On modal definability of Horn formulas

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Abstract

In this short paper we give a criterion of modal definability of a first-order universal Horn sentence with exactly one positive atom in terms of its graph. As a consequence we obtain that every modal logic axiomatized by a single modal Horn formula (i.e. of the form $\mathbf{K} + \phi$ where ϕ is a modal Horn formula) is Kripke complete.

Modal definability of first-order formulas has been intensively studied in modal logic, and even applied to automatic reasoning [9]. On the one hand, it has a nice Goldblatt-Thomason characterization [4], on the other hand, the problem "decide whether a first-order formula is modally definable" is in general undecidable [2]. But the cause of this undecidability is in the undecidability of first-order logic, so when we restrict attention to a fragment with decidable implication, we are likely to obtain an algorithmic criterion for modal definability, as in this paper. Also this research is motivated by scrutinizing Theorem 5.9 of [3] saying that if L_1 and L_2 are Kripke complete and Horn axiomatizable unimodal logics, then $L_1 \times L_2 = [L_1, L_2]$ and studying whether Horn axiomatizability implies Kripke completeness. We give the positive answer to the last question for the case of a single universal Horn sentence with exactly one positive atom, but in general this problem seems to be open.

Consider the classical first-order language $\mathcal{L}f_\Lambda$ in the signature consisting of only binary predicates R_λ indexed by a finite set Λ. *An atom* is a formula of the form $x_i R_\lambda x_j$, where x_i and x_j are object variables and $\lambda \in \Lambda$. *Universal Horn sentences* (in short, *Horn formulas*) are closed (i.e. without free variables) formulas of the form $\forall x_1 \ldots \forall x_n (\psi \to \phi)$, where ψ is a conjunction of atoms and ϕ is an atom. Allowing $\vee$ in ψ as in [3] is equivalent to considering conjunctions of such formulas. Universal Horn sentences can be represented by tuples of the form $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$, where $W^D = \{x_1, \ldots, x_n\}$ is a finite set, R^D_λ are binary relations on W^D, $\alpha, \beta \in W^D$ and $\lambda_0 \in \Lambda$. Such a tuple D, called *a diagram*, gives rise to the Horn formula

$$E^D = \forall x_1 \ldots \forall x_n \left(\bigwedge_{x_i R^D_\lambda x_j} x_i R_\lambda x_j \to \alpha R_{\lambda_0} \beta \right).$$

For a diagram D, define its size $|D| = \sum_{\lambda \in \Lambda} |R^D_\lambda|$, where $|R^D_\lambda|$ denotes the cardinality of R^D_λ. A diagram D is called *minimal* if there is no diagram D' of size less than $|D|$ such that $E^{D'} \equiv E^D$, where $\equiv$ denotes the predicate calculus equivalence. A diagram D is called *non-trivial* if E^D is not equivalent to $\top$.

We also consider the modal language $\mathcal{M}l_\Lambda$ with countably many propositional variables $p_1, p_2, \ldots$, unary modalities $\Diamond_\lambda$ and their duals $\Box_\lambda$, where $\lambda \in \Lambda$ and boolean connectives $\wedge, \vee, \neg, \to$. A *Kripke frame* is an $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$. A *valuation of propositional variables in* F is a map θ assigning to any p_i a set $\theta(p_i) \subseteq W$. A *Kripke model* *built on a frame* F is a pair $M = (F, \theta)$ where θ is a valuation of propositional variables in F. The *truth of modal formula* ϕ in a point x of Kripke model M is defined in the standard way. A modal formula ϕ is *valid* on a Kripke frame F (denoted $F \models \phi$) if ϕ is true in every point of every model M built on F.

An $\mathcal{L}f_\Lambda$-sentence E is called *modally definable* if there exists a modal formula ϕ such that, for any Kripke frame F, $F \models E$ iff $F \models \phi$. Here $\models$ on the left-hand side means the classical truth of an $\mathcal{L}f_\Lambda$-formula in $\mathcal{L}f_\Lambda$-structure, while $\models$ on the right-hand side means the validity of a modal formula in a Kripke frame. If this equivalence holds and, in addition, E is a universal Horn sentence, then ϕ is called a *modal Horn formula.*

Consider a finite $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$, where, for all $\lambda \in \Lambda$, $R_\lambda \subseteq W \times W$. A sequence $x_1, \lambda_1, x_2, \lambda_2, \ldots x_n$, where $x_i \in W^T$, $\lambda_i \in \Lambda$ and $(x_i, x_{i+1}) \in R_{\lambda_i}$ for $1 \le i \le n-1$ is called *a directed path from x_1 to x_n in F*. The definition of *an undirected path from x_1 to x_n* is obtained by replacing $(x_i, x_{i+1}) \in R_{\lambda_i}$ with $(x_i, x_{i+1}) \in R_{\lambda_i} \cup R^{-1}_{\lambda_i}$. An $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ is called *a directed tree* if there is a point $r \in W$ such that the following holds:

- $(R_\lambda)^{-1}(r) = \emptyset$ for all $\lambda \in \Lambda$,
- for every point $x \neq r$, there exists a unique directed path from r to x.

THEOREM 1. The Horn formula E^D corresponding to a minimal non-trivial diagram $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$ is modally definable iff the $\mathcal{L}f_\Lambda$-structure $(W^D, (R^D_\lambda : \lambda \in \Lambda))$ is a directed tree.

The proof of the 'if' direction is simple: if $(W^D, (R^D_\lambda : \lambda \in \Lambda))$ is a directed tree, then all points x_i except the root x_0 have a unique predecessor $x_{pr(i)}$ such that $x_{p(i)} R_{\lambda(i)} x_i$ for some $\lambda(i) \in \Lambda$. Assuming that the x are numbered in such a way that, for all i, $pr(i) < i$ and using the restricted universal quantifier

$$(\forall x_i \rhd_\lambda x_j) A \equiv \forall x_i (x_j R_\lambda x_i \to A),$$

we can rewrite E^D as

$$\forall x_0 (\forall x_1 \rhd_{\lambda(1)} x_0)(\forall x_2 \rhd_{\lambda(2)} x_{pr(2)}) \ldots (\forall x_n \rhd_{\lambda(n)} x_{pr(n)})(\alpha R_{\lambda_0} \beta).$$

This is obviously a Kracht formula [6], [7], so it is modally definable by a Sahlqvist formula. The proof of the 'only if' direction follows from lemmas 2 and 4 and the fact that all modally definable properties are preserved under disjoint unions and bounded morphisms (e.g. [1]). Together with the Sahlqvist completeness theorem it gives us that any modal logic axiomatizable by a single modal Horn formula is Kripke complete. The complexity of similar logics is studied in [5].

Consider two $\mathcal{L}f_\Lambda$-structures $F^1 = (W^1, (R^1_\lambda : \lambda \in \Lambda))$ and $F^2 = (W^2, (R^2_\lambda : \lambda \in \Lambda))$. A map $g : W^1 \to W^2$ is called a *homomorphism* from F^1 to F^2 if, for any $\lambda \in \Lambda$ and $a, b \in W^1$, $a R^1_\lambda b$ implies $f(a) R^2_\lambda f(b)$. For a finite $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ and a (diagram of a) Horn formula $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$ we define a *Horn closure* F^*_D in the following way. Set $F^0_D = F$. Let $F^{i-1}_D = (W, (R^{i-1}_\lambda : \lambda \in \Lambda))$ be already defined. Let $\mathcal{G}_i$ be the set of all homomorphisms from $(W^D, (R^D_\lambda : \lambda \in \Lambda))$ to F^{i-1}_D. Set $F^i_D = (W, (R^i_\lambda : \lambda \in \Lambda))$ where

$$R^i_{\lambda_0} = R^{i-1}_{\lambda_0} \cup \bigcup_{g \in \mathcal{G}_i} \{(g(\alpha), g(\beta))\}$$

and $R^i_\lambda = R^{i-1}_\lambda$ for $\lambda \neq \lambda_0$. Since W is finite, there exists n such that $F^n_D = F^{n+1}_D = F^{n+2}_D$, and so on. Then we set $F^*_D = F^n_D$ for such n. This construction generalizes the well-known transitive and symmetric closure.

An $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ and a diagram $D = (F, \alpha, \beta, \lambda)$ are called *connected* if any two different points of W may be connected by an undirected way.

LEMMA 2. Take a minimal non-trivial diagram $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$. If an $\mathcal{L}f_\Lambda$-structure $G^D = (W^D, (R^D_\lambda : \lambda \in \Lambda))$ is not connected then E^D is not preserved under disjoint unions.

Proof. First suppose that α and β belong to different connected components of G^D. Then take $F = G^D$ and its Horn closure F^*_D. Thus we have $F^*_D \models E^D$ but $F^*_D \sqcup F^*_D \not\models E^D$, and the lemma is proved.

Then consider the case where G^D is split into connected components $K_1, \dots, K_n$ and α and β belong to the same connected component, say, K_1. Note that since D is minimal, there is no homomorphism from K_2 to K_1, otherwise we can throw K_2 out of a diagram without affecting E^D semantically. Thus we have $K_1 \models E^D$, since there is no homomorphism from G^D in K_1 because of K_2. Put $D' = (K_1, \alpha, \beta, \lambda_0)$. Then $(G^D \setminus K_1)^*_{D'} \models E^D$ (since $E^{D'} \models E^D$) but $K_1 \sqcup (G^D \setminus K_1)^*_{D'} \not\models E^D$ (because of the identity homomorphism of G^D into itself and non-triviality of D). □

LEMMA 3. Consider two diagrams $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$ and $D' = (W^{D'}, (R^{D'}_\lambda : \lambda \in \Lambda), \alpha', \beta', \lambda'_0)$. Put $F = (W^D, (R^D_\lambda : \lambda \in \Lambda))$. Then $F^*_{D'} \models E^D$ implies $E^{D'} \models E^D$.

Proof. Take any $G = (W, (R_\lambda : \lambda \in \Lambda))$. Assume that $G \models E^{D'}$ and prove that $G \models E^D$. Take a homomorphism h from F to G. Now execute the process of construction of $F^*_{D'}$ and copy any its step by h into G, applying $G \models E^{D'}$ for each new edge. Finally we will obtain that $h(\alpha) R_{\lambda_0} h(\beta)$. □

LEMMA 4. Let D be a minimal non-trivial diagram. Then if $G^D = (W^D, (R^D_\lambda : \lambda \in \Lambda))$ contains a directed cycle or a point c with two incoming arrows then E^D is not preserved under bounded morphism.

Proof. First suppose that G^D contains a directed cycle. Then take $F = G^D$ and it unravelling $F^u = ((W^D)^u, ((R^D)^u_\lambda : \lambda \in \Lambda))$, where $(W^D)^u$ is the set of all directed pathes in F, with a natural bounded morphism $f : (W^D)^u \to W^D$, sending each path to its last point, and $(R^D)^u_\lambda$ defined in a standard way: for $x, y \in (W^D)^u$ $x(R^D)^u_\lambda y$ iff $y = x, \lambda, f(y)$. Since D is non-trivial, $F \not\models E^D$. But $F^u \models E^D$, since there is no homomorphism from G^D to the tree F^u because of a directed cycle in G^D, so the lemma is proved.

Now assume that G^B contains a vertex with at least two incoming edges. It means that there exist points $a, b, c \in W^D$ and $\lambda_1, \lambda_2 \in \Lambda$ such that $(a, c) \in R^D_{\lambda_1}$ and $(b, c) \in R^D_{\lambda_2}$. If $\lambda_1 \neq \lambda_2$, the same argument as for the directed cycle works: a point with two incoming arrows of different kinds cannot be embedded into the tree F^u.

But if $a \neq b$ and $\lambda_1 = \lambda_2$ it may still happen that there is a homomorphism h from G^B to F^u, in this case $h(a) = h(b)$. So we consider the set $\mathcal{T}$ of all directed trees T such that there exists a surjective homomorphism from G^B to T. We claim that there exists a directed tree $T_0 \in \mathcal{T}$ such that for all $T \in \mathcal{T}$ there exists a surjective homomorphism from T_0 to T.

Let $\sim$ be the smallest equivalence relation on W^D satisfying condition (cf. [8])

if there exists a, b, c, c' such that $aR^D_\lambda c$, $bR^D_\lambda c'$ and $c \sim c'$, then $a \sim b$.

Define $T_0 = (W^0, (R^0_\lambda : \lambda \in \Lambda))$ where $W^0 = W^D/\sim$, and for equivalence classes $A, B \in W^0$ $AR^0_\lambda B$ iff there exist $a \in A$ and $b \in B$ such that $aR_\lambda b$. In other words, T^0 is obtained from G^D by a sequence of following reductions: if there exist $a, b, c \in W^D$ such that $aR^D_\lambda c$ and $bR^D_\lambda c$, then join a and b into one point. The main property of $\sim$ is that for every homomorphism g from G^D to a directed tree T $a \sim b$ implies $g(a) = g(b)$, that is every such g factors through T_0.

Let h be the natural projection from G^B to T_0. Consider the diagram $D' = (T_0, h(\alpha), h(\beta), \lambda_0)$. In any case, a homomorphism from G_B to T_0 implies that $E_D \models E_{D'}$, and a vertex with two incoming edges in G^B implies that $|D'| < |D|$. Since D is minimal, $E^{D'} \not\models E^D$ and according to Lemma 3 it follows that $F^*_{D'} \not\models E^D$.

Now we can prove the lemma, since $(F^u)^*_{D'} \models E^D$ (use universal property of T_0), $F^*_{D'} \not\models E^D$ and f is a p-morphism not only from F^u to F, but also from $(F^u)^*_{D'}$ to $F^*_{D'}$. □

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