1 + Q_1 + \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right] \Phi_y + (\dots)_y,
\end{aligned} \tag{4.1.68}$$

where we have used the fact

$$\Phi_y \Psi_{1\tau} = (\Phi_y \Psi_1)_\tau - (\Phi_\tau \Psi_1)_y + \Psi_{1y}^2.$$

Integrating (4.1.68) with respect to y and τ and using Hölder inequality give

$$\begin{aligned}
&\int \left[\frac{2\mu(\Theta)}{3V} \Phi_y^2 - \Phi_y \Psi_1 \right](\tau, y) dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{Z}{2V} \Phi_y^2 dy d\tau \\
&\leq C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|\Psi\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_y, W_y)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int (N_1^2 + \bar{Q}_1^2 + Q_1^2) dy d\tau \\
&\quad + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|\sqrt{|U_{1y}^{S_3}} \Psi\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi |^2 dy d\tau.
\end{aligned} \tag{4.1.69}$$

By (2.62), (3.8) and (2.104) and the Cauchy inequality, one has

$$\int_{\frac{h}{\varepsilon}}^{\tau} \int (N_1^2 + \bar{Q}_1^2 + Q_1^2) dy d\tau \leq C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|\psi_{1y}\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \quad (4.1.70)$$

Now we estimate the last term on the right hand side of (4.1.69). By (4.1.42), we have

$$\begin{aligned} K_2 &:= \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1}(\tilde{\mathbf{G}}_{\tau}) d\xi \right|^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \left(\frac{u_1}{v} \tilde{\mathbf{G}}_{1y} \right) d\xi \right|^2 dy d\tau \\ &+ C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \left[\frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y}) \right] d\xi \right|^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} [Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)] d\xi \right|^2 dy d\tau \\ &+ C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} [2Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})] d\xi \right|^2 dy d\tau \\ &+ C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \xi_1^2 (|J_3|^2 + |J_4|^2 + |J_5|^2) d\xi dy d\tau := \sum_{i=1}^6 K_2^i. \end{aligned} \quad (4.1.71)$$

Then we can obtain

$$\begin{aligned} K_2^1 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \frac{\nu(|\xi|) |\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{\tau}|^2}{\mathbf{M}_{\star}} d\xi \right| \cdot \int \nu^{-1}(|\xi|) \xi_1^4 \mathbf{M}_{\star} d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_{\tau}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau. \end{aligned} \quad (4.1.72)$$

Similarly,

$$\begin{aligned} K_2^2 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_{1y}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} (|\mathbf{G}_y^{R_1}|^2 + |\mathbf{G}_y^{CD}|^2) d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Theta_{yy}^{R_1}, U_{1yy}^{R_1}, \Theta_{yy}^{CD}, U_{yy}^{CD})\|^2 d\tau \\ &\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int |(v_y, u_y, \theta_y)|^2 \cdot |(\Theta_y^{R_1}, U_{1y}^{R_1}, \Theta_y^{CD}, U_y^{CD})|^2 dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned} \quad (4.1.73)$$

Moreover,

$$\begin{aligned} K_2^3 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \frac{\nu(|\xi|) |\mathbf{L}_M^{-1} [\frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})]}{\mathbf{M}_{[2v_{\star}, 2u_{\star}, 2\theta_{\star}]}} d\xi \right| \cdot \int \nu^{-1}(|\xi|) \xi_1^4 \mathbf{M}_{[2v_{\star}, 2u_{\star}, 2\theta_{\star}]} d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |\frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_{1y})|^2}{\mathbf{M}_{[2v_{\star}, 2u_{\star}, 2\theta_{\star}]}} d\xi dy d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_{1y}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned} \quad (4.1.74)$$

From Lemma 3.2, we have

$$\begin{aligned} K_2^4 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_{\star}} d\xi \cdot \int \frac{|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\ &\leq C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_{\star}} d\xi dy d\tau. \end{aligned} \quad (4.1.75)$$

Using similar idea as for (4.1.60) and (4.1.62), we can obtain

$$\begin{aligned}
K_2^5 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) |Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})|^2}{\mathbf{M}_\star} d\xi dy d\tau \\
&\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi \int \frac{\nu(|\xi|) (|\mathbf{G}^{R_1}|^2 + |\mathbf{G}^{CD}|^2 + |\mathbf{G}^{S_3}|^2)}{\mathbf{M}_\star} d\xi dy d\tau \\
&\leq C(\delta + \varepsilon) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy d\tau,
\end{aligned} \tag{4.1.76}$$

and

$$\begin{aligned}
K_2^6 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|) (|J_3|^2 + |J_4|^2 + |J_5|^2)}{\mathbf{M}_\star} d\xi dy d\tau \\
&\leq C(\delta + \varepsilon) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.77}$$

Substituting the estimations of K_2^i ($i = 1, 2, \dots, 6$) into (4.1.71), we obtain

$$\begin{aligned}
K_2 &= \int_{\frac{h}{\varepsilon}}^{\tau} \int \int |\xi_1^2 (\Pi_1 - \Pi_1^{CD} - \Pi_1^{S_3}) d\xi|^2 dy d\tau \\
&\leq C(\delta + \varepsilon) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Phi_y, \Psi_y, \zeta, \phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
&+ C(\delta + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau.
\end{aligned} \tag{4.1.78}$$

Thus combining (4.1.69), (4.1.70) and (4.1.78) yields

$$\begin{aligned}
\|\Phi_y(\tau, \cdot)\|^2 &+ \int_{\frac{h}{\varepsilon}}^{\tau} \|\Phi_y\|^2 d\tau \leq C \|\Psi_1(\tau, \cdot)\|^2 + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\Psi_y, W_y, \zeta)\|^2 d\tau \\
&+ C(\delta + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\
&+ C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|\sqrt{|U_{1y}^{S_3}|} \Psi\|^2 d\tau + C[\delta + C_{h,T}\chi] \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau.
\end{aligned} \tag{4.1.79}$$

Step 3. Estimation on the non-fluid component.

The microscopic component $\tilde{\mathbf{G}}_1$ can be estimated by using the equation (3.15). Multiplying (3.15) by $\frac{v\tilde{\mathbf{G}}_1}{\mathbf{M}_\star}$ gives

$$\begin{aligned}
\left(\frac{v\tilde{\mathbf{G}}_1^2}{2\mathbf{M}_\star} \right)_{1\tau} - \frac{v\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} \mathbf{L}_M \tilde{\mathbf{G}}_1 &= v_\tau \frac{\tilde{\mathbf{G}}_1^2}{2\mathbf{M}_\star} + \left\{ \frac{u_1}{v} \tilde{\mathbf{G}}_y - \frac{1}{v} \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) + 2Q(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) \right. \\
&\quad \left. + Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + J_1 + J_2 - \mathbf{G}_\tau^{R_1} - \mathbf{G}_\tau^{CD} \right\} \frac{v\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} \\
&= \left\{ -\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1 + vJ_2 \right. \\
&\quad \left. - v\mathbf{G}_\tau^{R_1} - v\mathbf{G}_\tau^{CD} + u_1 \mathbf{G}_y^{R_1} + u_1 \mathbf{G}_y^{CD} \right\} \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_\star} + (\dots)_y.
\end{aligned} \tag{4.1.80}$$

The Cauchy inequality implies

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \left(-v \mathbf{G}_{\tau}^{R_1} - v \mathbf{G}_{\tau}^{CD} + u_1 \mathbf{G}_y^{R_1} + u_1 \mathbf{G}_y^{CD} \right) \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_{\star}} d\xi dy d\tau \\
& \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \sum_{|\alpha|=2} |\partial^{\alpha}(\Theta^{CD}, U^{CD}, \Theta^{R_1}, U^{R_1})|^2 dy d\tau \\
& \quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\Theta_y^{CD}, U_y^{CD}, \Theta_y^{R_1}, U_y^{R_1})|^2 \left[|(v_{\tau}, u_{\tau}, \theta_{\tau})|^2 + |(v_y, u_y, \theta_y)|^2 \right] dy d\tau \\
& \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \sum_{|\alpha'|=1} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.81}$$

Notice that $\mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) = \xi_1 \tilde{\mathbf{G}}_y - \sum_{j=0}^4 \langle \xi_1 \tilde{\mathbf{G}}_y, \chi_j \rangle \chi_j$. Then we have

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \mathbf{P}_1(\xi_1 \tilde{\mathbf{G}}_y) \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_{\star}} d\xi dy d\tau \\
& = \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \xi_1 (\tilde{\mathbf{G}}_{1y} + \mathbf{G}_y^{R_1} + \mathbf{G}_y^{CD}) \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_{\star}} - \sum_{j=0}^4 \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \langle \xi_1 \tilde{\mathbf{G}}_y, \chi_j \rangle \chi_j \frac{\tilde{\mathbf{G}}_1}{\mathbf{M}_{\star}} \\
& \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_y^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\
& \quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial_y(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}},
\end{aligned} \tag{4.1.82}$$

where we have used the fact that

$$|\langle \xi_1 \tilde{\mathbf{G}}_y, \chi_j \rangle|^2 \leq \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_y|^2}{\mathbf{M}_{\star}} d\xi.$$

Lemma 3.2-Lemma 3.5 and wave interaction estimates imply

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \int \int Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) \frac{v \tilde{\mathbf{G}}_1}{\mathbf{M}_{\star}} d\xi dy d\tau \\
& \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left(\int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi \right)^{\frac{1}{2}} \left(\int \frac{\nu(|\xi|) \tilde{\mathbf{G}}^2}{\mathbf{M}_{\star}} d\xi \int \frac{\tilde{\mathbf{G}}^2}{\mathbf{M}_{\star}} d\xi \right)^{\frac{1}{2}} dy d\tau \\
& \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int |(\Theta_y^{CD}, U_y^{CD}, \Theta_y^{R_1}, U_y^{R_1})|^4 dy d\tau \\
& \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.83}$$

Moreover, Lemma 2.5 and Cauchy's inequality and wave interaction estimates imply

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \int \int (J_1 + J_2) \frac{v \tilde{\mathbf{G}}_1}{\mathbf{M}_{\star}} d\xi dy d\tau \leq \frac{\tilde{\sigma}}{16} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_{\star}} d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau \\
& \quad + C\delta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.1.84}$$

Integrating (4.1.80) with respect to ξ, y and τ and using (3.23), (4.1.81)-(4.1.84) and the smallness of

$\chi, \delta, \varepsilon$ yield that

$$\begin{aligned} & \int \int \frac{\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)|\tilde{\mathbf{G}}_1|^2}{\mathbf{M}_\star} d\xi dy d\tau \\ & \leq C\delta \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta)\|^2 d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_y|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned} \quad (4.1.85)$$

On the other hand, from the fluid-type system (3.6), we can get an estimate for $\|(\Psi_\tau, W_\tau)\|^2$ as follows.

$$\begin{aligned} & \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi_\tau, W_\tau)\|^2 d\tau \leq C \int_{\frac{h}{\varepsilon}}^\tau \|(\Phi_y, \Psi_y, W_y, \Psi_{yy}, \zeta_y)\|^2 d\tau + C\delta \int_{\frac{h}{\varepsilon}}^\tau \|\sqrt{|U_{1y}^{S_3}}|(\Psi, W)\|^2 d\tau \\ & + C\varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|(\Psi, W)\|^2 d\tau + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau \\ & + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^\tau \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}. \end{aligned} \quad (4.1.86)$$

From (3.5), we have

$$\begin{aligned} & \int_{\frac{h}{\varepsilon}}^\tau \|\zeta\|^2 d\tau \leq C \int_{\frac{h}{\varepsilon}}^\tau \|W_y\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int |\Psi_y|^4 dy d\tau + C \int_{\frac{h}{\varepsilon}}^\tau \int |U_y \cdot \Psi|^2 dy d\tau \\ & \leq C \int_{\frac{h}{\varepsilon}}^\tau \|W_y\|^2 d\tau + C\chi^2 \int_{\frac{h}{\varepsilon}}^\tau \|\Psi_y\|^2 d\tau + C\delta \int_{\frac{h}{\varepsilon}}^\tau \|\sqrt{|U_{1y}^{S_3}}|\Psi\|^2 d\tau + C_{h,T} \varepsilon \int_{\frac{h}{\varepsilon}}^\tau \|\Psi\|^2 d\tau. \end{aligned} \quad (4.1.87)$$

In summary, collecting the estimates (4.1.66), (4.1.79), (4.1.85)-(4.1.87), we complete the proof of Proposition 3.1.

4.2 Proof of Proposition 3.2

Proof of Proposition 3.2. The proof is divided into the following five steps.

Step 1. Estimation on $\|(\phi, \psi, \zeta)(\tau, \cdot)\|^2$.

Similar to (4.1.10), we multiply (3.39)₁ by ϕ , (3.39)₂ by $\frac{V}{Z}\psi_1$, (3.39)₃ by ψ_i , (3.39)₄ by $\frac{2\zeta}{3Z^2}$ respectively and adding them together to have

$$\begin{aligned} & \left(\frac{\phi^2}{2} + \frac{V}{2Z}\psi_1^2 + \sum_{i=2}^3 \frac{\psi_i^2}{2} + \frac{\zeta^2}{3Z^2} \right)_\tau + \frac{4\mu(\Theta)}{3Z}\psi_{1y}^2 + \sum_{i=2}^3 \frac{\mu(\Theta)}{V}\psi_{iy}^2 + \frac{2\kappa(\Theta)}{3Z^2V}\zeta_y^2 \\ & = I_6(\phi, \psi, \zeta, \psi_y, \zeta_y) + \frac{V}{Z}\psi_1(N_5 - \bar{Q}_{1y} - Q_{1y}) + \sum_{i=2}^3 \psi_i(N_{i+4} - \bar{Q}_{iy} - Q_{iy}) \\ & + \frac{2\zeta}{3Z^2}[N_8 - \bar{Q}_{4y} - Q_{4y} + \sum_{i=1}^3 U_i(\bar{Q}_{iy} + Q_{iy})] + K_3 + (\cdots)_y, \end{aligned} \quad (4.2.1)$$

where

$$\begin{aligned} I_6(\phi, \psi, \zeta, \psi_y, \zeta_y) = & \left(\frac{V}{2Z} \right)_\tau \psi_1^2 + \left(\frac{1}{3Z^2} \right)_\tau \zeta^2 + \left(\frac{2}{3Z} \right)_y \psi_1 \zeta - \frac{V}{Z} \psi_1 H_1 - \sum_{i=2}^3 \psi_i H_i \\ & - \frac{2}{3Z^2} \zeta H_4 - \frac{4\mu(\Theta)}{3V} \left(\frac{V}{Z} \right)_y \psi_1 \psi_{1y} - \frac{\kappa(\Theta)}{V} \left(\frac{2}{3Z^2} \right)_y \zeta \zeta_y, \end{aligned} \quad (4.2.2)$$

and

$$\begin{aligned}
K_3 &= -\frac{V}{Z} \psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - \sum_{i=2}^3 \psi_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\
&\quad + \frac{2\zeta}{3Z^2} \left[-\frac{1}{2} \int \xi_1 |\xi|^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi + \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi \right. \\
&\quad \left. + \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right]. \tag{4.2.3}
\end{aligned}$$

Direct calculation shows that

$$\begin{aligned}
I_6 &\leq C|(V_\tau, Z_\tau, V_y, U_y, \Theta_y, Z_y)| [|(\phi_y, \psi_y, \zeta_y)|^2 + (\phi, \psi, \zeta)^2] \\
&\leq C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) |(\phi_y, \psi_y, \zeta_y)|^2 + C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) |(\phi, \psi, \zeta)|^2. \tag{4.2.4}
\end{aligned}$$

Thus, integrating (4.2.1) with respect to τ and y and using Cauchy inequality yield that

$$\begin{aligned}
\|(\phi, \psi, \zeta)(\tau, \cdot)\|^2 &+ \int_{\frac{h}{\varepsilon}}^\tau \|(\psi_y, \zeta_y)\|^2 d\tau \leq C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^\tau \|\phi_y\|^2 d\tau \\
&+ C(\beta + \delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta)\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \sum_{i=1}^4 (\bar{Q}_{iy}^2 + Q_{iy}^2) dy d\tau \\
&+ C \int_{\frac{h}{\varepsilon}}^\tau \int |(\psi, \zeta)| [|N_5| + \sum_{i=2}^3 |N_{i+4}| + |N_8|] dy d\tau + \int_{\frac{h}{\varepsilon}}^\tau \int K_3 dy d\tau. \tag{4.2.5}
\end{aligned}$$

By (3.31) and (3.33), we have

$$\int_{\frac{h}{\varepsilon}}^\tau \int \sum_{i=1}^4 (\bar{Q}_{iy}^2 + Q_{iy}^2) dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}. \tag{4.2.6}$$

From (3.43), (3.44) and (3.45), we have

$$\int_{\frac{h}{\varepsilon}}^\tau \int |(\psi, \zeta)| [|N_5| + \sum_{i=2}^3 |N_{i+4}| + |N_8|] dy d\tau \leq C_{h,T} \chi \int_{\frac{h}{\varepsilon}}^\tau \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \psi_{yy}, \zeta_{yy})\|^2 d\tau. \tag{4.2.7}$$

Now we estimate the microscopic term $\int_{\frac{h}{\varepsilon}}^\tau \int K_3 dy d\tau$ in (4.2.5). We only estimate study $K_{31} :=$

$-\int_{\frac{h}{\varepsilon}}^\tau \int \frac{V}{Z} \psi_1 \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi dy d\tau$ and $K_{32} := \frac{\zeta}{3Z^2} \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi$ because the other terms in $\int_{\frac{h}{\varepsilon}}^\tau \int K_3 dy d\tau$ can be estimated similarly.

For K_{31} , integration by parts with respect to y and the Cauchy inequality yield

$$\begin{aligned}
K_{31} &= \int_{\frac{h}{\varepsilon}}^\tau \int \left(\frac{V}{Z} \psi_1 \right)_y \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^\tau \|\psi_{1y}\|^2 d\tau + C(\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^\tau \|\psi_1\|^2 d\tau + C_\beta \int_{\frac{h}{\varepsilon}}^\tau \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi \right|^2 dy d\tau, \tag{4.2.8}
\end{aligned}$$

where the last term has been estimated in (4.1.78).

For K_{32} , we have

$$\begin{aligned}
K_{32} &= - \sum_{i=1}^3 \int_{\frac{h}{\varepsilon}}^{\tau} \int (\frac{\zeta \psi_i}{3Z^2})_y \int \xi_1 \xi_i \Pi_1 d\xi dy d\tau \\
&\leq C \sum_{i=1}^3 \int_{\frac{h}{\varepsilon}}^{\tau} \int (|\zeta_y \psi| + |\psi_y \zeta| + |Z_y \psi \zeta|) \\
&\quad \cdot \left[\int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi + \int \xi_1 \xi_i (\Pi_{11}^{CD} + \Pi_1^{S_3}) d\xi \right] dy d\tau \\
&\leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau + C_\beta (\delta + C_{h,T} \varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi, \zeta)\|^2 d\tau \\
&\quad + C_\beta \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3}) d\xi|^2 dy d\tau.
\end{aligned} \tag{4.2.9}$$

Substituting (4.2.6)-(4.2.9) and (4.1.78) into (4.2.5) and choosing $\beta, \varepsilon, \delta, \chi$ suitably small yield that

$$\begin{aligned}
\|(\phi, \psi, \zeta)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau &\leq C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_y\|^2 d\tau \\
+C_\beta (\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_{yy}, \zeta_{yy})\|^2 d\tau + C_{h,T,\beta} \varepsilon^{\frac{1}{2}} \\
+C(\delta + \chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau.
\end{aligned} \tag{4.2.10}$$

Step 2. Estimation on $\|\phi_y(\tau, \cdot)\|^2$.

To estimate the term $\int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_y\|^2 d\tau$, we firstly rewrite the equation (3.39)₂ as

$$\begin{aligned}
\frac{4}{3} \frac{\mu(\Theta)}{V} \phi_{y\tau} - \psi_{1\tau} + \frac{Z}{V} \phi_y &= -\frac{4}{3} \left(\frac{\mu(\Theta)}{V}\right)_y \psi_{1y} + \frac{2}{3V} \zeta_y + H_1 \\
&+ \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - N_5 + \bar{Q}_{1y} + Q_{1y},
\end{aligned} \tag{4.2.11}$$

by using the equation of conservation of the mass (3.39)₁. Multiplying (4.2.11) by ϕ_y , we get

$$\begin{aligned}
\left(\frac{2\mu(\Theta)}{3V} \phi_y^2 - \phi_y \psi_1\right)_\tau + (\phi_\tau \psi_1)_y + \frac{Z}{V} \phi_y^2 &= \psi_y^2 \\
+ \left[-\frac{4}{3} \left(\frac{\mu(\Theta)}{V}\right)_y \psi_{1y} + \frac{2}{3V} \zeta_y + H_1 + \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - N_5 + \bar{Q}_{1y} + Q_{1y}\right] \phi_y.
\end{aligned} \tag{4.2.12}$$

Integrating (4.2.12) with respect to τ, y and using (4.2.6), (4.2.7) and the Cauchy inequality yield

$$\begin{aligned}
\|\phi_y(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_y\|^2 d\tau &\leq C \|\psi_1(\tau, \cdot)\|^2 + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau \\
&+ C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|\psi_{1yy}\|^2 d\tau \\
&+ C_{h,T} \varepsilon^{\frac{1}{2}} + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi|^2 dy d\tau.
\end{aligned} \tag{4.2.13}$$

For the microscopic term $\int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right|^2 dy d\tau$, by (4.1.42), we have

$$\begin{aligned}
K_4 &:= \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right|^2 dy d\tau \\
&\leq C \left[\int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{\tau})_y d\xi \right|^2 dy d\tau + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (\mathbf{L}_M^{-1} \frac{u_1}{v} \tilde{\mathbf{G}}_{1y})_y d\xi \right|^2 dy d\tau \right. \\
&\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 [\mathbf{L}_M^{-1} \frac{1}{v} \mathbf{P}_1 (\xi_1 \tilde{\mathbf{G}}_{1y})]_y d\xi \right|^2 dy d\tau + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 [\mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1)]_y d\xi \right|^2 dy d\tau \\
&\quad + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 [\mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \mathbf{G}^{R_1} + \mathbf{G}^{CD} + \mathbf{G}^{S_3})]_y d\xi \right|^2 dy d\tau \\
&\quad \left. + \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 (J_{3y} + J_{4y} + J_{5y}) d\xi \right|^2 dy d\tau \right] := \sum_{i=1}^6 K_4^i.
\end{aligned} \tag{4.2.14}$$

Then we have

$$\begin{aligned}
K_4^1 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{y\tau} d\xi \right|^2 dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} \tilde{\mathbf{G}}_{\tau}, \mathbf{M}_y)\} d\xi \right|^2 dy d\tau \\
&\leq C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) |\tilde{\mathbf{G}}_{\tau}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau.
\end{aligned} \tag{4.2.15}$$

Similar estimates hold for K_4^i ($i = 2, 3$). Moreover,

$$\begin{aligned}
K_4^4 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_{1y}) d\xi \right|^2 dy d\tau \\
&\quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \left| \int \xi_1^2 \mathbf{L}_M^{-1} \{Q(\mathbf{L}_M^{-1} Q(\tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_1), \mathbf{M}_y)\} d\xi \right|^2 dy d\tau \\
&\leq C_{h,T\chi} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2)}{\mathbf{M}_{\star}} d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}},
\end{aligned} \tag{4.2.16}$$

$$K_4^5 \leq C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2)}{\mathbf{M}_{\star}} d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}, \tag{4.2.17}$$

and

$$\begin{aligned}
K_4^6 &\leq C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{|J_{3y}|^2 + |J_{4y}|^2 + |J_{5y}|^2}{\mathbf{M}_{\star}} d\xi dy d\tau \\
&\leq C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \phi_{yy}, \psi_{yy}, \zeta_{yy}\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.2.18}$$

Substituting (4.2.15)-(4.2.18) into (4.2.14) gives

$$\begin{aligned}
K_4 &\leq C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
&\quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \phi_{yy}, \psi_{yy}, \zeta_{yy}\|^2 d\tau \\
&\quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) (|\tilde{\mathbf{G}}_{\tau}|^2 + |\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2)}{\mathbf{M}_{\star}} d\xi dy d\tau.
\end{aligned} \tag{4.2.19}$$

Thus we have

$$\begin{aligned}
& \|\phi_y(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_y\|^2 d\tau \leq C\|\psi_1(\tau, \cdot)\|^2 + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_y, \zeta_y)\|^2 d\tau \\
& + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
& + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu^{-1}(|\xi|)}{\mathbf{M}_*} |\partial^\alpha \mathbf{G}|^2 d\xi dy d\tau \\
& + C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)(\sum_{|\alpha'|=1} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 + |\tilde{\mathbf{G}}_1|^2)}{\mathbf{M}_*} d\xi dy d\tau.
\end{aligned} \tag{4.2.20}$$

We now turn to the time derivatives. To estimate $\|(\phi_\tau, \psi_\tau, \zeta_\tau)\|^2$, we need to use the system (3.27). By multiplying (3.27)₁ by ϕ_τ , (3.27)₂ by $\psi_{1\tau}$, (3.27)₃ by $\psi_{i\tau}$ ($i = 2, 3$) and (3.27)₄ by ζ_τ respectively, and adding them together, after integrating with respect to τ and y , we have

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_\tau, \psi_\tau, \zeta_\tau)(\tau, \cdot)\|^2 d\tau \leq C(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau \\
& + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_*} |\tilde{\mathbf{G}}_y|^2 d\xi dy d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.2.21}$$

Step 3. Estimation on $\|(\phi_y, \psi_y, \zeta_y)(\tau, \cdot)\|^2$.

Multiplying (3.46)₁ by ϕ_y , (3.46)₂ by $\frac{V}{Z}\psi_{1y}$, (3.46)₃ by ψ_{iy} , and (3.46)₄ by $\frac{2}{3Z^2}\zeta_y$, adding them together gives

$$\begin{aligned}
& \left(\frac{\phi_y^2}{2} + \frac{V}{2Z}\psi_{1y}^2 + \sum_{i=2}^3 \frac{\psi_{iy}^2}{2} + \frac{\zeta_y^2}{3Z^2}\right)_\tau + \frac{4\mu(\Theta)}{3Z}\psi_{1yy}^2 + \sum_{i=2}^3 \frac{\mu(\Theta)}{V}\psi_{iyy}^2 + \frac{\kappa(\Theta)}{3Z^2V}\zeta_{yy}^2 \\
& = I_7(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \psi_{yy}, \zeta_{yy}) - (N_5 - \bar{Q}_{1y} - Q_{1y})\left(\frac{V\psi_{1y}}{Z}\right)_y - \sum_{i=2}^3 (N_{i+4} - \bar{Q}_{iy} - Q_{iy})\psi_{iyy} \\
& - \left(\frac{2\zeta_y}{3Z^2}\right)_y [N_8 - \bar{Q}_{4y} - Q_{4y} + \sum_{i=1}^3 U_i(\bar{Q}_{iy} + Q_{iy})] + K_5 + (\dots)_y,
\end{aligned} \tag{4.2.22}$$

where

$$\begin{aligned}
& I_7(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y, \psi_{yy}, \zeta_{yy}) \\
& = \left(\frac{V}{2Z}\right)_\tau \psi_{1y}^2 + \left(\frac{1}{3Z^2}\right)_\tau \zeta_y^2 + \left(\frac{2}{3Z}\right)_y \psi_{1y} \zeta_y - \frac{V}{Z} \psi_{1y} H_5 - \sum_{i=2}^3 \psi_{iy} H_{i+4} \\
& - \frac{2}{3Z^2} \zeta_y H_8 - \left(\frac{4\mu(\Theta)\psi_{1y}}{3V}\right)_y \left(\frac{V}{Z}\right)_y \psi_{1y} - \left(\frac{4\mu(\Theta)}{3V}\right)_y \psi_{1y} \frac{V}{Z} \psi_{1yy} \\
& - \sum_{i=2}^3 \left(\frac{\mu(\Theta)}{V}\right)_y \psi_{iy} \psi_{iyy} - \left(\frac{\kappa(\Theta)\zeta_y}{V}\right)_y \left(\frac{2}{3Z^2}\right)_y \zeta_y - \left(\frac{\kappa(\Theta)}{V}\right)_y \zeta_y \frac{2}{3Z^2} \zeta_{yy} \\
& \leq \beta \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 + C_\beta (\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)\|^2,
\end{aligned} \tag{4.2.23}$$

and

$$\begin{aligned}
K_5 & = \left(\frac{V\psi_{1y}}{Z}\right)_y \int \xi_1^2 (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi + \sum_{i=2}^3 \psi_{iyy} \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \\
& + \left(\frac{2\zeta_y}{3Z^2}\right)_y \left[\int \xi_1 \frac{|\xi|^2}{2} (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi - \sum_{i=1}^3 \psi_i \int \xi_1 \xi_i \Pi_{1y} d\xi \right. \\
& \left. - \sum_{i=1}^3 U_i \int \xi_1 \xi_i (\Pi_1 - \Pi_{11}^{CD} - \Pi_1^{S_3})_y d\xi \right].
\end{aligned} \tag{4.2.24}$$

Integrating (4.2.22) with respect to τ, y , and substituting (4.2.6), (4.2.7) and (4.2.19) into (4.2.22) and (4.2.24), choosing β, χ, ε small enough, we have

$$\begin{aligned}
& \|(\phi_y, \psi_y, \zeta_y)(\tau, \cdot)\|^2 + \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_{yy}, \zeta_{yy})\|^2 d\tau \\
& \leq \beta \int_{\frac{h}{\varepsilon}}^{\tau} \|\phi_{yy}\|^2 d\tau + C_{\beta}(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
& \quad + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\
& \quad + C_{h,T}\chi \sum_{|\alpha|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau.
\end{aligned} \tag{4.2.25}$$

Again, to recover $\|\phi_{yy}\|^2$ in the dissipation rate, applying ∂_y to (3.27)₂, we get

$$\psi_{1y\tau} + (p-P)_{yy} = -\frac{4}{3} \left[\frac{\mu(\Theta)}{V} U_{1y} - \frac{\mu(\Theta^{S_3})}{V^{S_3}} U_{1y}^{S_3} \right]_{yy} - \int \xi_1 \xi_i \Pi_{11yy}^{CD} d\xi - \bar{Q}_{1yy} - Q_{1yy} - \int \xi_1^2 \tilde{\mathbf{G}}_{yy} d\xi. \tag{4.2.26}$$

Note that

$$(p-P)_{yy} = -\frac{p}{v} \phi_{yy} + \frac{2}{3v} \zeta_{yy} - \frac{1}{v} (p-P) V_{yy} - \frac{\phi}{v} P_{yy} - \frac{2v_y}{v} (p-P)_y - \frac{2P_y}{v} \phi_y. \tag{4.2.27}$$

Multiplying (4.2.26) by $-\phi_{yy}$ and using (4.2.27) imply

$$\begin{aligned}
& - \int \psi_{1y} \phi_{yy}(\tau, y) dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \frac{p}{2v} \phi_{yy}^2 dy d\tau \\
& \leq C(\delta + C_{h,T}\chi) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta, \phi_y, \psi_y, \zeta_y)\|^2 d\tau \\
& \quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\psi_{1yy}, \zeta_{yy})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau,
\end{aligned} \tag{4.2.28}$$

where we have used the fact that

$$\int_{\frac{h}{\varepsilon}}^{\tau} \int (|\bar{Q}_{1yy}|^2 + |Q_{1yy}|^2) dy d\tau \leq C_{h,T} \varepsilon^{\frac{1}{2}}. \tag{4.2.29}$$

To estimate $\|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2$ and $\|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2$, we use the system (3.27) again. Applying ∂_y to (3.27), and multiplying the four equations of (3.27) by $\phi_{y\tau}, \psi_{1y\tau}, \psi_{iy\tau}$ ($i = 2, 3$), $\zeta_{y\tau}$ respectively, then adding them together and integrating with respect to τ and y , we have

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2 d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{yy}, \psi_{yy}, \zeta_{yy})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
& \quad + C_{h,T}\chi \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau + C(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau \\
& \quad + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau.
\end{aligned} \tag{4.2.30}$$

Similarly, we can obtain

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{\tau\tau}, \psi_{\tau\tau}, \zeta_{\tau\tau})\|^2 d\tau \leq C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{y\tau}, \psi_{y\tau}, \zeta_{y\tau})\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}} \\
& \quad + C_{h,T}\chi \sum_{|\alpha|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha} (\phi, \psi, \zeta)\|^2 d\tau + C(\delta + C_{h,T}\varepsilon^{\frac{1}{2}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau \\
& \quad + C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_{\star}} |\partial^{\alpha} \tilde{\mathbf{G}}|^2 d\xi dy d\tau.
\end{aligned} \tag{4.2.31}$$

A suitable linear combination of (4.2.25)-(4.2.31) gives

$$\begin{aligned}
& \|(\phi_y, \psi_y, \zeta_y)(\tau, \cdot)\|^2 - \int \psi_{1y} \phi_{yy} dy + \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 d\tau \\
& \leq C \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau + C \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^{\alpha'} \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\
& \quad + C_{h,T\chi} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_1|^2 d\xi dy d\tau + C(\delta + C_{h,T\varepsilon^{\frac{1}{2}}}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|^2 d\tau \\
& \quad + C_{h,T\chi} \sum_{|\alpha'|=1} \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^{\alpha'}(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.2.32}$$

Step 4: Estimation on the non-fluid component:

To close the a priori estimate, we also need to estimate the derivatives on the non-fluid component $\tilde{\mathbf{G}}$, i.e., $\partial^\alpha \tilde{\mathbf{G}}$, ($|\alpha| = 1, 2$). For this, from (3.11), we obtain

$$v\tilde{\mathbf{G}}_\tau - v\mathbf{L}_M\tilde{\mathbf{G}} = u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y) - v\left[\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3})\right] + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1. \tag{4.2.33}$$

Applying ∂_y on (4.2.33), we have

$$\begin{aligned}
v\tilde{\mathbf{G}}_{y\tau} - (v\mathbf{L}_M\tilde{\mathbf{G}})_y & = \left\{ u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y) - v\left[\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3})\right] \right. \\
& \quad \left. + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1 \right\}_y - v_y\tilde{\mathbf{G}}_\tau.
\end{aligned} \tag{4.2.34}$$

Multiplying (4.2.34) by $\frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star}$ gives

$$\begin{aligned}
\left(\frac{v\tilde{\mathbf{G}}_y^2}{2\mathbf{M}_\star}\right)_\tau - \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star}(v\mathbf{L}_M\tilde{\mathbf{G}})_y & = \left\{ u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y) - v\left[\frac{1}{v}\mathbf{P}_1(\xi_1\mathbf{M}_y) - \frac{1}{V^{S_3}}\mathbf{P}_1^{S_3}(\xi_1\mathbf{M}_y^{S_3})\right] \right. \\
& \quad \left. + 2vQ(\tilde{\mathbf{G}}, \mathbf{G}^{S_3}) + vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + vJ_1 \right\}_y \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} - v_y\tilde{\mathbf{G}}_\tau \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} + v_\tau \frac{\tilde{\mathbf{G}}_y^2}{2\mathbf{M}_\star}.
\end{aligned} \tag{4.2.35}$$

Then Lemma 3.2-Lemma 3.5 and Cauchy inequality give

$$\begin{aligned}
& - \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} (v\mathbf{L}_M\tilde{\mathbf{G}})_y d\xi dy d\tau \\
& = - \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} \left(v\mathbf{L}_M\tilde{\mathbf{G}}_y + 2v_yQ(\mathbf{M}, \tilde{\mathbf{G}}) + 2vQ(M_y, \tilde{\mathbf{G}}) \right) d\xi dy d\tau \\
& \geq \frac{7\tilde{\sigma}}{8} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau - C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau \\
& \quad - C\varepsilon \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_y, \psi_y, \zeta_y)\|^2 d\tau - C_{h,T} \varepsilon^{\frac{1}{2}},
\end{aligned} \tag{4.2.36}$$

and

$$\begin{aligned}
& \int_{\frac{h}{\varepsilon}}^{\tau} \int \int -v_y\tilde{\mathbf{G}}_\tau \frac{\tilde{\mathbf{G}}_y}{\mathbf{M}_\star} + v_\tau \frac{\tilde{\mathbf{G}}_y^2}{2\mathbf{M}_\star} + (u_1\tilde{\mathbf{G}}_y - \mathbf{P}_1(\xi_1\tilde{\mathbf{G}}_y))_y \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
& \leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_\tau^2}{\mathbf{M}_\star} d\xi dy d\tau \\
& \quad + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)\tilde{\mathbf{G}}_{yy}^2}{\mathbf{M}_\star} d\xi dy d\tau.
\end{aligned} \tag{4.2.37}$$

Note that

$$\begin{aligned}
J_6 &\triangleq -v \left[\frac{1}{v} \mathbf{P}_1(\xi_1 \mathbf{M}_y) - \frac{1}{V^{S_3}} \mathbf{P}_1^{S_3}(\xi_1 \mathbf{M}_y^{S_3}) \right] \\
&= -\frac{3}{2\theta} \mathbf{P}_1 \left[\xi_1 \mathbf{M} \left(\frac{|\xi - u|^2}{2\theta} (\theta_y - \Theta_y^{S_3}) + \xi \cdot (u_y - U_y^{S_3}) \right) \right] - v \left[\frac{3}{2v\theta} \mathbf{P}_1[\xi_1 \mathbf{M} \left(\frac{|\xi - u|^2}{2\theta} \Theta_y^{S_3} + \xi \cdot U_y^{S_3} \right)] \right. \\
&\quad \left. - \frac{3}{2V^{S_3} \Theta^{S_3}} \mathbf{P}_1^{S_3}[\xi_1 \mathbf{M}^{S_3} \left(\frac{|\xi - U^{S_3}|^2}{2\Theta^{S_3}} \Theta_y^{S_3} + \xi \cdot U_y^{S_3} \right)] \right]. \tag{4.2.38}
\end{aligned}$$

Then Lemma 3.2-Lemma 3.5 and wave interaction estimates imply that

$$\begin{aligned}
&\int_{\frac{h}{\varepsilon}}^{\tau} \int \int (vJ_1 + J_6)_y \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
&\leq \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau \\
&\quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi, \psi, \zeta)\|_{H^1(dy)}^2 d\tau + C_{h,T\varepsilon} \frac{1}{2}, \tag{4.2.39}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{h}{\varepsilon}}^{\tau} \int \int (vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}))_y \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
&= \int_{\frac{h}{\varepsilon}}^{\tau} \int \int (v_y Q(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}) + 2vQ(\tilde{\mathbf{G}}, \tilde{\mathbf{G}}_y)) \frac{\tilde{\mathbf{G}}_y}{2\mathbf{M}_\star} d\xi dy d\tau \\
&\leq C_{h,T\varepsilon} \frac{1}{2} + \frac{\tilde{\sigma}}{32} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_y^2}{\mathbf{M}_\star} d\xi dy d\tau + C(\chi + \delta) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|) \tilde{\mathbf{G}}_1^2}{\mathbf{M}_\star} d\xi dy d\tau. \tag{4.2.40}
\end{aligned}$$

Thus, integrating (4.2.35) with respect to ξ, y and τ and using (4.2.36), (4.2.37) and (4.2.39), (4.2.40), we obtain

$$\begin{aligned}
&\int \int \frac{|\tilde{\mathbf{G}}_y|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_y|^2 d\xi dy d\tau \\
&\leq C_{h,T\varepsilon} \frac{1}{2} + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_{yy}|^2 d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{yy}, \zeta_{yy})\|^2 d\tau \\
&\quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_\tau|^2) d\xi dy d\tau \\
&\quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} [\|(\phi, \psi, \zeta)\|^2 + \|(\phi, \psi, \zeta)_y\|^2] d\tau. \tag{4.2.41}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int \int \frac{|\tilde{\mathbf{G}}_\tau|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_\tau|^2 d\xi dy d\tau \\
&\leq C_{h,T\varepsilon} \frac{1}{2} + C \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\tilde{\mathbf{G}}_{y\tau}|^2 d\xi dy d\tau + C \int_{\frac{h}{\varepsilon}}^{\tau} \|(\phi_{y\tau}, \zeta_{y\tau})\|^2 d\tau \\
&\quad + C(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} (|\tilde{\mathbf{G}}_1|^2 + |\tilde{\mathbf{G}}_y|^2) d\xi dy d\tau \\
&\quad + CC(\delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} [\|(\phi, \psi, \zeta)\|^2 + \|(\phi, \psi, \zeta)_\tau\|^2] d\tau. \tag{4.2.42}
\end{aligned}$$

Step 5: Highest order estimates:

Finally, we estimate the highest order derivatives, that is, $-\int \psi_{1y} \phi_{yy} dy$ and $\int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)|\partial^\alpha \tilde{\mathbf{G}}|^2}{\mathbf{M}_\star} d\xi dy d\tau$ with $|\alpha| = 2$ in (4.2.32). To do so, it is sufficient to study $\int \int \frac{|\partial^\alpha \tilde{f}|^2}{\mathbf{M}_\star} d\xi dy$ ($|\alpha| = 2$) in view of (3.38) and (3.37). Using the same idea in [19], we obtain the estimation for the highest order derivative terms, i.e.,

$$\begin{aligned} & \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + \sum_{|\alpha|=2} \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} |\partial^\alpha \tilde{\mathbf{G}}|^2 d\xi dy d\tau \\ & \leq C(\eta_0 + \delta + C_{h,T\chi}) \int_{\frac{h}{\varepsilon}}^{\tau} \int \int \frac{\nu(|\xi|)}{\mathbf{M}_\star} (|\tilde{\mathbf{G}}_y|^2 + |\tilde{\mathbf{G}}_\tau|^2 + |\tilde{\mathbf{G}}_1|^2) d\xi dy d\tau \\ & \quad + C(\eta_0 + \delta + C_{h,T\chi}) \sum_{|\alpha|=0}^2 \int_{\frac{h}{\varepsilon}}^{\tau} \|\partial^\alpha(\phi, \psi, \zeta)\|^2 d\tau + C_{h,T} \varepsilon^{\frac{1}{2}}, \end{aligned} \quad (4.2.43)$$

where η_0 is defined in Lemma 3.3.

Noting that

$$\begin{aligned} -\int \psi_{1y} \phi_{yy} dy & \leq \beta \|\psi_{1y}\|^2 + C_\beta \|\phi_{yy}\|^2 \\ & \leq \beta \|\psi_{1y}\|^2 + C_\beta \sum_{|\alpha|=2} \int \int \frac{|\partial^\alpha \tilde{f}|^2}{2\mathbf{M}_\star}(\tau, y, \xi) d\xi dy + C_{h,T,\beta} \varepsilon^{\frac{1}{2}}, \end{aligned}$$

and combining the estimates (4.2.32), (4.2.41), (4.2.42) and (4.2.43), we complete the proof of Proposition 3.2.

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