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(6) **SIGNAL RECONSTRUCTION
AFTER NOISY NONLINEAR TRANSFORMATIONS.**

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Abstract

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A deterministic signal s in zero mean Gaussian noise N is observed through a zero memory nonlinearity $f(x)$. The reconstruction of the signal is considered when the nonlinearity, the noise covariance and the first or second order moments of the output process $f[s+N]$ are known. Arbitrary signals can be reconstructed for monotonic and certain odd, not necessarily monotonic, nonlinearities; included here are hard limiters, quantizers and infinite interval windows. Arbitrary signals can be reconstructed, up to a global sign, for two distinct classes of even nonlinearities; included here are $2v$ -th law devices and symmetric interval windows. ←

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I. INTRODUCTION

Consider a signal s observed through a nonlinear system and suppose that we want to reconstruct the signal from the output of the system. Unless the nonlinear system is one-to-one, an unrealistic situation, the signal cannot be uniquely reconstructed. As a simple example, consider the zero memory nonlinearity $f(x) = 1_{(a, \infty)}(x)$: The only information about $s(t)$ contained in the output $f[s(t)]$ is whether $s(t) \leq a$ or $s(t) > a$ and it is evident that there is an uncountably infinite number of signals having the same output. Intuitively, this situation may improve if there is an additive noise N at the input of the nonlinear system so that the observed output is $f[s(t) + N(t)]$. If the set of values of the random variable $N(t)$ is sufficiently large, knowledge of certain moments of the output $f[s(t) + N(t)]$ may provide more information on the signal than in the absence of noise. This is indeed true when the marginal distributions of the noise are Gaussian, in which case the signal can be uniquely reconstructed as shown in Theorem 1. The idea that additive input noise helps in reconstructing the signal was first raised by Grünbaum [1] and is investigated in this paper.

Specifically, we consider a deterministic signal s in additive stationary Gaussian noise N , with zero mean and covariance function R , observed through a zero memory nonlinearity f . We are concerned with the reconstruction of the signal from the knowledge of the nonlinearity f , the noise covariance R , and the first or second order moments of the output process $f(s+N)$. We show that the signal can be reconstructed in the following cases.

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(i) From the knowledge of the nonlinearity f , the variance $R(0)$ of the noise, and the first or second moment of the output, the signal is reconstructed for monotonic nonlinearities which need not be strictly monotonic (Theorem 1) as well as for some odd nonlinearities with non-negative Hermite coefficients (Theorem 2). Included here are hard and soft limiters and infinite interval windows.

(ii) From the knowledge of the nonlinearity f , the noise covariance function R whose zeros are assumed isolated, and the mean and correlation functions of the output, the signal is reconstructed up to a global sign for the following two classes of even nonlinearities: (a) bounded below or above and monotonic on the positive real line, (b) with nonnegative Hermite coefficients. Included here are $2v^{\text{th}}$ law devices and symmetric interval windows. Similar results are, in fact, established for nonlinearities symmetric around an arbitrary point. (Theorem 4).

Here we are assuming that the noise covariance function R is known. The problem of reconstructing the noise covariance R from the correlation function of the output process $f(N)$, i.e. the signal-free case, has been considered in [2],[3].

The stationarity assumption on the Gaussian noise is used only to make the signal reconstruction feasible from a practical viewpoint. Even though it is not stated so, Theorems 1 to 3 need only the one-dimensional distributions of the noise to be Gaussian, while Theorem 4 requires the bivariate distributions of the noise to be Gaussian. It should be clear that results similar to Theorems 1 to 4 would also be true for other noise processes with sufficiently smooth, symmetric first and second order distributions.

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The organization of the paper is as follows: The results on the reconstruction of the signal are presented and discussed in Section III. The derivations in Section III are kept to a minimum by collecting the essential elements of the proofs into propositions which are presented in Section II and could be of independent interest.

II. CERTAIN MOMENT PROPERTIES OF FUNCTIONS OF GAUSSIAN RANDOM VARIABLES

In this section we study, for use in the following section, certain properties of the moment functions

$$\mu_f(x) = E[f(\xi+x)] \quad (1)$$

and

$$\Gamma_f(x,y;\rho) = E[f(\xi+x)f(\eta+y)] \quad (2)$$

where ξ and η are jointly Gaussian random variables with means zero, variances σ^2 , correlation coefficient ρ , joint density $\phi(x,y;\rho)$ and marginal densities $\phi(x;\sigma)$; f is a (nonconstant) real-valued Borel measurable function on the real line. As the notation indicates, we treat f and σ as fixed and we consider the dependence of μ on $x \in (-\infty, \infty)$ and of Γ on $x, y \in (-\infty, \infty)$ and $\rho \in [-1, 1]$.

The following moment inequality simplifies the conditions under which μ and Γ exist

$$E|f(\xi+x)| \leq \exp[x^2/2\sigma^2] E^{1/2}[f^2(\xi)]. \quad (3)$$

Thus

$$E[f^2(\xi)] < \infty \Rightarrow E|f(\xi+x)| < \infty \text{ for all } x.$$

This can be obtained as follows:

$$\begin{aligned} E|f(\xi+x)| &= \int_{-\infty}^{\infty} |f(u+x)|\phi(u;\sigma)du = \int_{-\infty}^{\infty} |f(y)|\phi(y-x;\sigma)dy \\ &= \exp[x^2/2\sigma^2] \int_{-\infty}^{\infty} |f(y)|\phi^{1/2}(y;\sigma)\phi^{1/2}(y-2x;\sigma)dy \\ &\leq \exp[x^2/2\sigma^2] \left\{ \int_{-\infty}^{\infty} f^2(y)\phi(y;\sigma)dy \cdot \int_{-\infty}^{\infty} \phi(y-2x;\sigma)dy \right\}^{1/2} \\ &= \exp[x^2/2\sigma^2] E^{1/2}[f^2(\xi)]. \end{aligned}$$

It then follows from (3) that $\mu_f(x)$ exists for all x provided $E[f^2(\xi)] < \infty$, and $\Gamma_f(x,y;\rho)$ exists for all x,y and $\rho \in [-1,1]$ provided $E[f^4(\xi)] < \infty$.

We first derive certain analytical properties of μ_f .

Proposition 1. If $E[f^2(\xi)] < \infty$, then for all $-\infty < x < \infty$, $\mu_f(x)$ has the power series representation

$$\mu_f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad (4)$$

where

$$a_k = \frac{1}{k! \sigma^{2k}} \int_{-\infty}^{\infty} f(x) H_{k,\sigma}(x) \phi(x;\sigma) dx$$

and the convergence is absolute. $\mu_f(x)$ is infinitely differentiable and $\mu_f^{(n)}(x)$ has the integral representation

$$\mu_f^{(n)}(x) = \frac{1}{\sigma^{2n}} \int_{-\infty}^{\infty} f(y+x) H_{n,\sigma}(y) \phi(y;\sigma) dy. \quad (5)$$

Proof. The Hermite polynomials $\{H_{k,\sigma}(\xi)\}_{k=0}^{\infty}$ are the orthogonalization (not orthonormalization) of the sequence $\{\xi^k\}_{k=0}^{\infty}$ via the Gram-Schmidt procedure.

$\{H_{k,\sigma}(x)\}_{k=0}^{\infty}$ is a complete orthogonal set in $L_2(\phi(x;\sigma)dx)$ and

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} H_{k,\sigma}(x) = \exp\left(zx - \frac{1}{2} \sigma^2 z^2\right)$$

$$E[H_{j,\sigma}(\xi) H_{k,\sigma}(\xi)] = k! \sigma^{2k} \delta_{jk}$$

$$E[H_{k,\sigma}(\xi+x)] = x^k \quad (6)$$

where (6) is shown in [1]. Since $f(x) \in L_2(\phi(x;\sigma)dx)$, we have

$$f(\xi) = \sum_{k=0}^{\infty} a_k H_{k,\sigma}(\xi)$$

in quadratic mean, where

$$E[f(\xi)H_{k,\sigma}(\xi)] = a_k E[H_{k,\sigma}^2(\xi)]$$

or

$$\int_{-\infty}^{\infty} f(y)H_{k,\sigma}(y)\phi(y;\sigma)dy = a_k k! \sigma^{2k} .$$

Now inequality (3) implies that

$$f(y+x) = \sum_{k=0}^{\infty} a_k H_{k,\sigma}(y+x)$$

in $L_1(\phi(y;\sigma)dy)$ for all x and hence

$$\begin{aligned} \mu(x) &= E[f(\xi+x)] = \sum_{k=0}^{\infty} a_k E[H_{k,\sigma}(\xi+x)] \\ &= \sum_{k=0}^{\infty} a_k x^k \end{aligned}$$

where the last step follows by (6). The absolute convergence of the series (4) is seen as follows. We have

$$\infty > E[f^2(\xi)] = \sum_{k=0}^{\infty} a_k^2 E[H_{k,\sigma}^2(\xi)] = \sum_{k=0}^{\infty} a_k^2 \sigma^{2k} k! ,$$

so that $a_k^2 \sigma^{2k} k! \rightarrow 0$ as $k \rightarrow \infty$ and for large k , $|a_k| \leq \epsilon \sigma^{-k} / \sqrt{k!}$. Thus it suffices to show that for each x

$$\sum_{k=0}^{\infty} \frac{(|x|/\sigma)^k}{\sqrt{k!}} < \infty$$

which follows easily from the ratio test. The power series representation (4) implies that $\mu(x)$ is infinitely differentiable, has a finite number of extrema on each finite interval, and is not constant on any interval.

The derivation of (5) proceeds as follows. From (4) we have

$$\begin{aligned} \mu^{(n)}(x) &= \sum_{k=n}^{\infty} a_k \frac{k!}{(k-n)!} x^{k-n} \\ &= \sum_{k=n}^{\infty} \frac{x^{k-n}}{\sigma^{2k} (k-n)!} \int_{-\infty}^{\infty} f(y)H_{k,\sigma}(y)\phi(y;\sigma)dy . \end{aligned}$$

Hence if for each fixed x

$$\sum_{k=n}^{\infty} \int_{-\infty}^{\infty} |f(y)| |H_{k,\sigma}(y)| \frac{|x|^{k-n}}{\sigma^{2k}(k-n)!} \phi(y;\sigma) dy < \infty \quad (7)$$

then by the bounded convergence theorem,

$$\mu^{(n)}(x) = \int_{-\infty}^{\infty} f(y) \left[\sum_{k=n}^{\infty} \frac{x^{k-n}}{\sigma^{2k}(k-n)!} H_{k,\sigma}(y) \right] \phi(y;\sigma) dy .$$

Now the term in brackets is equal to

$$\begin{aligned} \frac{\partial^n}{\partial x^n} \sum_{k=0}^{\infty} \frac{(x/\sigma^2)^k}{k!} H_{k,\sigma}(y) &= \frac{\partial^n}{\partial x^n} \exp\left[\frac{2xy-x^2}{2\sigma^2}\right] \\ &= \exp[y^2/2\sigma^2] \frac{\partial^n}{\partial x^n} \exp\left[-\frac{1}{2\sigma^2}(y-x)^2\right] \\ &= \frac{1}{\phi(y;\sigma)} \sigma^{-2n} H_{n,\sigma}(y-x) \phi(y-x;\sigma) . \end{aligned}$$

Thus

$$\mu^{(n)}(x) = \frac{1}{\sigma^{2n}} \int_{-\infty}^{\infty} f(y) H_{n,\sigma}(y-x) \phi(y-x;\sigma) dy .$$

We now verify (7). The expression in (7) is equal to

$$\begin{aligned} &\sum_{k=n}^{\infty} \frac{|x|^{k-n}}{\sigma^{2k}(k-n)!} E[|f(\xi)| |H_{k,\sigma}(\xi)|] \\ &\leq \sum_{k=n}^{\infty} \frac{|x|^{k-n}}{\sigma^{2k}(k-n)!} E^{1/2}[f^2(\xi)] E^{1/2}[H_{k,\sigma}^2(\xi)] \\ &= E^{1/2}[f^2(\xi)] \sum_{k=n}^{\infty} \frac{|x|^{k-n} \sqrt{k!}}{\sigma^k (k-n)!} = \frac{1}{\sigma^n} E^{1/2}[f^2(\xi)] \sum_{j=0}^{\infty} \frac{(|x|/\sigma)^j \sqrt{(j+n)!}}{j!} \end{aligned}$$

and the sum in the last expression above is finite by the root test. \square

We now derive a series expansion of the second order moment function $\Gamma_f(x,y;\rho)$ in terms of the derivatives of the first order moment function $\mu_f(x)$.

Proposition 2. If $E[f^4(\xi)] < \infty$, then for all $-\infty < x, y < \infty$ and $-1 < \rho < 1$, we have

$$\Gamma_f(x, y; \rho) = \sum_{n=0}^{\infty} \frac{\rho^n \sigma^{2n}}{n!} \mu_f^{(n)}(x) \mu_f^{(n)}(y).$$

Proof. Since $E[f^4(\xi)] < \infty$, inequality (3) implies that $E[f^2(\xi+x)] < \infty$ for all $-\infty < x < \infty$ and thus

$$\Gamma(x, y; \rho) = E[f(\xi+x)f(\eta+y)]$$

is well defined and finite for all $-\infty < x, y < \infty$ and $-1 \leq \rho \leq 1$. Now for all $-\infty < u, v < \infty$ and $-1 < \rho < 1$ we have [4]

$$\frac{\phi(u, v; \rho)}{\phi(u; \sigma)\phi(v; \sigma)} = \sum_{n=0}^{\infty} \frac{\rho^n}{n! \sigma^{2n}} H_{n, \sigma}(u) H_{n, \sigma}(v),$$

and, since $\sum_{n=0}^{\infty} |\rho|^{2n} < \infty$, the convergence is also in $L_2(\phi(u; \sigma)\phi(v; \sigma) du dv)$.

Finally, since $f(u+x)f(v+y)$ is in $L_2(\phi(u; \sigma)\phi(v; \sigma) du dv)$, it follows that

$$\begin{aligned} \Gamma(x, y; \rho) &= \iint_{-\infty}^{\infty} f(u+x)f(v+y) \left[\frac{\phi(u, v; \rho)}{\phi(u; \sigma)\phi(v; \sigma)} \right] \phi(u; \sigma)\phi(v; \sigma) du dv \\ &= \sum_{n=0}^{\infty} \frac{\rho^n}{n! \sigma^{2n}} \iint_{-\infty}^{\infty} f(u+x)f(v+y) H_{n, \sigma}(u) H_{n, \sigma}(v) \phi(u; \sigma)\phi(v; \sigma) du dv \end{aligned}$$

and the result follows from (5) of Proposition 1. \square

We finally establish a certain lack of symmetry of the function Γ_f , which plays a crucial role in the reconstruction of the signal when the function f is symmetric.

Proposition 3. If the nonconstant function f is symmetric around some x_0 and satisfies (a) or (b):

- (a) f is bounded below or above, monotonic on $[x_0, \infty)$ and such that $E[f^4(\xi)] < \infty$,
- (b) $E[f^4(\xi+x_0)] < \infty$ and the coefficients $\{a_{2n}\}_{n=1}^{\infty}$ of $f(x+x_0)$ in its Hermite expansion are nonnegative,

then

$$\Gamma_f(x_0+x, x_0+y; \rho) \neq \Gamma_f(x_0-x, x_0+y; \rho) \quad \text{for all } x, y, \rho \neq 0. \quad (8)$$

$$\Gamma_f(x_0+x, x_0+y; \rho) \neq \Gamma_f(x_0+x, x_0-y; \rho)$$

It should be noted that when $\rho=0$, equality holds in (8) for all x, y .

Indeed, we have

$$\Gamma_f(x, y; 0) = \mu_f(x)\mu_f(y)$$

and for all x ,

$$\mu_f(x_0+x) - \mu_f(x_0-x) = \int_{-\infty}^{\infty} [f(u+x_0+x) - f(u+x_0-x)]\phi(u; \sigma)du = 0$$

since $f(u+x_0+x) - f(u+x_0-x)$ is an odd function of u .

Proof. Putting $g(x) = f(x+x_0)$, it is clear that g has all the properties that f has with $x_0=0$ and that

$$\begin{aligned} \Gamma_g(x, y; \rho) &= E[g(\xi+x)g(\eta+y)] \\ &= E[f(\xi+x_0+x)f(\eta+x_0+y)] \\ &= \Gamma_f(x_0+x, x_0+y; \rho). \end{aligned}$$

Thus it suffices to prove (8) in the case $x_0=0$.

(a) Assume f satisfies condition (a). Without loss of generality, we may replace the property that f is bounded below (or above) by the property that $f(x) \geq 0$ (or ≤ 0) for all x . Indeed, if f is bounded below, say, i.e. if $f(x) \geq A > -\infty$ for all x , putting $q(x) = f(x) - A$, we have $q(x) \geq 0$ and

$$\begin{aligned} \Gamma_q(x, y; \rho) - \Gamma_q(-x, y; \rho) &= \Gamma_f(x, y; \rho) - \Gamma_f(-x, y; \rho) \\ \Gamma_q(x, y; \rho) - \Gamma_q(x, -y; \rho) &= \Gamma_f(x, y; \rho) - \Gamma_f(x, -y; \rho). \end{aligned}$$

These equations follow from

$$\Gamma_q(x, y; \rho) - \Gamma_q(x, -y; \rho) = \Gamma_f(x, y; \rho) - \Gamma_f(x, -y; \rho) + AE[f(\eta+y) - f(\eta-y)]$$

and the fact that $[f(v+y) - f(v-y)]$ is an odd function of v which implies that $E[f(\eta+y)-f(\eta-y)] = 0$.

Thus, under Assumption (a), it suffices to prove the result when the nonconstant function f is nonnegative, even and nondecreasing on $[0, \infty)$.

When $\rho = \pm 1$, then $\xi = \pm \eta$ a.s. and

$$\Gamma(x, y; \pm 1) - \Gamma(x, -y; \pm 1) = \int_{-\infty}^{\infty} f(\pm u+x)[f(u+y)-f(u-y)]\phi(u; \sigma) du.$$

It is seen that the term in bracket is an odd function of u which is nonnegative on $[0, \infty)$. Since f is even, nonnegative and nondecreasing on $[0, \infty)$, it follows that for all $x, y \neq 0$, the integral is nonzero.

From now on we assume that $0 < \rho < 1$ (the case $-1 < \rho < 0$ can be treated similarly). Then

$$\Gamma(x, y; \rho) - \Gamma(x, -y; \rho) = \iint_{-\infty}^{\infty} f(u+x)[f(v+y)-f(v-y)]\phi(u, v; \rho) du dv.$$

Denoting by $\phi(v|u) = \phi(u, v; \rho)/\phi(u; \sigma)$ the conditional density of η given $\xi = u$, which is Gaussian with mean ρu and variance $\sigma^2(1-\rho^2)$, we have

$$\begin{aligned} \Gamma(x, y; \rho) - \Gamma(x, -y; \rho) &= \int_{-\infty}^{\infty} f(u+x)\phi(u; \sigma) \left\{ \int_{-\infty}^{\infty} f(z)[\phi(z-y|u) - \phi(z+y|u)] dz \right\} du \\ &= \int_{-\infty}^{\infty} f(u+x)\phi(u; \sigma) J(u; y) du, \end{aligned} \quad (9)$$

where

$$J(u, y) = \int_{-\infty}^{\infty} f(z)[\phi(z-y|u) - \phi(z+y|u)] dz.$$

Note that as a function of z , $[\phi(z-y|u) - \phi(z+y|u)]$ is antisymmetric around $z = \rho u$ and is positive on $(\rho u, \infty)$. Now since $f(z)$ is even and nondecreasing on $[0, \infty)$, we can write (see Fig. 1)

$$J(u,y) = \int_{-\infty}^{-\rho u} + \int_{-\rho u}^{\rho u} + \int_{\rho u}^{3\rho u} + \int_{3\rho u}^{\infty}$$

and conclude that for $u > 0$ we have

$$\int_{-\infty}^{-\rho u} + \int_{3\rho u}^{\infty} \geq 0 \quad \text{and} \quad \int_{-\rho u}^{\rho u} + \int_{\rho u}^{3\rho u} \geq 0$$

so that $J(u,y) \geq 0$ for $u > 0$, and similarly $J(u,y) \leq 0$ for $u < 0$. It is also clear that $J(0,y) = 0$, that $J(u,y)$ is an odd function of u , and that $J(u,y)$ is not identically zero as a function of u . Finally, since f is even, non-constant, nonnegative, and nondecreasing on $[0, \infty)$, it follows from (9) that for all $x, y \neq 0$, we have

$$\Gamma(x,y;\rho) - \Gamma(x,-y;\rho) \neq 0.$$

It is shown similarly that $\Gamma(x,y;\rho) - \Gamma(-x,y;\rho) \neq 0$ for all $x, y \neq 0$. Thus the proof of the proposition under Assumption (a) is complete.

(b) Assume now that f satisfies Assumption (b) (with $x_0 = 0$). Since f is even, we have $a_n = 0$ for all odd n so that

$$f(\xi) = \sum_{n=0}^{\infty} a_{2n} H_{2n,\sigma}(\xi) \tag{10}$$

in quadratic mean.

We first consider the case $\rho = 1$ (the case $\rho = -1$ being similar). Then $\xi = \eta$ a.s. and

$$\begin{aligned} \Gamma(x,y;1) &= E[f(\xi+x)f(\xi+y)] = \int_{-\infty}^{\infty} f(u+x)f(u+y)\phi(u;\sigma)du \\ &= \int_{-\infty}^{\infty} f(z)f(z-x+y)\phi(z-x;\sigma)dz \\ &= E\{f(\xi)[f(\xi-x+y)\phi(\xi-x;\sigma)\phi^{-1}(\xi;\sigma)]\} \end{aligned} \tag{11}$$

Now the random variable in bracket has a finite second moment for all x, y , since

$$\begin{aligned} \int_{-\infty}^{\infty} f^2(z-x+y) \frac{\phi^2(z-x;\sigma)}{\phi(z;\sigma)} dz &= \exp[x^2/\sigma^2] \int_{-\infty}^{\infty} f^2(z-x+y)\phi(z-2x;\sigma) dz \\ &= \exp[x^2/\sigma^2] E[f^2(\xi+x+y)] \\ &< \exp\left\{\frac{(x+y)^2+2x^2}{2\sigma^2}\right\} E^{1/2}[f^4(\xi)] < \infty \end{aligned}$$

where the inequality follows from (3) applied to f^2 . Since f has the expansion (10) in quadratic mean, we then have by (11)

$$\Gamma(x, y; 1) = \sum_{n=0}^{\infty} a_{2n} \int_{-\infty}^{\infty} H_{2n, \sigma}(z) f(z-x+y) \phi(z-x; \sigma) dz. \quad (12)$$

Similarly, we obtain for the integral in (12)

$$\begin{aligned} \int_{-\infty}^{\infty} H_{2n, \sigma}(z) f(z-x+y) \phi(z-x; \sigma) dz &= \int_{-\infty}^{\infty} f(u) H_{2n, \sigma}(u+x-y) \phi(u-y; \sigma) du \\ &= E\{f(\xi) [H_{2n, \sigma}(\xi+x-y) \phi(\xi-y; \sigma) \phi^{-1}(\xi; \sigma)]\} \\ &= \sum_{m=0}^{\infty} a_{2m} E[H_{2m, \sigma}(\xi) H_{2n, \sigma}(\xi+x-y) \phi(\xi-y; \sigma) \phi^{-1}(\xi; \sigma)] \\ &= \sum_{m=0}^{\infty} a_{2m} E[H_{2m, \sigma}(\xi+y) H_{2n, \sigma}(\xi+x)]. \end{aligned}$$

It then follows by (12) that

$$\Gamma(x, y; 1) = \sum_{n, m=0}^{\infty} a_{2n} a_{2m} E[H_{2n, \sigma}(\xi+x) H_{2m, \sigma}(\xi+y)]. \quad (13)$$

Since

$$H_{n, \sigma}(\xi+x) = \sum_{k=0}^n \binom{n}{k} x^{n-k} H_{k, \sigma}(\xi)$$

we can evaluate the expectation in (13), using the orthogonality of the Hermite polynomials, and obtain

$$\Gamma(x,y;1) = \sum_{n,m=0}^{\infty} a_{2n} a_{2m} \sum_{k=0}^{2\min(n,m)} k! \binom{2n}{k} \binom{2m}{k} \sigma^{2k} x^{2n-k} y^{2m-k},$$

and thus

$$\Gamma(x,y;1) - \Gamma(x,-y;1) = 2 \sum_{n,m=1}^{\infty} a_{2n} a_{2m} \sum_{k=0}^{2\min(n,m)} k! \binom{2n}{k} \binom{2m}{k} \sigma^{2k} x^{2n-k} y^{2m-k}.$$

Since $a_{2n} \geq 0$, $n \geq 1$, with at least one coefficient being strictly positive (since f is nonconstant), it is clear that

$$\Gamma(x,y;1) - \Gamma(x,-y;1) \neq 0 \text{ for all } x,y \neq 0.$$

Now let $-1 < \rho < 1$. Since f is even, we have by (4) of Proposition 1 that

$$\mu^{(n)}(x) = \sum_{2k \geq n} \frac{(2k)!}{(2k-n)!} a_{2k} x^{2k-n} \quad (14)$$

which is an even function of x for n even and odd for n odd. Then from Proposition 2, we obtain

$$\begin{aligned} \Gamma(x,y;\rho) - \Gamma(x,-y;\rho) &= \sum_{n=0}^{\infty} \frac{1}{n!} \rho^n \sigma^{2n} \mu^{(n)}(x) [\mu^{(n)}(y) - \mu^{(n)}(-y)] \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} \rho^{2n-1} \sigma^{2(2n-1)} \mu^{(2n-1)}(x) \mu^{(2n-1)}(y). \end{aligned} \quad (15)$$

Again, since $a_{2k} \geq 0$, $k \geq 1$, with at least one coefficient being strictly positive, we have by (14) that $\mu^{(2n-1)}(x) \neq 0$ for $x \neq 0$ for all n such that $\mu^{(2n-1)}(x)$ is not identically zero. It is then clear from (15) that

$$\Gamma(x,y;\rho) - \Gamma(x,-y;\rho) \neq 0 \text{ for all } x,y,\rho \neq 0.$$

The proof of $\Gamma(x,y;\rho) - \Gamma(-x,y;\rho) \neq 0$ for all $x,y,\rho \neq 0$ is similar and thus the proof of the proposition under Assumption (b) is complete. \square

III. THE RECONSTRUCTION OF THE SIGNAL

In this section, we consider the problem of reconstructing the real deterministic signal $s = \{s(t), -\infty < t < \infty\}$ from the knowledge of the nonlinearity f and the first or second order moments of the output process $f(s+N) = \{f[s(t)+N(t)], -\infty < t < \infty\}$, where the noise $N = \{N(t), -\infty < t < \infty\}$ is a real stationary Gaussian process with mean zero and known continuous covariance function $R(t)$ with $R(0) = \sigma^2$. Since f is known, so are the functions $\mu_f(x)$ and $\Gamma_f(x, y; \rho)$ introduced in Section II. Also known are one or more of the following functions: The moments $m_k(t)$ of the output process

$$m_k(t) = E\{f^k[s(t)+N(t)]\} = \mu_{fk}[s(t)], \quad k = 1, 2,$$

and the correlation function $C(t, \tau)$ of the output process

$$\begin{aligned} C(t, \tau) &= E\{f[s(t)+N(t)]f[s(\tau)+N(\tau)]\} \\ &= \Gamma_f[s(t), s(\tau); \frac{R(t-\tau)}{\sigma^2}]. \end{aligned}$$

We present several results on the reconstruction of the signal s for various classes of nonlinearities f . It is important to note that the propositions of Section II constitute the crux of the proofs of these results. By referring to these propositions, the derivations in this section become clear and simple.

We first consider monotonic nonlinearities for which reconstruction of the signal s is always possible, which is quite clear on intuitive grounds as well.

Theorem 1. The signal $s(t)$ can be reconstructed from the k -th moment $m_k(t)$, $k = 1$ or 2 , of the output process when the nonlinearity f is such that $f^k(x)$ is monotonic and nonconstant and $f^k(x) \in L_2(\phi(x; \sigma) dx)$.

Proof. By (5) of Proposition 1, we have for all x

$$\begin{aligned}\mu'_{fk}(x) &= \frac{1}{\sigma^2} \int_{-\infty}^{\infty} f^k(y+x) H_{1,\sigma}(y) \phi(y;\sigma) dy \\ &= \frac{1}{\sigma^2} \int_0^{\infty} y [f^k(x+y) - f^k(x-y)] \phi(y;\sigma) dy.\end{aligned}$$

If $f^k(y)$ is nonconstant and nondecreasing, say, then $f^k(x+y) - f^k(x-y) \geq 0$ and $\text{Leb}\{y : f^k(x+y) - f^k(x-y) > 0\} > 0$ for all x . It follows that $\mu'_{fk}(x) > 0$, $-\infty < x < \infty$, i.e. $\mu_{fk}(x)$ is strictly increasing. Similarly, $f^k(x)$ nonconstant and nonincreasing implies that $\mu_{fk}(x)$ is strictly decreasing. Hence $s(t)$ can be reconstructed from $m_k(t) = \mu_{fk}[s(t)]$. \square

Aside from the integrability conditions, the nonlinearities covered by Theorem 1 are those which are monotonic or whose absolute values are monotonic (values of k larger than 2 do not enlarge this class). Examples of monotonic nonlinearities satisfying the conditions of Theorem 1 are the hard-limiter, smooth-limiters, quantizers, v -law devices with v odd and infinite interval windows $1_{(-\infty, a)}(x)$ or $1_{(a, \infty)}(x)$.

It should be emphasized that the nonlinearity f is assumed to be monotonic but not strictly monotonic and thus the result of Theorem 1 is a substantial improvement over the no noise case. As a simple example that illustrates this point well, consider the monotonic nonlinearity $f(x) = 1_{(a, \infty)}(x)$, $-\infty < a < \infty$. In the absence of noise, all we would be able to conclude by observing the output $f[s(t)]$ would be whether $s(t) \leq a$ or $s(t) > a$. In sharp contrast to this, when there is additive Gaussian noise N the first order moment of $f[s(t)+N(t)]$ is sufficient to reconstruct the signal s .

We now turn our attention to nonlinearities that are not monotonic.

The first result in this direction concerns certain odd functions.

Theorem 2. The signal $s(t)$ can be reconstructed from the first moment $m_1(t)$ of the output process when the nonconstant nonlinearity f is odd, satisfies $f(x) \in L_2(\phi(x;\sigma)dx)$, and is such that the coefficients in its Hermite expansion are nonnegative.

Proof. By (4) of Proposition 1, we have for all $-\infty < x < \infty$,

$$\mu'_f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Since f is odd, $a_{2n} = 0$, and thus

$$\mu'_f(x) = \sum_{m=0}^{\infty} (2m+1) a_{2m+1} x^{2m} > 0$$

for all $x \neq 0$, since at least one of the nonnegative coefficients $\{a_{2m+1}\}$ is positive. It follows that $\mu_f(x)$ is strictly increasing and thus $s(t)$ can be reconstructed from $m_1(t) = \mu_f[s(t)]$. \square

It should be pointed out that one may easily construct examples of nonmonotonic nonlinearities f satisfying the conditions of Theorem 2. For example, $f(x) = x^3 - \sigma^2 x = 2\sigma^2 H_{1,\sigma}(x) + H_{3,\sigma}(x)$ is clearly nonmonotonic and satisfies the conditions of Theorem 2. In fact $\mu_f(x) = x(x^2 + 2\sigma^2)$ is strictly monotonic.

From now on we concentrate on nonlinearities f symmetric around an arbitrary point x_0 .

Theorem 3. If f is nonconstant, symmetric around some x_0 , and satisfies either (a) or (b):

(a) f is monotonic on $[x_0, \infty)$ and $f(x) \in L_2(\phi(x; \sigma) dx)$,

(b) $f(x+x_0) \in L_2(\phi(x; \sigma) dx)$ and the coefficients $\{a_{2n}\}_{n=1}^{\infty}$ in its Hermite expansion are nonnegative,

then $|s(t) - x_0|$ can be reconstructed from the first moment $m_1(t)$ of the output process.

Proof. We first note that the symmetry of f around x_0 implies that of $\mu_f(x)$:

$$\begin{aligned} \mu_f(x_0 - x) &= \int_{-\infty}^{\infty} f(x_0 - x + y) \phi(y; \sigma) dy = \int_{-\infty}^{\infty} f(x_0 + x - y) \phi(y; \sigma) dy \\ &= \int_{-\infty}^{\infty} f(x_0 + x + z) \phi(z; \sigma) dz = \mu_f(x_0 + x). \end{aligned}$$

(a) By (5) of Proposition 1 we have for all x

$$\mu_f'(x) = \frac{1}{\sigma^2} \int_0^{\infty} y [f(x+y) - f(x-y)] \phi(y; \sigma) dy.$$

Suppose f is nondecreasing on $[x_0, \infty)$. Then for every $x > x_0$, we have

$f(x+y) - f(x-y) \geq 0$ for all $y > 0$ and $\text{Leb}\{y: f(x+y) - f(x-y) > 0\} > 0$.

It follows that $\mu_f'(x) > 0$ for $x > x_0$ and thus $\mu_f(x)$ is strictly increasing on (x_0, ∞) . Similarly, if f is nonincreasing on $[x_0, \infty)$, $\mu_f(x)$ is strictly decreasing on $[x_0, \infty)$. Thus $\mu_f(x)$ is symmetric around x_0 and strictly monotonic on (x_0, ∞) .

(b) By (4) of Proposition 1 applied to $g(x) = f(x+x_0)$, we have for all x ,

$$\mu_g(x) = \sum_{n=0}^{\infty} a_n x^n$$

and

$$\mu_f(x) = \mu_g(x - x_0) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Since $g(x)$ is even, $a_{2n+1} = 0$, and thus

$$\mu_f(x) = \sum_{n=0}^{\infty} a_{2n} (x - x_0)^{2n}$$

is symmetric around x_0 and is strictly increasing on (x_0, ∞) , since $a_{2n} \geq 0, n \geq 1$, with at least one coefficient positive.

Since in both cases (a) and (b), $\mu_f(x)$ is symmetric around x_0 and strictly monotonic on (x_0, ∞) , $|s(t) - x_0|$ can be reconstructed from $m_1(t) = \mu_f[s(t)]$. \square

We remark that if the signal s in additive noise N can be observed through two nonlinearities f and \bar{f} , satisfying the conditions of Theorem 3 with distinct centers x_0 and \bar{x}_0 , then $s(t)$ can be reconstructed from the two moment functions $m_1(t)$ and $\bar{m}_1(t)$, since

$$|s(t) - x_0|^2 - |s(t) - \bar{x}_0|^2 = (\bar{x}_0 - x_0) [2s(t) - x_0 - \bar{x}_0].$$

It is clear that for the nonlinearities f satisfying the conditions of Theorem 3, the first order moment function $m_1(t)$ of the output process determines $|s(t) - x_0|$ and we are left with the problem of determining the sign of $s(t) - x_0$ for each t . This is now accomplished by using the correlation function $C(t, \tau)$ of the output process.

Theorem 4. Assume f is nonconstant, symmetric around some x_0 and satisfies either (a) or (b):

- (a) f is bounded below or above, monotonic on $[x_0, \infty)$ and such that $f^2(x) \in L_2(\phi(x; \sigma) dx)$,
- (b) $f^2(x+x_0) \in L_2(\phi(x; \sigma) dx)$ and the coefficients $\{a_{2n}\}_{n=1}^{\infty}$ of $f(x+x_0)$ in its Hermite expansion are nonnegative.

Assume the correlation function R of the noise N has at most a finite number of zeros on each finite interval. Then from the mean function m_1 and the correlation function C of the output process, a set of two signals s and $2x_0 - s$

can be reconstructed (there is no way to determine which, among these two, is the actual signal). When f is even, i.e. $x_0=0$, then the signal s is reconstructed up to a global sign.

Proof. (a) f clearly satisfies the conditions of Theorem 3(a) and thus the function $a(t) = |s(t)-x_0|$, $-\infty < t < \infty$, can be reconstructed from the mean function m_1 . For any t such that $a(t) = 0$, we have $s(t) = x_0$. Consider now two distinct points t and τ such that $a(t) \neq 0 \neq a(\tau)$ and $R(t-\tau) \neq 0$. We then have $s(t) = x_0 \pm a(t)$, $s(\tau) = x_0 \pm a(\tau)$ and

$$C(t,\tau) = \Gamma_f[s(t),s(\tau); \sigma^{-2}R(t-\tau)].$$

Therefore, $C(t,\tau)$ is equal to at least one of the four numbers

$\Gamma_f[x_0 \pm a(t), x_0 \pm a(\tau); \sigma^{-2}R(t-\tau)]$, which are related as follows (for brevity, we drop the dependence of Γ_f on R):

$$\Gamma_f[x_0 + a(t), x_0 + a(\tau)] \neq \Gamma_f[x_0 + a(t), x_0 - a(\tau)]$$

$$\Gamma_f[x_0 - a(t), x_0 + a(\tau)] \neq \Gamma_f[x_0 - a(t), x_0 - a(\tau)]$$

by Proposition 3, and

$$\Gamma_f[x_0 + a(t), x_0 + a(\tau)] = \Gamma_f[x_0 - a(t), x_0 - a(\tau)]$$

$$\Gamma_f[x_0 + a(t), x_0 - a(\tau)] = \Gamma_f[x_0 - a(t), x_0 + a(\tau)]$$

since for all x, y , $\Gamma_f(x_0 - x, x_0 - y) = \Gamma_f(x_0 + x, x_0 + y)$, a straightforward consequence of the symmetry of f around x_0 . It follows that if we choose $s(t) = x_0 + a(t)$, then exactly one of the two values

$$\Gamma_f[x_0 + s(t), x_0 + a(\tau)], \Gamma_f[x_0 + a(t), x_0 - a(\tau)]$$

will equal $C(t,\tau)$, and if we choose $s(t) = x_0 - a(t)$ then again exactly one of the two values

$$\Gamma_f[x_0 - a(t), x_0 + a(\tau)], \Gamma_f[x_0 - a(t), x_0 - a(\tau)]$$

will equal $C(t,\tau)$. Thus for each choice of $s(t)$, the value of $s(\tau)$ is uniquely determined by the correlation $C(t,\tau)$.

The reconstruction procedure is now clear: If $a(t) = 0$, then $s(t) = x_0$. Otherwise, fix an arbitrary point t_0 with $a(t_0) \neq 0$ and let t be such that $a(t) \neq 0$ and $R(t-t_0) \neq 0$. Then for each possible choice of $s(t_0)$, $s(t_0) = x_0 + a(t_0)$ or $s(t_0) = x_0 - a(t_0)$, $s(t)$ is uniquely determined by comparing $C(t_0, t)$ with the two numbers $\Gamma_f[s(t_0), x_0 + a(t), \sigma^{-2}R(t_0-t)]$. Thus, we determine two signals s_1 and s_2 which are related by $s_1(t) + s_2(t) = 2x_0$ for all t and which give rise to the known mean and correlation functions m_1 and C . It is also clear from the above that the actual signal s giving rise to the known mean and correlation functions m_1 and C must be identically equal on $(-\infty, \infty)$ to either s_1 or s_2 . Thus, the reconstructed set of two signals $\{s_1, s_2\}$ is the same as the set $\{s, 2x_0 - s\}$. In particular, when the nonlinearity f is even, i.e. $x_0 = 0$, then the actual signal s is reconstructed up to a global sign.

(b) Follows in a similar manner by using Theorem 3(b) and Proposition 3(b). \square

Some simple examples of nonlinearities f satisfying the conditions of Theorem 4(a) are

$$[(x-x_0)^{2n-b^2}], [(x-x_0)^{2n+b^2}]^{-1}, \exp[-a^2|x-x_0|^b], 1_{(a,b)}(x),$$

where $-\infty < a < b < \infty$ and $n=1, 2, \dots$. When $f(x) = x^2$, the result of Theorem 4 was obtained by Grünbaum[1], who was the first to consider this problem.

The two classes of nonlinearities considered in (a) and (b) of Theorem 4 are distinct, as the following examples show. For simplicity, assume $x_0 = 0$ and $\sigma = 1$.

$$(i) f(x) = (x^2-2)(x^2-4) = H_{4,1}(x) + 5$$

satisfies the conditions of Theorem 4(b) but not those of Theorem 4(a), since f is clearly nonmonotonic on $[0, \infty)$.

$$(ii) \quad f(x) = x^6 - 18x^4 + 114x^2 - 75 \\ = H_{6,1}(x) - 3H_{4,1}(x) + 51H_{2,1}(x).$$

Then $f'(x) = 6x[(x-6)^2 + 2] > 0$ for $x > 0$ so that f is monotonic on $[0, \infty)$ and satisfies the conditions of Theorem 4(a). On the other hand, the coefficient of $H_{4,1}(x)$ is negative and thus f does not satisfy the conditions of Theorem 4(b).

The results of Theorem 4 provide a substantial improvement over the no noise case. As a simple example that illustrates this point, consider an even nonlinearity which is strictly increasing on $[0, \infty)$: In the absence of noise, the output $f[s(t)]$ determines $|s(t)|$ and we are left with determining the sign of $s(t)$ for each t . Thus the number of distinct signals with the given absolute value is in general uncountably infinite. If s is known to be a continuous function, we then have 2^{N+1} distinct signals; the value of N can be read off the graph of the function $|s(t)|$ and may be finite or infinite (N is the number of points t such that $s(t) = 0$ and $s(\tau) \neq 0$ in some left neighborhood of t). In sharp contrast, in the presence of Gaussian noise N satisfying the condition of Theorem 4, the mean and correlation functions of the output process $f[s+N]$ determine s up to a global sign. This substantial improvement over the no noise case is due to the fact that the Gaussian noise takes all real values and thus, the output mean and correlation functions contain more information about s than just $f(s)$. Even more illuminating is the case where the nonlinearity f is a symmetric interval window $f(x) = 1_{[-a,a]}(x)$. In the absence of noise, we can only determine whether $|s(t)| \leq a$ or $|s(t)| > a$ for each t ; in the presence of the noise N , the signal s is determined up to a global sign.

It should be noted that the condition on the zeros of the noise covariance R imposed in Theorem 4 could be weakened in certain cases. For example, it is clear that the reconstruction procedure of Theorem 4 remains valid with no conditions on R in case the function $|s(t)-x_0|$, which is reconstructed from the output moment function m_1 , satisfies

$$|s(t)-x_0| > 0 \text{ a.e. for } a < t < b$$

and

$$|s(t)-x_0| = 0 \text{ for } t \notin (a,b)$$

for some $-\infty < a < b < \infty$. Next suppose that for some $-\infty < a < b < \infty$,

$$|s(t)-x_0| = 0 \text{ for } a \leq t \leq b$$

and

$$|s(t)-x_0| > 0 \text{ a.e. for } t \notin [a,b].$$

Then for the reconstruction procedure of Theorem 4 to be feasible, we need that $R(t) \neq 0$ for some $t > b-a$, which is much weaker than the condition imposed in Theorem 4. The main point here is that the reconstructed function $|s(t)-x_0|$ enables us to check whether or not a given noise covariance R allows the reconstruction procedure of Theorem 4 to apply.

For arbitrary covariances R , we can always reconstruct a set of at least two signals containing the actual signal. As an example, consider the case where $R(t) = 0$ for all $|t| \geq T$, e.g. triangular covariance. Denote by N the number of those open intervals of zeros of $|s(t)-x_0|$ whose length is $\geq 2T$. Then a set of 2^{N+1} signals can be reconstructed by applying the procedure described in the proof of Theorem 4 to each of the $(N+1)$ intervals over which $|s(t)-x_0| > 0$ a.e. This set contains the actual signal s and any two signals s_1, s_2 in the set satisfy $s_1(t) = s_2(t)$ or $s_1(t) + s_2(t) = 2x_0$ for each fixed t .

In case higher order moments of the output process $f[s(t)+N(t)]$ are available, results similar to those of Theorem 4 can be obtained for additional classes of nonlinearities f . In particular, if $f^2(x)$, rather than $f(x)$, satisfies the conditions of Theorem 4, the conclusion of Theorem 4 remains valid provided we use the second moment function

$$m_2(t) = E\{f^2[s(t) + N(t)]\}$$

and the fourth order correlation function

$$C_2(t, \tau) = E\{f^2[s(t) + N(t)]f^2[s(\tau)+N(\tau)]\}$$

of the output process. This extension covers certain, possibly discontinuous, asymmetric nonlinearities f such that f^2 is symmetric around some x_0 , as well as certain odd nonlinearities not covered by Theorems 1 and 2. For example, the nonlinearity (with $\sigma=1$ for simplicity)

$$f(x) = x^3 - 7x = H_{3,1}(x) - 4 H_{1,1}(x)$$

is not monotonic nor has nonnegative Hermite coefficients and hence the reconstruction of the signal s is not feasible by Theorems 1 and 2. On the other hand,

$$f^2(x) = (x^3 - 7x)^2 = H_{6,1}(x) + H_{4,1}(x) + 10 H_{2,1}(x) + 22$$

and thus $f^2(x)$ satisfies the conditions of Theorem 4(b). Hence the signal s can be reconstructed, up to a global sign, from the moment and correlation functions m_2 and C_2 of the output process.

Finally, when the first or second order distributions (rather than moments) of the output process are known, we have the following result, where $B(f)$ denotes the σ -field of Borel sets generated by f .

Theorem 5 (i) The signal s can be reconstructed from the first order distribution of the output if $B(f)$ contains an interval of the form $(-\infty, a)$ or (a, ∞) , where $-\infty < a < \infty$, or two finite intervals (a, b) and (c, d) with $a+b \neq c+d$.

(ii) If $B(f)$ contains a finite interval (a, b) , and if the noise correlation function R has at most a finite number of zeros on each finite interval, then a set of two signals s and $a+b-s$ can be reconstructed from the first and second order distributions of the output; and when $a=-b$, the signal s can be reconstructed up to a global sign.

(i) follows from Theorem 1 and the remark following Theorem 3, and (ii) follows from Theorem 4.

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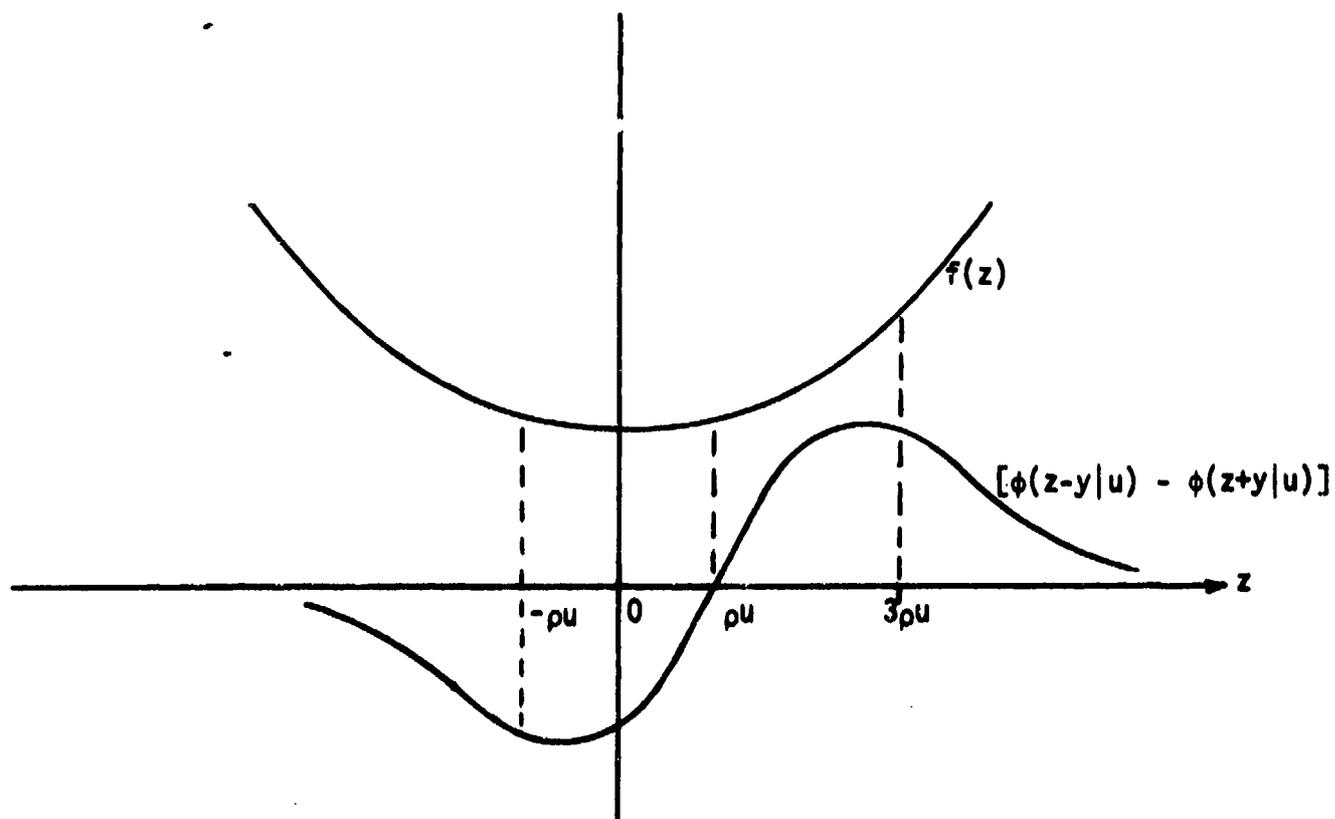


Fig. 1 The graph of $[\phi(z-y|u) - \phi(z+y|u)]$ and typical $f(z)$.

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20. Abstract continued.

reconstructed, up to a global sign, for two distinct classes of even nonlinearities; included here are 2ν -th law devices and symmetric interval windows.

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