Cycle and Cocycle Coverings of Graphs

Sean McGuinness

THOMPSON RIVERS UNIVERSITY, KAMLOOPS
BC, CANADA V2C5N3
E-mail: smcguinness@tru.ca

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Abstract: In this article, we show that for any simple, bridgeless graph G on n vertices, there is a family $\mathscr{C}$ of at most $n-1$ cycles which cover the edges of G at least twice. A similar, dual result is also proven for cocycles namely: for any loopless graph G on n vertices and ε edges having cogirth $g^* \geq 3$ and $k(G)$ components, there is a family of at most $\varepsilon - n + k(G)$ cocycles which cover the edges of G at least twice. J Graph Theory

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1. INTRODUCTION

For a graph G we let $n(G)=|V(G)|$ and $\varepsilon(G)=|E(G)|$. We let $k(G)$ denote the number of components of G. For a vertex $v \in V(G)$, we let $N_G(v)$ be the set of neighbors of v in G, and we let $E_v(G)$ be the set of edges in G incident to v. It should be noted that all graphs in this article will be allowed to have loops or multiple edges. The *circumference* of G is defined to be the length of a longest cycle in G, and is denoted by $c(G)$. The *girth* of G is the length of a shortest cycle of G, and is denoted by $g(G)$. In addition, the *cogirth*, $g^*(G)$, denotes the smallest size of a cocycle of G.

For disjoint subsets X and Y of vertices in a graph G, we denote by $[X,Y]$ the set of edges having one endvertex in X and the other in Y. A *cutset* in G is a set of edges $[X, V(G)\backslash X]$, for some non-empty $X \subset V(G)$. We also denote such a set by $\partial(X)$. A *cocycle* is a minimal cutset; that is, a cutset not properly containing another cutset. Furthermore, a cocycle $C^* = [X, V(G)\backslash X]$ is said to be *non-trivial* if $|X| \geq 2$ and $|V(G)\backslash X| \geq 2$.

For a subset of edges $X \subseteq E(G)$ (respectively, vertices $X \subseteq V(G)$) in a graph G, we let $G[X]$ be the subgraph induced by X.

For a matroid M, we let $r(M)$ denote the rank of M and we let $c(M)$ the length of a largest circuit (called the *circumference* of M). There is an interesting connection between various bounds on the size of a graph and the size of a matroid. To start with, for a simple graph G, we have an old, well-known bound for $\varepsilon(G)$ due to Erdős and Gallai [4] (see also [1, Section 3.4]).

Theorem 1.1 (Erdős–Gallai)**.** *For a simple graph G,*

$$\varepsilon(G) \leq \tfrac{1}{2}c(G)(n(G)-1).$$

Murty [9] showed that a similar looking bound holds for simple binary matroids having no F_7-minor.

Theorem 1.2 (Murty)**.** *Let M be a simple binary matroid having no F_7-minor. Then*

$$|E(M)| \leq \tfrac{1}{2}r(M)(r(M)+1)$$

and equality is attained if and only if $M \simeq M(K_{r+1})$.

In [8], I showed that the Erdős -Gallai bound and Murty bound have a common generalization.

Theorem 1.3 (McGuinness)**.** *Let M be a simple, connected binary matroid having no F_7-minor. Then*

$$|E(M)| \leq \tfrac{1}{2}r(M)c(M).$$

In the same article, I made the following conjecture, which seems natural in light of the above theorem.

Conjecture 1.4. *For a simple, connected binary matroid M having no F_7-minor, there exists a collection of at most $r(M)$ circuits which cover the elements of M at least twice.*

For the special case of graphic matroids, the above conjecture asserts that for any simple, 2-connected graph G, there is a collection of at most $n(G)-1$ cycles which cover the edges of G at least twice. In Section 3, we prove this special case and prove something stronger.

For a family of cycles or cocycles $\mathscr{C}$ of a graph G and $e \in E(G)$, let

$$\mathscr{C}(e) = \{C \in \mathscr{C} | e \in C\}, \quad \alpha_{\mathscr{C}}(e) = |\mathscr{C}(e)|.$$

A family of cycles (respectively, cocycles) $\mathscr{C}$ is said to be a ≥ 2-*cycle cover* (respectively, ≥ 2-*cocycle cover*) if $\alpha_{\mathscr{C}}(e)\geq 2$ for all $e\in E(G)$. The first main result of this article is the following:

Theorem 1.5. *Let G be a simple, bridgeless graph and let $x\in V(G)$. Then G has a ≥ 2-cycle cover $\mathscr{C}$ such that* (i) $|\mathscr{C}|\leq n(G)-1$ *and* (ii) *for edges e having both endvertices in $N_G(x)$, it holds that $\alpha_{\mathscr{C}}(e)=2$.*

Interestingly, Bondy [2] conjectured that the edges of G can be covered exactly twice with at most $n(G)-1$ cycles, the so-called *Small cycle double cover conjecture*.

Conjecture 1.6 (Bondy). *For a simple,* 2*-connected graph G, there is a collection of at most $n(G)-1$ cycles which cover the edges of G exactly twice.*

In [5], Erdős, Goodman, and Pósa conjectured the following (see also [3, Problem 6]):

Conjecture 1.7 (Erdős, Goodman, Posa). *For any simple,* 2*-connected graph G, there is a collection of at most $n(G)-1$ cycles which cover the edges of G.*

This conjecture was subsequently proven by Pyber [10]. In [6], Fan showed that $\lfloor (2n(G)-1)/3\rfloor$ cycles will suffice, this being the best possible. In light of this, Theorem 1.5 is somewhat surprising in that it implies that a ≥ 2-cover cover exists with at most $n(G)-1$ cycles.

2. CYCLES

In this section, we shall prove the first of the main theorems, Theorem 1.5. The main ingredients in the proof are a *switching lemma* by Fan [6], and Li's theorem on perfect path double covers [7].

We shall use the following definitions given by Fan [6]. Let xy be an edge in a simple graph G where $N_G(y)\subseteq N_G(x)\cup\{x\}$. Let C be a cycle of G containing x. A cycle C' is a *transformation* of C if one of the following holds:

(a) $C'=C$.

(b) $y\notin V(C)$, and for $w,z\in N_G(x)$ it holds that xw, $xz\in E(C)$ and C' is one of the cycles (see Fig. 1):

  (i) $C'=(C\backslash\{xz\})\cup\{xy,yz\}$.

  (ii) $C'=(C\backslash\{xw\})\cup\{xy,yw\}$.

  (iii) $C'=(C\backslash\{xz,\ xw\})\cup\{yz,yw\})$.

(c) $xy\in E(C)$ and there exist distinct $w,z\in N_G(x)\backslash\{y\}$ for which $yw,xz\in E(C)$. Then $C'=(C\backslash\{xz,yw\})\cup\{yz,xw\}$.

(d) $y\in V(C)$, and $xy\notin E(C)$. Suppose there are distinct $z_1,z_2\in N_G(x)\backslash\{y\}$ and distinct $w_1,w_2\in N_G(y)\backslash\{x\}$, such that $xz_1,xz_2,yw_1,yw_2\in E(C)$. Assume that for $i=1,2$ that z_i and w_i lie in the same component of $C\backslash\{x,y\}$. Then we have that C' is

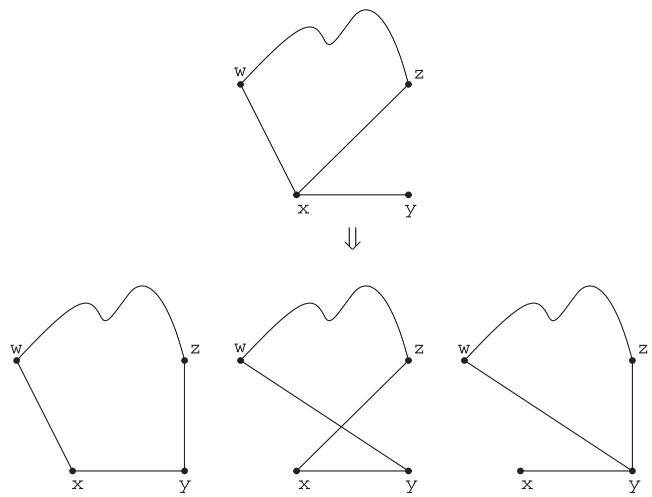

FIGURE 1. Cycle transformation (b).

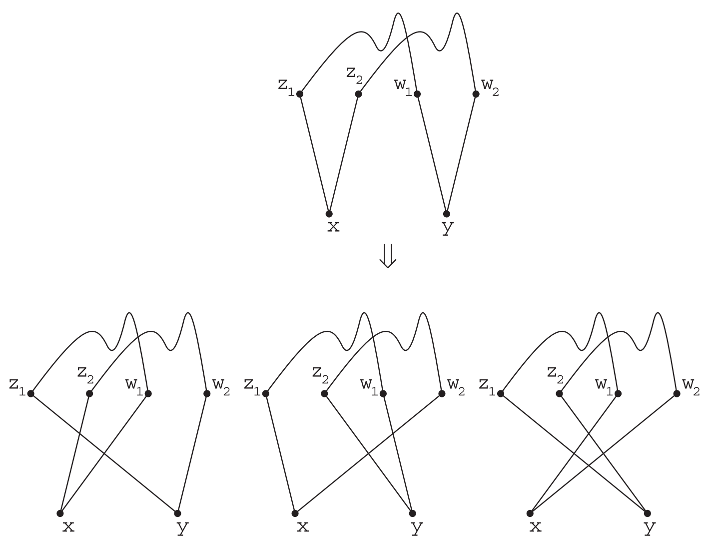

FIGURE 2. Cycle transformation (d).

one of the cycles (see Fig. 2):

(i) $C' = (C \backslash \{xz_1, yw_1\}) \cup \{yz_1, xw_1\}$;

(ii) $C' = (C \backslash \{xz_2, yw_2\}) \cup \{yz_2, xw_2\}$;

(iii) $C' = (C \backslash \{xz_1, yw_1, xz_2, yw_2\}) \cup \{yz_1, xw_1, yz_2, xw_2\}$.

The following lemma by Fan [6, Switching Lemma] is central to the proof of Theorem 1.5.

Lemma 2.1 (The Switching Lemma). *Let G be a simple graph and let $xy \in E(G)$ where $N_G(y) \subseteq N_G(x) \cup \{x\}$. Suppose that $\mathscr{C} = \{C_1, \ldots, C_m\}$ is a family of cycles such that $\alpha_{\mathscr{C}}(vy) \leq \alpha_{\mathscr{C}}(vx)$ for all $v \in N_G(x) \cap N_G(y)$. Let*

$$Z = \{z_1, z_2, \ldots, z_s\} \subseteq N_G(x) \cap N_G(y)$$

where $\alpha_{\mathscr{C}}(z_i y) < \alpha_{\mathscr{C}}(z_i x)$ for $i = 1, \ldots, s$. Then for any s integers $k_1, \ldots, k_s$ where $1 \leq k_i \leq \alpha_{\mathscr{C}}(z_i x) - \alpha_{\mathscr{C}}(z_i y)$, there is a family of cycles $\mathscr{C}' = \{C_1', C_2', \ldots, C_m'\}$ such that

(i) *C_j' is a transformation of C_j at xy, $j = 1, \ldots, m$.*
(ii) *$\alpha_{\mathscr{C}'}(z_i y) = \alpha_{\mathscr{C}}(z_i y) + k_i$ and $\alpha_{\mathscr{C}'}(z_i x) = \alpha_{\mathscr{C}}(z_i x) - k_i$, $i = 1, \ldots, s$.*
(iii) *$\alpha_{\mathscr{C}'}(xy) = \alpha_{\mathscr{C}}(xy) + \theta$ where $\theta \leq \sum_{i=1}^{s} k_i$, $\theta \equiv \sum_{i=1}^{s} k_i \pmod 2$.*
(iv) *$\alpha_{\mathscr{C}'}(e) = \alpha_{\mathscr{C}}(e)$ for all $e \in E(G) \backslash (\{xy\} \cup \{z_i x, z_i y \mid 1 \leq i \leq s\})$.*

We shall also make use of the following theorem of Li [7]:

Theorem 2.2 (Li). *Let G be a simple graph on n vertices. There is a family of paths $\mathscr{P}$ such that each edge belongs to exactly two paths of $\mathscr{P}$, and each vertex occurs exactly twice as an endvertex of paths in $\mathscr{P}$.*

The above theorem had originally been conjectured by Bondy [2].

Note that a single vertex v is considered itself to be a path (the *trivial path*) and for such a path, v is counted twice as an endvertex of this path. If G has no isolated vertices, then we can choose $\mathscr{P}$ so that it has no trivial paths in the above theorem. In this case, each vertex is the endvertex of two different paths in $\mathscr{P}$, and thus $|\mathscr{P}| = n$.

Let $xy \in E(G)$ and let $Z = N(y) \backslash (N(x) \cup \{x\})$. As defined in [6], we define an *edge-switching from y to x* to be the operation on G where we delete the edges yz, $z \in Z$, and add the edges xz, $z \in Z$. The resulting graph is denoted by $G[y \to x]$. See Figure 3.

Proof of Theorem 1.5. Assume that the theorem is false and let G be graph and let $x \in V(G)$ be such that the theorem is false for G and x. Assume that G has $n = n(G)$ vertices. Among such counterexamples choose G such that $\varepsilon(G)$ is minimum, and subject to $\varepsilon(G)$ being minimum, assume that $d_G(x)$ is maximum. Suppose first that $d_G(x) = n-1$. Let $G' = G \backslash \{x\}$. Since G is bridgeless, G' has no isolated vertices, and thus by Theorem 2.2 there is a family of $n-1$ non-trivial paths $\mathscr{P}$ where each edge of G' is covered exactly twice by paths in $\mathscr{P}$, and every vertex of G' is the endvertex of exactly two paths in $\mathscr{P}$. Let $\mathscr{P} = \{P_i | i = 1, 2, \ldots, n-1\}$, where P_i has endvertices u_i and v_i. For $i = 1, 2, \ldots, n-1$, let C_i be the cycle defined by $C_i = P_i \cup \{x, xu_i, xv_i\}$. Now $\mathscr{C} = \{C_1, C_2, \ldots, C_{n-1}\}$ is seen to be a collection of cycles where $\alpha_{\mathscr{C}}(e) = 2$ for all $e \in E(G)$, and $\mathscr{C}$ is seen to satisfy (i) and (ii) of the theorem. We conclude that $d_G(x) < n-1$.

Let $xy \in E(G)$. Suppose first that xy belongs to no triangle of G (that is, $N_G(x) \cap N_G(y) = \emptyset$). Let $G' = G/xy$ (that is, G contract xy) and let x' be the vertex formed when contracting the edge xy. We have that $\varepsilon(G') < \varepsilon(G)$ and G' is also seen to be simple and bridgeless. By assumption, there is a ≥ 2-cycle cover $\mathscr{C}'$ for G' where $|\mathscr{C}'| \leq n(G') - 1 = n-2$, and for each edge $e \in E(G')$, having both endvertices in $N_{G'}(x')$, it holds that $\alpha_{\mathscr{C}'}(e) = 2$. Let $\mathscr{C}'(x') = \{C' \in \mathscr{C}' | x' \in V(C')\}$, and

$$E_x' = \{x'v | v \in N_G(x) \backslash \{y\}\}, \quad E_y' = \{x'v | v \in N_G(y) \backslash \{x\}\}.$$

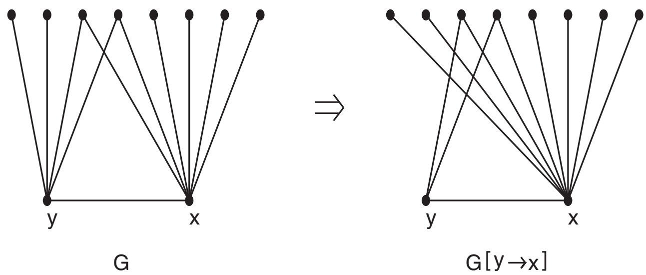

FIGURE 3. Edge-switching from y to x.

Let

$$\mathscr{C}'_1 = \{C' \in \mathscr{C}'(x') | C' \cap E'_x \neq \emptyset, C' \cap E'_y = \emptyset\},$$

$$\mathscr{C}'_2 = \{C' \in \mathscr{C}'(x') | C' \cap E'_x = \emptyset, C' \cap E'_y \neq \emptyset\},$$

$$\mathscr{C}'_3 = \mathscr{C}'(x') \backslash (\mathscr{C}'_1 \cup \mathscr{C}'_2).$$

Let

$$\mathscr{C}_1 = \mathscr{C}'_1, \quad \mathscr{C}_2 = \mathscr{C}'_2.$$

Furthermore, let $\mathscr{C}_3$ be the collection of cycles obtained from $\mathscr{C}'_3$ by replacing each cycle $C' \in \mathscr{C}'$ with a cycle in G where $E(C) = E(C') \cup \{xy\}$. If $|\mathscr{C}_3| \geq 2$, then $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3$ is seen to be a ≥ 2-cycle cover of G where $|\mathscr{C}| = |\mathscr{C}'| = n-2$ and for each edge e having endpoints in $N_G(x)$ it holds that $\alpha_{\mathscr{C}}(e) = \alpha_{\mathscr{C}'}(e) = 2$ (since e is also an edge with endpoints in $N_{G'}(x')$).

Suppose that $|\mathscr{C}_3| \leq 1$. Then, $|\mathscr{C}_1| \geq 1$ and $|\mathscr{C}_2| \geq 1$. Let $C \in \mathscr{C}_2$, and let P be a path of shortest length between x and $V(C)$ in $G \backslash \{xy\}$. Let $\{z\} = V(P) \cap V(C)$, and let P' be a path in C between y and z. Let C_1 be the cycle $C_1 = P \cup P' \cup \{xy\}$, and let C_2 be the cycle $C_2 = C \triangle C'$. See Figure 4. Let $\mathscr{C} = \mathscr{C}_1 \cup (\mathscr{C}_2 \backslash \{C\}) \cup \{C_1, C_2\} \cup \mathscr{C}_3$. Then it is seen that $\mathscr{C}$ is a ≥ 2-cycle cover for G, where $|\mathscr{C}| = |\mathscr{C}'| + 1 \leq n-1$. Furthermore, since P is a shortest path from x to $V(C)$, it contains no edge with both endpoints in $N_G(x)$. Thus for each edge e having endpoints in $N_G(x)$, it holds that $\alpha_{\mathscr{C}}(e) = \alpha_{\mathscr{C}'}(e) = 2$.

From the above, it holds that for each edge $xy \in E_x(G)$, $N_G(x) \cap N_G(y) \neq \emptyset$. Since x is not adjacent to every other vertex of G, there is an edge $xy \in E_x(G)$, where $N_G(y) \backslash N_G(x) \neq \emptyset$. Let $G' = G[y \to x]$. Given that xy belongs to a triangle in G, it is seen that G' is bridgeless. We have $\varepsilon(G') = \varepsilon(G)$, and $d_{G'}(x) > d_G(x)$. By assumption, there exists a ≥ 2-cycle cover $\mathscr{C}'$ for G for which $|\mathscr{C}'| \leq n-1$ and $\alpha_{\mathscr{C}'}(e) = 2$ for all edges e having both endpoints in $N_{G'}(x)$. Let $Z = N_G(y) \backslash N_G(x)$, and $G'' = G' \cup \{yz | z \in Z\}$. Then $\mathscr{C}'$ can be viewed as a collection of cycles of G'' where $\alpha_{\mathscr{C}'}(yz) = 0$ for all $z \in Z$. For each $v \in Z$, we have $\alpha_{\mathscr{C}'}(xv) > \alpha_{\mathscr{C}'}(yv) = 0$. Furthermore, for each $v \in N_{G''}(x) \cap N_{G''}(y) \backslash Z$, it holds that $\alpha_{\mathscr{C}'}(yv) = 2$ since $v, y \in N_{G'}(x)$, and hence $\alpha_{\mathscr{C}'}(xv) \geq \alpha_{\mathscr{C}'}(yv) = 2$. Thus by Lemma 2.1, we can switch the cycles in $\mathscr{C}'$ to obtain a collection of cycles $\mathscr{C}$ in G''

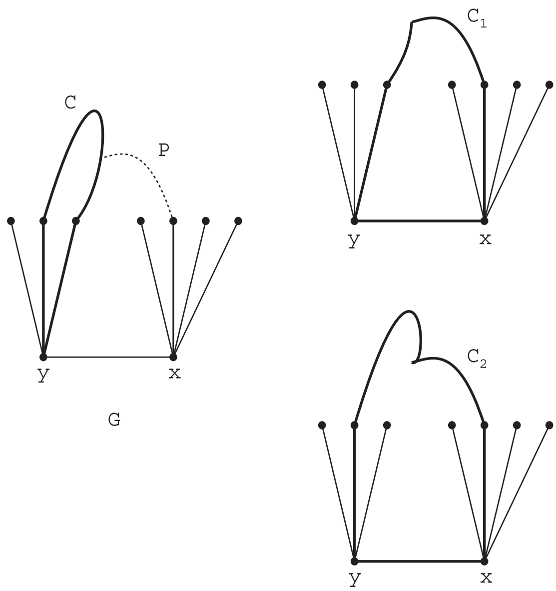

FIGURE 4. Splitting the cycle C.

where

$$\alpha_{\mathscr{C}}(yz) = \alpha_{\mathscr{C}'}(xz) \quad \text{for all } z \in Z,$$

$$\alpha_{\mathscr{C}}(xz) = 0 \quad \text{for all } z \in Z,$$

$$\alpha_{\mathscr{C}}(xy) \geq \alpha_{\mathscr{C}'}(xy),$$

$$\alpha_{\mathscr{C}}(e) = \alpha_{\mathscr{C}'}(e) \quad \text{for all } e \in E(G'') \setminus \{xz, yz | z \in Z\} \cup \{xy\}.$$

Now $\mathscr{C}$ is seen to be a ≥ 2-cycle cover of G where $|\mathscr{C}| = |\mathscr{C}'| \leq n-1$, and for each edge e having endpoints in $N_G(x)$, it holds that $\alpha_{\mathscr{C}}(e) = \alpha_{\mathscr{C}'}(e) = 2$. With this we arrive at a final contradiction. ■

3. COCYCLES

In this section, we prove that a "dual" result to Theorem 1.5 holds for cocycles.

For a graph G, let $r^*(G) = \varepsilon(G) - n(G) + k(G)$, which is the rank of the cocircuit matroid $M^*(G)$. We say that two cocycles $C_i^* = [X_i, Y_i]$, $i = 1, 2$ *cross* if $X_1 \cap X_2 \neq \emptyset, Y_1 \cap Y_2 \neq \emptyset$, and $X_i \cap Y_j \neq \emptyset$ for all $i, j \in \{1, 2\}$ where $i \neq j$. We write $C_i^* \curlywedge C_j^*$. For edges e and f, we say that a cocycle $C^* = [X, Y]$ *separates e and f* if e is an edge of $G[X]$ and f is an edge of $G[Y]$, or vice versa. Similarly, for vertices u and v we say that C^* *separates u and v* if $u \in X$ and $v \in Y$, or vice versa.

For edges $e = u_1v_1$ and $f = u_2v_2$ and cocycles $C_1^* = [X_1, V(G) \setminus X_1]$ and $C_2^* = [X_2, V(G) \setminus X_2]$ containing e, f, we say that the pair (C_1^*, C_2^*) *crosses* (e, f) if u_1 and u_2

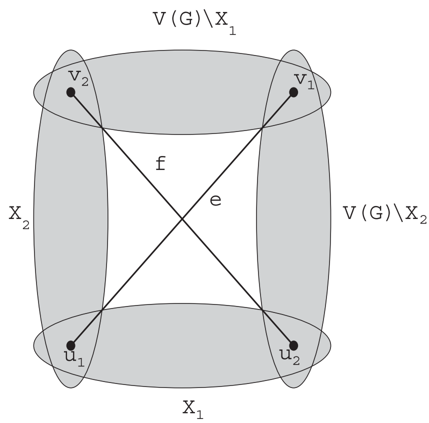

FIGURE 5. Pair of cocycles crossing (e,f).

are separated by exactly one of C_1^* or C_2^*. See Figure 5. Observe that if e and f are incident, then no pair of cocycles can cross (e,f).

For a ≥ 2-cocycle cover $\mathscr{C}^*$, we say that (e,f) is a *bad pair* for $\mathscr{C}^*$ if

(i) no cocycle of $\mathscr{C}^*$ separates e and f,
and
(ii) for some C_1^*, $C_2^* \in \mathscr{C}^*$, it holds that $\mathscr{C}^*(e) = \mathscr{C}^*(f) = \{C_1^*, C_2^*\}$ and (C_1^*, C_2^*) crosses (e,f).

For edges e and f and cocycles C_1^*, C_2^* satisfying (ii), we write $(e,f) \leftrightarrow (C_1^*, C_2^*)$ and refer to (C_1^*, C_2^*) as a *bad pair of cocyles*. We observe that if $(e,f) \leftrightarrow (C_1^*, C_2^*)$, then $C_1^* \curlywedge C_2^*$. If $\mathscr{C}^*$ is a ≥ 2-cocycle cover with no bad pairs of cocycles, then we say that $\mathscr{C}^*$ is *good*.

Lemma 3.1. *Let $C^* = [X, Y]$ be a cocycle in a 2-connected graph G and let $e \in E(G) \backslash C^*$. Then there exist cocycles C_1^*, C_2^* where $e \in C_i^*$, $i = 1, 2$, and $C^* \subset C_1^* \cup C_2^*$.*

Proof. Let $e \in E(G)$ where $e = u_1 u_2$ and $u_1, u_2 \in X$. Since G is 2-connected, there exist two internally disjoint paths $P_i = v_{i1} e_{i1} v_{i2} e_{i2} \ldots e_{i(k_i-1)} v_{ik_i}$, $i = 1, 2$, from $\{u_1, u_2\}$ to Y, where $u_i = v_{i1}$ and $v_{ik_i} \in Y_i$, for $i = 1, 2$ and $v_{ij} \in V(G)$, $e_{ij} \in E(G)$, $i = 1, 2$; $j = 1, \ldots, k_i$. Let $P_i' = v_{i1} e_{i1} v_{i2} e_{i2} \ldots e_{i(k_i-2)} v_{i(k_i-1)}$, $i = 1, 2$, be the portion of the path P_i lying in $G[X]$. Let T be a spanning tree of $G[X]$ containing P_1', P_2', and e, and let T_1, T_2 be the components of $T \backslash e$, where T_i contains P_i', $i = 1, 2$. Let $C_i^* = \partial(V(T_i))$, $i = 1, 2$. Then $e \in C_i^*$, $i = 1, 2$. Furthermore, $G[V(G) \backslash V(T_1)]$ is connected since $V(G) \backslash V(T_1) = V(T_2) \cup Y$, and $G[V(T_2)]$ is connected to $G[Y]$ by the edge $e_{2(k_2-1)}$. Likewise, $G[V(G) \backslash V(T_2)]$ is also connected. Thus C_i^*, $i = 1, 2$, are cocycles and $C^* \subset C_1^* \cup C_2^*$. ∎

We refer to the pair of cocycles (C_1^*, C_2^*) in the above lemma as an *e-splitting* of C^*. We now present the second of the main results in this article.

Theorem 3.2. *Let G be a loopless graph on n vertices having cogirth $g^* \geq 3$. Then there exists a good ≥ 2-cocycle cover $\mathscr{C}^*$ of G where $|\mathscr{C}^*| \leq r^*(G)$.*

Proof. By induction on $\varepsilon(G)$. Suppose $\varepsilon(G)=3$. Then $n(G)=2$ and G is a multiple 3-edge. In this case, $C^*=E(G)$ is itself a cocycle, and $\mathscr{C}^*=\{C^*,C^*\}$ is seen to be a good ≥ 2-cocycle cover with $r^*(G)=2$ cocycles. We shall assume that G is a loopless graph on n vertices where $\varepsilon(G)>3$, $g^*(G)\geq 3$, and the theorem is true for all loopless having cogirth at least 3 and less edges than G. We shall make some reductions (**A**)–(**E**) below on G.

(**A**) G is connected.

Proof. If G is not connected, then by assumption, the theorem would hold for each of its components. Taking $\mathscr{C}^*$ to be the union of the good ≥ 2-cocycle covers over the components over G would yield a good ≥ 2-cocycle cover of G with $|\mathscr{C}^*|\leq r^*(G)$. Thus we may assume that G is connected. ■

(**B**) G is 2-connected.

Proof. Suppose G is not 2-connected. Then we can express G as the union $G=G_1\cup G_2$ where G_1 and G_2 have exactly one vertex in common. Then G_i, $i=1,2$, is loopless, and $g^*(G_i)\geq 3$, $i=1,2$. By assumption, G_i has a good ≥ 2-cocycle cover $\mathscr{C}_i^*$ where $|\mathscr{C}_i^*|\leq r^*(G_i)$, $i=1,2$. Let $\mathscr{C}^*=\mathscr{C}_1^*\cup\mathscr{C}_2^*$. Then

$$|\mathscr{C}^*|\leq r^*(G_1)+r^*(G_2)=\sum_{i=1}^{2}(\varepsilon(G_i)-n(G_i)+1)=\varepsilon(G)-(n(G_1)+n(G_2))+2.$$

We also have $n(G)=n(G_1)+n(G_2)-1$. Thus we obtain

$$|\mathscr{C}^*|\leq \varepsilon(G)-(n(G)+1)+2=r^*(G).$$

Since each of the collections $\mathscr{C}_i^*$, $i=1,2$, is a good ≥ 2-cocycle cover of G_i, one sees that $\mathscr{C}^*$ is a good ≥ 2-cocycle cover of G. Thus we may assume that G is 2-connected. ■

(**C**) There is an edge $e\in E(G)$ which belongs to no non-trivial 3-cocycle.

Proof. Suppose this assertion is false, and assume that every edge of G belongs to a non-trivial 3-cocycle. Among all non-trivial 3-cocycles, let $C^*=\{e_1,e_2,e_3\}=[X,Y]$, where $e_i=u_iv_i$, $u_i\in X$, $v_i\in Y$, $i=1,2,3$, and C^* is chosen so that $|X|$ is minimum. Let $e_1'\in E(G[X])$, and let $C'^*=[X',Y']=\{e_1',e_2',e_3'\}$ be a non-trivial 3-cocycle containing e_1'. By the minimality of $|X|$, it holds that $X'\not\subset X$ and $Y'\not\subset X$. In particular, this means that $X'\cap Y\neq\emptyset$, and $Y'\cap Y\neq\emptyset$. We also observe that since $e_1'\in E(G[X])$, it must hold that $X'\cap X\neq\emptyset$ and $Y'\cap X\neq\emptyset$. Thus $C^*\curlywedge C'^*$. Given that $|C^*|=|C'^*|=3$ and $g^*(G)\geq 3$, it follows by elementary counting arguments that

$$|\partial(X\cap X')|=|\partial(X\cap Y')|=|\partial(Y\cap X')|=|\partial(Y\cap Y')|=3.$$

Now if $|X\cap X'|>1$, then $\partial(X\cap X')$ is a non-trivial 3-cocyle where $X\cap X'\subset X$. This contradicts thus choice of X. Thus $|X\cap X'|=1$, and similarly, $|X\cap Y'|=1$. Now it is readily seen that $|C^*|=|\partial(X)|\neq 3$, a contradiction. We conclude that at least one edge belongs to no non-trivial 3-cocycle. ■

Let $e=uv$ be an edge of G which belongs to no non-trivial 3-cocycle of G, where $3\leq d_G(u)\leq d_G(v)$. Let $A_u^*=\partial(\{u\})$, $A_v^*=\partial(\{v\})$, and $A_{uv}^*=\partial(\{u,v\})$.

(**D**) $d_G(u)=3$.

Proof. Suppose $d_G(u) \geq 4$. Let $H = G \backslash e$. Since e belongs to no non-trivial 3-cocycle and $4 \leq d_G(u) \leq d_G(v)$, it follows that e belongs to no 3-cocycle. Consequently, $g^*(H) \geq 3$. By assumption, there exists a good ≥ 2-cocycle cover $\mathscr{B}^* = \{B_i^* | i = 1, \ldots, k\}$, of H where $B_i^* = [X_i, Y_i]$, $u \in X_i$, $i = 1, \ldots, k$. Define cocycles C_i^*, $i = 1, \ldots, k$, of G in the following way: if $v \in X_i$, let $C_i^* = B_i^*$; otherwise, let $C_i^* = B_i^* \cup \{e\}$. Then C_i^*, $i = 1, \ldots, k$, are seen to be cocycles of G. If $e \in C_i^*$ for some $1 \leq i \leq k$, then let $C_{k+1}^* = A_u^*$. Then $\mathscr{C}^* = \{C_1^*, \ldots, C_{k+1}^*\}$ is seen to be a ≥ 2-cocycle cover with $k+1 \leq r^*(H)+1 = r^*(G)$ cocycles. To show that $\mathscr{C}^*$ is good, we suppose that (e_1, e_2) is a bad pair of edges for $\mathscr{C}^*$, and $(e_1, e_2) \leftrightarrow (C_i^*, C_j^*)$. Given that $C_i^* \curlywedge C_j^*$, it holds that $i, j \neq k+1$. Since $e \in C_{k+1}^*$, it follows that $e \neq e_1, e_2$. Now it is seen that (e_1, e_2) is a bad pair for $\mathscr{B}^*$, a contradiction. Thus $\mathscr{C}^*$ is good.

On the other hand, if $e \notin C_i^*$ for $i = 1, \ldots, k$, then let (C_{k1}^*, C_{k2}^*) be an e-splitting for C_k^* and define $\mathscr{C}^* = (\{C_1^*, \ldots, C_{k-1}^*\} \cup \{C_{k1}^*, C_{k2}^*\}$. Then $\mathscr{C}^*$ is seen to be a ≥ 2-cocycle cover of G where $|C^*| = k+1 \leq r^*(G)$. We observe that $e \in C_{kj}^*$, $j = 1, 2$, and $C_{k1}^* C_{k2}^*$, and as such e belongs to no bad pair for $\mathscr{C}^*$. Since $\mathscr{B}^*$ is good, there is no bad pair of edges (e_1, e_2) for $\mathscr{C}^*$ where $e_1, e_2 \in E(H)$. Consequently, $\mathscr{C}^*$ is good. ■

(**E**) $d_G(u) = 3$ and $d_G(v) = 3$.

Proof. By (**D**), we may assume that $d_G(u) = 3$. Suppose $d_G(v) \geq 4$. Let $E_u(G) = \{e_1, e_2, e_3\}$ where $e_1 = uu_1$ and $e_2 = uu_2$. Since G is 2-connected (by (**B**)), we may assume that $u_1 \neq v$. If $u_2 = v$, let $H = (G \backslash \{u\}) \cup \{f\}$ where $f = u_1 v$. Then $r^*(H) = r^*(G) - 1$ and $g^*(G) \geq 3$ (since e belongs to no non-trivial 3-cocycle). By assumption, there exists a good ≥ 2-cocycle cover $\mathscr{B}^* = \{B_1^*, \ldots, B_k^*\}$ of H. Now the cocycles of $\mathscr{B}^*$ can be modified slightly so as to obtain cocycles $C_1^*, \ldots, C_k^*$ of G which cover all the edges of $E(G) \backslash E_u(G)$ at least twice, and the edges of $E_u(G)$ at least once. Then $\{C_1^*, \ldots, C_k^*, A_u^*\}$ is seen to be a good ≥ 2-cocycle cover of G. Thus we may assume that $u_2 \neq v$, and similarly, $u_1 \neq u_2$.

Let $H = (G \backslash \{u\}) \cup \{f\}$ where $f = u_1 u_2$. Then $r^*(H) = r^*(G) - 1$, and $g^*(H) \geq 3$ (since e belongs to no non-trivial 3-cocycle of G). By assumption, there exists a good ≥ 2-cocycle cover $\mathscr{B}^* = \{B_1^*, \ldots, B_k^*\}$ of H where $k \leq r^*(H)$. Assume that $B_i^* = [X_i, Y_i]$, $i = 1, \ldots, k$. Since e_1 belongs to at least two cocycles of $\mathscr{B}^*$, we may assume that $e_1 \in B_1^*$ and $e_1 \in B_2^*$. We shall define cocycles C_i^*, $i = 1, 2, \ldots, k+1$, as follows: let

$$(C_1^*, C_2^*) = \begin{cases} ([X_1 \cup \{u\}, Y_1], [X_2, Y_2 \cup \{u\}]), & \text{if } v \in X_2 \cap Y_1; \\ ([X_1, Y_1 \cup \{u\}], [X_2 \cup \{u\}, Y_2]), & \text{otherwise.} \end{cases}$$

For $i = 3, \ldots, k$, let

$$C_i^* = \begin{cases} [X_i \cup \{u\}, Y], & \text{if } u_2 \in X_i; \\ [X_i, Y_i \cup \{u\}], & \text{if } u_2 \in Y_i. \end{cases}$$

Let $C_{k+1}^* = A_u^*$ and $\mathscr{C}^* = \{C_i^* | i = 1, \ldots, k+1\}$. Then $\mathscr{C}^*$ is seen to be a ≥ 2-cocycle cover of G with $k+1 \leq r^*(G)$ cocycles. Suppose that $\mathscr{C}^*$ is not good, and let (f_1, f_2) be a bad pair for $\mathscr{C}^*$ where $(f_1, f_2) \leftrightarrow (C_i^*, C_j^*)$. We have that $C_i^* \curlywedge C_j^*$, and hence $C_{k+1}^* \neq C_i^*$, C_j^*, and consequently $f_1, f_2 \in E(G) \backslash E_u(G)$. However, this means that (f_1, f_2) is a bad pair for $\mathscr{B}^*$, contradicting the fact that $\mathscr{B}^*$ is good. Thus $\mathscr{C}^*$ is good. ■

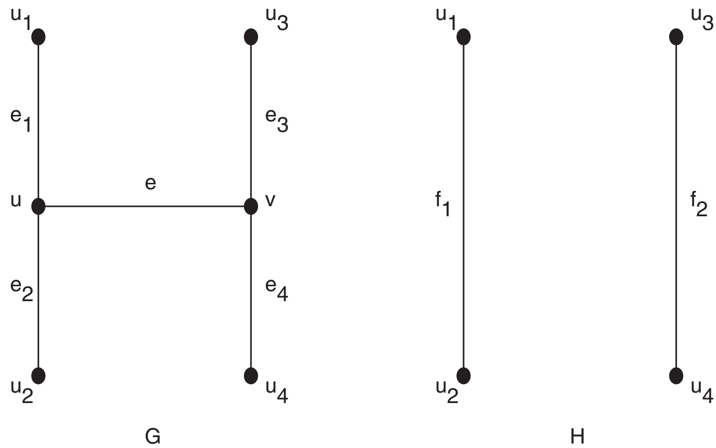

FIGURE 6. Deleting e to form H.

By (**E**), we may assume that $d_G(u)=d_G(v)=3$ for the remainder of the proof. Let $E_u(G)=\{e_1,e_2,e\}$ and $E_v(G)=\{e_3,e_4,e\}$. Let $e_i=uu_i$, $i=1,2$, and $e_i=vu_i$, $i=3,4$. We may assume that u_1, u_2, v are distinct vertices; otherwise, we can use the same arguments as in the proof of (**E**) to show that G has a good ≥ 2-cocycle cover. Likewise, we may assume that u_3, u_4, and v are distinct vertices. Let $H=(G\backslash\{u,v\})\cup\{f_1,f_2\}$ where $f_1=u_1u_2$ and $f_2=u_3u_4$. See Figure 6. Since e belongs to no non-trivial 3-cocycle of G, it follows that $g^*(H)\geq 3$. It is also seen that $r^*(H)=r^*(G)-1$. Thus by assumption, there is a good ≥ 2-cocycle cover $\mathscr{B}^*=\{B_1^*,\ldots,B_k^*\}$ of H where $B_i=[X_i,Y_i]$, $i=1,\ldots,k$. We may assume that $u_1\in X_i$, $i=1,\ldots,k$.

We define the following cutsets of G:

$$B_{i,\cdot}^* = [X_i, Y_i\cup\{u,v\}],$$
$$B_{i,u}^* = [X_i\cup\{u\}, Y_i\cup\{v\}],$$
$$B_{i,v}^* = [X_i\cup\{v\}, Y_i\cup\{u\}],$$
$$B_{i,uv}^* = [X_i\cup\{u,v\}, Y_i].$$

For $i=1,\ldots,k$, let $\boldsymbol{\alpha}_i=(a_{i1},a_{i2},a_{i3})\in\mathbb{Z}_2^3$, where for $j=1,2,3$

$$a_{ij}=\begin{cases}1, & \text{if } u_{j+1}\in X_i;\\ 0, & \text{if } u_{j+1}\in Y_i.\end{cases}$$

For $i=1,\ldots,k$ let

$$A_i^*=\begin{cases}B_{i,u}^*, & \text{if } \boldsymbol{\alpha}_i\neq(0,1,1),(1,1,1);\\ B_{i,v}^*, & \text{if } \boldsymbol{\alpha}_i=(0,1,1);\\ B_{i,uv}^*, & \text{if } \boldsymbol{\alpha}_i=(1,1,1).\end{cases}$$

See Figure 7.

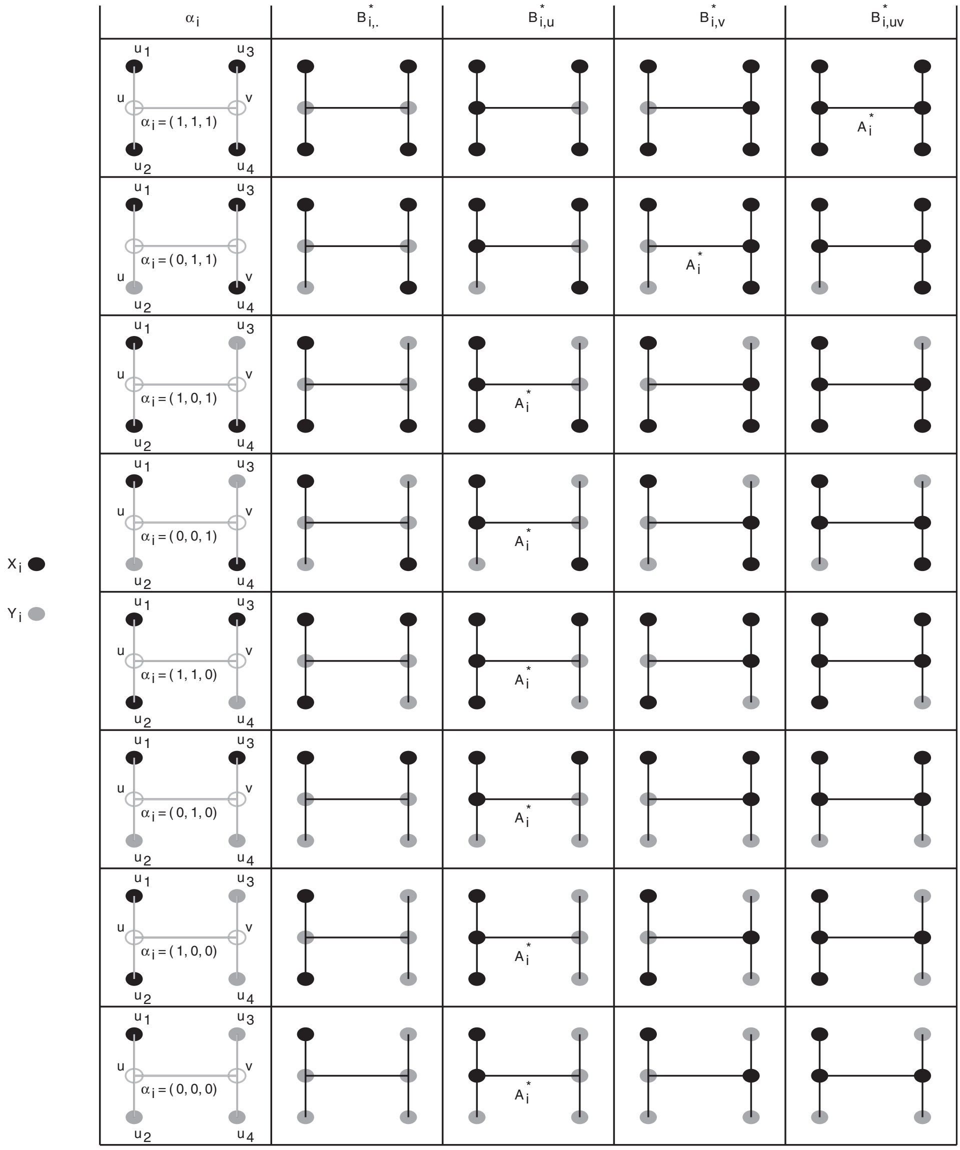

FIGURE 7. Table of cocycles.

Let $A^*_{k+1}=A^*_{uv}=\{e_1,e_2,e_3,e_4\}$. We shall show that, with one exception, the cocycles of $\mathscr{B}^*$ which contain f_1 and f_2 can be transformed into cocycles of G in such a way that e_1, e_2, e_3, e_4 are covered at least once, and e is covered at least twice. We shall consider three cases:

Case 1: $\{f_1,f_2\}\not\subset B^*_i$, $i=1,\ldots,k$.

Given that $\mathscr{B}^*$ is a ≥ 2 cover for H, we may assume that $f_1\in B^*_1$, B^*_2 and $f_2\in B^*_3$, B^*_4. Then

$$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2\in\{(0,0,0),(0,1,1)\},\ \boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4\in\{(1,0,1),(1,1,0)\}.$$

We define C_1^*, C_2^* in the following way:

$$(C_1^*, C_2^*) = \begin{cases} (B_{1,u}^*, B_{2,\cdot}^*), & \text{if } \alpha_1 = (0,0,0),\ \ \alpha_2 = (0,0,0); \\ (B_{1,u}^*, B_{2,v}^*), & \text{if } \alpha_1 = (0,0,0),\ \ \alpha_2 = (0,1,1); \\ (B_{1,v}^*, B_{2,u}^*), & \text{if } \alpha_1 = (0,1,1),\ \ \alpha_2 = (0,0,0); \\ (B_{1,v}^*, B_{2,uv}^*), & \text{if } \alpha_1 = (0,1,1),\ \ \alpha_2 = (0,1,1). \end{cases}$$

Similarly, we define C_3^* and C_4^* where

$$(C_3^*, C_4^*) = \begin{cases} (B_{3,uv}^*, B_{4,u}^*), & \text{if } \alpha_3 = (1,1,0),\ \ \alpha_4 = (1,1,0); \\ (B_{3,u}^*, B_{4,u}^*), & \text{if } \alpha_3 = (1,1,0),\ \ \alpha_4 = (1,0,1); \\ (B_{3,u}^*, B_{4,u}^*), & \text{if } \alpha_3 = (1,0,1),\ \ \alpha_4 = (1,1,0); \\ (B_{3,uv}^*, B_{4,u}^*), & \text{if } \alpha_3 = (1,0,1),\ \ \alpha_4 = (1,0,1). \end{cases}$$

For $i = 5, \dots, k+1$, let $C_i^* = A_i^*$. Then $\mathscr{C}^* = \{C_1^*, \dots, C_{k+1}^*\}$ is seen to be a ≥ 2-cocycle cover of G with $k+1 \leq r^*(G)$ cocycles. Suppose $\mathscr{C}^*$ is not good and let (g,h) be a bad pair of edges where $(g,h) \leftrightarrow (C_i^*, C_j^*)$, for some $i<j$. Since $\mathscr{B}^*$ is good, at least one of g or h belongs to $E_u(G) \cup E_v(G)$, and we may assume that this holds for g. Since $C_{k+1}^* = C_{uv}^*$, it separates e from any edge in $E(G) \backslash \{e_1, e_2, e_3, e_4\}$. Thus (e, e') is not a bad pair for all $e' \in E(G) \backslash \{e_1, e_2, e_3, e_4\}$. Moreover, (e, e_i), $i = 1,2,3,4$, is not a bad pair since $e \notin C_{k+1}^*$. Thus $g \neq e$, and hence $g \in \{e_1, e_2, e_3, e_4\} = C_{k+1}^*$. Consequently, $C_j^* = C_{k+1}^*$, and hence $h \in \{e_1, e_2, e_3, e_4\}$. Noting that g and h cannot be incident, we may assume that $g \in \{e_1, e_2\}$ and $h \in \{e_3, e_4\}$. It must hold that $C_i^* \in \{C_1^*, C_2^*, C_3^*, C_4^*\}$. However, by construction, none of the cocycles C_1^*, C_2^*, C_3^*, C_4^* contain edges from both $\{e_1, e_2\}$ and $\{e_3, e_4\}$. Thus (g,h) cannot be a bad pair. We conclude that $\mathscr{C}^*$ is good.

Case 2: There exists exactly one $B_i^* \in \mathscr{B}^*$ such that $\{f_1, f_2\} \subseteq B_i^*$.

We may assume that $\{f_1, f_2\} \subseteq B_1^*$, $f_1 \in B_2^*$, and $f_2 \in B_3^*$. Furthermore, we may assume without loss of generality that $\boldsymbol{\alpha}_1 = (0,1,0)$. We observe that $\boldsymbol{\alpha}_2 \in \{(0,0,0),(0,1,1)\}$ and $\boldsymbol{\alpha}_3 \in \{(1,0,1),(1,1,0)\}$. We shall define C_1^*, C_2^*, C_3^* in the following way:

$$(C_1^*, C_2^*, C_3^*) = \begin{cases} (B_{1,\cdot}^*, B_{2,u}^*, B_{3,u}^*), & \text{if } \boldsymbol{\alpha}_2 = (0,0,0),\ \ \boldsymbol{\alpha}_3 = (1,0,1); \\ (B_{1,v}^*, B_{2,u}^*, B_{3,u}^*), & \text{if } \boldsymbol{\alpha}_2 = (0,0,0),\ \ \boldsymbol{\alpha}_3 = (1,1,0); \\ (B_{1,u}^*, B_{2,v}^*, B_{3,u}^*), & \text{if } \boldsymbol{\alpha}_2 = (0,1,1),\ \ \boldsymbol{\alpha}_3 = (1,0,1); \\ (B_{1,uv}^*,\ B_{2,v}^*, B_{3,u}^*), & \text{if } \boldsymbol{\alpha}_2 = (0,1,1),\ \ \boldsymbol{\alpha}_3 = (1,1,0). \end{cases}$$

For $i = 4, \dots, k+1$, let $C_i^* = A_i^*$. Now $\mathscr{C}^* = \{C_1^*, \dots, C_{k+1}^*\}$ is seen to be a ≥ 2-cocycle cover of G with $k+1 \leq r^*(G)$ cocycles. Suppose $\mathscr{C}^*$ is not good, and (g,h) is a bad pair where $(g,h) \leftrightarrow (C_i^*, C_j^*)$, for some $i<j$. Similar to Case 1, we have that $C_j^* = C_{k+1}^*$, $C_i^* \in \{C_1^*, C_2^*, C_3^*\}$ and $g, h \in \{e_1, e_2, e_3, e_4\}$. We may assume that $g \in \{e_1, e_2\}$ and $h \in \{e_3, e_4\}$. Since there are only two cocycles in $\mathscr{C}^*$ which contain edges from both $\{e_1, e_2\}$ and $\{e_3, e_4\}$, namely C_1^* and C_{k+1}^*, it holds that $(g,h) \leftrightarrow (C_1^*, C_{k+1}^*)$. Observe that in all cases,

C_3^* separates g and h. Thus (g,h) cannot be a bad pair, a contradiction. We conclude that $\mathscr{C}^*$ is good.

Case 3: At least two cocycles of $\mathscr{B}^*$ contain both f_1 and f_2.

We may assume that $\{f_1,f_2\}\subseteq B_i^*$, $i=1,2$, and $\boldsymbol{\alpha}_1=(0,1,0)$. We have that $\boldsymbol{\alpha}_2\in\{(0,1,0),(0,0,1)\}$. Suppose first that $\boldsymbol{\alpha}_2=(0,1,0)$. Let $C_1^*=B_{1,u}^*$, $C_2^*=B_{2,v}^*$, and for $i=3,\dots,k+1$, let $C_i^*=A_i^*$. Then $\mathscr{C}^*=\{C_1^*,\dots,C_{k+1}^*\}$ is seen to be a ≥ 2-cocycle cover of G with $k+1\leq r^*(G)$ cocycles. If $\mathscr{C}^*$ is not good, then there is a bad pair of edges (g,h). As in Cases 1 and 2, we have that $g,h\in\{e_1,e_2,e_3,e_4\}$, and we may assume that $g\in\{e_1,e_2\}$ and $h\in\{e_3,e_4\}$. Since $g,h\in C_{k+1}^*=A_{uv}^*=\{e_1,e_2,e_3,e_4\}$, and $\{e_1,e_2,e_3,e_4\}\subset C_1^*\cup C_2^*$, it follows that either $(g,h)\leftrightarrow(C_1^*,C_{k+1}^*)$ and $(g,h)=(e_2,e_3)$, or $(g,h)\leftrightarrow(C_2^*,C_{k+1}^*)$ and $(g,h)=(e_1,e_4)$. However, (e_2,e_3) is not a bad pair since C_2^* separates e_2 and e_3. Similarly, (e_1,e_4) is not a bad pair since C_1^* separates e_1 and e_4. This yields a contradiction, and consequently $\mathscr{C}^*$ is good.

If $\boldsymbol{\alpha}_2=(0,0,1)$, let $C_1^*=B_{1,\cdot}^*$, and $C_2^*=B_{2,u}^*$. Let $C_i^*=A_i^*, i=3,\dots,k+1$, and $\mathscr{C}^*=\{C_1^*,\dots,C_{k+1}^*\}$. To show that $\mathscr{C}^*$ is a ≥ 2-cocycle cover, we first note that the cocycles C_1^*,C_2^*,C_{k+1}^* cover the edges e_1, e_2, e_3, e_4 twice, but cover e only once. Note that (B_1^*,B_2^*) crosses (f_1,f_2). If B_i^* separates f_1 from f_2 for some $i\geq 3$, then $\boldsymbol{\alpha}_i=(1,0,0)$ and $A_i=B_{i,u}^*$. In this case, $e\in B_{i,u}^*$ and $\mathscr{C}^*$ is seen to be a ≥ 2-cocycle cover. If $\mathscr{C}^*$ is not good and (g,h) is a bad pair, then as before, $g\in\{e_1,e_2\}$, $h\in\{e_3,e_4\}$, and $(g,h)\leftrightarrow(C_j^*,C_{k+1}^*)$ where $j\in\{1,2\}$. Then $(e_1,e_3)\leftrightarrow(C_1^*,C_{k+1}^*)$ or $(e_2,e_4)\leftrightarrow(C_2^*,C_{k+1}^*)$. However, C_i^* separates e_1 and e_3, and it also separates e_2 and e_4. Thus in this case $\mathscr{C}^*$ is a good ≥ 2-cocycle cover. Henceforth, we may assume that no cocycle of $\mathscr{B}^*$ separates f_1 and f_2.

Since $\mathscr{B}^*$ is good, (f_1,f_2) cannot be a bad pair. Thus, either $f_1\in B_{j^*}^*$ or $f_2\in B_{j^*}^*$ for some $3\leq j^*\leq k$. In particular, $j^*\neq(1,1,1)$. By construction,

$$C_{j^*}^*=A_{j^*}^*=\begin{cases} B_{j^*,u}^*, & \text{if } \boldsymbol{\alpha}_{j^*}\neq(0,1,1);\\ B_{j^*,v}^*, & \text{if } \alpha_{j^*}=(0,1,1).\end{cases}$$

Hence $e\in C_{j^*}$ and e belongs to at least two cocyles of $\mathscr{C}^*$. Consequently, $\mathscr{C}^*$ is a ≥ 2 cocycle cover of G with at most $k+1\leq r^*(G)$ cocycles. If $\mathscr{C}^*$ is not good then as before, there is a bad pair (g,h) where we may assume $g\in\{e_1,e_2\}$ and $h\in\{e_3,e_4\}$. Thus either $(g,h)\leftrightarrow(C_1^*,C_{k+1}^*)$ and $(g,h)=(e_1,e_3)$, or $(g,h)\leftrightarrow(C_2^*,C_{k+1}^*)$ and $(g,h)=(e_2,e_4)$. Now (C_1^*,C_{k+1}^*) can not be a bad pair of cocycles since $C_1^*C_{k+1}^*$. Suppose (C_2^*,C_{k+1}^*) is a bad pair and $(e_2,e_4)\leftrightarrow(C_2^*,C_{k+1}^*)$. Then $e_2,e_4\notin C_{j^*}^*$, and by definition of A_{j^*}, it must hold that $\alpha_{j^*}\in\{(0,1,1),(1,1,0),(1,0,0)\}$. However, for each choice of α_{j^*}, it is seen that $C_j^*=A_{j^*}$ separates e_2 and e_4. Thus (C_2^*,C_{k+1}^*) cannot be a bad pair, and consequently $\mathscr{C}^*$ is good.

The proof of the theorem now follows from the consideration of Cases 1–3 above.

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