NON-CLASSICAL DUAL METHODS IN EQUILIBRIUM PROBLEMS FOR THIN ELASTIC PLATES

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Summary

A dual functional is constructed for the Dirichlet problem for plates with transverse shear deformation, which makes it possible to obtain bilateral error estimates for approximate solutions. In the absence of an elastic foundation, the classical dual functional is maximized in spaces of functions that satisfy certain differential restrictions. The non-classical dual functional proposed here is maximized in spaces of functions with no such restrictions, regardless of whether or not an elastic foundation is present. A numerical example is given to illustrate the efficiency of this dual method.

1. Introduction

Elastic plates are important structural elements in modern engineering, finding application in a wide variety of domains that range from building to electronics. The computation of stresses in plates requires the development of approximate methods with sharp error estimates. In what follows we propose such a method and illustrate its efficiency. The technique is based on the simultaneous solution of two extremal problems, namely the direct (original) one and its dual. This type of approach has been discussed extensively in the literature; see, for example, (**1** to **6**).

Kirchhoff's original model for bending of plates (**7**) has subsequently been refined to take account of transverse shear deformation. A detailed mathematical analysis of the new theory can be found in (**8**). The existence of weak solutions to this model was proved in (**9**, **10**). In this paper we construct and analyse dual variational problems, which generate bilateral estimates for the solutions in the sense explained below. The classical approach to this task, presented in (**11**), is very suitable when the plate lies on an elastic foundation. In the absence of such a foundation, the classical method fails and we need to maximize the intervening dual functionals in spaces of functions that

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satisfy a number of differential restrictions in their domain. This makes the numerical realization of the corresponding algorithms very complicated. Here we indicate a new, non-classical, scheme for obtaining dual functionals whose maxima are sought in function spaces with no restrictions. This scheme is based on ideas developed in (**6**) for the stationary heat equation.

Dual methods are more difficult to formulate in the case of large (nonlinear) deformation models, where the total potential of an elastic plate is generally non-convex. Advances in that direction can be found, for example, in (**12** to **14**).

2. Prerequisites

Consider a thin elastic plate that occupies a domain $\bar{S} \times [-h_0/2, h_0/2] \subset \mathbb{R}^3$, where S is a bounded domain in $\mathbb{R}^2$ and h_0, with $0 < h_0 = \text{const} \ll \operatorname{diam} S$, is called the thickness. The plate may, or may not, lie on an elastic foundation that opposes its bending. Let $v(x') = \big(v_1(x'), v_2(x'), v_3(x')\big)^T$ be the displacement vector at a point $x' = (x, x_3)$, where $x = (x_1, x_2)$ and the superscript T denotes matrix transposition. In the transverse shear deformation model investigated in (**8**) it is assumed that

$$v_\alpha(x') = x_3 u_\alpha(x), \quad \alpha = 1, 2, \quad v_3(x') = u_3(x).$$

In what follows, for a three-component vector-valued function $p = (p_1, p_2, p_3)^T$ we write $\bar{p} = (p_1, p_2)^T$. The vector field $u = (\bar{u}^T, u_3)^T$ satisfies the equilibrium equation

$$Au + Ku = q \quad \text{in } S, \tag{2.1}$$

where q is a combination of the forces and moments acting on the plate and its faces, A is the (3×3)-matrix differential operator

$$A = \begin{pmatrix} -h^2\mu\Delta - h^2(\lambda+\mu)\partial_1^2 + \mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1 \\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta - h^2(\lambda+\mu)\partial_2^2 + \mu & \mu\partial_2 \\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta \end{pmatrix},$$

$\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2$, λ and μ are the Lamé constants of the material, and $h^2 = h_0^2/12$.

The interaction of the plate with the elastic foundation is described by a (3×3)-matrix K of the form

$$K = \begin{pmatrix} \bar{K} & 0 \\ 0 & k_{33} \end{pmatrix},$$

where $k_{33} \geqslant 0$ and $\bar{K}$ is a symmetric, positive semi-definite (2×2)-matrix. If there is no elastic foundation, then $K = 0$.

Below we assume that the boundary ∂S of S is a simple, closed, piecewise smooth curve. For simplicity, we restrict our attention to the Dirichlet problem (D), which, in its classical formulation, consists in finding $u \in C^2(S) \cap C(\bar{S})$ such that

$$Au + Ku = q \quad \text{in } S, \quad u = f \quad \text{on } \partial S, \tag{2.2}$$

where q and f are given functions.

Our analysis makes use only of real vector-valued functions. For convenience, we use the same notation for spaces, inner products, and norms of such functions as for scalar functions. The inner product and norm in $L^2(S)$ are denoted by $(\cdot\,,\cdot)_{0;S}$ and $\|\cdot\|_{0;S}$; if $S = \mathbb{R}^2$, then we write $(\cdot\,,\cdot)_0$ and $\|\cdot\|_0$, respectively.

Let $H_m(\mathbb{R}^2)$, $m \in \mathbb{R}$, be the standard Sobolev space (**7**) of three-component distributions $u \in \mathcal{S}'(\mathbb{R}^2)$ with norm

$$\|u\|_m^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^m |\tilde{u}(\xi)|^2 \, d\xi,$$

where $\tilde{u}(\xi)$ is the Fourier transform of $u(x)$. In what follows, equivalent norms are denoted by the same symbol. For example, the norm of $u \in H_1(\mathbb{R}^2)$ can also be defined by

$$\|u\|_1^2 = \|u\|_0^2 + \sum_{i=1}^{3} \|\nabla u_i\|_0^2.$$

For any $m \in \mathbb{R}$, the spaces $H_m(\mathbb{R}^2)$ and $H_{-m}(\mathbb{R}^2)$ are dual with respect to the duality generated by the inner product in $L^2(\mathbb{R}^2)$ (**7**).

For $m \in \mathbb{R}$, $\mathring{H}_m(S)$ is the subspace of $H_m(\mathbb{R}^2)$, of all u with $\operatorname{supp} u \subset \bar{S}$. For any $m \in \mathbb{R}$, $H_m(S)$ is the space of the restrictions to S of all $u \in H_m(\mathbb{R}^2)$, and is equipped with the norm

$$\|u\|_{m;S} = \inf_{v \in H_m(\mathbb{R}^2): v|_S = u} \|v\|_m.$$

The norms of $u \in \mathring{H}_1(S)$ and $u \in H_1(S)$ are equivalent to

$$\left\{ \|u\|_{0;S}^2 + \sum_{i=1}^{3} \int_S |\nabla u_i(x)|^2 \, dx \right\}^{1/2}.$$

For any $m \in \mathbb{R}$, $\mathring{H}_m(S)$ and $H_{-m}(S)$ are dual with respect to the duality generated by $(\cdot\,, \cdot)_{0;S}$.

Let $H_m(\partial S)$, $m \in \mathbb{R}$, be the standard Sobolev space of three-component distributions on ∂S (**7**). For any $m \in \mathbb{R}$, $H_m(\partial S)$ and $H_{-m}(\partial S)$ are dual with respect to the duality generated by the inner product $(\cdot\,, \cdot)_{0;\partial S}$ in $L^2(\partial S)$. The norm on $H_m(\partial S)$ is denoted by $\|\cdot\|_{m;\partial S}$. Let γ be the trace operator that maps $H_1(S)$ continuously onto $H_{1/2}(\partial S)$.

In the sequel we use the bilinear forms

$$a(u, v) = 2 \int_S E(u, v) \, dx, \quad a_K(u, v) = a(u, v) + (Ku, v)_{0;S},$$

where

$$\begin{aligned} 2E(u, v) &= h^2 E_0(u, v) + h^2 \mu (\partial_2 u_1 + \partial_1 u_2)(\partial_2 v_1 + \partial_1 v_2) \\ &\quad + \mu[(u_1 + \partial_1 u_3)(v_1 + \partial_1 v_3) + (u_2 + \partial_2 u_3)(v_2 + \partial_2 v_3)], \\ E_0(u, v) &= (\lambda + 2\mu)\big[(\partial_1 u_1)(\partial_1 v_1) + (\partial_2 u_2)(\partial_2 v_2)\big] + \lambda\big[(\partial_1 u_1)(\partial_2 v_2) + (\partial_2 u_2)(\partial_1 v_1)\big]. \end{aligned}$$

It is not difficult to see (**8**) that if $u \in C^2(S) \cap C^1(\bar{S})$ and $v \in C^1(S) \cap C(\bar{S})$, then

$$(Au, v)_0 = a(u, v) - (Tu, v)_{0;\partial S},$$

where T is the differential operator of the boundary moments and shear force; see (**8**).

The last equality suggests that the variational problem (D) should consist in finding $u \in H_1(S)$ such that

$$a_K(u, v) = (q, v)_{0;S} \quad \forall\, v \in \mathring{H}_1(S), \quad \gamma u = f. \tag{2.3}$$

THEOREM 2.1. *For any $q \in H_{-1}(S)$ and any $f \in H_{1/2}(\partial S)$, problem* (2.3) *has a unique solution $u \in H_1(S)$, which satisfies the estimate*

$$\|u\|_{1;S} \leqslant c(\|q\|_{-1;S} + \|f\|_{1/2;\partial S}).$$

The proof of this assertion can be found in (**9**) for a positive definite matrix K, and in (**10**) for $K = 0$.

Throughout the paper, we denote by the generic symbol c (with or without subscripts) any positive constant occurring in various estimates, which is independent of the functions in these estimates.

Below we intend to develop a technique for constructing a posteriori error estimates for approximate solutions to (2.3). Problem (2.3) with $f = 0$ can also be formulated in an abstract variational form. Let H be a real separable Hilbert space, $b(u, v)$ a bilinear form on H, and $L(v)$ a linear functional on H. We want to find $u \in H$ such that

$$b(u, v) = L(v) \quad \forall v \in H. \tag{2.4}$$

If $b(u, v)$ satisfies the inequalities

$$|b(u, v)| \leqslant c_1\|u\|\|v\| \quad \forall u, v \in H,$$
$$b(u, u) \geqslant c_2\|u\|^2 \quad \forall u \in H,$$

that is, if $b(u, v)$ is continuous and coercive on H, then, by the Lax–Milgram lemma (**1**), equation (2.4) has a unique solution $u_0 \in H$, which minimizes in H the energy functional

$$J^0(u) = \frac{1}{2}b(u, u) - L(u);$$

in other words,

$$J^0(u_0) = \inf_{u \in H} J^0(u).$$

It is easily verified that the energy functional can also be written in the form

$$J^0(u) = \frac{1}{2}\big[b(u - u_0, u - u_0) - b(u_0, u_0)\big],$$

which implies that $J^0(u) - J^0(u_0) = \frac{1}{2}b(u - u_0, u - u_0)$. The coerciveness of $b(u, v)$ now yields

$$\|u - u_0\|^2 \leqslant c_2^{-1}b(u - u_0, u - u_0) = 2c_2^{-1}\big[J^0(u) - J^0(u_0)\big].$$

If $\{u_n\}_{n=1}^{\infty}$ is a sequence of approximate solutions to (2.4) constructed by the Galerkin method, then this last inequality leads to the global error estimates

$$\begin{aligned} &\|u_n - u_0\|^2 \leqslant 2c_2^{-1}\big[J^0(u_n) - J^0(u_0)\big], \\ &b(u - u_0, u - u_0) \leqslant 2\big[J^0(u_n) - J^0(u_0)\big]. \end{aligned} \tag{2.5}$$

However, estimates (2.5) are not used very often since the value of $J^0(u_0)$ is normally unknown. But if we are able to find a number J_0 such that $J_0 \leqslant J^0(u_0)$, then (2.5) can be replaced by the much more convenient pair

$$\begin{aligned} &\|u_n - u_0\| \leqslant \big[2c_2^{-1}\big(J^0(u_n) - J_0\big)\big]^{1/2}, \\ &b(u - u_0, u - u_0) \leqslant 2\big[J^0(u_n) - J_0\big]. \end{aligned} \tag{2.6}$$

How useful estimates (2.6) are depends on the magnitude of $J^0(u_0) - J_0$; to minimize this number, we aim to produce a scheme for computing a numerical sequence $\{J_n\}_{n=1}^{\infty}$ such that $J_n \leqslant J^0(u_0)$ and $\lim_{n\to\infty} J_n = J^0(u_0)$. In problem (D) we then define a dual functional for $J^0(u)$, whose maximum on the appropriate function space coincides with $J^0(u_0)$. If such a functional has been obtained, we can easily construct the desired sequence $\{J_n\}_{n=1}^{\infty}$.

3. The dual functional problem

First, we consider the simpler problem where $f = 0$. In this case, (D) reduces to problem (D_0) of finding $u \in \mathring{H}_1(S)$ such that

$$a_K(u, v) = (q, v)_{0;S} \quad \forall v \in \mathring{H}_1(S). \tag{3.1}$$

We introduce the matrix differential operators

$$\operatorname{Div} u = \begin{pmatrix} \partial_1 u_1 + \partial_2 u_3 \\ \partial_2 u_2 + \partial_1 u_3 \end{pmatrix}, \quad \operatorname{Def} \bar{u} = \begin{pmatrix} \partial_1 u_1 \\ \partial_2 u_2 \\ \partial_2 u_1 + \partial_1 u_2 \end{pmatrix}$$

and the matrix

$$R = h^2 \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

The classical version of (D_0) can now be written in the equivalent form

$$\begin{aligned} -\operatorname{Div}(R \operatorname{Def} \bar{u}) + \mu(\bar{u} + \nabla u_3) + \bar{K}\bar{u} &= \bar{q} \quad \text{in } S, \\ -\mu \operatorname{div}(\bar{u} + \nabla u_3) + k_{33} u_3 &= q_3 \quad \text{in } S, \\ u &= 0 \quad \text{on } \partial S, \end{aligned}$$

where $u \in C^2(S) \cap C(\bar{S})$ and $q = (\bar{q}^T, q_3)^T$, $\bar{q} = (q_1, q_2)^T$. In this new setting, $a(u, v)$ becomes

$$a(u, v) = (R \operatorname{Def} \bar{u}, \operatorname{Def} \bar{v})_{0;S} + \mu(\bar{u} + \nabla u_3, \bar{v} + \nabla v_3)_{0;S}.$$

As is well known (**2**), the weak solution u^* of (3.1) minimizes in $\mathring{H}_1(S)$ the energy functional

$$J^0(u) = \frac{1}{2} a_K(u, u) - (q, u)_{0;S}.$$

We denote by $H(\overline{\operatorname{Div}}, S)$ the space of all five-component vector-valued functions $\theta = (\zeta^T, \eta^T)^T$, $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T$, $\eta = (\eta_1, \eta_2)^T$, in S such that $\zeta,\ \eta,\ \operatorname{Div}\zeta,\ \operatorname{div}\eta \in L^2(S)$. For $\theta,\ \theta' \in H(\overline{\operatorname{Div}}, S)$, we define the inner product and norm

$$\begin{aligned} (\theta, \theta')_{\overline{\operatorname{Div}},S} &= (\zeta, \zeta')_{0;S} + (\eta, \eta')_{0;S} + (\operatorname{Div}\zeta, \operatorname{Div}\zeta')_{0;S} + (\operatorname{div}\eta, \operatorname{div}\eta')_{0;S}, \\ \|\theta\|^2_{\overline{\operatorname{Div}},S} &= (\theta, \theta)_{\overline{\operatorname{Div}},S}. \end{aligned}$$

It is clear that $H(\overline{\operatorname{Div}}, S)$ is a Hilbert space.

Consider the set

$$\mathcal{U} = \{\upsilon = (\varphi^T, \psi^T)^T : \varphi,\ \psi,\ \mathrm{Div}\,\varphi + \bar{q},\ \mathrm{div}\,\psi + q_3 \in L^2(S)\}.$$

If $u^* \in \mathring{H}_1(S)$ is the weak solution of (D_0), we write

$$\varphi^* = R\ \mathrm{Def}\,\bar{u}^*, \quad \psi^* = \mu(\bar{u}^* + \nabla u_3^*), \quad \upsilon^* = (\varphi^{*T}, \psi^{*T})^T. \tag{3.2}$$

It is obvious that $\varphi^*,\ \psi^* \in L^2(S)$ and that

$$\mathrm{Div}\,\varphi^* + \bar{q} = \mathrm{Div}(R\ \mathrm{Def}\,\bar{u}^*) + \bar{q} = \mu(\bar{u}^* + \nabla u_3^*) + \bar{K}\bar{u}^* \in L^2(S),$$
$$\mathrm{div}\,\psi^* + q_3 = k_{33}u_3 \in L^2(S),$$

which implies that $\upsilon^* \in \mathcal{U}$. For $\upsilon = (\varphi^T, \psi^T)^T \in \mathcal{U}$, we define the five-component vector-valued function $\theta = (\zeta^T, \eta^T)^T$, where $\theta = \upsilon - \upsilon^*$, $\zeta = \varphi - \varphi^*$, and $\eta = \psi - \psi^*$. We see that $\zeta,\ \eta \in L^2(S)$ and that

$$\mathrm{Div}\,\zeta = \mathrm{Div}\,\varphi - \mathrm{Div}\,\varphi^* = (\mathrm{Div}\,\varphi + \bar{q}) - (\mathrm{Div}\,\varphi^* + \bar{q}) \in L^2(S),$$
$$\mathrm{div}\,\eta = \mathrm{div}\,\psi - \mathrm{div}\,\psi^* = (\mathrm{div}\,\psi + q_3) - (\mathrm{div}\,\psi^* + q_3) \in L^2(S),$$

from which we conclude that $\theta \in H(\overline{\mathrm{Div}}, S)$. Conversely, if $\theta \in H(\overline{\mathrm{Div}}, S)$, then $\upsilon^* + \theta \in \mathcal{U}$, so

$$\mathcal{U} = \{\upsilon^* + \theta : \theta \in H(\overline{\mathrm{Div}}, S)\}.$$

For a positive definite matrix K, in (**11**) we defined on $\mathcal{U}$ the functional

$$J_0(\upsilon) = -\frac{1}{2}\big[(R^{-1}\varphi, \varphi)_{0;S} + \mu^{-1}\|\psi\|^2_{0;S} + \big(\bar{K}^{-1}(\bar{q} + \mathrm{Div}\,\varphi - \psi), \bar{q} + \mathrm{Div}\,\varphi - \psi\big)_{0;S}$$
$$+ \big(k_{33}^{-1}(q_3 + \mathrm{div}\,\psi), q_3 + \mathrm{div}\,\psi\big)_{0;S}\big]$$

and proved the following assertion.

THEOREM 3.1.

$$\inf_{u \in \mathring{H}_1(S)} J^0(u) = J^0(u^*) = J_0(\upsilon^*) = \sup_{\upsilon \in \mathcal{U}} J_0(\upsilon).$$

We intend to generalize Theorem 3.1 to the case where K is only positive semi-definite, as we have assumed in our problem, and, therefore, K^{-1} may not exist.

Let

$$R^{-1} = \begin{pmatrix} \bar{R}^{-1} & 0 \\ 0 & r_{33}^{-1} \end{pmatrix}, \quad \bar{R}^{-1} = \begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{21} & \tilde{r}_{22} \end{pmatrix},$$

and let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be functions in $C^1(\bar{S})$ such that the matrix

$$K_{(\alpha,\beta)} = K - \begin{pmatrix} -\partial_1\beta_1 + \tilde{r}_{11}\beta_1^2 & \tilde{r}_{12}\beta_1\beta_2 & \alpha_1 \\ \tilde{r}_{21}\beta_1\beta_2 & -\partial_2\beta_2 + \tilde{r}_{22}\beta_2^2 & \alpha_2 \\ \alpha_1 & \alpha_2 & -\mathrm{div}\,\alpha + \mu^{-1}(\alpha_1^2 + \alpha_2^2) \end{pmatrix}$$

is positive definite, uniformly in S (see Remark 3.5 below). For any $\upsilon = (\varphi^T, \psi^T)^T$, we write

$$q_{(\alpha,\beta)}(\upsilon) = q + \begin{pmatrix} \mathrm{Div}\,\varphi - \psi - \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}\bar{R}^{-1}\bar{\varphi} \\ \mathrm{div}\,\psi - \mu^{-1}(\alpha_1\psi_1 + \alpha_2\psi_2) \end{pmatrix}.$$

We also write

$$\varphi^*_{(\alpha,\beta)} = \varphi^* + (\beta_1 u_1^*, \beta_2 u_2^*, 0)^T, \quad \psi^*_{(\alpha,\beta)} = \psi^* + \alpha u_3^*, \quad \upsilon^*_{(\alpha,\beta)} = (\varphi^{*T}_{(\alpha,\beta)}, \psi^{*T}_{(\alpha,\beta)})^T, \tag{3.3}$$

with φ^* and ψ^* given by (3.2). Finally, we define the dual functional J_0 of J^0, which acts on $\upsilon = (\varphi^T, \psi^T)^T \in \mathcal{U}$ according to the formula

$$J_0(\upsilon) = -\frac{1}{2}\{(R^{-1}\varphi, \varphi)_{0;S} + \mu^{-1}\|\psi\|^2_{0;S} + \big(K^{-1}_{(\alpha,\beta)} q_{(\alpha,\beta)}(\upsilon), q_{(\alpha,\beta)}(\upsilon)\big)_{0;S}\}.$$

THEOREM 3.2.

$$\inf_{u\in \mathring{H}_1(S)} J^0(u) = J^0(u^*) = J_0(\upsilon^*_{(\alpha,\beta)}) = \sup_{\upsilon\in\mathcal{U}} J_0(\upsilon). \tag{3.4}$$

Proof. It was mentioned above that the first equality in (3.4) holds. Also, it is evident that $\mathcal{U} = \{\upsilon_{(\alpha,\beta)} + \theta\colon \theta \in H(\overline{\mathrm{Div}}, S)\}$. We now claim that the maximum of $J_0(\upsilon) = J_0(\upsilon^*_{(\alpha,\beta)} + \theta)$, as υ ranges over $\mathcal{U}$ (or as θ ranges over $H(\overline{\mathrm{Div}}, S)$), occurs when $\theta = 0$. A straightforward calculation shows that

$$u^* = K^{-1}_{(\alpha,\beta)} q_{(\alpha,\beta)}(\upsilon^*_{(\alpha,\beta)}). \tag{3.5}$$

Let $\theta = (\zeta^T, \eta^T)^T$; then

$$q_{(\alpha,\beta)}(\upsilon^*_{(\alpha,\beta)} + \theta) = q_{(\alpha,\beta)}(\upsilon^*_{(\alpha,\beta)}) + q^0_{(\alpha,\beta)}(\theta),$$

where

$$q^0_{(\alpha,\beta)}(\theta) = \begin{pmatrix} \mathrm{Div}\,\zeta - \eta - \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \bar{R}^{-1}\bar{\zeta} \\ \mathrm{div}\,\eta - \mu^{-1}(\alpha_1\eta_1 + \alpha_2\eta_2) \end{pmatrix}.$$

We have

$$\begin{aligned} J_0(\upsilon^*_{(\alpha,\beta)} + \theta) = J_0(\upsilon^*_{(\alpha,\beta)}) + J_0(\theta) - \{&(R^{-1}\varphi^*_{(\alpha,\beta)}, \zeta)_{0;S} + \mu^{-1}(\psi^*_{(\alpha,\beta)}, \eta)_{0;S} \\ &+ \big(u^*, q^0_{(\alpha,\beta)}(\theta)\big)_{0;S}\}. \end{aligned}$$

Given that $u^* \in \mathring{H}_1(S)$ and $C_0^\infty(S)$ is dense in $\mathring{H}_1(S)$ (**7**), we approximate u^* by functions in $C_0^\infty(S)$, integrate by parts, and arrive at

$$(\mathrm{Def}\,\bar{u}^*, \zeta)_{0;S} = -(\bar{u}^*, \mathrm{Div}\,\zeta)_{0;S}, \quad (\nabla u_3^*, \eta)_{0;S} = -(u_3^*, \mathrm{div}\,\eta)_{0;S};$$

hence

$$\begin{aligned} &(R^{-1}\varphi^*_{(\alpha,\beta)}, \zeta)_{0;S} + \mu^{-1}(\psi^*_{(\alpha,\beta)}, \eta)_{0;S} + \big(u^*, q^0_{(\alpha,\beta)}(\theta)\big)_{0;S} \\ &\quad = \big(\mathrm{Def}\,\bar{u}^* + R^{-1}(\beta_1 u_1^*, \beta_2 u_2^*, 0)^T, \zeta\big)_{0;S} + (\bar{u}^* + \nabla u_3^* + \mu^{-1} u_3^* \alpha, \eta)_{0;S} \\ &\qquad + \left(\bar{u}^*, \mathrm{Div}\,\zeta - \eta - \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \bar{R}^{-1}\bar{\zeta}\right)_{0;S} + \big(u_3^*, \mathrm{div}\,\eta - \mu^{-1}(\alpha_1\eta_1 + \alpha_2\eta_2)\big)_{0;S} = 0 \end{aligned}$$

and

$$J_0(\upsilon^*_{(\alpha,\beta)} + \theta) \leqslant J_0(\upsilon^*_{(\alpha,\beta)}) \quad \forall\, \theta \in H(\overline{\mathrm{Div}}, S),$$

with equality holding if and only if $\theta = 0$. Direct calculation now shows that $J_0(v^*_{(\alpha,\beta)}) = J^0(u^*)$, which completes the proof.

Let $\{u_n\}_{n=1}^{\infty} \subset \mathring{H}_1(S)$ be a minimizing sequence for $J^0(u)$ and $\{v_n\}_{n=1}^{\infty} \subset \mathcal{U}$ a maximizing one for $J_0(v)$; then

$$\|u_n - u^*\|_1 \leqslant c[J^0(u_n) - J_0(v_n)]^{1/2} \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

Now consider the general problem (D) with $f \in H_{1/2}(\partial S)$ prescribed on the boundary. We select $F \in H_1(S)$ so that $\gamma F = f$ and seek the solution $u \in H_1(S)$ of (2.3) as $u = F + w$; therefore, $w \in \mathring{H}_1(S)$ satisfies the variational equation

$$a_K(w, v) = (q, v)_{0;S} - a_K(F, v) \quad \forall v \in \mathring{H}_1(S). \tag{3.7}$$

By (2.6), for an approximate solution w_n of (3.7) and $v \in \mathcal{U}$,

$$\|w_n - w^*\|_1 \leqslant c[J^0(w_n) - J_0(v)]^{1/2},$$

where w^* is the exact solution of (3.7). Since $u_n = F + w_n$ and $u^* = F + w^*$ are, respectively, an approximate solution and the exact solution of (D), we have the error estimate

$$\|u_n - u^*\|_1 = \|w_n - w^*\|_1 \leqslant c[J^0(w_n) - J_0(v)]^{1/2}.$$

If $q \in L^2(S)$, then we do not need to construct an extension F of f from ∂S to S to realize this scheme. The exact solution $u^* \in H_1(S)$ of (D) minimizes $J^0(u)$ on the space $\mathcal{U}_f = \{u \in H_1(S)\colon \gamma u = f\}$. For $v \in \mathcal{U}$, we define the functional J_0 by

$$J_0(v) = -\frac{1}{2}\big[(R^{-1}\varphi, \varphi)_{0;S} + \mu^{-1}\|\psi\|^2_{0;S} + \big(K^{-1}_{(\alpha,\beta)} q_{(\alpha,\beta)}(v), q_{(\alpha,\beta)}(v)\big)_{0;S}\big] + (\tau_\nu v,\ f)_{0;\partial S}$$

$$-\frac{1}{2}\big[(\beta_1 f_1^2, \nu_1)_{0;\partial S} + (\beta_2 f_2^2, \nu_2)_{0;\partial S} + (f_3^2, \alpha_1\nu_1 + \alpha_2\nu_2)_{0;\partial S}\big],$$

where $\nu = (\nu_1, \nu_2)^T$ is the outward unit normal to ∂S and

$$\tau_\nu = \begin{pmatrix} \nu_1 & 0 & \nu_2 & 0 & 0 \\ 0 & \nu_2 & \nu_1 & 0 & 0 \\ 0 & 0 & 0 & \nu_1 & \nu_2 \end{pmatrix}.$$

To show that $(\tau_\nu v,\ f)_{0;\partial S}$ is well defined, we need to check that $\tau_\nu v \in H_{-1/2}(\partial S)$. Since $q \in L^2(S)$, it follows that for any $v = (\varphi^T, \psi^T)^T \in \mathcal{U}$, we have $\varphi,\ \psi,\ \operatorname{Div}\varphi,\ \operatorname{div}\psi \in L^2(S)$, which implies that $v \in H(\overline{\operatorname{Div}}, S)$. The question concerning $\tau_\nu v$ is now settled by the next assertion.

LEMMA 3.3. *The mapping $v \mapsto \tau_\nu v$ from $C^1(\bar{S})$ to $C^1(\partial S)$ can be extended to a linear and continuous mapping $\tau_\nu\colon H(\overline{\operatorname{Div}}, S) \to H_{-1/2}(\partial S)$.*

Proof. As in (**7**), let $v = (\varphi^T, \psi^T)^T \in C^1(\bar{S})$ and $f \in H_{1/2}(\partial S)$, and let $F \in H_1(S)$ be such that $\gamma F = f$. Consider the linear functional π defined for $f \in H_{1/2}(\partial S)$ by

$$\pi(f) = (\operatorname{Div}\varphi, \bar{F})_{0;S} + (\varphi, \operatorname{Def}\bar{F})_{0;S} + (\operatorname{div}\psi, F_3)_{0;S} + (\psi, \nabla F_3)_{0;S}.$$

Integrating by parts, we see that if $F \in \mathring{H}_1(S)$, then the right-hand side in the above equality is zero, so the definition of $\pi(f)$ is consistent. Integration by parts also shows that

$$\pi(f) = (\tau_\nu v,\ f)_{0;\partial S}. \tag{3.8}$$

The proof is completed by observing that the obvious inequalities

$$|\pi(f)| \leqslant c\|v\|_{\overline{\mathrm{Div}},S}\|f\|_{1;S} \leqslant c\|v\|_{\overline{\mathrm{Div}},S}\|f\|_{1/2;\partial S}$$

imply that π is continuous on $H_{1/2}(\partial S)$ and that $\|\tau_\nu v\|_{-1/2;\partial S} \leqslant c\|v\|_{\overline{\mathrm{Div}},S}$.

THEOREM 3.4. *If $u^* \in H_1(S)$ is the solution of* (2.3), *$q \in L^2(S)$, and $v^*_{(\alpha,\beta)}$ is defined by* (3.3), *then*

$$\inf_{u\in\mathcal{U}_f} J^0(u) = J^0(u^*) = J_0(v^*_{(\alpha,\beta)}) = \sup_{v\in\mathcal{U}} J_0(v).$$

The proof of this assertion is based on (3.8) and is similar to that of Theorem 3.2.

REMARK 3.5. If K is positive definite, then the function parameters α and β can be set equal to zero. In the general case (when K is merely positive semi-definite), a simple way to construct α and β is to choose them in the form $\alpha = \sigma^2 x$ and $\beta = \sigma x$; then

$$K_{(\alpha,\beta)} = K + \sigma\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\sigma\end{pmatrix} - \sigma^2\begin{pmatrix}\tilde{r}_{11}x_1^2 & \tilde{r}_{12}x_1x_2 & x_1\\ \tilde{r}_{21}x_1x_2 & \tilde{r}_{22}x_2^2 & x_2\\ x_1 & x_2 & \mu^{-1}\sigma^2(x_1^2+x_2^2)\end{pmatrix},$$

and it is clear that, for sufficiently small $\sigma > 0$, the matrix $K_{(\alpha,\beta)}$ is positive definite, uniformly in S. This, however, may lead to some ill-conditioning in the numerical implementation of the scheme.

REMARK 3.6. Dual functionals can also be constructed in certain mixed problems, in particular, in problems where the boundary includes segments of the coordinate axes, on which Neumann-type conditions are prescribed, with Dirichlet data on the rest of the boundary.

4. Numerical example

We illustrate the usefulness of the dual methods by means of numerical results in the case of a square steel floor clamped around its edge. Specifically, the values of the various physical parameters are, in kg, cm and sec,

$$\lambda = 1{\cdot}141 \times 10^6, \quad \mu = 8{\cdot}262 \times 10^5, \quad K = 0, \quad h_0 = 7,$$

and $S = [-100, 100] \times [-100, 100]$.

The direct and dual problems are solved by means of the Galerkin method. In the direct problem, the basis functions are

$$(\chi P_{ij}, 0, 0)^T, \quad (0, \chi P_{ij}, 0)^T, \quad (0, 0, \chi P_{ij})^T, \quad i, j = 0, 1, 2, \ldots,$$

where $\chi(x_1, x_2) = (10^4 - x_1^2)(10^4 - x_2^2)$ vanishes on ∂S and $P_{ij} = x_1^i x_2^j$. The basis functions in the dual problem are constructed from those in the direct problem by means of formulae (3.2) and

(3.3) with

$$\alpha(x_1, x_2) = \tfrac{2}{125}\,(x_1, x_2), \quad \beta(x_1, x_2) = 470(x_1, x_2),$$

for which the matrix $K_{(\alpha,\beta)}$ is positive definite. The terms u_n and v_n of the approximating sequences are computed by means of the basis functions derived as above by taking all polynomials P_{ij} of degrees $i + j = 0, \ldots, n$. The computations, performed with MATHEMATICA, involve some symbolic manipulation and the use of infinite numerical precision in certain parts.

To test the accuracy of the method, first we consider the (physically realistic) problem with the exact solution

$$u_1^* = -10^{-10} x_1 \chi, \quad u_2^* = -10^{-10} x_2 \chi, \quad u_3^* = -10^{-24} \chi^3. \tag{4.1}$$

This solution is used to compute q, which is then used as data in the dual method to recover the function u^*.

Table 1 shows the approximate values of the direct and dual functionals for various degrees n of the polynomials P_{ij}. The value of the energy functional at the exact solution is $J^0(u^*) = -1{\cdot}28567 \times 10^9$. Table 2 lists the differences between the corresponding values of J^0 and J_0, and the H_1-norm and C-norm of the error term, the latter being defined as

$$|u_n - u^*|_C = \max_{x \in \bar{S},\ i=1,2,3} |u_{n,i}(x) - u_i^*(x)|.$$

The figures in Table 2 suggest an upper bound for the constant on the right-hand side in estimate (3.6) of

$$c = 1{\cdot}6756 \times 10^{-3}. \tag{4.2}$$

Table 1 Approximate values of the direct and dual functionals for the first problem, (4.1)

n	$J^0(u_n)$	$J_0(v_n)$
1	$-1{\cdot}28544 \times 10^9$	$-3{\cdot}04160 \times 10^{10}$
3	$-1{\cdot}28563 \times 10^9$	$-4{\cdot}13613 \times 10^9$
5	$-1{\cdot}28567 \times 10^9$	$-1{\cdot}32390 \times 10^9$
7	$-1{\cdot}28567 \times 10^9$	$-1{\cdot}28573 \times 10^9$

Table 2 Errors for the first problem

n	$J^0(u_n) - J_0(v_n)$	$\|u_n - u^*\|_1$	$\|u_n - u^*\|_C$
1	$2{\cdot}91306 \times 10^{10}$	92·7466	1·24614
3	$2{\cdot}85050 \times 10^9$	40·0509	0·64980
5	$3{\cdot}82278 \times 10^7$	2·68976	0·03602
7	$5{\cdot}46616 \times 10^4$	0·39176	0·00419

Table 3 Results for the second problem

n	$J^0(u_n)$	$J_0(v_n)$	$J^0(u_n) - J_0(v_n)$	$\|u_n - u^*\|_1$
1	$-1{\cdot}22360 \times 10^5$	$-2{\cdot}60676 \times 10^6$	$2{\cdot}48440 \times 10^6$	$2{\cdot}64110$
3	$-1{\cdot}22634 \times 10^5$	$-1{\cdot}25099 \times 10^5$	$2{\cdot}46538 \times 10^3$	$8{\cdot}31989 \times 10^{-2}$
5	$-1{\cdot}22635 \times 10^5$	$-1{\cdot}22638 \times 10^5$	$2{\cdot}57174$	$2{\cdot}68713 \times 10^{-3}$
7	$-1{\cdot}22635 \times 10^5$	$-1{\cdot}22635 \times 10^5$	$1{\cdot}07600 \times 10^{-3}$	$5{\cdot}49643 \times 10^{-5}$

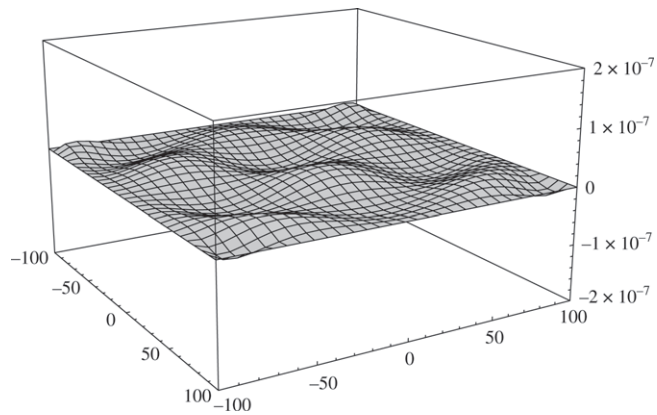

Fig. 1 Graph of $u_{7,1}$

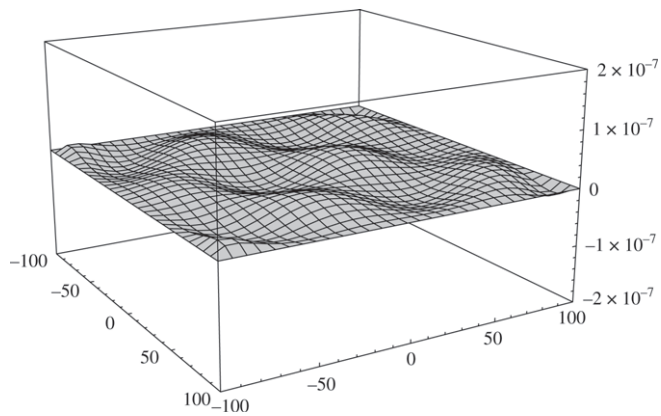

Fig. 2 Graph of $u_{7,2}$

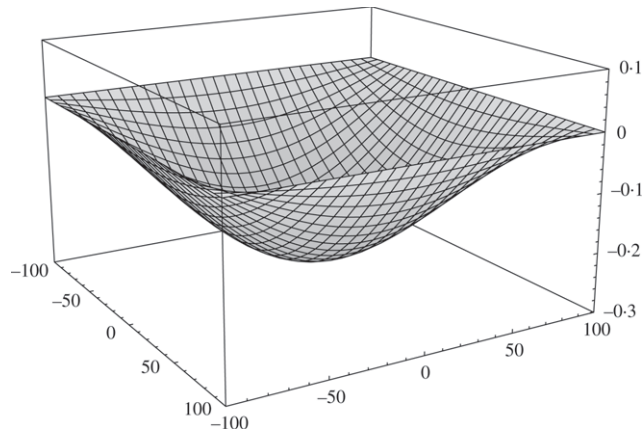

Fig. 3 Graph of $u_{7,3}$

Next, we consider a load that, after averaging across thickness (**8**), yields

$$q_1(x_1, x_2) = \frac{10^4}{\pi} \sin\left(\frac{\pi}{200} x_1\right) \cos\left(\frac{\pi}{200} x_2\right),$$

$$q_2(x_1, x_2) = \frac{10^4}{\pi} \cos\left(\frac{\pi}{200} x_1\right) \sin\left(\frac{\pi}{200} x_2\right),$$

$$q_3(x_1, x_2) = -100 \cos\left(\frac{\pi}{200} x_1\right) \cos\left(\frac{\pi}{200} x_2\right).$$

Table 3 shows the approximate values of the direct and dual functionals computed for this new problem with the same parameters as in the first example. The table also lists the difference between J^0 and J_0 and the estimates of the H_1-norm of the error, obtained by means of (3.6) and (4.2). The graphs of the three components of the approximate solution u_7 are shown in Figs 1 to 3.

The above numerical results clearly illustrate that the simultaneous use of both the direct and dual functionals produces a high degree of computational accuracy even when the matrix K is not positive definite.

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