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On the Well-ordered S-matrix

Shoichi HORI

Department of Physics, Kanazawa University

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With the help of the relation between T-product and S-product given by Wick, a new formula for the S-matrix has been obtained, which expresses the S-matrix directly as S-product and from which several already known results can be derived much easily.

§ 1. Introduction

The evaluation of the S-matrix should be performed for the present in the form of power series expansion of the coupling constant, each term of which becomes finite, as proved by Dyson[1], after the renormalizations of mass and charge are carried out. The convergence of the series, however, remains now to be proved. For this purpose, more detailed knowledge about the structure of the S-matrix—finite part of the S-matrix—will be required. Therefore it is desirable to express the S-matrix in a simpler (closed) form and it is also desirable in order to make it possible to evaluate energy eigenvalues of the bound state from the S-matrix by means of analytic continuation and to find a clue to the future theory as well.

Recently Feynman[2] succeeded to write the S-matrix in a compact form, using his operational calculus. For calculating its matrix element, however, we must again transform it into a sum of the normal products, namely, into the usual power series. We will try from the beginning to express it as an S-product—a sum of ordered products.

§ 2. Well-ordering of the S-matrix

Recently Wick[3] has introduced algebraic operations named T-product and S-product which make the treatment of the S-matrix much easier. Using his T-product symbol, one can write the S-matrix as,

$$S=\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n\ T(H(x_1)H(x_2)\cdots H(x)), \tag{1}$$

where $H(x)$ is the Hamiltonian density of the system in consideration.

Since T-product is a linear operation, we can change the order of T-product operation and since $H(x)$ contains even numbers (including zero) of Fermi field variables in the case of physical interest, $H(x)$'s in the T-product commute each other. Therefore we obtain

$$S=T\left[\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\left(\int_{-\infty}^{\infty}H(x)dx\right)^n\right]$$

or

$$S=T[e^{-i\int_{-\infty}^{\infty}H(x)dx}].$$

Considering the meaning of the T-product, this expression is in complete accordance with Feynman's result regarding the time as a disentangling parameter.

To fix the idea, we will confine ourselves to the quantum electrodynamics. Further we will consider the transformation function $S[\sigma]$ instead of the S-matrix. The S-matrix can be obtained immediately by letting $\sigma\to\infty$; $S=S[\infty]$.

The relation between T-product and S-product was given by Wick[3] :

$$T(UVW\cdots XYZ)=:UVW\cdots XYZ:+:U^{\cdot}V^{\cdot}W\cdots XYZ:+\cdots$$
$$+:UVW\cdots XY^{\cdot}Z^{\cdot}:+\cdots+:U^{\cdot}V^{\cdot\cdot}W^{\cdot\cdot\cdot}\cdots X^{\cdot\cdot\cdot}Y^{\cdot}Z^{\cdot\cdot}: \qquad (3)$$

where on the right hand side the summation is over all possible contractions with respect to $UVW\cdots XYZ$.

In the case of the quantum electrodynamics the Hamilian density is given by,

$$H_i(x)=-j_\mu(x)A_\mu(x), \qquad (4)$$

where

$$j_\mu(x)=ie:\bar{\psi}(x)\gamma_\mu\psi(x): \qquad (5)$$

and then we can readily see that

$$S[\sigma]=\sum_{n=1}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\ T(j_{\mu1}(x_1)\cdots j_{\mu n}(x_n))\times$$
$$T(A_{\mu1}(x_1)\cdots A_{\mu n}(x_n)). \qquad (6)$$

When the second T-product (with respect to A_μ) in the Eq. (6) is transformed into S-product, the terms which contain no contraction can be written as,

$$\sum_{n=1}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\ T(j_{\mu1}(x_1)\cdots j_{\mu n}(x_n)):A_{\mu1}(x_1)\cdots A_{\mu n}(x_n):$$

or

$$S_A\left[\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\ T(j_{\mu1}(x_1)\cdots j_{\mu n}(x_n))A_{\mu1}(x_1)\cdots A_{\mu n}(x_n)\right]. \qquad (7)$$

Here we have used a notation S_A instead of colons to express S-product only with respect to A_μ.

Because

$$A_\mu(x)A_\nu(y)=1/2\ \delta_{\mu\nu}\ D_F(x-y), \qquad (8)$$

introducing the operator

$$\Delta[\sigma]=\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\, D_F(x-y)\frac{\delta^2}{\delta A_\mu(x)\delta A_\mu(y)}, \tag{9}$$

we can write the terms which contain a contraction,

$$S_A\left[\sum_{n=0}^{\infty}\Delta[\sigma]\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\; T(j_{\mu 1}(x_1)\cdots j_{\mu n}(x_n))A_{\mu 1}(x_1)\cdots A_{\mu n}(x_n)\right], \tag{10}$$

and the terms containing two contractions,

$$S_A\left[\sum_{n=0}^{\infty}\frac{1}{2!}\Delta^2[\sigma]\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\; T(j_{\mu 1}(x_1)\cdots j_{\mu n}(x_n))A_{\mu 1}(x_1)\cdots A_{\mu n}(x_n)\right].$$

The factor $1/2!$ is necessay because the same terms with two contractions arise in two ways. In general the terms with r contractions can be written as,

$$S_A\left[\sum_{n=0}^{\infty}\frac{1}{r!}\Delta^r[\sigma]\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\; T(j_{\mu 1}(x_1)\cdots j_{\mu n}(x_n))A_{\mu 1}(x_1)\cdots A_{\mu n}(x_n)\right].$$

Finally they are collected to be

$$S_A\left[\sum_{n=0}^{\infty}\Big(\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{1}{r!}\Delta^r[\sigma]\Big)\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-\infty}^{\sigma}dx_n\times\right.$$
$$\left.T(j_{\mu 1}(x_1)\cdots j_{\mu n}(x_n))A_{\mu 1}(x_1)\cdots A_{\mu n}(x_n),\right],$$

where $\left[\frac{n}{2}\right]=\frac{n}{2}$ for even n and $\frac{n-1}{2}$ for odd n. We may replace this upper limit of the summation by infinity, because the alteration merely give rise to a vanishig effect. Therefore we obtain

$$S[\sigma]=S_A\left[e^{\Delta[\sigma]}\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\sigma}dx_1\cdots\int_{-8}^{\sigma}dx_n\; T(j_{\mu 1}(x_1)\cdots j_{\mu n}(x_n))\times\right.$$
$$\left.A_{\mu 1}(x_1)\cdots A_{\mu n}(x_n)\right]. \tag{11}$$

In this case the definition of the functional derivatives with respect to the operator A_μ exhibites no serious difficulty, for the functional derivatives always appear in the S-symbol.

Next, noting that

$$\psi_\alpha(x)\bar{\psi}_\beta(y)=\frac{1}{2}S_{F\alpha\beta}(x-y) \tag{12}$$

and definig the operator

$$\Sigma[\sigma]=\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\; S_{F\alpha\beta}(x-y)\frac{\delta^2}{\delta\psi_\alpha(x)\delta\bar{\psi}_\beta(y)}, \tag{13}$$

we can treat the first T-product (with respect to the electron field) in the entirely similar manner with only exception of precaution to the sign. Combinations such as $\bar{\psi}(x)\psi(x)$ or $j_\mu(x)$ commute each other in the S-symbol not changing its sign. Hence we can

always arrange $\psi(x)$ and $\bar{\psi}(y)$ in such order as they stand; $\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)$. After $\Sigma[\sigma]$ operates, this factor changes into $\bar{\psi}(x)S_F\psi(y)$ which again contains a $\bar{\psi}$ and a ψ and commute with j_μ or each other in the S-symbol, and so on. Thus we need only define that the factor arising by the differentiation with respect to $\psi(x)$ must be written always on the left side of the factor arising by the differentiation of $\bar{\psi}(y)$. The situation may be more explicitly manifested* by writing (13) as

$$\Sigma[\sigma]=\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\frac{\delta}{\delta\psi(x)}S_F(x-y)\frac{\delta}{\delta\bar{\psi}(y)}. \tag{14}$$

Eventually $S[\sigma]$ can be expressed as an S-product**:

$$S[\sigma]=:e^{\Sigma[\sigma]}e^{\Delta[\sigma]}\sum_{n=0}^{\infty}\frac{i^n}{n!}\left[\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx\right]^n:$$

$$=:e^{\Sigma[\sigma]}e^{\Delta[\sigma]}\,e^{i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx}:. \tag{15}$$

It is the main feature of $S[\sigma]$ written in the form (15) that every $A_\mu(x)$ in the Eq. (15) are commutable and $\bar{\psi}(x)$, $\psi(x)$ are anticommutable each other, resyectively and commutation relations which should be appear are all involved in the operator $e^{\Sigma[\sigma]}$ and $e^{\Delta[\sigma]}$. Therefore it will be very opportune for the discussion of many problems about S-matrix. The operator $e^{\Sigma[\sigma]}$ and $e^{\Delta[\sigma]}$ play the rôle to combine the fundamental graph (Fig. 1) representing $H_i(x)=-ij_\mu(x)A_\mu(x)$ in every possible ways.

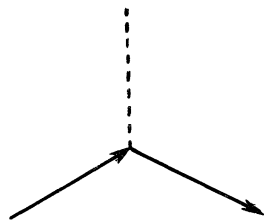

Fig. 1.

We can easily shown that even if Hamiltonian density in consideration contains not only $H_i(x)$ (4) but the mass renormalization term,

$$H_s(x)=-\delta mc^2:\bar{\psi}(x)\psi(x):$$

or the Gupta's term[4] or both, we can write similarly

$$S[\sigma]=:e^{\Sigma[\sigma]}e^{\Delta[\sigma]}e^{-i\int_{-\infty}^{\sigma}H(x)dx}: \tag{16}$$

But the latter cases need some remarks.

§ 3. Remarks on the Gupta's term

We can automatically take into account the effect of the charge renormalization by adding the Gupta's term[4] to the Hamiltonan density $H(x)$. Since this term contains the time derivatives of the potential $A_\mu(x)$, a normal dependent term necessarily appears in the Hamiltonian. Further we would have to modify $\Delta[\sigma]$ because of the presence of $F_{\mu\nu}(x)$ in $H(x)$. If we use nevertheless the same expression for $\Delta[\sigma]$, the result obtained by

* The definition $\delta^2/\delta\,\psi(x)\delta\bar{\psi}(y)=-\delta^2/\delta\psi(y)\delta\bar{\psi}(x)$, may be also available.

** In the Eq. (15) the Eqs. of motion for the field variables should be taken in consideration after $e^{\Sigma[\sigma]}$ and $e^{\Delta[\sigma]}$ have already operated.

applying $\Delta[\sigma]$ can be summarised as follows: First we rewrite $F_{\rho\sigma}(x)=(\delta_{\mu\rho}\frac{\partial}{\partial x_\sigma}-\delta_{\mu\sigma}\frac{\partial}{\partial x_\rho})$ $A_\mu(x)$. Second when $\Delta[\sigma]$ is applied, $F_{\rho\sigma}(x)$ differentiated functionally with respect to $A_\mu(x)$ bequeathes $(\delta_{\mu\rho}\frac{\partial}{\partial x_\sigma}-\delta_{\mu\sigma}\frac{\partial}{\partial x_\rho})$ and this always operate to $D_F(x-y)$ contained in $\Delta[\sigma]$. For example,

$$\begin{aligned}
&\Delta[\sigma]\ \exp\left[i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx+\frac{if}{4}\int_{-\infty}^{\sigma}F_{\rho\sigma}(x)F_{\rho\sigma}(x)dx\right]\\
&=\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\left\{ij_\mu(x)+\frac{if}{2}F_{\rho\sigma}(x)\left(\delta_{\mu\rho}\frac{\partial}{\partial x_\sigma}-\delta_{\mu\sigma}\frac{\partial}{\partial x_\rho}\right)\right\}D_F(x-y)\times\\
&\times\left\{ij_\mu(y)+\frac{if}{2}\left(\delta_{\mu\lambda}\frac{\overleftarrow{\partial}}{\partial y_\kappa}-\delta_{\mu\kappa}\frac{\overleftarrow{\partial}}{\partial y_\lambda}\right)F_{\lambda\kappa}(y)\right\}\times\\
&\times\exp\left[i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx+\frac{if}{4}\int_{-\infty}^{\sigma}F_{\rho\sigma}(x)F_{\rho\sigma}(x)\ dx\right]\\
&=\left[-\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\ j_\mu(x)D_F(x-y)\ j_\mu(y)+\right.\\
&\quad+f\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\ j_\mu(x)\frac{\partial}{\partial y_\rho}D_F(x-y)F_{\rho\mu}(y)\\
&\quad\left.-\frac{f^2}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\ F_{\rho\mu}(x)\frac{\partial^2}{\partial x_\rho\,\partial y_\lambda}D_F(x-y)F_{\lambda\mu}(y)\right]\times\\
&\quad\times\exp\left[-i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx+\frac{if}{4}\int^{\sigma}F_{\rho\sigma}\,F_{\rho\sigma}dx\right].
\end{aligned}\tag{17}$$

If Δ's no more operate from the left, the second and the third term in the first factor vanishes due to the Eq. of motion (cf. the foot note on p. 804): The result above mentioned amounts to the difinitions,

$$A_\mu^{\cdot}(x)F_{\rho\sigma}^{\cdot}(y)=\left(\delta_{\mu\rho}\frac{\partial}{\partial y_\sigma}-\delta_{\mu\sigma}\frac{\partial}{\partial y_\rho}\right)D_F(x-y),\tag{18}$$

$$F_{\mu\nu}^{\cdot}(x)F_{\rho\sigma}^{\cdot}(y)=\left(\delta_{\alpha\mu}\frac{\partial}{\partial x_\nu}-\delta_{\alpha\nu}\frac{\partial}{\partial x_\mu}\right)\left(\delta_{\alpha\rho}\frac{\partial}{\partial y_\sigma}-\delta_{\alpha\sigma}\frac{\partial}{\partial x_\rho}\right)D_F(x-y).\tag{19}$$

The Eqs. (18) and (19) are correct except that to the Eq. (19) should be added a singular term containing a δ-function. The normal dependent term plays the part of compensating the singular expression. Eventually we need only retain the first term of the Gupta's term,

$$H_p=-\frac{1}{2}fF_{\mu\nu}(x)\ F_{\mu\nu}(x)$$

with $\Delta[\sigma]$ not modified.

§ 4. The diffusion equations

Now we replace the charge e by $\varepsilon = ab^{\frac{1}{2}}e$ and rewrite

$$\sqrt{b}\,A_\mu(x)\to A_\mu(x),\quad \sqrt{a}\,\bar{\psi}(x)\to\bar{\psi}(x),\quad \sqrt{a}\,\psi(x)\to\psi(x)$$

Then we have instead of (8) and (12)

$$A_\mu^{\cdot}(x)A_\nu^{\cdot}(y)=\frac{1}{2}b\,\delta_{\mu\nu}\,D_F(x-y) \tag{20}$$

$$\psi_\alpha^{\cdot}(x)\,\bar{\psi}_\beta^{\cdot}(y)=\frac{1}{2}a\,S_{F\alpha\beta}\,(x-y) \tag{21}$$

and instead of $S[\sigma]$,

$$S[a,b;\sigma]=:e^{a\Sigma[\sigma]}e^{b\Delta[\sigma]}e^{-i\int_{-\infty}^{\sigma}H(x)dx}:. \tag{22}$$

Let $\boldsymbol{A}_\mu(x)$ be an arbitrary c-number function and $\bar{\boldsymbol{\psi}}(x)$, $\boldsymbol{\psi}(x)$ be a set of an arbitrary c-number spinor functions and $\bar{\Delta}[\sigma]$ and $\bar{\Sigma}[\sigma]$ be the same operators as $\Delta[\sigma]$ and $\Sigma[\sigma]$ replaced A_μ and $\bar{\psi}$, ψ by $\boldsymbol{A}_\mu$ and $\bar{\boldsymbol{\psi}}$, $\boldsymbol{\psi}$, and

$$\boldsymbol{S}[a,b;\sigma]=e^{a\bar{\Sigma}[\sigma]}e^{b\bar{\Delta}[\sigma]}e^{-i\int_{-\infty}^{\sigma}\boldsymbol{H}(x)dx}. \tag{23}$$

$S[a,b;\sigma]$ can be obtained from $\boldsymbol{S}[a,b;\sigma]$ by replacing $\boldsymbol{A}_\mu$, $\bar{\boldsymbol{\psi}}$, $\boldsymbol{\psi}$ by A_μ, $\bar{\psi}$, ψ and taking its S-product. One can readily see that

$$\frac{\partial\boldsymbol{S}[a,b;\sigma]}{\partial a}=\bar{\Sigma}[\sigma]\boldsymbol{S}[a,b;\sigma], \tag{24}$$

$$\frac{\partial\boldsymbol{S}[a,b;\sigma]}{\partial b}=\bar{\Delta}[\sigma]\boldsymbol{S}[a,b;\sigma]. \tag{25}$$

It might seem paradoxical at first sight the S-matrix involving divergences (15) and the renormalized (16) satisfy the same diffusion equations. This is because the initial conditions at $a=0$ and $b=0$ differ in the both cases. The Eqs. (24) and (25) correspond to the diffusion equations obtained by Tomonaga and Fukuda[5], but in their work above consideration about the difference $S[a,b;\sigma]$ and between $\boldsymbol{S}[a,b;\sigma]$ was made only implicitly.

Next, since

$$\begin{aligned}&\Delta[\sigma]e^{i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx}=\\ &=-\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\,j_\mu(x)D_F(x-y)\,j_\mu(y)e^{i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu dx},\end{aligned} \tag{26}$$

operating $\Delta[\sigma]$ again, we obtain

$$\begin{aligned}&\Delta^2[\sigma]e^{i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx}\\ &=\left\{-\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\,j_\mu(x)D_F(x-y)j_\mu(y)\right\}^2e^{i\int_{-\infty}^{\sigma}j_\mu(x)A_\mu(x)dx},\end{aligned}$$

and similarly in general,

$$\Delta^n[\sigma]e^{i\int_{-\infty}^{\sigma} j_\mu(x)A_\mu(x)dx} =$$
$$=\left\{-\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\, j_\mu(x)D_F(x-y)j_\mu(y)\right\}^n e^{i\int_{-\infty}^{\sigma} j_\mu(x)A_\mu(x)dx}.$$

Thus we have

$$S[a, b; \sigma]=:e^{a\Sigma[\sigma]}e^{-\frac{1}{2}\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\, j_\mu(x)D_F(x-y)\, j_\mu(y)}\times$$
$$e^{i\int_{-\infty}^{\sigma} j_\mu(x)A_\mu(x)dx}:. \tag{27}$$

In the Eq. (27) there is no more $\Delta[\sigma]$ contained, so that we may replace $A_\mu(x)$ by $\mathfrak{A}_\mu(x)$ which denotes only real field (and external field which we have hitherto omitted out of our consideration) and may be treated as if it was c-number function*. Returing back S-Product into T-Product we obtain

$$S[a, b; \sigma]=T(e^{-\frac{1}{2}b\int_{-\infty}^{\sigma}dx\int_{-\infty}^{\sigma}dy\, j_\mu(x)D_F(x-y)j_\mu(y)}e^{i\int_{-\infty}^{\sigma} j_\mu(x)\mathfrak{A}_\mu(x)dx}). \tag{28}$$

Regarding the time as a disentangling parameter and remembering that $D_F(x)$ is equal to $-\frac{i}{2\pi}\delta_+(s_{12}^2)$, we find that the Eq. (28) is identical to that obtained by Feynman $(\sigma\to\infty)$. The same result was obtained by Glauber[6] but in the case of c-number $j_\mu(x)$. The Eq. (25) can easily afford to derive the Feynman's diffusion equation**

$$\frac{\partial S}{\partial b}=-\frac{1}{2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\; D_F(x-y)\frac{\delta^2 S}{\delta\mathfrak{A}_\mu(x)\delta\mathfrak{A}_\mu(y)},$$

which is essentially the same as the Eq. (25).

In conclusion the author wishes to express his hearty thanks to Prof. S. Ozaki for his helpful discussions. He is also very grateful to the Press the Chubu Nippon for the financial aid.

* It operates to the state vector and creates or annihilates a photon.

** In the Eq. (26) S should be replaced by $\boldsymbol{S}$ in the case of not using the Feynman's operator calculus

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