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CLIQUE POLYNOMIALS OF THRESHOLD GRAPHS

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We give a characterization of the clique polynomials of threshold graphs and show that every threshold graph is completely determined by its clique polynomial within the class of threshold graphs.

1. INTRODUCTION

Throughout this paper we assume that $G = (V, E)$ is a simple graph. A k-clique of G is a complete subgraph of G with k vertices. For $S \subset V$, $\langle S \rangle$ denotes the graph induced by the vertices in S, while $G - S$ denotes the graph induced by vertices in $V - S$. If $S = \{u\}$ we shall write $G - u$ for $G - S$. By $N(u)$ we denote the graph induced by the neighbors of the vertex u. The join of graphs G and H is denoted by $G + H$.

Many kinds of graph polynomials have been introduced and extensively studied in the literature. Clique polynomials have been first introduced in [**2**], and then indepedently in [**1**].

Definition 1.1. *The clique polynomial of a graph G, denoted by $C(G;x)$ is defined by*

$$C(G;x) = \sum_{k=0}^{n} a_k(G)x^k,$$

where $a_0(G) = 1$ and $a_k(G)$ is the number of k-cliques of G. For $V(G) = \emptyset$ we define $C(G;x) = 1$.

The following result is an easy consequence of the above definition.

Theorem 1.2. *Let G and H be vertex-disjoint graphs and u an arbitrary vertex of G. Then*

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(i) $C(G \cup H; x) = C(G; x) + C(H; x) - 1;$

(ii) $C(G + H; x) = C(G; x) \cdot C(H; x);$

(iii) $C(G; x) = C(G - u; x) + x \cdot C(N(u); x).$

2. THRESHOLD GRAPHS

In [**1**], C. Hoede and X. Li posed the question of characterizing clique polynomials. Here we shall give a characterization of the clique polynomials of threshold graphs. We shall also show that non-isomorphic threshold graphs cannot have the same clique polynomial.

Definition 2.1. *Graph $G = (V, E)$ with n vertices is called threshold graph if its vertices can be ordered in the sequence $v_1, v_2, \ldots, v_n$ such that for the vertex v_i $(i = 2, \ldots, n)$ one of the following conditions holds:*

A: v_i is adjacent to all vertices v_j $(j = 1, \ldots, i-1)$;

N: v_i is not adjacent to any vertex v_j $(j = 1, \ldots, i-1)$.

The sequence $v_1, v_2, \ldots, v_n$ will be called the threshold sequence of the graph G. We shall say that the vertices u and v of G have the same character if in the threshold sequence of G the same condition (A or N) holds for both u and v.

Theorem 2.2. *If G and H are threshold graphs, then G is isomorphic to H if and only if $C(G; x) = C(H; x)$.*

Theorem 2.3. *The polynomial $P(x)$ with integral non-negative coefficients,* $\deg P = n \geq 1$, *is the clique polynomial of some threshold graph if and only if $P(0) = 1$ and*

$$\sum_{i=0}^{r} \frac{P^{(i)}(-1)}{i!} \leq 0 \qquad (r = 0, 1, \ldots, n-1),$$

where $P^{(i)}(x)$ denotes the i^{th} derivative of $P(x)$.

3. PROOFS

Proof of Theorem 2.2. Suppose that there exist non-isomorphic threshold graphs G and H such that $C(G; x) = C(H; x)$. We may further suppose that the sum of the numbers of their vertices $|G| + |H|$ is minimal. Let $g_1, g_2, \ldots, g_{|G|}$ and $h_1, h_2, \ldots, h_{|H|}$ be the corresponding threshold sequences of these graphs. If vertices $g_{|G|}$ and $h_{|H|}$ have the same character then $G - g_{|G|}$ and $H - h_{|H|}$ would be non-isomorphic threshold graphs with the same clique polynomial, giving contradiction to the minimality of the choice of G and H.

Therefore, we may suppose that $g_{|G|}$ is adjacent to vertices $g_1, \ldots, g_{|G|-1}$ of G and that for some $k, k \leq |H|$, h_k is adjacent to vertices $h_1, \ldots, h_{k-1}$, while $h_{k+1}, \ldots, h_{|H|}$ are isolated vertices of H. Then

$$C(G;x) = (1+x) \cdot C(G - g_{|G|}; x)$$

and $(1+x) \mid C(G;x)$, while

$$C(H;x) = (|H| - k) \cdot x + (1+x) \cdot C(\langle h_1, \ldots, h_{k-1} \rangle; x),$$

hence $C(H;x)$ is not divisible by $1+x$ and we have a contradiction. □

Proof of Theorem 2.3. ($\Rightarrow$) Suppose that $P(x)$ is the clique polynomial of threshold graph G with k vertices. We prove theorem by induction on k.

If $k=1$ then $P(x) = x+1$, and clearly $P(x)$ satisfies given conditions.

If $k \geq 1$ then let $v_1, \ldots, v_k$ be the threshold sequence of G. If v_k is adjacent to vertices $v_1, \ldots, v_{k-1}$, then $P(x) = (1+x) \cdot C(G - v_k; x)$. From this, it follows that $P(0) = C(G - v_k; 0) = 1$, $P(-1) = 0$, and

$$P^{(i)}(x) = (1+x) \cdot C^{(i)}(G - v_k; x) + i \cdot C^{(i-1)}(G - v_k; x),$$

hence $P^{(i)}(-1) = i \cdot C^{(i-1)}(G - v_k; -1)$, for $i \geq 1$. Therefore

$$\begin{aligned} \sum_{i=0}^{r} \frac{P^{(i)}(-1)}{i!} &= \frac{P(-1)}{0!} + \sum_{i=1}^{r} \frac{i \cdot C^{(i-1)}(G - v_k; -1)}{i!} \\ &= \sum_{i=0}^{r-1} \frac{C^{(i)}(G - v_k; -1)}{i!} \leq 0 \quad (r = 1, 2, \ldots, n-1). \end{aligned}$$

Similarly, if v_k is an isolated vertex of G, then $P(x) = x + C(G - v_k; x)$, and $P(0) = C(G - v_k; 0) = 1$, while

$$\begin{aligned} \sum_{i=0}^{r} \frac{P^{(i)}(-1)}{i!} &= -1 + \frac{C(G - v_k; -1)}{0!} + 1 + \frac{C'(G - v_k; -1)}{1!} \\ &\qquad + \sum_{i=2}^{r} \frac{C^{(i)}(G - v_k; -1)}{i!} \\ &= \sum_{i=0}^{r} \frac{C^{(i)}(G - v_k; -1)}{i!} \leq 0 \quad (r = 0, 1, \ldots, n-1). \end{aligned}$$

($\Leftarrow$) We shall use induction on the degree $\deg P = n$. If $n = 1$ then $P(x) = ax + b$, and from $P(0) = 1$ it follows that $b = 1$, hence $P(x)$ is clique polynomial of the graph consisting of a isolated vertices.

Let

$$P(x) = a_n(x+1)^n + a_{n-1}(x+1)^{n-1} + \ldots + a_1(x+1) + a_0.$$

Then $a_k = \dfrac{P^{(k)}(-1)}{k!}$ for $k \geq 0$, and the conditions of the theorem now become $\sum\limits_{i=0}^{r} a_i \leq 0$, for all $r = 0, 1, \ldots, n-1$. Now

$$P(x) = (x+1)\big(a_n(x+1)^{n-1} + \ldots + a_2(x+1) + a_1 + a_0\big) - a_0 x = (x+1)\cdot Q(x) - a_0 x.$$

Since $Q(x)$ obviously satisfies given conditions and deg $Q < n$, it is the clique polynomial of some threshold graph T. From the conditions of theorem for $r = 0$ it follows that $a_0 \leq 0$ and therefore $P(x)$ is the clique polynomial of the threshold graph formed from T by adding first a vertex adjacent to all vertices of T, and then $-a_0$ isolated vertices. □

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