PRICING PATH-DEPENDENT OPTIONS IN SOME BLACK-SCHOLES MARKET, FROM THE DISTRIBUTION OF HOMOGENEOUS BROWNIAN FUNCTIONALS.

T. FUJITA, F. PETIT, AND M. YOR

ABSTRACT. We give some explicit formulae for the prices of two path-dependent options which combine Brownian averages and penalizations. Because these options are based on both the maximum and local time of Brownian motion, the obtention of their prices necessitates some involved study of homogeneous Brownian functionals, which may be of interest by themselves.

1. MOTIVATION, INTRODUCTION

1.1. To $(B_t, t \geq 0)$, a one-dimensional Brownian motion, we associate its maximum, minimum and local time processes, which we denote respectively by:

$$M_t = \sup_{s \leq t} B_s, \quad I_t = \inf_{s \leq t} B_s, \quad L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{(|B_s| \leq \varepsilon)}\, ds.$$

We also consider $A_t^+ = \int_0^t 1_{(B_s \geq 0)}\, ds$, the time spent in $\mathbb{R}_+$ by B up to time t.

The primary objective of this work was to obtain an explicit expression for the price of the two exotic financial options:

$$(A) \qquad \phi_t^{a,b,l}(K) = \mathbb{E}[1_{(L_t < l)}\, (\exp(aM_t + bB_t) - K)^+]$$

(and more generally when $(B_u, u \geq 0)$ is replaced by $(B_u^{(\delta)} \equiv B_u + \delta u,\ u \geq 0)$ for some drift δ),
and

$$(B) \qquad \psi_t^m(K) = \mathbb{E}\left[\left(\int_0^t 1_{(\exp(B_s) \geq 1)} ds - K\right)^+ 1_{(M_t < m)}\right]$$
$$= \mathbb{E}\big[(A_t^+ - K)^+ 1_{(M_t < m)}\big] \qquad (0 \leq K < t).$$

Clearly, these are two path dependent options, involving a barrier which, in the case (A), penalizes too much time spent by the process

Date: April 25, 2003.

1991 *Mathematics Subject Classification.* Primary 60J55, 60J65, 91B70.

Key words and phrases. Brownian motion, Brownian supremum, Brownian local time, options with barrier.

$(\exp(B_t))$ in the neighborhood of 1, whereas in case (B), it penalizes high values of $(\exp(B_t))$. We believe that such combinations of averages and penalizations may be attractive from a practical viewpoint.

In Section 4, we answer these questions by finding the joint law of $(M_{S_\theta}, B_{S_\theta}, L_{S_\theta})$ and also $(A^+_{S_\theta}, M_{S_\theta})$ for S_θ an exponential time, with parameter θ, independent of B, and then invert the corresponding Laplace transform.

1.2. Of course, the study of the law of $(M_{S_\theta}, I_{S_\theta}, B_{S_\theta}, L_{S_\theta})$ and/or $(A^+_{S_\theta}, B_{S_\theta})$ has been done, at least partially, in a number of papers ([18], Chapter XII), but it seems worthwhile to gather some of these results, and indicate a number of connections with other Brownian studies. This has been a second motivation to write the present paper, and, in Section 2, we present a list of formulae involving the above Brownian functionals and their laws, which may be helpful for various purposes.

1.3. We now say a few words about the methods which we use in this paper. Thanks to the scaling property of Brownian motion (i.e. for $c > 0$, $(B_{ct}; t \geq 0) \overset{d}{=} (\sqrt{c} B_t; t \geq 0)$), it suffices to consider the case $\theta = \frac{1}{2}$ in our study. We then denote simply S for $S_{\frac{1}{2}}$.
We rely essentially upon the path decomposition of $(B_u, u \leq S)$ into the two independent components $(B_u, u \leq g_S)$ and $(B_{S-u}, u \leq S - g_S)$, where $g_t = \sup\{s \leq t; B_s = 0\}$. We recall some further well-known facts:
(ii) $\sqrt{g_S} \overset{d}{=} \sqrt{S - g_S} \overset{d}{=} |N|$, where N is a standard Gaussian variable;
(iii) $\left(b(u) \equiv \dfrac{1}{\sqrt{g_S}} B_{u g_S}; u \leq 1 \right)$ is a standard Brownian bridge, independent of g_S;
(iv) $\left(m_v \equiv \dfrac{1}{\sqrt{S - g_S}} |B_{g_S + v(S - g_S)}|; v \leq 1 \right)$ is a standard Brownian meander, independent of the σ-field $\sigma\{B_u, u \leq g_S; S\}$.

1.4. In our solution of (A) and (B), the joint law, on one hand, of $(\sigma_- \equiv \sup_{s \leq g_S} |B_s|, L_S)$, and, on the other hand, of $(\sigma_+ \equiv \sup_{g_S \leq s \leq S} |B_s|, |B_S|)$, plays an important role. These laws are "well-known", and given by, denoting N an independent standard Gaussian variable:

$$\begin{aligned} (\mathrm{C}) \qquad & \mathbb{P}[\sigma_- \leq a; L_S \in dl] \\ & = \mathbb{P}[|N| \sup_{u \leq 1} |b(u)| \leq a; |N| l_1(b) \in dl] \\ & = \exp(-l \coth a)\, dl \end{aligned}$$

(where $l_1(b)$ is the local time of b at level 0 and time 1),

(D)
$$\begin{aligned}&\mathbb{P}[\sigma_+ \le b; |B_S| \in da]\\ &= \mathbb{P}[|N| \sup_{u\le 1} |m_u| \le b; |N| m_1 \in da]\\ &= \frac{\sinh(b-a)}{\sinh b} 1_{(a\le b)}\, da.\end{aligned}$$

We devote Section 3 to a detailed discussion of (C) and (D). We consider Section 3 as a warm-up, which an expert reader might skip.

1.5. In Section 5, we present another approach to the joint law of (M_t, L_t), whereas, in Section 6, we are naturally led to study the joint law of $(B_{S_\theta}, L^0_{S_\theta}, L^a_{S_\theta})$.

2. Table of results

We first recall our notation:

Brownian motion $(B_s)_{s\ge 0}$, $\sigma_- = \sup_{s\le g_S} |B_s|$, $\sigma_+ = \sup_{g_S\le s\le S} |B_s|$, $L_t = L^0_t$,

Brownian motion with drift $(B^{(\delta)}_s \equiv B_s + \delta s)_{s\ge 0}$,

Brownian bridge $(b(u))_{0\le u\le 1}$,

Brownian meander $(m_u)_{0\le u\le 1}$, $m^* \equiv \sup_{u\le 1} m_u$.

The following Table guides the reader to the precise point in our paper where the law of the random variable in the first column is discussed.

Random variables	Explicit formula
(σ_-, L_S)	(C)
$(\sigma_+, \lvert B_S\rvert)$	(D) and (D')
$\sup_{u\le 1} \lvert b(u)\rvert$	(E)
(m^*, m_1)	(F) and (F')
$(L^{(\delta)}_{S_\theta},\ B^{(\delta)}_{S_\theta},\ M^{(\delta)}_{S_\theta})$	(4.1)
$\mathbb{P}[L^{(\delta)}_{S_\theta} \ge l,\ M^{(\delta)}_{S_\theta} \le m]$	(4.2)
$\mathbb{E}[\phi^{a,0,l}_{S^{(\delta)}_\theta}(K)] = \mathbb{E}\left[(\exp(aM^{(\delta)}_{S_\theta}) - K)^+ 1_{(L^{(\delta)}_{S_\theta}\le l)}\right]$	(4.3)
$\mathbb{P}[M_t < m, L_t \in dl]$	(5.1)
$\mathbb{E}[M^i_1 L^j_1]$	(5.2)
$\mathbb{P}\left[L_t \in dl,\ B_t \in da,\ M_t \le m\right]$	(5.3)
$\mathbb{E}[e^{-\frac{\mu^2}{2} A^+_{S_\theta}} 1_{(M_{S_\theta}\ge m)}]$	(4.5)
$\mathbb{P}[A^+_t \in dz, M_t \le m]$	(4.6)
$\psi^m_t(0) = \mathbb{E}[A^+_t\, 1_{(M_t\le m)}]$	(4.7)
$\mathbb{E}[e^{-\frac{\alpha^2}{2} A^+_{S_\theta}}\, e^{-\frac{\beta^2}{2} A^-_{S_\theta}} 1_{(M_{S_\theta}\le m)}, B_{S_\theta} \in da]$	(4.8)
$(B_{S_\theta}, L^0_{S_\theta}, L^a_{S_\theta} = 0) \qquad (a>0)$	(4.1) and (6.1)
$(B_{S_\theta}, L^0_{S_\theta}, L^a_{S_\theta}, L^a_{S_\theta} > 0) \qquad (a>0)$	(6.2)
$(L^0_{S_\theta}, L^a_{S_\theta}, L^a_{S_\theta} > 0) \qquad (a>0)$	(6.3)

3. On the Brownian meander and its maximum

3.1. Before we discuss (D) in depth, we note that an elementary computation shows it is equivalent to:

$$\text{(D')} \qquad \mathbb{P}[\sigma_+ \in db; |B_S| \in da] = \frac{\sinh a}{\sinh^2 b} 1_{(a \leq b)}\, da\, db.$$

3.2. We now manipulate (D') in order to translate it in terms of only the Brownian meander m taken at time 1, i.e. m_1, and its maximum $m^* \equiv \sup_{u \leq 1} m_u$.

Indeed, from (D'), (ii) and (iv), we deduce that, for $h : \mathbb{R}_+ \to \mathbb{R}_+$, and $f : [0,1] \to \mathbb{R}_+$, two Borel functions,

$$\mathbb{E}\left[h(|N|m^*)\, f\left(\frac{m_1}{m^*}\right)\right] = \mathbb{E}\left[h(\sigma_+)\, f\left(\frac{|B_S|}{\sigma_+}\right)\right]$$

$$= \int_0^\infty db\, \frac{h(b)}{\sinh^2 b} \int_0^b da\, \sinh a\, f\left(\frac{a}{b}\right)$$

$$= \int_0^\infty db\, \frac{b\, h(b)}{\sinh^2 b} \int_0^1 du\, \sinh(bu)\, f(u). \tag{3.1}$$

3.3. Let us consider now the following function:

$$\varphi_u(b) = \frac{b}{\sinh b}\, \frac{\sinh(bu)}{u \sinh b}.$$

$\varphi_u(b)$ is the value of the Laplace transform, in $\frac{b^2}{2}$, of the sum:

$$\hat{T}_u \equiv T_{0 \to 1} + \tilde{T}_{u \to 1} \tag{3.2}$$

where $T_{a \to c}$ denotes the first hitting time of c for a three-dimensional Bessel process, starting from a, and in (3.2), the two hitting times on the right-hand side are assumed to be independent. We recall that, for $0 < a < c < \infty$:

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2} T_{a \to c})] = \left(\frac{\lambda c}{\sinh(\lambda c)}\right) / \left(\frac{\lambda a}{\sinh(\lambda a)}\right).$$

3.4. With the help of the preceding remark, we are now able to rewrite the identity (3.1) in terms, on one hand, of the pair $(m^*, \frac{m_1}{m^*})$, and on the other hand, of $\hat{T}_u$ (as defined in (3.2)). Indeed, the left-hand side of (3.1) may be rewritten as:

$$\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\nu^2/2}\, \mathbb{E}\left[h(\nu m^*)\, f\left(\frac{m_1}{m^*}\right)\right] d\nu$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty h(b)\, \mathbb{E}\left[\frac{1}{m^*}\, e^{-\frac{b^2}{2(m^*)^2}}\, f\left(\frac{m_1}{m^*}\right)\right] db,$$

whereas the right-hand side of (3.1) may be rewritten as:

$$\int_0^1 du\, f(u)\, u \int_0^\infty db\, h(b)\, \mathbb{E}\left[\exp(-\frac{b^2}{2}\hat{T}_u)\right].$$

Equating these two expressions, which are valid for every positive Borel functions f and h, we easily obtain : for every pair of positive Borel functions f and k,

$$\sqrt{\frac{2}{\pi}}\,\mathbb{E}\left[\frac{1}{m^*}\,k\left(\frac{1}{(m^*)^2}\right)\,f\left(\frac{m_1}{m^*}\right)\right] = \int_0^1 \mathbb{E}[k(\hat{T}_u)]\,u\,f(u)\,du, \tag{3.3}$$

which, upon changing $f(u)$ in $\frac{f(u)}{u}$, leads to:

$$\sqrt{\frac{2}{\pi}}\,\mathbb{E}\left[\frac{1}{m_1}\,k\left(\frac{1}{(m^*)^2}\right)\,f\left(\frac{m_1}{m^*}\right)\right] = \int_0^1 \mathbb{E}[k(\hat{T}_u)]\,f(u)\,du. \tag{3.4}$$

3.5. We now give a simple probabilistic interpretation of the identity (3.4), with the help of Williams' path decomposition of Brownian motion $(B_u; u \leq T_0)$ starting at level 1, and considered up to its first passage time at 0. Precisely, let $M = \sup_{u \leq T_0} B_u$. Then:
(i) $\frac{1}{M}$ is uniformly distributed on $[0,1]$;
(ii) there exists a unique time $\rho\,(< T_0)$ at which $B_\rho = M$, and, conditionally on $M = m$, the two processes $(\tilde{B}_u \equiv B_{T_0-u},\ u \leq T_0 - \rho)$ and $(B_u, u \leq \rho)$ are two independent three-dimensional Bessel processes considered up to their first hitting time of m.
Another ingredient we shall need to reinterpret (3.4) is the absolute continuity relationship

$$\sqrt{\frac{2}{\pi}}\,\mathbb{E}\left[\frac{1}{m_1}F(m_u, u \leq 1)\right] = \mathbb{E}_0^{(3)}\left[F\left(\frac{1}{\sqrt{\gamma_1}}R_{u\gamma_1}, u \leq 1\right)\right] \tag{3.5}$$

where, on the right-hand side, $\mathbb{P}_0^{(3)}$ denotes the law of the three-dimensional Bessel process $(R_u; u \geq 0)$ starting from 0, and $\gamma_1 = \sup\{u; R_u = 1\}$.
Formula (3.5) follows easily from a combination of Imhof's relation:

$$\mathbb{E}\left[F(m_u, u \leq 1)\right] = \sqrt{\frac{\pi}{2}}\,\mathbb{E}_0^{(3)}\left[\frac{1}{R_1}F\left(R_u, u \leq 1\right)\right] \tag{3.6}$$

and the "comeander's relation" (i.e. absolute continuity between the law of the "comeander" $(\frac{1}{\sqrt{\gamma_1}}R_{u\gamma_1}, u \leq 1)$ and $\mathbb{P}_0^{(3)}$:

$$\mathbb{E}_0^{(3)}\left[F\left(\frac{1}{\sqrt{\gamma_1}}R_{u\gamma_1}, u \leq 1\right)\right] = \mathbb{E}_0^{(3)}\left[\frac{1}{(R_1)^2}F\left(R_u, u \leq 1\right)\right] \tag{3.7}$$

(for a discussion of these, see, e.g. Yor ([19], Lecture 4)). We now transform (3.4) thanks to (3.5), and we obtain:

$$\mathbb{E}_0^{(3)}\left[f\left(\frac{1}{R^*_{\gamma_1}}\right)\,k\left(\frac{\gamma_1}{(R^*_{\gamma_1})^2}\right)\right] = \int_0^1 \mathbb{E}[k(\hat{T}_u)]\,f(u)\,du, \tag{3.4}_3$$

or, with the help of D. Williams' time reversal theorem:

$$(B_u; u \leq T_0) \stackrel{d}{=} (R_{\gamma_1 - u}; u \leq \gamma_1) \tag{3.8}$$

equation $(3.4)_3$ becomes

$$\mathbb{E}_1\left[f\left(\frac{1}{M}\right)\, k\left(\frac{T_0}{M^2}\right)\right] = \int_0^1 \mathbb{E}[k(\hat{T}_u)]\, f(u)\, du, \tag{3.4}_1$$

and now, $(3.4)_1$ is clearly a consequence of (i) and (ii) above (modulo the relationship due to Brownian scaling) :

$$(\frac{T_0}{M^2}, M) \stackrel{d}{=} (T_{\frac{1}{M}\to 1} + \tilde{T}_{0\to 1}, M)$$

with the same notation as in (3.2).
Note. Obviously, the subscripts 1 and 3 added to (3.4) refer to the dimensions 1 and 3.

3.6. We now present a slightly different derivation of (D) which does not hinge so directly on Imhof's type relations, but rather on the Biane-Yor integral representation:

$$\int_0^\infty dt\, \mathbb{P}^{(t)} = \int_0^\infty dl\, \mathbb{P}^{\tau_l} \circ \int_{-\infty}^{+\infty} da\, {}^r(\mathbb{P}_a^{T_0}),$$

(for these notations, see [4]. See also [1] and [13]).
For any positive Borel function f, we have:

$$\mathbb{P}[\sigma_+ \leq b; f(|B_S|)] = \mathbb{E}[f(|B_S|)\, 1_{(\sup_{t \leq S - g_S} B_{S-t} \leq b)}]$$

$$= \frac{1}{2}\int_{\mathbb{R}} f(|a|)\, \mathbb{E}_a[1_{(\sup_{t\leq T_0}|B_t|\leq b)} e^{-\frac{1}{2}T_0}]\, da = \int_0^{+\infty} f(a)\, \mathbb{E}_a[1_{(\sup_{t\leq T_0} B_t \leq b)} e^{-\frac{1}{2}T_0}]\, da$$

$$= \int_0^{+\infty} f(a)\, \mathbb{E}_1[1_{(\sup_{t\leq T_0} B_t \leq \frac{b}{a})} e^{-\frac{a^2}{2}T_0}]\, da$$

$$= \int_0^b f(a)\, \mathbb{Q}_0^2\left[e^{-\frac{a^2}{2}\int_0^1 Y_x dx}\, \mathbb{Q}_{Y_1}^0\left[e^{-\frac{a^2}{2}\int_0^{+\infty} Y_x dx}\, 1_{(T_0 \leq \frac{b}{a} - 1)}\right]\right] da$$

$$= \int_0^b f(a)\, \mathbb{Q}_0^2\left[e^{-\frac{a^2}{2}\int_0^1 Y_x dx}\, e^{-\frac{aY_1}{2}\coth(b-a)}\right] da = \int_0^b f(a)\, \frac{\sinh(b-a)}{\sinh b} da$$

since

$$\mathbb{Q}_l^0\left[e^{-\frac{a^2}{2}\int_0^{+\infty} Y_x dx}\, 1_{(T_0 \leq \beta)}\right] = \int_0^\beta \mathbb{Q}_l^0\left[e^{-\frac{a^2}{2}\int_0^{T_0} Y_x dx} / T_0 = t\right] \mathbb{Q}_l^0[T_0 \in dt]$$

$$= \exp(-\frac{al}{2}\coth(a\beta)),$$

where $\mathbb{Q}_x^d$ denotes the law of the d-dimensional squared Bessel process starting at x ([18], Chapter XI).

3.7. In this section, we relate (C) and (D) with the theta series expansions:

$$\text{(E)} \qquad \mathbb{P}\left[\sup_{u\leq 1}|b(u)|\leq x\right] = \mathbb{P}_0^{(3)}\left[\frac{\pi^2}{4}T_1 \leq x^2\right] = \sum_{k\in\mathbb{Z}}(-1)^k\,\exp(-2k^2\,x^2)$$

(for instance, see [8], [5] p. 74, [19] p. 86)

$$\text{(F)} \qquad \mathbb{P}[m^*\leq x, m_1\leq a] = \sum_{k\in\mathbb{Z}}(e^{-\frac{(2kx)^2}{2}} - e^{-\frac{(2kx+a)^2}{2}}),\ \ 0<a\leq x.$$

(see [10] and [2]).
We now compare (F) with formula (3.4):
- we first deduce from (F) that

$$\text{(F')} \qquad \mathbb{P}[m^*\in dx, m_1\in da] = H(x,a)\,1_{(a\leq x)}\,dx\,da,$$

with

$$H(x,a) = \sum_{k\in\mathbb{Z}} 2k(1-(2kx+a)^2)\,e^{-\frac{(2kx+a)^2}{2}}.$$

Thus, the left-hand side of (3.4) is equal to:

$$\sqrt{\frac{2}{\pi}}\int_0^\infty da\int_a^\infty dx\,H(x,a)\,\frac{1}{a}\,k(\frac{1}{x^2})\,f(\frac{a}{x}),$$

which, after elementary manipulations, and comparison with the right-hand side, leads to:

$$\mathbb{P}[\hat{T}_u\in dy] = \frac{dy}{\sqrt{2\pi}\,u\,y^{\frac{3}{2}}}\,H(\frac{1}{\sqrt{y}},\frac{u}{\sqrt{y}}), \quad \text{hence}$$

$$\text{(3.9)} \qquad \mathbb{P}[\hat{T}_u\in dy] = \frac{dy}{\sqrt{2\pi}\,u\,y^{\frac{3}{2}}}\,\sum_{k\in\mathbb{Z}} 2k(1-\frac{(2k+u)^2}{y})e^{-\frac{(2k+u)^2}{2y}}.$$

As a thorough check on (3.9), let us recall that:

$$\text{(3.10)} \qquad \mathbb{E}[\exp(-\frac{\lambda^2}{2}\hat{T}_u)] = \left(\frac{\lambda}{\sinh\lambda}\right)^2 / \left(\frac{\lambda u}{\sinh(\lambda u)}\right)$$

and we should
(i) either recover (3.10) from (3.9);
(ii) or, starting from (3.10), and developping $\sinh\alpha$, etc, then recover (3.9);
(iii) or, using again another method, check that (3.9) is indeed obtained from convoluting the densities of $T_{0\to 1}$ and $T_{u\to 1}$.

To check (i), we write, from (3.9):

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2}\hat{T}_u)] = \frac{1}{\sqrt{2\pi}u}\,\sum_{k\in\mathbb{Z}}\int_0^{+\infty} 2k\,y^{-\frac{3}{2}}\,(1-\frac{(2k+u)^2}{y})e^{-\frac{(2k+u)^2}{2y}}e^{-\frac{\lambda^2}{2}y}\,dy$$

$$= \frac{\sqrt{2}}{u\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{\lambda^2}{2}y} \frac{\partial}{\partial y} \left(-\sum_{k\in\mathbb{Z}} \frac{2k}{\sqrt{y}} e^{-\frac{(2k+u)^2}{2y}} \right) dy$$

$$= -\lambda^2 \frac{\sqrt{2}}{u\sqrt{\pi}} \sum_{k\in\mathbb{Z}} \int_0^{+\infty} e^{-\frac{\lambda^2}{2}y} \frac{k}{\sqrt{y}} e^{-\frac{(2k+u)^2}{2y}} dy = -\frac{2\lambda}{u} \sum_{k\in\mathbb{Z}} k\, e^{-\lambda|2k+u|}$$

$$= -\frac{2\lambda}{u} \sum_{k\geq 1} k\, e^{-\lambda(2k+u)} + \frac{2\lambda}{u} \sum_{n\geq 1} n\, e^{-\lambda(2n-u)}$$

$$= \frac{4\lambda}{u} \sinh(\lambda u) \sum_{k\geq 1} k\, (e^{-2\lambda})^k = \frac{\lambda \sinh(\lambda u)}{u \sinh^2 \lambda}$$

Checking (ii) is essentially the same as for (i).

For the point (iii), we first recall the well-known Poisson summation formula:

$$\forall x > 0,\ \forall t \in \mathbb{R},\ \sum_{k\in\mathbb{Z}} e^{-x^2(2k\pi - t)^2/2} = \frac{1}{x\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{int}\, e^{-n^2/2x^2} \tag{3.11}$$

and the Fourier developments ([9])

$$\forall k \in \mathbb{Z}^*,\ \forall t \in]-\pi, \pi[,\ \sum_{\substack{n\in\mathbb{Z} \\ n^2\neq k^2}} \frac{(-1)^n}{n^2 - k^2} e^{int} = \frac{t\,(-1)^k}{k} \sin(kt) + \frac{(-1)^k}{2k^2} \cos(kt) \tag{3.12}$$

$$\forall t \in]-\pi, \pi[,\ \sum_{n\in\mathbb{Z}^*} \frac{(-1)^n}{n^2} e^{int} = \frac{t^2}{2} - \frac{\pi^2}{6}. \tag{3.13}$$

From [5] page 74, we have:
$\mathbb{P}[T_{0\to 1} \leq x] = \sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{n^2\pi^2 x}{2}}$
and

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2} T_{u\to 1})] = \frac{\sinh(\lambda u)}{u \sinh \lambda} = \frac{1}{u} \sum_{n\geq 0} \left(e^{-\lambda(2n+1-u)} - e^{-\lambda(2n+1+u)} \right)$$

$$= \frac{1}{u\sqrt{2\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2}{2}t}}{t^{\frac{3}{2}}} \sum_{n\geq 0} \left((2n+1-u)\, e^{-\frac{(2n+1-u)^2}{2t}} - (2n+1+u)\, e^{-\frac{(2n+1+u)^2}{2t}} \right) dt$$

$$= \frac{1}{u\sqrt{2\pi}} \int_0^{+\infty} \frac{e^{-\frac{\lambda^2}{2}t}}{t^{\frac{3}{2}}} \sum_{k\in\mathbb{Z}} (2k+1-u)\, e^{-\frac{(2k+1-u)^2}{2t}} dt$$

$$= \frac{\pi}{2u} \int_0^{+\infty} \sum_{k\in\mathbb{Z}^*} (-1)^{k-1} k \sin(k\pi u)\, e^{-\frac{k^2\pi^2 t}{2}} e^{-\frac{\lambda^2}{2}t} dt \text{ from (3.11).}$$

Then we have ($u \in]0,1[$):

$$\mathbb{P}[\hat{T}_u \leq y] = \mathbb{P}[T_{0\to 1} \leq y - T_{u\to 1}]$$

$$= \tfrac{\pi}{2u} \int_0^y \sum_{n\in\mathbb{Z}} (-1)^n \, e^{-\frac{n^2\pi^2(y-t)}{2}} \sum_{k\in\mathbb{Z}^*} (-1)^{k-1} \, k \, \sin(k\pi u) \, e^{-\frac{k^2\pi^2 t}{2}} \, dt$$

$$= -\sum_{k\in\mathbb{Z}^*} \frac{\pi k y \sin(k\pi u)\, e^{-\frac{k^2\pi^2 y}{2}}}{u} + \sum_{k\in\mathbb{Z}^*} \frac{(-1)^{k-1}\, k \, \sin(k\pi u)\, e^{-\frac{k^2\pi^2 y}{2}}}{\pi u} \sum_{\substack{n\in\mathbb{Z} \\ n^2\neq k^2}} \frac{(-1)^n}{n^2-k^2}$$

$$+ \tfrac{1}{\pi u} \sum_{n\in\mathbb{Z}} (-1)^n \, e^{-\frac{n^2\pi^2 y}{2}} \sum_{\substack{k\in\mathbb{Z} \\ k^2\neq n^2}} \frac{(-1)^k \, k \sin(k\pi u)}{n^2-k^2}$$

$$= -\sum_{k\in\mathbb{Z}^*} \frac{\pi k y \sin(k\pi u)\, e^{-\frac{k^2\pi^2 y}{2}}}{u} - \sum_{k\in\mathbb{Z}^*} \frac{\sin(k\pi u)\, e^{-\frac{k^2\pi^2 y}{2}}}{2k\pi u}$$

$$+ \sum_{n\in\mathbb{Z}^*} e^{-\frac{n^2\pi^2 y}{2}} (\cos(n\pi u) + \tfrac{1}{2n\pi u} \sin(n\pi u)) + 1 \quad \text{from (3.12) and (3.13)}$$

$$= -\sqrt{\tfrac{2}{\pi}} \tfrac{1}{u\sqrt{y}} \sum_{k\in\mathbb{Z}} (2k+u) e^{-\frac{(2k+u)^2}{2y}} + \sum_{n\in\mathbb{Z}} e^{-in\pi u} e^{-n^2\pi^2 y/2} \text{ from (3.11)}$$

$$= -\sqrt{\tfrac{2}{\pi}} \tfrac{1}{u\sqrt{y}} \sum_{k\in\mathbb{Z}} 2k e^{-\frac{(2k+u)^2}{2y}} \text{ using (3.11) again.}$$

Deriving the preceding equality in y, we recover (3.9).

4. Computations at independent exponential time

4.1. The option problem (A) : the law of the triplet $(M_{S_\theta}^{(\delta)}, B_{S_\theta}^{(\delta)}, L_{S_\theta}^{(\delta)})$.

Proposition 4.1. *For every positive Borel function f, we have:*

$$\mathbb{E}[f(B_{S_\theta}) 1_{(L_{S_\theta}\geq l)} 1_{(M_{S_\theta}\leq m)} e^{-\frac{\delta^2}{2} S_\theta}]$$

$$= \frac{\theta^2}{\sqrt{\theta^2+\delta^2}} \exp(-\frac{l\sqrt{\theta^2+\delta^2}}{2}(1+\coth(m\sqrt{\theta^2+\delta^2}))\, \exp(-m\sqrt{\theta^2+\delta^2})$$

$$\left(\int_0^m f(a)\, \sinh((m-a)\sqrt{\theta^2+\delta^2})\, da + \sinh(m\sqrt{\theta^2+\delta^2}) \int_{-\infty}^0 f(a)\, e^{a\sqrt{\theta^2+\delta^2}}\, da \right).$$

Proof. From excursion theory ([19], p. 73), we have, for every positive Borel functions f and h:

$$\mathbb{E}[f(B_{S_\theta}) h(L_{S_\theta}) 1_{(M_{S_\theta}\leq m)} e^{-\frac{\delta^2}{2} S_\theta}]$$

$$= \mathbb{E}[h(L_{S_\theta})\, 1_{(M_{g_{S_\theta}}\leq m)}\, e^{-\frac{\delta^2}{2} g_{S_\theta}}]\, \mathbb{E}[f(B_{S_\theta})\, 1_{(\sup_{t\leq S_\theta - g_{S_\theta}} B_{S_\theta - t} \leq m)}\, e^{-\frac{\delta^2}{2}(S_\theta - g_{S_\theta})}].$$

We now consider each of these expectations:

a) $\mathbb{E}[h(L_{S_\theta})\, 1_{(M_{g_{S_\theta}}\leq m)}\, e^{-\frac{\delta^2}{2} g_{S_\theta}}] = \theta \int_0^{+\infty} h(l)\, \mathbb{E}[1_{(M_{\tau_l}\leq m)} e^{-\frac{\theta^2+\delta^2}{2}\tau_l}]\, dl$

$$= \theta \int_0^{+\infty} h(l)\, \exp(-\frac{l\sqrt{\theta^2+\delta^2}}{2}(1+\coth(m\sqrt{\theta^2+\delta^2}))\, dl.$$

b) $\mathbb{E}[f(B_{S_\theta})1_{(\sup_{t\le S_\theta - g_{S_\theta}} B_{S_\theta - t}\le m)} e^{-\frac{\delta^2}{2}(S_\theta - g_{S_\theta})}] = \theta \int_{\mathbb{R}} f(a)\mathbb{E}_a[1_{(M_{T_0}\le m)} e^{-\frac{\theta^2+\delta^2}{2}T_0}]da.$

Consequently, we obtain the expected result thanks to Ray-Knight theorems.
Notice that, thanks to (D) and the scaling property of Brownian motion, we may also write directly:

$$\mathbb{E}[f(|B_{S_\theta}|)1_{(\sup_{t\le S_\theta - g_{S_\theta}} B_{S_\theta - t}\le m)} e^{-\frac{\delta^2}{2}(S_\theta - g_{S_\theta})}] = \mathbb{E}[f(\frac{|B_S|}{\theta})1_{(\sup_{t\le S-g_S} B_{S-t}\le m\theta)} e^{-\frac{\delta^2}{2\theta^2}(S-g_S)}]$$

$$= \mathbb{E}[f(\frac{m_1|N|}{\theta})1_{(|N|m^*\le m\theta)} e^{-\frac{\delta^2 N^2}{2\theta^2}}] = \frac{\theta}{\sqrt{\theta^2+\delta^2}}\mathbb{E}[f(\frac{m_1|N|}{\sqrt{\theta^2+\delta^2}})1_{(|N|m^*\le m\sqrt{\theta^2+\delta^2})}]$$

$$= \theta \int_0^m \frac{\sinh((m-a)\sqrt{\theta^2+\delta^2})}{\sinh(m\sqrt{\theta^2+\delta^2})} f(a)\, da.$$

□

Remark 4.2. Let us consider now $(B_t^{(\delta)} \equiv B_t + \delta\, t,\ t \ge 0)$, the Brownian process with drift δ, and $L_t^{(\delta)}$ its local time at level 0 at time t. From the Cameron-Martin relationship, we obtain, for every positive Borel function f:

$$\mathbb{E}\left[1_{(L_{S_\theta}^{(\delta)}\ge l)}\, 1_{(M_{S_\theta}^{(\delta)}\le m)}\, f(B_{S_\theta}^{(\delta)})\right] = \mathbb{E}\left[e^{-\delta B_{S_\theta}}\, e^{-\frac{\delta^2}{2}S_\theta}\, f(B_{S_\theta})1_{(L_{S_\theta}\ge l)}\, 1_{(M_{S_\theta}\le m)}\right]$$

$$= \frac{\theta^2}{\sqrt{\theta^2+\delta^2}} \exp\left(-\frac{\sqrt{\theta^2+\delta^2}}{2}\left(l+2m+l\coth(m\sqrt{\theta^2+\delta^2})\right)\right)$$

$$\left(\int_0^m f(a)\, e^{-\delta a}\, \sinh((m-a)\sqrt{\theta^2+\delta^2})\, da + \sinh(m\sqrt{\theta^2+\delta^2}) \int_{-\infty}^0 f(a)\, e^{a(\sqrt{\theta^2+\delta^2}-\delta)}\, da\right).$$

Hence,

$$\int_0^{+\infty} e^{-\frac{\theta^2}{2}t}\, \mathbb{E}\left[1_{(L_t^{(\delta)}\ge l)}\, 1_{(M_{S_\theta}^{(\delta)}\le m)}\, f(B_t^{(\delta)})\right] dt$$

$$= \frac{1}{\sqrt{\theta^2+\delta^2}} \exp\left(-\frac{l\sqrt{\theta^2+\delta^2}}{1-e^{-2m\sqrt{\theta^2+\delta^2}}}\right)$$

$$\cdot \int_{-\infty}^m f(a)\, e^{-\delta a}\, \left(e^{-|a|\sqrt{\theta^2+\delta^2}} - e^{-(2m-a)\sqrt{\theta^2+\delta^2}}\right) da.$$

Consequently, we obtain:

$$\mathbb{P}\left[L_{S_\theta}^{(\delta)} \in dl,\, B_{S_\theta}^{(\delta)} \in da,\, M_{S_\theta}^{(\delta)} \le m\right] \qquad (m \ge 0) \tag{4.1}$$

$$= \frac{\theta^2\, e^{-\delta a}\, \exp(-\frac{l\sqrt{\theta^2+\delta^2}}{1-e^{-2m\sqrt{\theta^2+\delta^2}}})}{2(1-e^{-2m\sqrt{\theta^2+\delta^2}})} \left(e^{-|a|\sqrt{\theta^2+\delta^2}} - e^{-(m+|a-m|)\sqrt{\theta^2+\delta^2}}\right) dl\, da$$

In particular, we have:

$$\mathbb{P}\left[L_{S_\theta}^{(\delta)} \ge l,\, M_{S_\theta}^{(\delta)} \le m\right] = \left(1 - e^{-m(\delta+\sqrt{\theta^2+\delta^2})}\right) \exp\left(-\frac{l\sqrt{\theta^2+\delta^2}}{1-e^{-2m\sqrt{\theta^2+\delta^2}}}\right). \tag{4.2}$$

Proposition 4.3. *If $\delta + \sqrt{\theta^2+\delta^2} > 1$ and $a > 0$, we have:*

$$\mathbb{E}\left[(\exp(aM_{S_\theta}^{(\delta)}) - K)^+ 1_{(L_{S_\theta}^{(\delta)} \le l)}\right] = K\left(e^{-l\sqrt{\theta^2+\delta^2}} - e^{-\frac{l\sqrt{\theta^2+\delta^2}}{1-K^{-2\sqrt{\theta^2+\delta^2}/a}}}\right) \tag{4.3}$$

$$+\frac{2\sqrt{\theta^2+\delta^2}}{(2\sqrt{\theta^2+\delta^2}-a)\,K^{(2\sqrt{\theta^2+\delta^2}-a)/a}} \sum_{k\ge1} \frac{(-l\sqrt{\theta^2+\delta^2})^k}{(k-1)!}\, {}_2F_1(k+1, 1-\tfrac{a}{2\sqrt{\theta^2+\delta^2}}; 2-\tfrac{a}{2\sqrt{\theta^2+\delta^2}}; \tfrac{1}{K^{2\sqrt{\theta^2+\delta^2}/a}})$$

$$-\frac{a}{(\delta-a+\sqrt{\theta^2+\delta^2})\,K^{(\delta-a+\sqrt{\theta^2+\delta^2})/a}} \sum_{k\ge1} \frac{(-l\sqrt{\theta^2+\delta^2})^k}{k!}\, {}_2F_1(k, \tfrac{1}{2}+\tfrac{\delta-a}{2\sqrt{\theta^2+\delta^2}}; \tfrac{3}{2}+\tfrac{\delta-a}{2\sqrt{\theta^2+\delta^2}}; \tfrac{1}{K^{2\sqrt{\theta^2+\delta^2}/a}}),$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ denotes the hypergeometric function (see [12]*).*

Proof. Noticing that, for $a > 0$, we have

$$(e^{aM} - K)^+ = a \int_{\frac{\ln K}{a}}^{+\infty} e^{ax}\, 1_{(x<M)}\, dx, \tag{4.4}$$

we may compute

$$\mathbb{E}\left[(\exp(aM_{S_\theta}^{(\delta)}) - K)^+ 1_{(L_{S_\theta}^{(\delta)} \le l)}\right] = a\,\mathbb{E}\left[\int_{\frac{\ln K}{a}}^{+\infty} e^{ax}\, 1_{(x<M_{S_\theta}^{(\delta)})} 1_{(L_{S_\theta}^{(\delta)} \le l)}\, dx\right]$$

$$= a \int_{\frac{\ln K}{a}}^{+\infty} e^{ax} \left(1 - \mathbb{P}[M_{S_\theta}^{(\delta)} \le x] - \mathbb{P}[L_{S_\theta}^{(\delta)} \ge l] + \mathbb{P}[L_{S_\theta}^{(\delta)} \ge l,\, M_{S_\theta}^{(\delta)} \le x]\right) dx$$

$$= a \int_{\frac{\ln K}{a}}^{+\infty} e^{ax} \left(e^{-x(\delta+\sqrt{\theta^2+\delta^2})} - e^{-l\sqrt{\theta^2+\delta^2}} + (1 - e^{-x(\delta+\sqrt{\theta^2+\delta^2})})\, e^{-\frac{l\sqrt{\theta^2+\delta^2}}{1-e^{-2x\sqrt{\theta^2+\delta^2}}}}\right) dx$$

$$= \sum_{k\ge1} \frac{(-l\sqrt{\theta^2+\delta^2})^k}{k!} \left(B_k^K(\frac{\sqrt{\theta^2+\delta^2}}{a}) - A_k^K(\frac{\sqrt{\theta^2+\delta^2}}{a}, \frac{\delta}{a} - 1 + \frac{\sqrt{\theta^2+\delta^2}}{a})\right),$$

denoting

$$B_k^K(\alpha) = \int_{\ln K}^{+\infty} e^x((1-e^{-2\alpha x})^{-k}-1)dx = K(1-(1-K^{-2\alpha})^{-k})+2k\alpha A_{k+1}^K(\alpha, 2\alpha-1)$$

and

$$A_k^K(\alpha,\beta) = \int_{\ln K}^{+\infty} e^{-\beta x}(1-e^{-2\alpha x})^{-k}\, dx = \frac{1}{\beta K^\beta}\, {}_2F_1(k, \frac{\beta}{2\alpha}; 1+\frac{\beta}{2\alpha}; \frac{1}{K^{2\alpha}}).$$

□

4.2. **The option problem (B) : the law of the pair $(A^+_{S_\theta}, M_{S_\theta})$.** For the second option pricing problem, denoting $A_t^\pm = \int_0^t 1_{(\pm B_s>0)}ds$, we are interested in the computation of:

Proposition 4.4.

(4.5)

$$\mathbb{E}[e^{-\frac{\mu^2}{2}A^+_{S_\theta}} 1_{(M_{S_\theta}\geq m)}] = \frac{\frac{\theta^2}{\theta^2+\mu^2}(1-e^{-m\sqrt{\theta^2+\mu^2}}) + e^{-m\sqrt{\theta^2+\mu^2}}\,\frac{\theta}{\sqrt{\theta^2+\mu^2}}}{\cosh(m\sqrt{\mu^2+\theta^2}) + \frac{\theta}{\sqrt{\mu^2+\theta^2}}\sinh(m\sqrt{\mu^2+\theta^2})}$$

Inverting the Laplace transform, we obtain, for every $t>0$:

$$\mathbb{P}[A_t^+ \in dz, M_t \leq m]/dz = \frac{1}{\pi\sqrt{z(t-z)}} \qquad (0<z<t) \tag{4.6}$$

$$+\frac{\sqrt{2}}{\pi^{\frac{3}{2}}}\sum_{k\geq 1} 2mk \int_0^z \frac{e^{-\frac{(2mk)^2}{2s}}}{s^{\frac{3}{2}}\sqrt{(t-z)(z-s)}}\, T_k(2\frac{z-s}{t-s}-1)\, ds$$

$$-\frac{1}{\pi}\sum_{k\geq 0}(2k+1)\int_0^z \frac{e^{-\frac{(2k+1)^2m^2}{2s}}}{\sqrt{s}(t-s)^{\frac{3}{2}}}\, P_k(2\frac{z-s}{t-s}-1)\, ds,$$

where T_k and P_k are respectively the Tchebychev and Legendre polynomials.

Remark 4.5. Consequently, we may compute for instance:

$$\psi_t^m(0) = \mathbb{E}[A_t^+\, 1_{(M_t\leq m)}] = \frac{t}{2} + m\sqrt{\frac{t}{2\pi}}e^{-\frac{2m^2}{t}}\psi(2,\frac{3}{2};\frac{2m^2}{t}) \tag{4.7}$$

$$-\frac{t}{2\pi}e^{-\frac{m^2}{2t}}[\sqrt{\pi}\psi(\frac{1}{2},\frac{1}{2};\frac{m^2}{2t}) + \psi(\frac{1}{2},-\frac{1}{2};\frac{m^2}{2t})]$$

$$-\frac{t}{2\pi}e^{-\frac{9m^2}{2t}}[\sqrt{\pi}\psi(\frac{1}{2},\frac{1}{2};\frac{9m^2}{2t}) - \psi(\frac{1}{2},-\frac{1}{2};\frac{9m^2}{2t})],$$

where $\psi(\alpha,\gamma;z)$ is the confluent hypergeometric function of the second type ([12]).

Proof of Proposition 4.4.

$$\mathbb{E}[e^{-\frac{\mu^2}{2}A^+_{S_\theta}}1_{(M_{S_\theta}\geq m)}] = \mathbb{E}[e^{-\frac{\mu^2}{2}A^+_{S_\theta}}1_{(S_\theta\geq T_m)}] = \frac{\theta^2}{2}\mathbb{E}[\int_{T_m}^{+\infty} e^{-\frac{\theta^2}{2}t}e^{-\frac{\mu^2}{2}A^+_t}\,dt]$$

$$= \frac{\theta^2}{2}\mathbb{E}[e^{-\frac{\theta^2}{2}A^-_{T_m}}\,e^{-\frac{\mu^2+\theta^2}{2}A^+_{T_m}}]\,\mathbb{E}_m[\int_0^{+\infty} e^{-\frac{\theta^2}{2}u}\,e^{-\frac{\mu^2}{2}A^+_u}\,du].$$

Thanks to the Brownian scaling property and Ray-Knight theorem:

$$\mathbb{E}[e^{-\frac{\theta^2}{2}A^-_{T_m}}\,e^{-\frac{\mu^2+\theta^2}{2}A^+_{T_m}}] = \mathbb{Q}^2_0[e^{-\frac{(\mu^2+\theta^2)m^2}{2}\int_0^1 Y_x dx}\,\mathbb{Q}^0_{Y_1}[e^{-\frac{\theta^2 m^2}{2}\int_0^{+\infty} Y_x dx}]]$$

$$= \frac{1}{\cosh(m\sqrt{\mu^2+\theta^2}) + \frac{\theta}{\sqrt{\mu^2+\theta^2}}\sinh(m\sqrt{\mu^2+\theta^2})}$$

and, by scaling again

$$\frac{\theta^2}{2}\,\mathbb{E}_m[\int_0^{+\infty} e^{-\frac{\theta^2}{2}u}\,e^{-\frac{\mu^2}{2}A^+_u}\,du]$$

$$= \frac{\theta^2}{2}\,\mathbb{E}_m[\int_0^{T_0} e^{-\frac{\theta^2}{2}u}\,e^{-\frac{\mu^2}{2}A^+_u}\,du] + \frac{\theta^2}{2}\,\mathbb{E}_m[\int_{T_0}^{+\infty} e^{-\frac{\theta^2}{2}u}\,e^{-\frac{\mu^2}{2}A^+_u}\,du]$$

$$= \frac{\theta^2}{\theta^2+\mu^2}\mathbb{E}_m[1-e^{-\frac{\theta^2+\mu^2}{2}T_0}] + \mathbb{E}_m[e^{-\frac{\theta^2+\mu^2}{2}T_0}]\,\frac{\theta^2}{2}\,\mathbb{E}_0[\int_0^{+\infty} e^{-\frac{\theta^2}{2}u}\,e^{-\frac{\mu^2}{2}A^+_u}\,du]$$

$$= \frac{\theta^2}{\theta^2+\mu^2}\mathbb{E}_1[1-e^{-\frac{(\theta^2+\mu^2)m^2}{2}T_0}] + \mathbb{E}_1[e^{-\frac{(\theta^2+\mu^2)m^2}{2}T_0}]\,\mathbb{E}_0[e^{-\frac{\mu^2}{2}A^+_{S_\theta}}]$$

$$= \frac{\theta^2}{\theta^2+\mu^2}(1-e^{-m\sqrt{\theta^2+\mu^2}}) + e^{-m\sqrt{\theta^2+\mu^2}}\,\frac{\theta}{\sqrt{\theta^2+\mu^2}},$$

from P. Lévy's arcsine law.
To invert the Laplace transform, we have used the following identities ([7], [12] and [15] page 304):

$$\frac{1}{\sqrt{2\pi t}}\int_0^t e^{-\frac{\mu^2 x}{2}}P_k(\frac{2x}{t}-1)dx = \frac{(-1)^k}{\mu}e^{-\frac{\mu^2 t}{4}}I_{k+\frac{1}{2}}(\frac{\mu^2 t}{4})$$

and

$$\frac{1}{\pi}\int_0^t \frac{e^{-\frac{\mu^2 x}{2}}}{\sqrt{tx-x^2}}\,T_k(\frac{2x}{t}-1)dx = (-1)^k\,e^{-\frac{\mu^2 t}{4}}I_k(\frac{\mu^2 t}{4}),$$

where I_ν denotes the modified Bessel function of the first kind of index ν. □

Remark 4.6. Using the same method, we get:

$$\mathbb{E}[e^{-\frac{\alpha^2}{2}A^+_{S_\theta}}\, e^{-\frac{\beta^2}{2}A^-_{S_\theta}} 1_{(M_{S_\theta}\leq m)}, B_{S_\theta}\in da]/da \tag{4.8}$$

$$= \frac{\theta^2(\sinh((m-a)\sqrt{\theta^2+\alpha^2})1_{(0<a<m)} + e^{a\sqrt{\theta^2+\beta^2}}\sinh(m\sqrt{\theta^2+\alpha^2})1_{(a<0)})}{\sqrt{\theta^2+\alpha^2}\cosh(m\sqrt{\theta^2+\alpha^2}) + \sqrt{\theta^2+\beta^2}\sinh(m\sqrt{\theta^2+\alpha^2})},$$

from which we can deduce the following identity for Brownian motion with drift δ:

$$\frac{2}{\theta^2}\mathbb{E}[e^{-\frac{\mu^2}{2}A^{(\delta),+}_{S_\theta}} 1_{(M^{(\delta)}_{S_\theta}\leq m)}] = \frac{\frac{e^{mb}-e^{-m\delta}}{b+\delta} + \frac{e^{-mb}-e^{-m\delta}}{b-\delta} + \frac{e^{mb}-e^{-mb}}{a-\delta}}{b\cosh(mb)+a\sinh(mb)},$$

where $a=\sqrt{\theta^2+\delta^2}$ and $b=\sqrt{\theta^2+\delta^2+\mu^2}$.

5. A direct approach to the joint law of (M_t, L_t)

Proposition 5.1. *Let H_p denotes the p-th Hermite polynomial (see* [12]*):*

$$(-1)^p\, e^{-x^2}\, H_p(x) = G^{(p)}(x) \text{ where } G(x) = e^{-x^2}.$$

Then, we have:

$$\mathbb{P}[M_t < m, L_t \in dl]/dl \tag{5.1}$$

$$= \sqrt{\frac{2}{\pi t}} \sum_{0\leq k\leq n} \frac{C^k_n}{k!}(\frac{l}{\sqrt{2t}})^k \left(G^{(k)}(\frac{2nm+l}{\sqrt{2t}}) - G^{(k)}(\frac{(2n+1)m+l}{\sqrt{2t}})\right).$$

Proof. From (4.2), we have:

$$\frac{2}{\theta^2}\mathbb{P}\left[L_{S_\theta}\in dl,\, M_{S_\theta}\leq m\right]/dl = \frac{2}{\theta}\frac{1-e^{-m\theta}}{1-e^{-2m\theta}}\, \exp\left(-\frac{l\theta}{1-e^{-2m\theta}}\right)$$

$$= \frac{2}{\theta}(1-e^{-m\theta})\, e^{-l\theta} \sum_{n\geq 0} L_n(l\theta)e^{-2mn\theta}$$

$$= 2\sum_{n\geq 0}\sum_{k=0}^{n} \frac{C^k_n}{k!}(-l)^k \left(\theta^{k-1}e^{-\theta(l+2mn)} - \theta^{k-1}e^{-\theta(l+(2n+1)m)}\right)$$

where L_n is the Laguerre polynomial of degree n (see [12], formula (4.17.3)).
We recall that for every integer $\nu \geq -1$, we have:

$$\forall \lambda>0, \forall y>0,\ \lambda^\nu e^{-\lambda y} = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} e^{-\frac{\lambda^2}{2}t}\, e^{-\frac{y^2}{2t}}\, \frac{H_{\nu+1}(\frac{y}{\sqrt{2t}})}{(2t)^{1+\frac{\nu}{2}}}\, dt.$$

Equation (5.1) immediately follows. □

Remark 5.2. We note that:

$$\mathbb{P}[M_t < m, L_t < l] = \mathbb{P}[t < T_m, t < \tau_l] = \mathbb{P}[t < T_m \wedge \tau_l].$$

Thus, we need to find the law of $T_m \wedge \tau_l$.

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2}(T_m \wedge \tau_l))] = \mathbb{E}[\exp(-\frac{\lambda^2}{2}T_m)\, 1_{(T_m \leq \tau_l)}] + \mathbb{E}[\exp(-\frac{\lambda^2}{2}\tau_l)\, 1_{(\tau_l < T_m)}]$$

$$= \mathbb{E}[\exp(-\frac{\lambda^2}{2}T_m)\, 1_{(L_{T_m} \leq l)}] + \mathbb{E}[\exp(-\frac{\lambda^2}{2}\tau_l)\, 1_{(M_{\tau_l} < m)}]$$

and both the joint laws of (T_m, L_{T_m}) on one hand, (τ_l, M_{τ_l}) are known. Thanks to excursion theory, we have ([18], Chapter XII, exercise 4.11):

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2}\tau_l)\, 1_{(M_{\tau_l} < m)}] = \mathbb{E}[\exp(-\frac{\lambda^2}{2}\tau_l^+)\, 1_{(M_{\tau_l} < m)}]\ \mathbb{E}[\exp(-\frac{\lambda^2}{2}\tau_l^-)]$$

$= \exp(-\frac{\lambda l}{2}(1 + \coth(\lambda m)))$.
On the other hand, using the Brownian scaling property, the second Ray-Knight theorem, and classical results on squared Bessel processes ([18], Chapter XI):

$$\mathbb{E}[\exp(-\frac{\lambda^2}{2}T_m)\, 1_{(L_{T_m} \leq l)}] = \mathbb{E}[\exp(-\frac{\lambda^2 m^2}{2}T_1)\, 1_{(mL_{T_1} \leq l)}]$$

$$= \mathbb{Q}_0^2 \left[\exp(-\frac{\lambda^2 m^2}{2}\int_0^1 Y_x dx)\, 1_{(Y_1 \leq l/m)}\ \exp(-\frac{\lambda m}{2}Y_1)\right]$$

$$= \int_0^{l/m} \frac{\lambda m}{\sinh(\lambda m)}\ \exp(-\frac{\lambda m}{2}z(1 + \coth(\lambda m)))\, dz$$

$$= \exp(-\lambda m)\left(1 - \exp(-\frac{\lambda l}{2}(1 + \coth(\lambda m)))\right).$$

Then, we recover (4.2), in the case $\delta = 0$:

$$\frac{\lambda^2}{2}\int_0^{+\infty} \mathbb{P}[M_t < m, L_t < l]\, e^{-\frac{\lambda^2}{2}t}\, dt$$

$$= 1 - \mathbb{E}[\exp(-\frac{\lambda^2}{2}(T_m \wedge \tau_l))] = \left(1 - e^{-\lambda m}\right)\left(1 - \exp(-\frac{\lambda l}{1 - e^{-2m\lambda}})\right).$$

Remark 5.3. From (5.1), we may deduce the following moment formula:

$$(5.2)\quad \forall i \in I\!N, \forall j \in I\!N,\ \mathbb{E}[M_1^i\, L_1^j] = \frac{i!\, j!}{2^{\frac{i+j}{2}}\Gamma(1+\frac{i+j}{2})}\ \sum_{k=1}^{2j+1} \frac{(-1)^{[\frac{k-1}{2}]}\, C_j^{[\frac{k}{2}]}}{k^i}.$$

Remark 5.4. In the same way, inverting the Laplace transform in (4.1), we may obtain the trivariate law:

$$\mathbb{P}\left[L_t \in dl,\, B_t \in da,\, M_t \leq m\right]/dl\, da \tag{5.3}$$

$$= \tfrac{1}{2t\sqrt{\pi}} \sum_{0\leq k\leq n} \tfrac{C_n^k}{k!} (\tfrac{l}{\sqrt{2t}})^k \left(G^{(k+1)}(\tfrac{(2n+1)m+l+|a-m|}{\sqrt{2t}}) - G^{(k+1)}(\tfrac{2nm+l+|a|}{\sqrt{2t}}) \right).$$

6. Local times at two different levels

As we have

$$\mathbb{P}[M_t < a, L_t^0 \in dl, B_t \in dx] = \mathbb{P}[L_t^a = 0, L_t^0 \in dl, B_t \in dx], \tag{6.1}$$

it also seemed natural to consider the law of the triplet (B_t, L_t^0, L_t^a), where $a > 0$.

Proposition 6.1. *Let $a > 0$. Then:*

$$\mathbb{P}\left[B_{S_\theta} \in dx, L^0_{S_\theta} \in dy, L^a_{S_\theta} \in dz,\, 1_{(L^a_{S_\theta}>0)}\right] \tag{6.2}$$

$$= \frac{\theta^3\, e^{-a\theta}}{2(1-e^{-2a\theta})^2} \exp\left(-\frac{\theta(y+z)}{1-e^{-2a\theta}}\right) \left\{ \left(e^{-\theta|x-a|} - e^{-\theta(a+|x|)}\right) I_0\Big[\frac{\theta\sqrt{yz}}{\sinh(a\theta)}\Big] \right.$$

$$\left. + \left(e^{-\theta|x|} - e^{-\theta(a+|x-a|)}\right) \left(\sqrt{\frac{y}{z}}\, I_1\Big[\frac{\theta\sqrt{yz}}{\sinh(a\theta)}\Big] - \frac{\theta\, y}{2\,\sinh(a\theta)} \right) \right\} dx\, dy\, dz,$$

from where we deduce

$$\mathbb{P}\left[L^0_{S_\theta} \in dy, L^a_{S_\theta} \in dz,\, 1_{(L^a_{S_\theta}>0)}\right] \tag{6.3}$$

$$= \frac{\theta^2\, \exp\Big(-\frac{\theta(y+z)}{1-e^{-2a\theta}}\Big)}{2\,\sinh(a\theta)\,(1+e^{-a\theta})} \left\{ I_0\Big[\frac{\theta\sqrt{yz}}{\sinh(a\theta)}\Big] + \sqrt{\frac{y}{z}}\, I_1\Big[\frac{\theta\sqrt{yz}}{\sinh(a\theta)}\Big] - \frac{\theta\, y}{2\,\sinh(a\theta)} \right\} dy\, dz.$$

Proof. We consider the function:

$u(x) = \frac{2}{\theta^2}\, \mathbb{E}_x \left[\exp(-\lambda B_{S_\theta} - \frac{\alpha^2}{2} L^a_{S_\theta} - \frac{\beta^2}{2} L^0_{S_\theta})\right].$

By Feynman Kac's formula, we obtain: $\left\{ \begin{array}{l} \theta^2 u = u" + 2\, e^{-\lambda x} \\ u'(a+) - u'(a-) = \alpha^2\, u(a) \\ u'(0+) - u'(0-) = \beta^2\, u(0) \end{array} \right.$

which gives, with $C = \frac{2}{\theta^2-\lambda^2}$

$$u(x) = \left\{ \begin{array}{ll} Ce^{-\lambda x} + C_1 e^{-\theta x} & x \geq a \\ Ce^{-\lambda x} + C_2 e^{\theta x} + C_3 e^{-\theta x} & 0 \leq x \leq a \\ Ce^{-\lambda x} + C_4 e^{\theta x} & x \leq 0 \end{array} \right.$$

with the conditions

$$\begin{cases} C_1 e^{-\theta a} = C_2 e^{\theta a} + C_3 e^{-\theta a} \\ C_2 + C_3 = C_4 \\ C_2 e^{\theta a} + (C_1 - C_3) e^{-\theta a} = -\frac{\alpha^2}{\theta} \left(C e^{-\lambda a} + C_1 e^{-\theta a} \right) \\ (C_2 - C_3 - C_4)\theta = \beta^2 (C + C_4) \end{cases} .$$

Then

$$\frac{2}{\theta^2}\, \mathbb{E}_0 \left[\exp(-\lambda B_{S_\theta} - \frac{\alpha^2}{2} L^a_{S_\theta} - \frac{\beta^2}{2} L^0_{S_\theta}) \right] = u(0) = C + C_4 \tag{6.4}$$

$$= \tfrac{2}{\theta^2 - \lambda^2} \tfrac{(\alpha^2+2\theta)e^{\theta a} - \alpha^2 e^{-\lambda a}}{(2\theta+\alpha^2+\beta^2)e^{\theta a} + \frac{\alpha^2\beta^2}{\theta}\sinh(a\theta)} = \int_{\mathbb{R}} e^{-\lambda x}\, \tfrac{(\alpha^2+2\theta)e^{\theta(a-|x|)} - \alpha^2 e^{-\theta|x-a|}}{\theta(2\theta+\alpha^2+\beta^2)e^{\theta a} + \alpha^2\beta^2 \sinh(a\theta)} dx,$$

from where we obtain:

$$\tfrac{2}{\theta^2}\, \mathbb{E}_0 \left[\exp(-\lambda B_{S_\theta} - \tfrac{\alpha^2}{2} L^a_{S_\theta} - \tfrac{\beta^2}{2} L^0_{S_\theta})\, 1_{(L^a_{S_\theta} > 0)} \right]$$

$$= \int_{\mathbb{R}} e^{-\lambda x}\, \theta\, e^{-\theta|x|}\, \tfrac{\theta + \frac{\beta^2}{2}(1 - e^{-2a\theta}) - (\theta + \frac{\beta^2}{2})(1 - e^{-\theta(a + |x-a| - |x|)})}{\left(\theta + \frac{\beta^2}{2}(1 - e^{-2a\theta})\right)\left(\frac{\alpha^2}{2}\left[\theta + \frac{\beta^2}{2}(1 - e^{-2a\theta})\right] + \theta(\theta + \frac{\beta^2}{2})\right)} dx.$$

Inverting the double Laplace transform (see [7], p. 197), we obtain (6.2). □

Remark 6.2. Notice that we may recover (4.1) from (6.4), thanks to (6.1).

References

[1] J. Bertoin, and J. W. Pitman. Path transformation connecting Brownian bridge, excursion and meander. *Bull. Sci. Math.*, vol. 118 (2), p. 147–166, 1994.

[2] J. Bertoin, L. Chaumont, and J. W. Pitman. Path transformations of first passage bridges. In preparation, 2003.

[3] Ph. Biane, J. W. Pitman and M. Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc.* (N.S.) 38 (2001), n° 4, 435–465 (electronic).

[4] Ph. Biane, and M. Yor. Sur la loi des temps locaux browniens pris en un temps exponentiel. *Sém. Proba. XXIII*, *Lect. Notes Math.*, vol. 1321, p. 454–466, Berlin Heidelberg New York, Springer, 1988.

[5] Ph. Biane, and M. Yor. Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Maths.*, 2ème série, vol. 111, p. 23–101, 1987.

[6] A. Borodin, and P. Salminen. *Handbook of Brownian motion: facts and formulae*, Birkhaüser, second edition, 2002.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. . Tricomi. Bateman manuscript project, *Tables of integral transforms*, Chapters IV and V, based, in part, on notes left by Harry Bateman, McGraw-Hill Book Co., New York, 1954.

[8] K. L. Chung. A cluster of great formulas. *Acta Math. Acad. Sci. Hungar.* 39 (1982), no. 1-3, 65–67.

[9] H. Cartan. *Théorie élémentaire des fonctions analytiques*. Hermann, Paris, 1961.

[10] R. Durrett and D.L. Iglehart. Functionals of Brownian meander and Brownian excursion. *Ann. Prob. 5*, p. 130–135, 1977.

[11] J.P Imhof. Density factorizations for Brownian motion and the three-dimensional Bessel processes and applications. *J. App. Prob. 21*, p. 500–510, 1984.
[12] N. N. Lebedev. *Special functions and their applications.* Dover Publications, New York, 1972. Translated and edited by Richard A. Silverman.
[13] C. Leuridan. Une démonstration élémentaire d'une identité de Biane et Yor. *Séminaire de Probabilités XXX*, 255–260, Lecture Notes in Math., 1626, Springer, Berlin, 1996.
[14] P. Messulam and M. Yor. On D. Williams' "pinching method" and some applications. *J. London Math. Soc.* (2) 26 (1982), n° 2, 348–364.
[15] G. Petiau. *La théorie des fonctions de Bessel exposée en vue de ses applications à la physique mathématique.* CNRS, IHP, 1955.
[16] J. W. Pitman and M. Yor. Asymptotic laws of planar Brownian motion. *Ann. Probab.* 14, no. 3, 733–779, 1986.
[17] J. W. Pitman and M. Yor. Path decomposition of a Brownian bridge related to the ratio of is maximum and amplitude. *Studia Sc. Math. Hungar.* 35, 457–474, 1999.
[18] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Springer, third edition, 1999.
[19] M. Yor. Local times and excursions for Brownian motion: a concise introduction. *Lecciones en Matemáticas. Numero 1*, lecture 4, Universidad Central de Venezuela, 1994-1995.

Takahiko Fujita, Graduate School of Commerce and Management, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo, 186-8601, Japan
E-mail address: fujita@math.hit-u.ac.jp

Frédérique Petit, Laboratoire de Probabilités, Université Paris VI, casier 188, 4, Place Jussieu, 75252 Paris Cedex 05, France
E-mail address: fpe@ccr.jussieu.fr

Marc Yor, Laboratoire de Probabilités, Université Paris VI, casier 188, 4, Place Jussieu, 75252 Paris Cedex 05, France