

The Equivariant Noncommutative Atiyah-Patodi-Singer Index Theorem *

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Abstract

In [Wu], the noncommutative Atiyah-Patodi-Singer index theorem was proved. In this paper, we extend this theorem to the equivariant case.

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1 Introduction

In [APS], Atiyah-Patodi-Singer proved their famous Atiyah-Patodi-Singer index theorem for manifolds with boundary. In [D], Donnelly extended this theorem to the equivariant case by modifying the Atiyah-Patodi-Singer original method. In [Z], Zhang got this equivariant Atiyah-Patodi-Singer index theorem by using a direct geometric method^[LYZ]. In [Wu], Wu proved the Atiyah-Patodi-Singer index theorem in the framework of noncommutative geometry. To do so, he introduced the total eta invariant (called the higher eta invariant in [Wu]) which is the generalization of the classical Atiyah-Patodi-Singer eta invariants^[APS], then proved its regularity by using the Getzler symbol calculus^[G1] as adopted in [BF] and computed its radius of convergence. Subsequently, he proved the variation formula of eta cochains, using which he got the noncommutative Atiyah-Patodi-Singer index theorem. In [G2], using superconnection, Getzler gave another proof of the noncommutative Atiyah-Patodi-Singer index theorem, which was more difficult, but avoided mention of the operators b and B of cyclic cohomology.

The purpose of this paper is to extend the noncommutative Atiyah-Patodi-Singer index theorem to the equivariant case.

The paper is organized as follows: In Section 2.1, we define the equivariant eta cochains and prove their regularity at infinity. In Section 2.2 we decompose the

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equivariant eta cochains into two parts and estimate the second part. The first part will be estimated in Section 2.3. In Section 2.4, we consider the convergence of the total eta invariant.

Let N be an odd dimensional spin manifold and G be a compact Lie group acting on N by orientation-preserving isometries. let

$$C_G^\infty(N) = \{f \in C^\infty(N) | f(g.x) = f(x), \text{ for any } g \in G \text{ and } x \in N\}.$$

Suppose that Dirac operator D is invertible with λ the smallest eigenvalue of $|D|$, and $p = p^* = p^2 \in \mathcal{M}_r(C_G^\infty(N))$ is an idempotent which satisfying $\|dp\| < \lambda$. Let $\eta^G(p(D \otimes I_r)p)$ is the equivariant Atiyah-Patodi-Singer eta invariant associated to $p(D \otimes I_r)p$ which is the Dirac operator with coefficients from $F = p(\mathbf{C}^r)$. $\eta^G(D)$ is the equivariant total eta invariants defined in Section 2. $\text{Ch}(p)$ is the Chern character of p defined in [GS]. In Section 3, using the superconnection method in [G2], we prove the formula

$$\frac{1}{2}\eta^G(p(D \otimes I_r)p) = \langle \eta^G(D), \text{Ch}(p) \rangle, \quad (1.1)$$

In section 4, we define the equivariant Chern-Connes character on manifolds with boundary and discuss its radius of convergence.

In section 5, we prove our main results. Using (1.1), we express the equivariant index of the Dirac operator with the coefficient from G -vector bundle $p(\mathbf{C}^r)$ over the cone as a pair of the equivariant Chern-Connes character and $\text{Ch}(p)$.

2 The Equivariant Total Eta Invariants

2.1 The Equivariant Eta Cochains

Let N be a compact oriented odd dimensional Riemannian manifold without boundary with a fixed spin structure and S be the bundle of spinors on N . Denote by D the associated Dirac operator on $H = L^2(N; S)$, the Hilbert space of L^2 -sections of the bundle S . Let $c(df) : S \rightarrow S$ denote the Clifford action with $f \in C^\infty(N)$. Suppose that G is a connected Lie group acting on N by orientation-preserving isometries and $g \in G$ has a lift $\widetilde{dg} : \Gamma(S) \rightarrow \Gamma(S)$ (see [LYZ]), then we have \widetilde{dg} commutes with the Dirac operator and \widetilde{dg} is a bounded operator.

Let $A = C^\infty(N)$, then the data (A, H, D, G) defines a finitely (hence θ -summable) equivariant unbounded Fredholm module in the sense of [KL] (for details see [CH],[FGV] and [KL]). Similar to [CH] or [W], for equivariant θ -summable Fredholm module (A, H, D, G) , we can define the equivariant cochain $\widetilde{\text{ch}}_k^G(tD, D)$ (k is even) by the formula:

$$\begin{aligned} & \widetilde{\text{ch}}_k^G(tD, D)(f^0, \dots, f^k)(g) \\ & := t^k \sum_{i=0}^k (-1)^i \langle f^0, c(df^1), \dots, c(df^i), D, c(df^{i+1}), \dots, c(df^k) \rangle_t(g), \end{aligned} \quad (2.1.1)$$

where $f^0, \dots, f^k \in C^\infty(N)$, $g \in G$. If A_i ($0 \leq i \leq n$) are operators on H , we define:

$$\langle A_0, \dots, A_n \rangle_t(g) = \int_{\Delta_n} \text{Tr}(A_0 e^{-t^2 s_1 D^2} A_1 e^{-t^2 (s_2 - s_1) D^2} \dots A_n e^{-t^2 (1 - s_n) D^2} \widetilde{dg}) ds, \quad (2.1.2)$$

where $\Delta_n = \{(s_1, \dots, s_n) \mid 0 \leq s_1 \leq \dots \leq s_n \leq 1\}$ is the simplex in \mathbf{R}^n .

Formally, the equivariant total η -invariant of the Dirac operator D is defined to be a sequence of even equivariant cochains on $C^\infty(N)$, by the formula:

$$\eta_k^G(D) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \widetilde{\text{ch}}_k^G(tD, D) dt, \quad (2.1.3)$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Then $\eta_0^G(D)(1)(g)$ is the half of the equivariant eta invariants defined in [APS], [D] and [Z]. In order to prove that the above definition is well defined, it is necessary to check the integrality near the two ends of the integration. Firstly, the regularity at infinity comes from the following lemma.

Lemma 2.1 *For $f^0, \dots, f^k \in C^\infty(N)$ and $g \in G$, we have*

$$\widetilde{\text{ch}}_k^G(tD, D)(f^0, \dots, f^k)(g) = O(t^{-2}), \quad \text{as } t \rightarrow \infty. \quad (2.1.4)$$

Proof. Since \widetilde{dg} is a bounded operator, our proof is similar to the proof of Lemma 2 in [CH]. \square

2.2 Expansion of The Equivariant Eta Cochains

In [W], Wu proved the regularity at zero of (2.1.3) in the $g = id$ case by using the Getzler symbol calculus. In what follows, we will give a proof of the regularity of (2.1.3) at zero in the general case by using the method in [CH] and [F].

Firstly, recall some Lemmas in [CH] and [F].

Let H be a Hilbert space. For $q \geq 0$, denote by $\|\cdot\|_q$ Schatten p -norm on Schatten ideal L^p (for details, see [S]). $L(H)$ denotes the Banach algebra of bounded operators on H .

Lemma 2.2 ([CH],[F]) (i) $\text{Tr}(AB) = \text{Tr}(BA)$, for $A, B \in L(H)$ and $AB, BA \in L^1$.
(ii) For $A \in L^1$, we have $|\text{Tr}(A)| \leq \|A\|_1$, $\|A\| \leq \|A\|_1$.
(iii) For $A \in L^q$ and $B \in L(H)$, we have: $\|AB\|_q \leq \|B\| \|A\|_q$, $\|BA\|_q \leq \|B\| \|A\|_q$.
(iv) (Hölder Inequality) If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $p, q, r > 0$, $A \in L^p$, $B \in L^q$, then $AB \in L^r$ and $\|AB\|_r \leq \|A\|_p \|B\|_q$.

Lemma 2.3 ([CH],[F]) *For any $u > 0$, $t > 0$ and any order l differential operator B , we have:*

$$\|e^{-utD^2} B\|_{u-1} \leq C_l u^{-\frac{l}{2}} t^{-\frac{l}{2}} (\text{tr}[e^{-\frac{tD^2}{2}}])^u. \quad (2.2.1)$$

Lemma 2.4 ([CH],[F]) *Let B_1, B_2 be positive order p, q pseudodifferential operators respectively, then for any $s, t > 0, 0 \leq u \leq 1$, we have the following estimate:*

$$\|B_1 e^{-ustD^2} B_2 e^{-(1-u)stD^2}\|_{s^{-1}} \leq C_{p,q} s^{-\frac{p+q}{2}} t^{-\frac{p+q}{2}} (\text{tr}[e^{-\frac{tD^2}{4}}])^s. \quad (2.2.2)$$

Let B be an operator and l be a positive interger. Write

$$B^{[l]} = [D^2, B^{[l-1]}], \quad B^{[0]} = B.$$

Lemma 2.5 ([CH],[F]) *Let B a finite order differential operator, then for any $s > 0$, we have:*

$$e^{-sD^2} B = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l B^{[l]} e^{-sD^2} + (-1)^N s^N B^{[N]}(s), \quad (2.2.3)$$

where $B^{[N]}(s)$ is given by

$$B^{[N]}(s) = \int_{\Delta_N} e^{-u_1 s D^2} B^{[N]} e^{-(1-u_1) s D^2} du_1 du_2 \cdots du_N. \quad (2.2.4)$$

Similar to Lemma 5, we have:

Lemma 2.6 *Let B a finite order differential operator, then for any $s > 0$, we have:*

$$B e^{-sD^2} = \sum_{l=0}^{N-1} \frac{1}{l!} s^l e^{-sD^2} B^{[l]} + s^N B_1^{[N]}(s), \quad (2.2.5)$$

where $B_1^{[N]}(s)$ is given by

$$B_1^{[N]}(s) = \int_{\Delta_N} e^{-(1-u_1) s D^2} B^{[N]} e^{-u_1 s D^2} du_1 du_2 \cdots du_N. \quad (2.2.6)$$

In order to prove that (2.1.3) is well defined, it is enough to prove that when $t \rightarrow 0$, we have the estimate

$$\widetilde{\text{ch}}_k^G(tD, D)(f^0, \dots, f^k)(g) \sim O(t); \quad \text{i.e.} \quad \widetilde{\text{ch}}_k^G(\sqrt{t}D, D)(f^0, \dots, f^k)(g) \sim O(t^{\frac{1}{2}}). \quad (2.2.7)$$

By (2.1.5), we have for $i = 0, \dots, k$, the i -th term of $\widetilde{\text{ch}}_k^G(\sqrt{t}D, D)(f^0, \dots, f^k)(g)$ up to sign is:

$$\begin{aligned} & t^{\frac{k}{2}} \langle f^0, c(df^1), \dots, c(df^i), D, c(df^{i+1}), \dots, c(df^k) \rangle_{\sqrt{t}}(g) \\ &= t^{\frac{k}{2}} \int_{\Delta_k} \text{Tr}[f^0 e^{-s_1 t D^2} c(df^1) e^{-(s_2 - s_1) t D^2} c(df^2) \cdots c(df^i) (s_{i+1} - s_i) D e^{-(s_{i+1} - s_i) t D^2} \\ & \quad \cdot c(df^{i+1}) \cdots c(df^k) e^{-(1-s_k) t D^2} \widetilde{dg}] ds_1 \cdots ds_k. \end{aligned} \quad (2.2.8)$$

In what follows, we will compute the expression of (2.2.8) by the above lemmas. By Lemma 2.6, we have:

$$\begin{aligned}
& c(df^{i+1})e^{-(s_{i+2}-s_{i+1})tD^2} \dots c(df^k)e^{-(1-s_k)tD^2} \\
&= \sum_{\lambda_{i+1}, \dots, \lambda_k=0}^{N-1} \frac{(1-s_{i+1})^{\lambda_{i+1}} \dots (1-s_k)^{\lambda_k} t^{\lambda_{i+1}+\dots+\lambda_k}}{\lambda_{i+1}! \dots \lambda_k!} \\
&\quad \cdot e^{-(1-s_{i+1})tD^2} [c(df^{i+1})]^{[\lambda_{i+1}]} \dots [c(df^k)]^{[\lambda_k]} \\
&+ \sum_{i+1 \leq q \leq k} \sum_{\lambda_{q+1}, \dots, \lambda_k=0}^{N-1} \frac{(1-s_q)^N (1-s_{q+1})^{\lambda_{q+1}} \dots (1-s_k)^{\lambda_k} t^{N+\lambda_{q+1}+\dots+\lambda_k}}{\lambda_{q+1}! \dots \lambda_k!} \\
&\quad \cdot c(df^{i+1})e^{-(s_{i+2}-s_{i+1})tD^2} \dots c(df^{q-1})e^{-(s_q-s_{q-1})tD^2} \\
&\quad \cdot \{[c(df^q)]_1^N((1-s_q)t)\} [c(df^{q+1})]^{[\lambda_{q+1}]} \dots [c(df^k)]^{[\lambda_k]}. \tag{2.2.9}
\end{aligned}$$

By Lemma 2.5, we have:

$$\begin{aligned}
& f^0 e^{-s_1 t D^2} c(df^1) e^{-(s_2-s_1)tD^2} c(df^2) \dots c(df^i) e^{-(s_{i+1}-s_i)tD^2} \\
&= \sum_{\lambda_1, \dots, \lambda_i=0}^{N-1} \frac{(-1)^{\lambda_1+\dots+\lambda_i} s_1^{\lambda_1} \dots s_i^{\lambda_i} t^{\lambda_1+\dots+\lambda_i}}{\lambda_1! \dots \lambda_i!} f^0 [c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} e^{-s_{i+1}tD^2} \\
&+ \sum_{1 \leq q \leq i} \sum_{\lambda_1, \dots, \lambda_{q-1}=0}^{N-1} \frac{(-1)^{\lambda_1+\dots+\lambda_{q-1}+N} s_1^{\lambda_1} \dots s_{q-1}^{\lambda_{q-1}} s_q^N t^{\lambda_1+\dots+\lambda_{q-1}+N}}{\lambda_1! \dots \lambda_{q-1}!} f^0 [c(df^1)]^{[\lambda_1]} \\
&\dots [c(df^{q-1})]^{[\lambda_{q-1}]} \{[c(df^q)]^{[N]}(s_q t)\} e^{-(s_{q+1}-s_q)tD^2} \dots c(df^i) e^{-(s_{i+1}-s_i)tD^2}. \tag{2.2.10}
\end{aligned}$$

By (2.2.8) and (2.2.10), we have:

$$\begin{aligned}
& t^{\frac{k}{2}} (s_{i+1} - s_i) \text{Tr} [f^0 e^{-s_1 t D^2} c(df^1) e^{-(s_2-s_1)tD^2} c(df^2) \dots c(df^i) D e^{-(s_{i+1}-s_i)tD^2} \\
&\quad c(df^{i+1}) \dots c(df^k) e^{-(1-s_k)tD^2} \widetilde{dg}] \\
&= \sum_{\lambda_1, \dots, \lambda_i=0}^{N-1} \frac{(-1)^{\lambda_1+\dots+\lambda_i} s_1^{\lambda_1} \dots s_i^{\lambda_i} (s_{i+1} - s_i) t^{\lambda_1+\dots+\lambda_i+\frac{k}{2}}}{\lambda_1! \dots \lambda_i!} \text{Tr} \{ f^0 [c(df^1)]^{[\lambda_1]} \\
&\quad \dots [c(df^i)]^{[\lambda_i]} e^{-s_{i+1}tD^2} D c(df^{i+1}) \dots c(df^k) e^{-(1-s_k)tD^2} \widetilde{dg} \} \\
&+ \sum_{1 \leq q \leq i} \sum_{\lambda_1, \dots, \lambda_{q-1}=0}^{N-1} \frac{(-1)^{\lambda_1+\dots+\lambda_{q-1}+N} s_1^{\lambda_1} \dots s_{q-1}^{\lambda_{q-1}} s_q^N (s_{i+1} - s_i) t^{\lambda_1+\dots+\lambda_{q-1}+N+\frac{k}{2}}}{\lambda_1! \dots \lambda_{q-1}!} \\
&\quad \cdot \text{Tr} \{ f^0 [c(df^1)]^{[\lambda_1]} \dots [c(df^{q-1})]^{[\lambda_{q-1}]} \{[c(df^q)]^{[N]}(s_q t)\} e^{-(s_{q+1}-s_q)tD^2} \dots \\
&\quad \cdot c(df^i) e^{-(s_{i+1}-s_i)tD^2} D c(df^{i+1}) \dots c(df^k) e^{-(1-s_k)tD^2} \widetilde{dg} \} \\
&= \sum_{0 \leq \lambda_1, \dots, \lambda_k \leq N-1} \frac{(-1)^{|\lambda|} s_1^{\lambda_1} \dots s_i^{\lambda_i} (s_{i+1} - 1)^{\lambda_{i+1}} \dots (s_k - 1)^{\lambda_k} t^{|\lambda|+\frac{k}{2}} (s_{i+1} - s_i)}{\lambda!}
\end{aligned}$$

$$\cdot \text{Tr}\{f^0[c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} D e^{-tD^2} [c(df^{i+1})]^{[\lambda_{i+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg}\} + A_1^i + A_2^i, \quad (2.2.11)$$

where the last equality comes from (2.2.9) and

$$\begin{aligned} A_1^i &= \sum_{1 \leq q \leq i} \sum_{\lambda_1, \dots, \lambda_{q-1}=0}^{N-1} \frac{(-1)^{\lambda_1 + \dots + \lambda_{q-1} + N} (s_{i+1} - s_i) s_1^{\lambda_1} \dots s_{q-1}^{\lambda_{q-1}} s_q^N t^{\lambda_1 + \dots + \lambda_{q-1} + N + \frac{k}{2}}}{\lambda_1! \dots \lambda_{q-1}!} \\ &\quad \cdot \text{Tr}\{f^0[c(df^1)]^{[\lambda_1]} \dots [c(df^{q-1})]^{[\lambda_{q-1}]} \{[c(df^q)]^{[N]}(s_q t)\} e^{-(s_{q+1} - s_q)tD^2} \dots \\ &\quad \cdot c(df^i) e^{-(s_{i+1} - s_i)tD^2} D c(df^{i+1}) \dots c(df^k) e^{-(1-s_k)tD^2} \widetilde{dg}\}; \end{aligned} \quad (2.2.12)$$

$$\begin{aligned} A_2^i &= \sum_{\lambda_1, \dots, \lambda_i=0}^{N-1} \sum_{i+1 \leq q \leq k} \sum_{\lambda_{q+1}, \dots, \lambda_k=0}^{N-1} \frac{(-1)^{\lambda_1 + \dots + \lambda_i}}{\lambda_1! \dots \lambda_i! \lambda_{q+1}! \dots \lambda_k!} (s_{i+1} - s_i) s_1^{\lambda_1} \dots s_i^{\lambda_i} \\ &\quad \cdot (1-s_q)^N (1-s_{q+1})^{\lambda_{q+1}} \dots (1-s_k)^{\lambda_k} t^{\lambda_1 + \dots + \lambda_i + N + \lambda_{q+1} + \dots + \lambda_k + \frac{k}{2}} \\ &\quad \cdot \text{Tr}\{f^0[c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} e^{-s_{i+1}tD^2} D c(df^{i+1}) e^{-(s_{i+2} - s_{i+1})tD^2} \dots c(df^{q-1}) \\ &\quad \cdot e^{-(s_q - s_{q-1})tD^2} \{[c(df^q)]_1^{[N]}((1-s_q)t)\} [c(df^{q+1})]^{[\lambda_{q+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg}\}. \end{aligned} \quad (2.2.13)$$

Using Lemma 2.2 (i) by taking appropriate bounded operators A, B , we have:

$$\begin{aligned} &\text{Tr}\{f^0[c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} D e^{-tD^2} [c(df^{i+1})]^{[\lambda_{i+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg}\} \\ &= \text{Tr}\{[c(df^{i+1})]^{[\lambda_{i+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg} f^0[c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} D e^{-tD^2}\} \\ &= \text{Tr}\{D_i^\lambda D e^{-tD^2}\}, \end{aligned} \quad (2.2.14)$$

where we use the notation

$$D_i^\lambda = [c(df^{i+1})]^{[\lambda_{i+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg} f^0[c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]}. \quad (2.2.15)$$

Using (2.1.1), (2.2.11), (2.2.12), (3.13) and (2.2.14), then we have:

Corollary 2.7 *Let $\dim M = n = 2m + 1$, for $f_j \in C^\infty(M)$, $0 \leq j \leq k$, k is even and $g \in G$, then*

$$\begin{aligned} &\widetilde{\text{ch}}_k^G(\sqrt{t}D, D)(f^0, \dots, f^k)(g) \\ &= \sum_{i=0}^k (-1)^i \sum_{0 \leq \lambda_1, \dots, \lambda_k \leq N-1} \frac{(-1)^{|\lambda|} C t^{|\lambda| + \frac{k}{2}}}{\lambda!} \text{Tr}\{D_i^\lambda D e^{-tD^2}\} + \sum_{i=0}^k (-1)^i \int_{\Delta_k} (A_1^i + A_2^i) ds, \end{aligned} \quad (2.2.16)$$

with the constant

$$C = \sum_{j_{i+1}=0}^{\lambda_{i+1}} \dots \sum_{j_k=0}^{\lambda_k} \left(\prod_{s=i+1}^k \binom{\lambda_s}{j_s} \right) (-1)^{\lambda_s - j_s} \frac{1}{\lambda_1 + 1} \dots \frac{1}{\sum_{j=1}^{i-1} \lambda_j + i - 1} \frac{1}{\sum_{j=1}^i \lambda_j + i}$$

$$\frac{1}{\sum_{j=1}^i \lambda_j + i + 1} \frac{1}{\sum_{j=1}^i \lambda_j + j_{i+1} + i + 2} \cdots \frac{1}{\sum_{j=1}^i \lambda_j + j_{i+1} + \cdots + j_k + k + 1},$$

Now we give the estimate of $\widetilde{\text{ch}}_k^G(\sqrt{t}D, D)$ and let $N = n + 2 - k = 2m + 3 - k$ in (2.2.16). We have:

Theorem 2.8 1) If $k \leq \dim N + 1 = 2m + 2$, then when $t \rightarrow 0^+$, we have:

$$\begin{aligned} & \widetilde{\text{ch}}_k^G(\sqrt{t}D, D)(f^0, \dots, f^k)(g) \\ &= \sum_{i=0}^k (-1)^i \sum_{0 \leq \lambda_1, \dots, \lambda_k \leq N-1} \frac{(-1)^{|\lambda|} C t^{|\lambda| + \frac{k}{2}}}{\lambda!} \text{Tr}\{D_i^\lambda D e^{-tD^2}\} + O(t^{\frac{1}{2}}). \end{aligned} \quad (2.2.17)$$

2) If $k > 2m + 2$, then when $t \rightarrow 0^+$, we have:

$$\widetilde{\text{ch}}_k^G(\sqrt{t}D, D)(f^0, \dots, f^k)(g) \sim O(t^{\frac{1}{2}}). \quad (2.2.18)$$

Proof. 1) In order to prove (2.2.17), we only prove that when $t \rightarrow 0^+$, $\int_{\Delta_k} A_2^i ds \sim O(t^{\frac{1}{2}})$ (similar $\int_{\Delta_k} A_1^i ds \sim O(t^{\frac{1}{2}})$). By (2.2.13), then

$$\begin{aligned} \left| \int_{\Delta_k} A_2^i ds \right| &\leq \int_{\Delta_k} |A_2^i| ds \\ &\leq \sum_{\lambda_1, \dots, \lambda_i=0}^{N-1} \sum_{i+1 \leq q \leq k} \sum_{\lambda_{q+1}, \dots, \lambda_k=0}^{N-1} \frac{A_{\lambda_1, \dots, \lambda_i, \lambda_{q+1}, \dots, \lambda_k}}{\lambda_1! \cdots \lambda_i! \lambda_{q+1}! \cdots \lambda_k!}, \end{aligned} \quad (2.2.19)$$

where

$$\begin{aligned} A_{\lambda_1, \dots, \lambda_i, \lambda_{q+1}, \dots, \lambda_k} &= \int_{\Delta_k} (s_{i+1} - s_i) s_1^{\lambda_1} \cdots s_i^{\lambda_i} (1 - s_q)^N (1 - s_{q+1})^{\lambda_{q+1}} \cdots (1 - s_k)^{\lambda_k} \\ &\quad \cdot t^{\lambda_1 + \cdots + \lambda_i + N + \lambda_{q+1} + \cdots + \lambda_k + \frac{k}{2}} \left| \text{Tr}\{f^0 [c(df^1)]^{[\lambda_1]} \cdots [c(df^i)]^{[\lambda_i]} \right. \\ &\quad \cdot e^{-s_{i+1}tD^2} D c(df^{i+1}) e^{-(s_{i+2} - s_{i+1})tD^2} \cdots c(df^{q-1}) e^{-(s_q - s_{q-1})tD^2} \\ &\quad \left. \{[c(df^q)]_1^{[N]} ((1 - s_q)t)\} [c(df^{q+1})]^{[\lambda_{q+1}]} \right. \\ &\quad \left. \cdots [c(df^k)]^{[\lambda_k]} \widetilde{dg} \right| ds. \end{aligned} \quad (2.2.20)$$

By Lemma 2.2, Lemma 2.4 and Lemma 2.6, then:

$$\begin{aligned} A_{\lambda_1, \dots, \lambda_i, \lambda_{q+1}, \dots, \lambda_k} &\leq \int_{\Delta_k} (s_{i+1} - s_i) s_i^{\lambda_1 + \cdots + \lambda_i} (1 - s_q)^N (1 - s_{q+1})^{\lambda_{q+1}} \cdots (1 - s_k)^{\lambda_k} \\ &\quad \cdot t^{\lambda_1 + \cdots + \lambda_i + N + \lambda_{q+1} + \cdots + \lambda_k + \frac{k}{2}} \left| \text{Tr}\left\{ \left(f^0 [c(df^1)]^{[\lambda_1]} \cdots [c(df^i)]^{[\lambda_i]} e^{-s_i t D^2} \right) \left(D e^{-(s_{i+1} - s_i) t D^2} \right) \right. \right. \\ &\quad \left. \left. \cdot \left(c(df^{i+1}) e^{-(s_{i+2} - s_{i+1}) t D^2} \right) \cdots \left(c(df^{q-1}) e^{-(s_q - s_{q-1}) t D^2} \right) \right. \right. \\ &\quad \left. \left. \cdot \left(\{[c(df^q)]_1^{[N]} ((1 - s_q)t)\} [c(df^{q+1})]^{[\lambda_{q+1}]} \cdots [c(df^k)]^{[\lambda_k]} \widetilde{dg} \right) \right\} \right| ds \end{aligned}$$

$$\begin{aligned}
&\leq C_0 \int_{\Delta_q} \int_{\Delta_N} (s_{i+1} - s_i) s_i^{\lambda_1 + \dots + \lambda_i} (1 - s_q)^{N + \lambda_{q+1} + \dots + \lambda_k + k - q} t^{N + \lambda_1 + \dots + \lambda_i + \lambda_{q+1} + \dots + \lambda_k + \frac{k}{2}} \\
&\quad \cdot \left\| \left(f^0 [c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} e^{-s_i t D^2} \right) \left(D e^{-(s_{i+1} - s_i) t D^2} \right) \left(c(df^{i+1}) e^{-(s_{i+2} - s_{i+1}) t D^2} \right) \right. \\
&\quad \quad \dots \left(c(df^{q-1}) e^{-(s_q - s_{q-1}) t D^2} \right) \cdot \left(e^{-(1-u_1)(1-s_q) t D^2} [c(df^q)]^{[N]} e^{-u_1(1-s_q) t D^2} \right. \\
&\quad \quad \quad \left. \left. \cdot [c(df^{q+1})]^{[\lambda_{q+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg} \right) \right\|_1 du_1 \dots du_N ds_1 \dots ds_q \\
&\leq C_0 \int_{\Delta_q} \int_{\Delta_N} t^{N + \lambda_1 + \dots + \lambda_i + \lambda_{q+1} + \dots + \lambda_k + \frac{k}{2}} \left\| s_i^{\lambda_1 + \dots + \lambda_i} f^0 [c(df^1)]^{[\lambda_1]} \dots [c(df^i)]^{[\lambda_i]} e^{-s_i t D^2} \right\|_{\frac{1}{s_i}} \\
&\quad \cdot \left\| (s_{i+1} - s_i) D e^{-\frac{(s_{i+1} - s_i)}{2} t D^2} \right\| \left\| e^{-\frac{(s_{i+1} - s_i)}{2} t D^2} \right\| \left\| \frac{1}{s_{i+1} - s_i} \left\| c(df^{i+1}) e^{-(s_{i+2} - s_{i+1}) t D^2} \right\|_{\frac{1}{s_{i+2} - s_{i+1}}} \right. \\
&\quad \quad \dots \left\| c(df^{q-1}) e^{-(s_q - s_{q-1}) t D^2} \right\|_{\frac{1}{s_q - s_{q-1}}} \left\| (1 - s_q)^{N + \lambda_{q+1} + \dots + \lambda_k + k - q} e^{-(1-u_1)(1-s_q) t D^2} \right. \\
&\quad \quad \quad \left. \cdot [c(df^q)]^{[N]} e^{-u_1(1-s_q) t D^2} [c(df^{q+1})]^{[\lambda_{q+1}]} \dots [c(df^k)]^{[\lambda_k]} \widetilde{dg} \right\|_{\frac{1}{1-s_q}} \\
&\leq \overline{C} \int_{\Delta_q} \int_{\Delta_N} t^{\frac{N + \lambda_1 + \dots + \lambda_i + \lambda_{q+1} + \dots + \lambda_k + k - 1}{2}} s_i^{\frac{\lambda_1 + \dots + \lambda_i}{2}} \sqrt{s_{i+1} - s_i} (1 - s_q)^{\frac{N + \lambda_{q+1} + \dots + \lambda_k + k - q}{2}} \\
&\quad \cdot \text{Tr} \left\{ e^{-\frac{t D^2}{4}} \right\} du_1 \dots du_N ds_1 \dots ds_q, \tag{2.2.21}
\end{aligned}$$

where C_0, \overline{C} are constants and the second inequality comes from integrating respect to s_{q+1}, \dots, s_k and (2.2.6). In the third inequality we use Lemma 2.2 (iv) and the last inequality comes from Lemma 2.2 and Lemma 2.4. By the Weyl asymptotics on the heat kernel we have when $t \rightarrow 0$,

$$\text{Tr} \left\{ e^{-\frac{t D^2}{4}} \right\} \sim O(t^{-\frac{n}{2}}). \tag{2.2.22}$$

By (2.2.22) and $N = n + 2 - k$, So (2.2.21) $\sim O(t^{\frac{1}{2}})$.

2) If $k > \dim N + 1$, then

$$\begin{aligned}
&\left| t^{\frac{k}{2}} \int_{\Delta_k} (s_{i+1} - s_i) \text{Tr} [f^0 e^{-s_1 t D^2} c(df^1) e^{-(s_2 - s_1) t D^2} c(df^2) \dots c(df^i) \right. \\
&\quad \left. \cdot D e^{-(s_{i+1} - s_i) t D^2} c(df^{i+1}) \dots c(df^k) e^{-(1-s_k) t D^2} \widetilde{dg}] ds \right| \\
&\leq t^{\frac{k}{2}} \int_{\Delta_k} \left\| \left(f^0 e^{-s_1 t D^2} c(df^1) \right) \left(e^{-(s_2 - s_1) t D^2} c(df^2) \right) \dots \left(e^{-(s_i - s_{i-1}) t D^2} c(df^i) \right) \right. \\
&\quad \cdot \left((s_{i+1} - s_i) D e^{-\frac{s_{i+1} - s_i}{2} t D^2} \right) \left(e^{-\frac{s_{i+1} - s_i}{2} t D^2} c(df^{i+1}) \right) \dots \\
&\quad \quad \cdot \left((e^{-(s_k - s_{k-1}) t D^2} c(df^k)) \left(e^{-(1-s_k) t D^2} \right) \widetilde{dg} \right) \Big|_1 ds \\
&\leq t^{\frac{k}{2}} \int_{\Delta_k} \sqrt{s_{i+1} - s_i} t^{-\frac{1}{2}} \text{Tr} \left\{ e^{-\frac{t D^2}{4}} \right\} ds \sim O(t^{-\frac{-n+k-1}{2}}) \sim O(t^{\frac{1}{2}}),
\end{aligned}$$

where we use Lemma 2.2, Weyl estimate and condition $k > \dim N + 1$. \square

2.3 Clifford Asymptotics for Heat Kernels

By Theorem 2.8, in order to prove the regularity of the equivariant eta cochains, it is enough to prove that when $t \rightarrow 0$,

$$t^{|\lambda|+\frac{k}{2}}\mathrm{Tr}\{D_i^\lambda D e^{-tD^2}\} \sim O(t^{\frac{1}{2}}).$$

Similar to Theorem 1.1 in [Z], we have the following lemma.

Lemma 2.9

$$\mathrm{Tr}\{D_i^\lambda D e^{-tD^2}\} = \int_M \mathrm{Tr}\{(D_i^\lambda)_x [D \exp(-tD^2)(x, y)]|_{y=g \cdot x}\} dx. \quad (2.3.1)$$

Proposition 2.10 *If g has no fixed points on N , then*

$$\lim_{t \rightarrow 0} t^{-\frac{1}{2}} \int_M \mathrm{Tr}\{(D_i^\lambda)_x [D \exp(-tD^2)(x, g \cdot x)]\} dx = 0. \quad (2.3.2)$$

Proof. We introduce an auxiliary Grassmann variable z as in [BF]. By Duhamel principle, we have

$$\exp(-t(D^2 - zD)) = \exp(-tD^2) + zt D \exp(-tD^2). \quad (2.3.3)$$

Since g has no fixed points, $d(x, g \cdot x) > \delta$ for some constant $\delta > 0$. So there exist positive constants C_i ($i = 1, 2, 3, 4$) and positive integers m_1, m_2 such that $t \rightarrow 0$,

$$\|(D_i^\lambda)_x \exp(-tD^2)(x, g \cdot x)\| \leq \frac{C_1}{t^{\frac{n}{2}+m_1}} \exp(-\frac{C_2}{t}); \quad (2.3.4)$$

$$\|(D_i^\lambda)_x \exp(-t(D^2 - zD))(x, g \cdot x)\| \leq \frac{C_3}{t^{\frac{n}{2}+m_2}} \exp(-\frac{C_4}{t}), \quad (2.3.5)$$

then similar to Corollary 1.4 in [Z], we prove this Proposition. \square

Since g is an isometry, the fixed point set F of g consists of components F_1, \dots, F_k , each of even codimension. If U is an open neighborhood of F , then by Proposition 2.10, we have

$$\begin{aligned} & \lim_{t \rightarrow 0} t^{-\frac{1}{2}} \int_M \mathrm{Tr}\{(D_i^\lambda)_x [D \exp(-tD^2)(x, g \cdot x)]\} dx \\ &= \lim_{t \rightarrow 0} t^{-\frac{1}{2}} \int_U \mathrm{Tr}\{(D_i^\lambda)_x [D \exp(-tD^2)(x, g \cdot x)]\} dx. \end{aligned} \quad (2.3.6)$$

We may assume $k = 1$ and $\mathrm{codim} F = 2n'$. Denote by $N(F)$ the normal bundle to F , similar to Theorem 2.2 in [LYZ] we need only to prove that

$$\lim_{t \rightarrow 0} t^{-\frac{1}{2}} \left| \int_F \int_{N_\xi(\varepsilon)} t^{|\lambda|+\frac{k}{2}} \mathrm{Tr}\{(D_i^\lambda)_x [D \exp(-tD^2)(x, g \cdot x)]\} dN_\xi d\xi \right| \leq C, \quad (2.3.7)$$

for some constant $C > 0$. Here $N_\xi(\varepsilon) = \{v \in N_\xi(F) \mid \|v\| < \varepsilon\}$.

Similar to [LYZ],[Y], for $\xi \in F$, we choose an open neighborhood U of ξ and the orthogonal frame over U (see [LYZ], pp.574). Consider the oriented orthonormal frame field $E^{g \cdot x}$ defined over the patch U by requiring that $E^{g \cdot x}(g \cdot x) = E(g \cdot x)$ and that $E^{g \cdot x}$ is parallel along geodesics through $g \cdot x$. Choose a spin frame field $\sigma : U \rightarrow \text{Spin}(M)$ such that $\pi\sigma = (E_1^{g \cdot x}, \dots, E_n^{g \cdot x})$ where $\pi : \text{Spin}(M) \rightarrow \text{SO}(M)$ is the two-fold covering over $\text{SO}(M)$. For $x \in U$, let $K(x)$, $g^*(x)$, $T_i^\lambda(x) \in \text{Hom}(I, I)$ (I is the canonical spinors space as in [LYZ].) be defined through the equivalence relations:

$$D\exp(-tD^2)(x, g \cdot x)[(\sigma(g \cdot x), u)] = [(\sigma(x), K(x)u)]; \quad (2.3.8)$$

$$\widetilde{dg}[(\sigma(x), v)] = [(\sigma(g \cdot x), g^*(x)v)]; \quad (2.3.9)$$

$$(D_i^\lambda)_x[(\sigma(x), w)] = [(\sigma(g \cdot x), T_i^\lambda(x)w)]. \quad (2.3.10)$$

Then similar to Lemma 4.1 in [LYZ], we have

Lemma 2.11

$$\text{Tr}\{(D_i^\lambda)_x[D\exp(-tD^2)(x, g \cdot x)]\} = \text{Tr}(T_i^\lambda(x)K(x)). \quad (2.3.11)$$

Proof. Let $\{b_j\}$ be the basis of I , then $\{[(\sigma(x), b_j)]\}$ is the basis of $\Gamma(S)|_U$. Then

$$\begin{aligned} & \text{Tr}\{(D_i^\lambda)_x[D\exp(-tD^2)(x, g \cdot x)]\} \\ &= \sum_j \langle (D_i^\lambda)_x[D\exp(-tD^2)(x, g \cdot x)](\sigma(g \cdot x), b_j), (\sigma(g \cdot x), b_j) \rangle \\ &= \sum_j \langle (\sigma(g \cdot x), T_i^\lambda(x)K(x)b_j), (\sigma(g \cdot x), b_j) \rangle \\ &= \sum_j \langle T_i^\lambda(x)K(x)b_j, b_j \rangle \\ &= \text{Tr}(T_i^\lambda(x)K(x)). \quad \square \end{aligned}$$

As in [LYZ],[Y], We define $\chi(x^\alpha D_x^\beta e^\gamma) = |\beta| - |\alpha| + |\gamma|$, $\alpha, \beta \in \mathbf{Z}^n$, $\gamma \in \mathbf{Z}_2^n$. We also define $\chi(t) = -2$ and $\chi(z) = 1$, then we have

$$\chi(zx^\alpha D_x^\beta e^\gamma) = 1 + |\beta| - |\alpha| + |\gamma|. \quad (2.3.12)$$

Lemma 2.12 *If $\lambda \neq 0$, $k \neq 0$ and k is even, then $\chi(t^{|\lambda| + \frac{k}{2}} T_i^\lambda) \leq -2 + 2n'$.*

Proof. By Lemma 3.6 in [CH], $\chi([c(df_i)]^{[\lambda i]}) \leq 2\lambda_i$ and the simple argument in [LYZ] shows that

$$g^* = \prod_{\alpha=n-2n'+1}^n e_\alpha \cdot d_1 + d_2 \quad (2.3.13)$$

where $\chi(d_1) \leq 0$ and $\chi(d_2) \leq 2n' - 2$. So $\chi(t^{|\lambda| + \frac{k}{2}} T_i^\lambda) \leq -2(|\lambda| + \frac{k}{2}) + 2n' + 2|\lambda| = 2n' - k \leq 2n' - 2$. \square

Lemma 2.13([Z]) *Suppose $i \leq [\frac{n}{2}] + 2$. If W is an odd element and $\chi(W) \leq 2i - 2 + 2n'$, then*

$$\lim_{t \rightarrow 0} \frac{1}{t^{\frac{3}{2}}} \left| \int_{N_\xi(\varepsilon)} \frac{e^{-\frac{d(x', g \cdot x')^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \text{Tr}(W(0; x')) t^i dx' \right| \leq C' \quad (2.3.14)$$

for some constant $C' > 0$; where in the $W(x''; x')$, x'' stands for tangential coordinates and x' stands for normal coordinates.

Theorem 2.14 *When $t \rightarrow 0$,*

$$t^{|\lambda| + \frac{k}{2}} \text{Tr}\{D_i^\lambda D e^{-tD^2}\} \sim O(t^{\frac{1}{2}}). \quad (2.3.15)$$

Proof. By Lemma 2.11, we only need to prove that when $t \rightarrow 0$,

$$t^{|\lambda| + \frac{k}{2}} \int_{N_\xi(\varepsilon)} \text{Tr}(T_i^\lambda(0, x') K(0, x')) dx' \sim O(t^{\frac{1}{2}}). \quad (2.3.16)$$

Set

$$h(x) = 1 + \frac{1}{2} z \sum_{i=1}^n x_i c(e_i), \quad (2.3.17)$$

where (x_1, \dots, x_n) is the normal coordinates under the frame $E_1^{g \cdot x}, \dots, E_n^{g \cdot x}$ and we consider h as $h\phi$ where ϕ is a cut function about (x_1, \dots, x_n) . By [Z], we have

$$hc(e_i)h^{-1} = c(e_i) + (\chi = 0); \quad h(D^2 - zD)h^{-1} = D^2 + zu, \quad (2.3.18)$$

where $\chi(u) \leq 0$, u contains no z and the equality

$$ztD \exp(-tD^2)(x, y) = h^{-1}(x) \exp(-t(D^2 + zu)(x, y)) h(y) - \exp(-tD^2)(x, y) \quad (2.3.19)$$

where

$$\exp(-tD^2)(x, y) = \frac{e^{-\frac{d(x, y)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \left(\sum_{i=0}^{[\frac{n}{2}] + 2} U_i t^i + o(t^{[\frac{n}{2}] + 2}) \right); \quad (2.3.20)$$

$$\exp(-t(D^2 + zu))(x, y) = \frac{e^{-\frac{d(x, y)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \left(\sum_{i=0}^{[\frac{n}{2}] + 2} (U_i + zV_i) t^i + o(t^{[\frac{n}{2}] + 2}) \right), \quad (2.3.21)$$

where $\chi(U_i) \leq 2i$, $\chi(V_i) \leq 2(i - 1)$ and U_i, V_i contains no z .

So we get:

$$tK(x) = \frac{e^{-\frac{d(x, g \cdot x)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \left[\sum_{i=0}^{[\frac{n}{2}] + 2} \left(\frac{1}{2} \sum_j ((dg - I)x)_j c(e_j) \right) U_i t^i + o(t^{[\frac{n}{2}] + 2}) \right. \\ \left. + \left(\sum_{i=0}^{[\frac{n}{2}] + 2} W_i t^i + o(t^{[\frac{n}{2}] + 2}) \right) \right], \quad (2.3.22)$$

where W_i is an odd element and $\chi(W_i) \leq 2(i-1)$. Because we integrate about the orthogonal coordinates, so in (2.3.22) we use the orthogonal coordinates again.

i) If $k = 0$, then $D_i^\lambda = \widetilde{dg}f^0$ and (2.3.15) is the result in [Z]. If $\lambda = 0$,

$$t^{\frac{k}{2}}D_i^\lambda = t^{\frac{k}{2}}[c(df^{i+1})] \cdots [c(df^k)]\widetilde{dg}f^0[c(df^1)] \cdots [c(df^i)].$$

By (2.3.13), we have

$$t^{\frac{k}{2}}T_i^\lambda = A_1 \prod_{\alpha=n-2n'+1}^n e_\alpha + A_2, \quad (2.3.23)$$

where $A_1 \in (\chi \leq 0)$ and $A_2 \in (\chi \leq 2n' - 2)$. So by (2.3.13), (2.3.22), (2.3.23) and Lemma 2.13, we get (2.3.15).

ii) we let $\lambda \neq 0$ and $k \neq 0$. By (2.3.22) and Lemma 2.12, we have

$$t^{|\lambda|+\frac{k}{2}+1}T_i^\lambda(x)K(x) = \frac{e^{-\frac{d(x,g \cdot x)^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \left[\sum_{i=0}^{[\frac{n}{2}]+2} \widetilde{W}_i t^i + o(t^{[\frac{n}{2}]+2}) \right], \quad (2.3.24)$$

where \widetilde{W}_i is an odd element and $\chi(\widetilde{W}_i) \leq 2i - 2 + 2n'$. Also by Lemma 2.13 we prove this Theorem. \square

2.4 The Convergence of The Total Eta Invariant

Let $C^1(N)$ be Banach algebra of once differentiable function on N with the norm

$$\|f\|_1 := \sup_{x \in N} |f(x)| + \sup_{x \in N} \|df(x)\|.$$

Let

$$\phi^G = \{\phi_0^G, \dots, \phi_{2q}^G, \dots\}$$

be an equivariant even cochains sequence in the bar complex of $C^1(N)$, then

$$\|\phi_{2q}^G\| = \sup_{\|f_i\| \leq 1; 0 \leq i \leq 2q} \{\|\phi_{2q}^G(f_0, \dots, f_{2q})\|_{C(G)}\}.$$

Definition 2.15 The radius of convergence of ϕ^G is defined to be that of the power series $\sum q! \|\phi_{2q}^G\| z^q$. The space of cochains sequence with radius of convergence at least $r > 0$ is denoted by $C_r^{\text{even}, G}(C^1(N))$ (similarly define $C_r^{\text{odd}, G}(C^1(N))$).

In general, the sequence

$$\eta^G(D) = \{\dots, \eta_{2q}^G(D), \eta_{2q+2}^G(D), \dots\}$$

which called total eta invariant is not an entire cochain.

Proposition 2.16 Suppose that D is invertible with λ the smallest positive eigenvalue of $|D|$. Then the equivariant total eta invariant $\eta^G(D)$ has radius of convergence r satisfying the inequality: $r \geq 4\lambda^2 > 0$ i.e. $\eta^G(D) \in C_{4\lambda^2}^{\text{even}, G}(C^1(N))$.

Proof. We can assume $q > (n + 2)/2$ for only considering convergence, where $n = \dim N$. For a constant $\delta > 0$, we have

$$\sum_q q! \|\eta_{2q}^G(D)\| z^q \leq \sum_q q! \int_{1+\delta}^{\infty} \|\widetilde{\text{ch}}_{2q}^G(tD, D)\| dt z^q + \sum_q q! \int_0^{1+\delta} \|\widetilde{\text{Ch}}_{2q}^G(tD, D)\| dt z^q.$$

As the proof of the proposition 1.5 in [W], the sequence

$$\sum_q q! \int_{1+\delta}^{\infty} \|\widetilde{\text{ch}}_{2q}^G(tD, D)\| dt z^q$$

has a radius $r \geq 4\lambda^2$ of convergence; while similar to the proof of Theorem 2.8 (2), we have

$$\sum_q q! \int_0^{1+\delta} \|\widetilde{\text{ch}}_{2q}^G(tD, D)\| dt z^q$$

is an entire sequence. So this proposition is correct. \square

For the idempotent $p \in \mathcal{M}_r(C^\infty(N))$, its Chern character $\text{Ch}(p)$ in entire cyclic homology is defined by the formula (for more details see [GS]):

$$\text{Ch}(p) = \text{Tr}(p) + \sum_{k \geq 1} \frac{(-1)^k (2k)!}{k!} \text{Tr}_{2k} \left(\left(p - \frac{1}{2} \right) \otimes \bar{p}^{\otimes 2k} \right) \quad (2.4.1)$$

where

$$\text{Tr}_{2k} : \mathcal{M}_r(C^\infty(N)) \otimes (\mathcal{M}_r(C^\infty(N))/\mathcal{M}_r(\mathbf{C}))^{\otimes 2k} \rightarrow C^\infty(N) \otimes (C^\infty(N)/\mathbf{C})^{\otimes 2k}$$

is the generalized trace map. Let

$$\|dp\| = \|[D, p]\| = \sum_{i,j} \|dp_{i,j}\| \quad (2.4.2)$$

where $p_{i,j}$ ($1 \leq i, j \leq r$) is the entry of p .

Proposition 2.17 *We assume that $\|dp\| < \lambda$, then the pairing $\langle \eta^G(D), \text{Ch}(p) \rangle$ is well-defined.*

Proof. By (2.4.2), using the same method as Proposition 2.16, we can prove this proposition. \square

3 The Proof of (1.1)

In this section, we will prove (1.1) by using the method in [G2]. Let

$$C_G^\infty(N) = \{f \in C^\infty(N) | f(g.x) = f(x), \text{ for any } g \in G \text{ and } x \in N\}.$$

Suppose that D is invertible with λ the smallest eigenvalue of $|D|$, and $p = p^* = p^2 \in \mathcal{M}_r(C_G^\infty(N))$ is a idempotent which satisfying $\|dp\| < \lambda$. Let

$$p(D \otimes I_r)p : p(H \otimes \mathbf{C}^r) = L^2(N, S \otimes p(\mathbf{C}^r)) \rightarrow L^2(N, S \otimes p(\mathbf{C}^r))$$

be the Dirac operator with coefficients from $F = p(\mathbf{C}^r)$. We denote

$$\widetilde{dg} \otimes I_r : L^2(N, S \otimes p(\mathbf{C}^r)) \rightarrow L^2(N, S \otimes p(\mathbf{C}^r))$$

still by g . Since $p \in \mathcal{M}_r(C_G^\infty(N))$, we have

$$g[p(D \otimes I_r)p] = [p(D \otimes I_r)p]g.$$

Theorem 3.1 *Under the assumption as above, we have*

$$\frac{1}{2}\eta^G(p(D \otimes I_r)p) = \langle \eta^G(D), \text{Ch}(p) \rangle, \quad (3.1)$$

where the left term is the equivariant Atiyah-Patodi-Singer eta invariant.

Let

$$\mathbf{D} = \begin{bmatrix} 0 & -D \otimes I_r \\ D \otimes I_r & 0 \end{bmatrix}; p = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}; \sigma = i \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}; g = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix},$$

be operators from $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$ to itself, then \mathbf{D} is skew-adjoint and

$$\mathbf{D}\sigma = -\sigma\mathbf{D}; \quad \sigma p = p\sigma.$$

Moreover $\mathbf{D}e^{t\mathbf{D}^2}$ and $e^{t\mathbf{D}^2}$ ($t > 0$) are traceclass, so $(C_G^\infty(N), H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r, \mathbf{D})$ is a degree-1 Fredholm module in the sense of [G2] (see Definition 2.1 in [G2]). For $u \in [0, 1]$, Let

$$D_u = (1 - u)D + u[pDp + (1 - p)D(1 - p)] = D + u(2p - 1)[D, p],$$

then

$$\mathbf{D}_u = \begin{bmatrix} 0 & -D_u \\ D_u & 0 \end{bmatrix} = \mathbf{D} + u(2p - 1)[\mathbf{D}, p].$$

We consider a family of Fredholm modules on $[0, 1] \times \mathbf{R} \times [0, \infty)$, parameterized by (u, s, t) ,

$$\widetilde{\mathbf{D}} = t^{\frac{1}{2}}\mathbf{D}_u + s\sigma(p - \frac{1}{2}),$$

then $\widetilde{\mathbf{D}}^* = -\widetilde{\mathbf{D}}$ and $g\widetilde{\mathbf{D}} = \widetilde{\mathbf{D}}g$. Let $A = d + \widetilde{\mathbf{D}}$ be a superconnection on the trivial infinite dimensional superbundle with base $[0, 1] \times \mathbf{R} \times [0, \infty)$ and fibre $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$. By [G2], we have

$$(d + \widetilde{\mathbf{D}})^2 = t\mathbf{D}_u^2 - s^2/4 - (1 - u)t^{\frac{1}{2}}s\sigma[\mathbf{D}, p] + ds\sigma(p - \frac{1}{2}) + t^{\frac{1}{2}}du(2p - 1)[\mathbf{D}, p] + \frac{1}{2}t^{-\frac{1}{2}}dt\mathbf{D}_u. \quad (3.2)$$

We also consider A as A_t , which is a family superconnection parameterized by t on trivial superbundle with base $[0, 1] \times \mathbf{R}$ and fibre $H \otimes \mathbf{C}^r \oplus H \otimes \mathbf{C}^r$. By Duhamel principle and $gA_t = A_tg$, then

$$\begin{aligned} \int_0^{+\infty} \text{Str}(ge^{A^2}) &= \int_0^{+\infty} \int_0^1 \text{Str} \left[ge^{s_0 A_t^2} \frac{1}{2} t^{-\frac{1}{2}} dt \mathbf{D}_u e^{(1-s_0) A_t^2} \right] ds_0 \\ &= \int_0^{+\infty} \text{Str} \left[g \frac{dA_t}{dt} e^{A_t^2} \right] dt. \end{aligned}$$

By $gA_t = A_tg$, similar to Theorem 9.23 in [BGV], we have,

$$\begin{aligned} d \int_0^{+\infty} \text{Str}(ge^{A^2}) &= d \int_0^{+\infty} \text{Str} \left[g \frac{dA_t}{dt} e^{A_t^2} \right] dt \\ &= \lim_{t \rightarrow +\infty} \text{Ch}_g(A_t) - \lim_{t \rightarrow 0} \text{Ch}_g(A_t). \end{aligned}$$

Using Duhamel principle and similar to the proof of Lemma 1.1 in [Wu], we have

$$\lim_{t \rightarrow +\infty} \text{Ch}_g(A_t) = 0. \quad (3.3)$$

Using Duhamel principle, as the proof of Section 2.3, we get

$$\lim_{t \rightarrow 0} \text{Ch}_g(A_t) = 0. \quad (3.4)$$

Let $\Gamma_u = \{u\} \times \mathbf{R} \subset [0, 1] \times \mathbf{R}$ be a contour oriented in the direction of increasing s and $\gamma_s = [0, 1] \times \{s\}$ be a contour oriented in the direction of increasing u . By Stokes theorem, then

$$0 = \int_{[0,1] \times \mathbf{R}} d \int_0^{+\infty} \text{Str}(ge^{A^2}) = \left(\int_{\Gamma_1} - \int_{\Gamma_0} - \int_{\gamma_{+\infty}} + \int_{\gamma_{-\infty}} \right) \left[\int_0^{+\infty} \text{Str}(ge^{A^2}) \right].$$

As the proof of (3.3), we have for some constant $C > 0$,

$$\int_{\gamma_s} \int_0^{+\infty} \text{Str}(ge^{A^2}) \sim O(e^{-cs^2}).$$

So

$$\int_{\Gamma_0} \int_0^{+\infty} \text{Str}(ge^{A^2}) = \int_{\Gamma_1} \int_0^{+\infty} \text{Str}(ge^{A^2}).$$

By Duhamel principle and (3.2), when $u = 0$, we have

$$\begin{aligned} \int_{\Gamma_0} \int_0^{+\infty} \text{Str}(ge^{A^2}) &= \sum_{k=0}^{\infty} \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-s^2/4} \\ &\quad \times \int_{\Delta_k} \text{Str} \left\{ e^{t_0 t \mathbf{D}^2} \left[-t^{\frac{1}{2}} s \sigma[\mathbf{D}, p] + ds \sigma \left(p - \frac{1}{2} \right) + (d\sqrt{t}) \mathbf{D} \right] \right. \\ &\quad \left. \times e^{t_1 t \mathbf{D}^2} \dots \left[-t^{\frac{1}{2}} s \sigma[\mathbf{D}, p] + ds \sigma \left(p - \frac{1}{2} \right) + (d\sqrt{t}) \mathbf{D} \right] e^{t_k t \mathbf{D}^2} g \right\} dt_0 \dots dt_k. \end{aligned} \quad (3.5)$$

Expanding (3.5) in powers of s , we will integrate about $dsdt$ and

$$\int_{-\infty}^{+\infty} e^{-s^2/4} s^{2k+1} ds = 0,$$

so we only keep terms with one factor of ds , one factor of $d\sqrt{t}$, and odd number of factors of σ , using (2.2), we get (3.5) equals

$$\begin{aligned} & \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} e^{-s^2/4} s^{2l} ds \int_0^{+\infty} t^l d\sqrt{t} \left\{ \sum_{i=0}^{2l} \sum_{j=0}^{2l-i} \times \right. \\ & \left[-\langle 1, \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{i \text{ times}}, \sigma(p - \frac{1}{2}), \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{j \text{ times}}, \mathbf{D}, \right. \\ & \left. \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{(2l-i-j) \text{ times}} \rangle_{\sqrt{t}}(g) + \langle 1, \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{i \text{ times}}, \right. \\ & \left. \mathbf{D}, \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{j \text{ times}}, \sigma(p - \frac{1}{2}), \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{(2l-i-j) \text{ times}} \rangle_{\sqrt{t}}(g) \right] \}. \quad (3.6) \end{aligned}$$

The following equality is the equivariant case of Lemma 2.2 (2) in [GS]. Assume $gD = Dg$ and $gA_i = A_i g$ for $0 \leq i \leq n$, then

$$\langle A_0, \dots, A_n \rangle_D(g) = \sum_{i=0}^n (-1)^{(|A_0| + \dots + |A_i|)(|A_{i+1}| + \dots + |A_n|)} \langle 1, A_{i+1}, \dots, A_n, A_0, \dots, A_i \rangle_D(g). \quad (3.7)$$

By (3.7), Lemma 3.2 in [G2] and $\mathbf{D} = \begin{bmatrix} 0 & -D \\ D & 0 \end{bmatrix}$, then (3.6) equals

$$\begin{aligned} & - \sum_{l=0}^{\infty} \int_{-\infty}^{+\infty} e^{-s^2/4} s^{2l} ds \int_0^{+\infty} t^l \sum_{j=0}^{2l} \times \\ & \langle \sigma(p - \frac{1}{2}), \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{j \text{ times}}, \mathbf{D}, \underbrace{\sigma[\mathbf{D}, p], \dots, \sigma[\mathbf{D}, p]}_{(2l-j) \text{ times}} \rangle_{\sqrt{t}}(g) d\sqrt{t} \\ & = - \sum_{l=0}^{\infty} \frac{2l!}{l!} \sqrt{4\pi} \int_0^{+\infty} \sum_{j=0}^{2l} \\ & \times \langle \sigma(p - \frac{1}{2}), \underbrace{\sigma[t\mathbf{D}, p], \dots, \sigma[t\mathbf{D}, p]}_{j \text{ times}}, \mathbf{D}, \underbrace{\sigma[t\mathbf{D}, p], \dots, \sigma[t\mathbf{D}, p]}_{(2l-j) \text{ times}} \rangle_t(g) dt \end{aligned}$$

$$\begin{aligned}
&= -\sum_{l=0}^{\infty} \frac{2l!}{l!} \sqrt{4\pi} \int_0^{+\infty} \sum_{j=0}^{2l} (-1)^j \\
&\quad \times \langle \sigma(p - \frac{1}{2}), \underbrace{[t\mathbf{D}, p], \dots, [t\mathbf{D}, p]}_{j \text{ times}}, \mathbf{D}, \underbrace{[t\mathbf{D}, p], \dots, [t\mathbf{D}, p]}_{(2l-j) \text{ times}} \rangle_t(g) dt \\
&= -\sum_{l=0}^{\infty} (-1)^l \frac{2l!}{l!} \sqrt{4\pi} \times 2\sqrt{-1} \int_0^{+\infty} \sum_{j=0}^{2l} (-1)^j \\
&\quad \times \langle (p - \frac{1}{2}), \underbrace{[tD, p], \dots, [tD, p]}_{j \text{ times}}, D, \underbrace{[tD, p], \dots, [tD, p]}_{(2l-j) \text{ times}} \rangle_t(g) dt \\
&= -4\sqrt{-1}\pi [\langle \eta^G(D), \text{Ch}(p) \rangle(g) - \frac{1}{2} \langle \eta^G(D), \text{rk}(p) \text{Ch}_*(1) \rangle(g)]. \quad (3.8)
\end{aligned}$$

When $u = 1$, using Duhamel principle, then

$$\begin{aligned}
&\int_{\Gamma_1} \int_0^{+\infty} \text{Str}(ge^{A^2}) = \int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-s^2/4} \sum_{k=0}^{\infty} \int_{\Delta_k} \text{Str} \left\{ e^{t_0 t \mathbf{D}_1^2} \right. \\
&\quad \times \left[ds\sigma(p - \frac{1}{2}) + d\sqrt{t} \mathbf{D}_1 \right] e^{t_1 t \mathbf{D}_1^2} \dots \left[ds\sigma(p - \frac{1}{2}) + d\sqrt{t} \mathbf{D}_1 \right] e^{t_k t \mathbf{D}_1^2} g \left. \right\} dt_0 \dots dt_k. \quad (3.9)
\end{aligned}$$

By the same reason as $u = 0$, we have $k = 2$ and (3.9) equals

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \int_0^{+\infty} e^{-s^2/4} \left\{ \int_{\Delta_2} \text{Str} \left[e^{t_0 t \mathbf{D}_1^2} ds\sigma(p - \frac{1}{2}) e^{t_1 t \mathbf{D}_1^2} d\sqrt{t} \mathbf{D}_1 e^{t_2 t \mathbf{D}_1^2} g \right] dt_0 dt_1 dt_2 \right. \\
&\quad \left. + \int_{\Delta_2} \text{Str} \left[e^{t_0 t \mathbf{D}_1^2} d\sqrt{t} \mathbf{D}_1 e^{t_1 t \mathbf{D}_1^2} ds\sigma(p - \frac{1}{2}) e^{t_2 t \mathbf{D}_1^2} g \right] dt_0 dt_1 dt_2 \right\} \\
&= -\int_{-\infty}^{+\infty} e^{-s^2/4} ds \int_0^{+\infty} \text{Str} \left[\sigma(p - \frac{1}{2}) \mathbf{D}_1 e^{t \mathbf{D}_1^2} g \right] d\sqrt{t} \\
&= -2i \int_{-\infty}^{+\infty} e^{-s^2/4} ds \int_0^{+\infty} \text{Tr} \left[(p - \frac{1}{2}) D_1 e^{-t D_1^2} g \right] d\sqrt{t} \quad (3.10)
\end{aligned}$$

Let

$$D_p = p(D \otimes I_r)p : p(H \otimes \mathbf{C}^r) \rightarrow p(H \otimes \mathbf{C}^r).$$

Lemma 3.2

$$\int_0^{+\infty} \text{Tr}[g D_1 e^{-t D_1^2}] d\sqrt{t} = \int_0^{+\infty} \text{Tr}[g D e^{-t D^2}] d\sqrt{t}. \quad (3.11)$$

Proof. Let $A = (2p - 1)dp$, using the property of trace, then

$$\frac{d}{du} \eta^G(D_u)(g) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{d}{du} \text{Tr}[g D_u e^{-t D_u^2}] d\sqrt{t}$$

$$\begin{aligned}
&= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \text{Tr}[gAe^{-tD_u^2} - 2tgAD_u^2e^{-tD_u^2}]d\sqrt{t} \\
&= \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \frac{d}{dt}[t^{\frac{1}{2}}\text{Tr}(gAe^{-tD_u^2})]dt \\
&= \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}\text{Tr}(gAe^{-tD_u^2})|_0^{+\infty} = 0.
\end{aligned}$$

In the last equality, considering $\chi(A) = 1$, we use the similar trick in [Z]. \square

By (3.11), then (3.10) equals

$$\begin{aligned}
&= -2\pi i \left\{ \frac{2}{\sqrt{\pi}} \int_0^{+\infty} \text{Tr}[gD_p e^{-tD_p^2} g] d\sqrt{t} - \frac{1}{2} \frac{2}{\sqrt{\pi}} \text{Tr}[gD e^{-tD^2}] d\sqrt{t} \right\} \\
&= -2\pi i [\eta^G(D_p) - \frac{1}{2} \eta^G(D \otimes I_r)] \\
&= -2\pi i [\eta^G(D_p) - \frac{1}{2} \text{rk}(p) \eta^G(D)] \\
&= -2\pi i [\eta^G(D_p) - \text{rk}(p) \langle \eta^G(D), \text{Ch}_*(1) \rangle]. \tag{3.12}
\end{aligned}$$

Since (3.8) equals (3.12), then we get Theorem 3.1.

4 The Equivariant Chern-Connes Character On Manifolds With Boundary

Let M be an even-dimensional compact spin manifolds with boundary $\partial M = N$ endowed with a metric which is a product in a collar neighborhood of N . Denote by D (D_N) the Dirac operator acting on the spinors bundle on M (N). Suppose that G is a compact Lie group acting on M by orientation-preserving isometries. Let $C_G^\infty(M) = \{f \in C^\infty(M) | f \text{ is independent of the normal coordinate } x_n \text{ near the boundary and } f|_N \in C_G^\infty(N)\}$.

Definition 4.1 The equivariant Chern-Connes character on M , $\tau^G = \{\tau_0^G, \tau_2^G, \dots, \tau_{2q}^G \dots\}$ is defined by

$$\begin{aligned}
&\tau_{2q}^G(f^0, f^1, \dots, f^{2q})(g) := -\eta_{2q}^G(D_N)(f^0|_N, f^1|_N, \dots, f^{2q}|_N)(g) + \\
&\frac{1}{(2q)!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} \hat{A}(TF_i) \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} f^0 df^1 \wedge \dots \wedge df^{2q}, \tag{4.1}
\end{aligned}$$

where $\{F_1, \dots, F_k\}$ are components of the fixed point set of g acting on M . Ω is the curvature matrix of the normal bundle $N(F_i)$ and θ is a function matrix on F_i (For details, see [LYZ]). $f_i \in C_G^\infty(M)$ ($0 \leq i \leq 2q$), $\eta_{2q}^G(D_N)$ is the equivariant η -cochain defined in Section 2 and $\hat{A}(TF_i)$ is the \hat{A} -polynomial of curvature R_{F_i} of F_i .

Proposition 4.2 *The equivariant Chern-Connes character is $b - B$ closed (for the definitions of b , B , see [FGV]), i.e.,*

$$b\tau_{2q-2}^G + B\tau_{2q}^G = 0. \quad (4.2)$$

Proof. Firstly, we have the equivariant version of the corollary 2.5 in [GS]. Let $gD_N = D_Ng$ for $g \in G$ and $f^i \in C_G^\infty(N)$ ($0 \leq i \leq 2q + 1$), we have

$$\begin{aligned} -\frac{d}{dt}\text{ch}_{2q+1}^G(tD_N)(f^0, \dots, f^{2q+1})(g) &= \widetilde{\text{bch}}_{2q}^G(tD_N, D_N)(f^0, \dots, f^{2q+1})(g) \\ &\quad + B\widetilde{\text{ch}}_{2q+2}^G(tD_N, D_N)(f^0, \dots, f^{2q+1})(g). \end{aligned} \quad (4.3)$$

Similar to the discussion in [CM], we have

$$\lim_{t \rightarrow \infty} \text{ch}_{2q+1}^G(tD_N)(f^0, \dots, f^{2q+1})(g) = 0. \quad (4.4)$$

By (4.3) and (4.4), then

$$\frac{1}{\Gamma(\frac{1}{2})} \lim_{t \rightarrow 0} \text{ch}_{2q+1}^G(tD_N)(f^0, \dots, f^{2q+1})(g) = [b\eta_{2q}^G(D_N) + B\eta_{2q+2}^G(D_N)](f^0, \dots, f^{2q+1})(g). \quad (4.5)$$

Similar to the computation in [CH], we get

$$\begin{aligned} \frac{1}{\Gamma(\frac{1}{2})} \lim_{t \rightarrow 0} \text{ch}_{2q+1}^G(tD_N)(f^0, \dots, f^{2q+1})(g) &= \frac{1}{(2q+1)!(2\pi\sqrt{-1})^{q+1}} \sum_{i=1}^k \int_{\partial F_i} \widehat{A}(T\partial F_i) \\ &\quad \times \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(\partial F_i)) \right] \right\}^{-1} f^0 d_N f^1 \wedge \dots \wedge d_N f^{2q+1}. \end{aligned} \quad (4.6)$$

By (4.5) and (4.6), similar to the discussion of [Wu], we get (4.2). \square

Remark By $f \in C_G^\infty(M)$, so b^G (B^G) defined in [KL] is b (B).

Let $C_G^1(M)$ be the Banach algebra, the completion of $C_G^\infty(M)$ under the norm $\|\cdot\|_1$ defined in Section 2. Let $\phi^G = \{\phi_0^G, \dots, \phi_{2q}^G, \dots\}$ be the an equivariant even cochains sequence in the bar complex of $C_G^1(M)$. We call that ϕ^G have radius of convergence at least $r > 0$ relative to N if ϕ^G can be written as a sum of two cochains $\phi^G = \phi^{(1),G} + \phi^{(2),G}$ with $\phi^{(1),G}$ entire and $\phi^{(2),G}$ supported on N such that $\phi^{(2),G} \in C_r^{*,G}(C_G^1(N))$. Then we have a corollary of Proposition 2.6 and Proposition 2.7.

Proposition 4.3 *The equivariant Chern-Connes character τ^G has radius of convergence at least $4\lambda^2$, where λ is the smallest positive eigenvalue of the invertible operator D_N . For a selfadjoint idempotent $p \in M_r(C_G^\infty(M))$ such that $\|d(p|_N)\| < \lambda$, then the pairing $\langle \tau^G, \text{Ch}(p) \rangle(g)$ is well-defined.*

5 The Index Pairing

In this section, we will give the main result of this paper.

Let M be a smooth connected compact manifold with smooth compact boundary N . Assume M has even dimension, is oriented and spin. Let

$$C_1(N) = N \times (0, 1]; \quad Z = M \cup_{N \times \{1\}} C_1(N),$$

and \mathcal{U} be a collar neighborhood of N in M . For $\varepsilon > 0$, we take a metric g^ε of Z such that on $\mathcal{U} \cup_{N \times \{1\}} C_1(N)$

$$g^\varepsilon = \frac{dr^2}{\varepsilon} + r^2 g^N.$$

Let $S = S^+ \oplus S^-$ be spinors bundle associated to (Z, g^ε) and H^∞ be the set $\{\xi \in \Gamma(Z, S) \mid \xi \text{ and its derivatives are zero near the vertex of cone}\}$. Denote by $L_c^2(Z, S)$ the L^2 -completion of H^∞ (similar define $L_c^2(Z, S^+)$ and $L_c^2(Z, S^-)$). Let

$$D_\varepsilon : H^\infty \rightarrow H^\infty; \quad D_{+, \varepsilon} : H_+^\infty \rightarrow H_-^\infty,$$

be the Dirac operators associated to (Z, g^ε) which are Fredholm operators for the sufficient small ε . Suppose that G is a compact connected Lie group acting on M by orientation-preserving isometries.

(H₁) Assume that the boundary Dirac operator D_N is invertible and $p = p^* = p^2 \in \mathcal{M}_r(C_G^\infty(M)) \subset \mathcal{M}_r(C^\infty(Z))$ such that $\|d(p|_N)\| < \lambda$, where λ is the smallest positive eigenvalue of $|D_N|$.

Consider

$$D_{p, \varepsilon}^+ := p(D_\varepsilon^+ \otimes I_r)p : p(L_c^2(Z, S^+) \otimes \mathbf{C}^r) \rightarrow p(L_c^2(Z, S^-) \otimes \mathbf{C}^r),$$

which is the Dirac operator with the coefficient from G -vector bundle $p(\mathbf{C}^r)$ over Z . We also assume that

(H₂) For any $g \in G$, there are lifts of g :

$$\begin{aligned} g_1 &: L^2(N, S_N \otimes \text{Im}(p|_N)) \rightarrow L^2(N, S_N \otimes \text{Im}(p|_N)); \\ g_2 &: L_c^2(Z, S \otimes \text{Im}(p)) \rightarrow L_c^2(Z, S \otimes \text{Im}(p)), \end{aligned}$$

which commute with D_N and $D_{p, \varepsilon}$ respectively.

Under the assumption **(H₂)**, we define

$$\text{Ind}_g D_{p, \varepsilon}^+ = \text{Tr}g|_{\ker D_{p, \varepsilon}^+} - \text{Tr}g|_{\ker D_{p, \varepsilon}^-}.$$

Similar to the discussion of [W, p.165], by **(H₁)** then

$$D_{N, p|_N} = p|_N(D_N \otimes I_r)p|_N : L^2(N, S_N \otimes \text{Im}(p|_N)) \rightarrow L^2(N, S_N \otimes \text{Im}(p|_N))$$

is invertible. Let $\dim M = 2m$. If we take the connection pd of the bundle $\text{Im}(p)$, by $g = \text{id}$ on $\text{Im}p|_{N_q}$, we get (see [FGV])

$$\text{Ch}_g(\text{Im}(p)) = \sum_{k=0}^{\infty} \left(-\frac{1}{2\pi\sqrt{-1}}\right)^k \frac{1}{k!} \text{Tr}[p(dp)^{2k}]. \quad (5.1)$$

So by Theorem 3.3 in [Z] (also see [D]), Theorem 3.1 and (5.1), we get

Theorem 5.1 *Under the assumption (\mathbf{H}_1) and (\mathbf{H}_2) , then*

$$\begin{aligned} \text{Ind}_g D_{p,\varepsilon}^+ &= \sum_{r=0}^m \sum_{i=1}^k \frac{(-1)^r}{r!(2\pi\sqrt{-1})^r} \int_{F_i} \widehat{A}(TF_i) \times \\ &\left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2r}] - \langle \eta^G(D_N)(g), \text{Ch}(p) \rangle. \end{aligned} \quad (5.2)$$

Let

$$\begin{aligned} \widehat{\tau}_{2q}^G(g)(f^0, f^1, \dots, f^{2q})(g) &:= \frac{1}{(2q)!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} \widehat{A}(TF_i) \times \\ &\left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} f^0 \wedge df^1 \wedge \dots \wedge df^{2q}, \end{aligned} \quad (5.3)$$

then by the Stokes theorem, we have

$$\begin{aligned} &\langle \widehat{\tau}^G, \text{Ch}(p) \rangle(g) \\ &= \langle \widehat{\tau}_0^G, \text{tr}(p) \rangle(g) + \sum_{q \geq 1} \langle \widehat{\tau}_{2q}^G, \frac{(-1)^q (2q)!}{q!} \text{Tr}((p - \frac{1}{2}) \otimes \overline{p}^{\otimes 2q}) \rangle(g) \\ &= \sum_{q \geq 0} \frac{(-1)^q}{q!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} \widehat{A}(TF_i) \times \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2q}] \\ &\quad - \sum_{q \geq 1} \frac{(-1)^q}{q!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} \widehat{A}(TF_i) \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[\frac{1}{2}(dp)^{2q}] \\ &= \sum_{q \geq 0} \frac{(-1)^q}{q!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} \widehat{A}(TF_i) \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2q}] \\ &\quad - \frac{1}{2} \sum_{q \geq 1} \frac{(-1)^q}{q!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} d\{\widehat{A}(TF_i) \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2q-1}]\} \\ &= \sum_{q \geq 0} \frac{(-1)^q}{q!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{F_i} \widehat{A}(TF_i) \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2q}] \end{aligned}$$

$$-\frac{1}{2} \sum_{q \geq 1} \frac{(-1)^q}{q!(2\pi\sqrt{-1})^q} \sum_{i=1}^k \int_{\partial F_i} \widehat{A}(T\partial F_i) \times \left\{ \text{Pf} \left[2\sinh(\Omega/4\pi + \frac{\sqrt{-1}\theta}{2})(N(\partial F_i)) \right] \right\}^{-1} \text{Tr}[p(dp)^{2q-1}].$$

So, suppose that g acting on N has no fixed points, then by Theorem 5.1 and (4.1), we have

Theorem 5.2 *Suppose that g acting on N has no fixed points. Under the assumption (\mathbf{H}_1) and (\mathbf{H}_2) , then*

$$\text{Ind}_g D_{p,\varepsilon}^+ = \langle \tau^G(D), \text{Ch}(p) \rangle(g). \quad (5.3)$$

Remark: Theorem 5.1 and 5.2 are easily to extend to the case of twisting a G -vector bundle.

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