A Note on Cross Correlation Distribution of Ternary m-Sequences

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Abstract. In this note, we prove a conjecture proposed by Tao Zhang, Shuxing Li, Tao Feng and Gennian Ge, IEEE Transaction on Information Theory, vol. 60, no. 5, May 2014. This conjecture is about the cross correlation distribution of ternary m-sequences.
Index Terms— cross correlation; decimation; ternary m-sequences

1 Introduction

Let ω be a 3-rd complex root of unity. Let $\{a_t\}$ and $\{b_t\}$ be two ternary sequences of period N. The cross correlation function of $\{a_t\}$ and $\{b_t\}$ is defined by

$$C_{a,b}(\tau) = \sum_{t=0}^{N-1} \omega^{a_{t+\tau}-b_t},\ \tau \in \mathbb{Z}/(N).$$

Let n be a positive integer. Let χ be an additive character of $GF(3^n)$ which is defined by $\chi(x) = \omega^{\mathrm{Tr}_1^n(x)}$, where $\mathrm{Tr}_1^n(x) = x + x^3 + \cdots + x^{3^{n-1}}$ is the trace function from $GF(3^n)$ to $GF(3)$. Generally, for a positive integer $r|n$, we denote by $\mathrm{Tr}_r^n(x)$ the trace function from $GF(3^n)$ to $GF(3^r)$, which is defined by $\mathrm{Tr}_r^n(x) = x + x^{3^r} + \cdots + x^{3^{r(\frac{n}{r}-1)}}$.

Let $\{a_t\}$ be a ternary m-sequence of period $3^n - 1$ and let $\{b_t\}$ be its d-decimation where $\gcd(d, 3^n - 1) = 1$. Let α be a primitive element of $GF(3^n)$. Then we denote $C_{a,b}(\tau)$ by $C_d(z)$, where $z = \alpha^\tau$. Clearly, we have $C_d(z) + 1 = \sum_{x\in GF(3^n)} \chi(zx - x^d)$. And we define $S_d(z) := \sum_{x\in GF(3^n)} \chi(zx - x^d)$. Hence computing the cross correlation distribution of m-sequences is equivalent to compute the values distribution of Weil sum $S_d(z)$.

The cross correlation distribution of m-sequences is an important topic in sequences, coding theory and communications. It essentially arises in many contexts with various names, please refer to the appendix of [1] for more details. Many results on this topic have been reported. Please see [2] for an exhaustive survey. In [2], Zhang et al. also determined the distribution of cross correlation of some ternary m-sequences and got some interesting results about the cross correlation of some binary m-sequences. One of their main results can be presented as follows.

Theorem 1. *[2, Theorem II.5] Let r be a positive integer such that $\gcd(r, 3) = 1$. Let $n = 3r$, $d = 3^r + 2$ or $d = 3^{2r} + 2$. Let s be a ternary m-sequence of period $3^n - 1$. Then the cross correlation values between s and its d-decimation are showed as in Tables 1 and 2.*

Zhang et al. conjectured that Theorem 1 is also right if $\gcd(r, 3) = 3$. In this paper we prove their conjecture for any positive integer r. And our technique is generalized from theirs.

In the rest of this paper, we always assume that r is a positive integer, $d = 3^r+2$ or $3^{2r}+2$, and $n = 3r$. Let $E = GF(3^n)$ and $F = GF(3^r)$. It is easy to verify that $\gcd(d, 3^n - 1) = 1$.

Table 1. The distribution of the case r even

Cross Correlation Value	Occurs Times
-1	$\frac{3^{3r}+3^{2r}}{2} - 3^r$
$3^{2r} - 1$	3^r
$3^{\frac{3r}{2}} - 1$	$\frac{3^{3r-1}+3^{2r-1}}{2}$
$-3^{\frac{3r}{2}} - 1$	$\frac{3^{3r-1}+3^{2r-1}}{2}$
$2 \cdot 3^{\frac{3r}{2}} - 1$	$\frac{3^{3r-1}+3^{2r-1}}{4}$
$-2 \cdot 3^{\frac{3r}{2}} - 1$	$\frac{3^{3r-1}+3^{2r-1}}{4}$

Table 2. The distribution of the case r odd

Cross Correlation Value	Occurs Times
-1	$2 \cdot 3^{3r-1} + 3^{2r-1} - 3^r$
$3^{2r} - 1$	3^r
$3^{\frac{3r+1}{2}} - 1$	$\frac{3^{3r-1}+3^{2r-1}}{2}$
$-3^{\frac{3r+1}{2}} - 1$	$\frac{3^{3r-1}+3^{2r-1}}{2}$

2 A Proof

Before proving the conjecture, let us review the sketch of the proof of Theorem 1 in [2]. At first, a suitable element was chosen to construct field extension from F to E. Hence every element in E can be expressed by some elements in the subfield F with the aforementioned element. And then $S_d(z)$ can be expressed by some exponential sums over F. Finally, $S_d(z)$ was computed by using some characterizations of quadratic Weil sum [2, Lemma II.2] and quadratic Gauss sum [2, Lemma II.1].

In the following, we will prove the conjecture. The main difference between our proof and the one in [2] is that we choose different elements to construct field extension. Then we can remove the restriction $\gcd(r, 3) = 1$ and the discussion of cases $r \equiv 2, 1 \pmod 3$ in the proof of [2].

The rest of the paper is split into two cases according to the value of d.

Case $d = 3^r + 2$:

Let u be an element of F such that $\mathrm{Tr}_1^r(u-1) = 1$. Hence $x^3 - x - (u-1)^3$ is an irreducible polynomial over F. And let α be a root of $x^3 - x - (u-1)^3 = 0$. Then we can get $E = F(\alpha)$, which means that for any $x \in E$, it can be uniquely expressed as $x = x_0 + x_1\alpha + x_2\alpha^2$, where $x_0, x_1, x_2 \in F$.

Lemma 1. *Let $x = x_0 + x_1\alpha + x_2\alpha^2$ and $z = z_0 + z_1\alpha + z_2\alpha^2$ be two elements of E. Then we have*

$$\mathrm{Tr}_r^n(x^d) = ((u-1)^3 + 1)x_2^3 + x_2^2 x_1 + x_2^2 x_0 + 2x_2 x_1^2 + 2x_1^3$$

and

$$\mathrm{Tr}_r^n(zx) = 2(z_2 + z_0)x_2 + 2z_1 x_1 + 2z_2 x_0.$$

Proof. Noting that $\alpha^3 = \alpha + (u-1)^3$, we can deduce that $\alpha^{3^r} = \alpha + \mathrm{Tr}_1^r((u-1)^3) = \alpha + 1$. Thus $x^{3^r} = x_0 + x_1(\alpha + 1) + x_2(\alpha + 1)^2$ and $x^{3^{2r}} = x_0 + x_1(\alpha + 2) + x_2(\alpha + 2)^2$. Hence we have

$$\begin{aligned}\mathrm{Tr}_r^n(x^d) = &\ x^d + (x^d)^{3^r} + (x^d)^{3^{2r}} \\ = &\ x^{3^r} x^2 + x^{3^{2r}} (x^{3^r})^2 + x(x^{3^{2r}})^2 \\ = &\ (x_0 + x_1(\alpha+1) + x_2(\alpha+1)^2)(x_0 + x_1\alpha + x_2\alpha^2)^2 \\ &+ (x_0 + x_1(\alpha+2) + x_2(\alpha+2)^2)(x_0 + x_1(\alpha+1) + x_2(\alpha+1)^2)^2 \\ &+ (x_0 + x_1\alpha + x_2\alpha^2)(x_0 + x_1(\alpha+2) + x_2(\alpha+2)^2)^2 \\ = &\ ((u-1)^3+1)x_2^3 + x_2^2 x_1 + x_2^2 x_0 + 2x_2 x_1^2 + 2x_1^3.\end{aligned}$$

The last step is got from a complicated but not difficult computation. Comparing with the above equation, it is easier to verify that

$$\begin{aligned}\mathrm{Tr}_r^n(zx) = &\ zx + (zx)^{3^r} + (zx)^{3^{2r}} \\ = &\ (z_0 + z_1\alpha + z_2\alpha^2)(x_0 + x_1\alpha + x_2\alpha^2) \\ &+ (z_0 + z_1(\alpha+1) + z_2(\alpha+1)^2)(x_0 + x_1(\alpha+1) + x_2(\alpha+1)^2) \\ &+ (z_0 + z_1(\alpha+2) + z_2(\alpha+2)^2)(x_0 + x_1(\alpha+2) + x_2(\alpha+2)^2) \\ = &\ 2(z_2+z_0)x_2 + 2z_1x_1 + 2z_2x_0.\end{aligned}$$

□

According to Lemma 1, we can get that

$$\mathrm{Tr}_1^n(x^d) = \mathrm{Tr}_1^r(\mathrm{Tr}_r^n(x^d)) = \mathrm{Tr}_1^r(x_2^2x_1 + x_2^2x_0 + 2x_2x_1^2 + ux_2 + 2x_1)$$

and

$$\mathrm{Tr}_1^n(zx) = \mathrm{Tr}_1^r(\mathrm{Tr}_r^n(zx)) = \mathrm{Tr}_1^r(2(z_2+z_0)x_2 + 2z_1x_1 + 2z_2x_0).$$

Define $\chi_F(x) = \omega^{\mathrm{Tr}_1^r(x)}$, for $x \in F$. Then we can get

$$\begin{aligned}&S_d(z) \\ &= \sum_{x\in E} \omega^{\mathrm{Tr}_1^n(zx - x^d)} \\ &= \sum_{x_0,x_1,x_2\in F} \chi_F(x_2x_1^2 + (2z_1 - x_2^2 + 1)x_1 + (2z_2 + 2z_0 - u)x_2 + (2z_2 - x_2^2)x_0) \\ &= \sum_{x_1,x_2\in F} \chi_F(x_2x_1^2 + (2z_1 - x_2^2+1)x_1 + (2z_2+2z_0-u)x_2) \sum_{x_0\in F}\chi_F((2z_2 - x_2^2)x_0) \\ &= 3^r \cdot \sum_{x_1\in F, x_2\in M} \chi_F(x_2x_1^2 + (2z_1 - x_2^2 + 1)x_1 + (2z_2 + 2z_0 - u)x_2),\end{aligned} \tag{1}$$

where

$$M = \{x_2 \in F | x_2^2 = -z_2\}.$$

Following from similar arguments in [2], we can deduce the following results by Eq. (1), [2, Lemma II.1] and [2, Lemma II.2]. The details are omitted here.

- If r is even, then $S_d(z)$ takes six values, namely 0, 3^{2r}, $3^{\frac{3r}{2}}$, $-3^{\frac{3r}{2}}$, $2\cdot 3^{\frac{3r}{2}}$ and $-2\cdot 3^{\frac{3r}{2}}$. And the number of occurrences of the first two values are $\frac{3^{3r}+3^{2r}}{2} - 3^r$ and 3^r respectively.

- If r is odd, then $S_d(z)$ takes four values, namely 0, 3^{2r}, $3^{\frac{3r+1}{2}}$ and $-3^{\frac{3r+1}{2}}$. And the number of occurrence of the second value is 3^r.

Then according to [2, Lemma II.3] and [2, Lemma II.4], we can solve the number of occurrences of all the values. The result is the same as Zhang et al. conjectured in [2]. Here a remark is as follows. In the original form of [2, Lemma II.4], the authors assumed that $\gcd(r, 3) = 1$. However, this result can be easily generalized to any positive integer r.

Case $d = 3^{2r} + 2$**:**

Let u be an element of F such that $\mathrm{Tr}_1^r(1-u) = 1$ and let α be a root of $x^3 - x = (1-u)^3$. Then we also can get

$$\mathrm{Tr}_1^n(x^d) = \mathrm{Tr}_1^r(\mathrm{Tr}_r^n(x^d)) = \mathrm{Tr}_1^r(2x_2^2x_1 + x_2^2x_0 + 2x_2x_1^2 + ux_2 + x_1)$$

and

$$\mathrm{Tr}_1^n(zx) = \mathrm{Tr}_1^r(\mathrm{Tr}_r^n(zx)) = \mathrm{Tr}_1^r(2(z_2 + z_0)x_2 + 2z_1x_1 + 2z_2x_0).$$

Then similarly as the first case, we can confirm the conjecture in this case.

3 Conclusion

In this note, we completely determine the distribution of cross correlation values of a ternary m-sequence with period $3^{3r} - 1$ and its d-decimation, where $d = 3^r + 2$ or $d = 3^{2r} + 2$. Hence we confirm the conjecture presented in [2].

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