Wave-corpuscle mechanics for elementary charges

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Abstract

It is well known that a concept of point charge interacting with electromagnetic (EM) field has a problem. To address that problem we introduce a concept of *wave-corpuscle* to describe spinless elementary charges interacting with the classical EM field. Every charge interacts only with the EM field and is described by a complex valued wave function over 4-dimensional space time continuum. A system of many charges interacting with the EM field is defined by a local, gauge and Lorentz invariant Lagrangian with a key ingredient - a nonlinear self-interaction term providing for a cohesive force assigned to every charge. An ideal wave-corpuscle is an exact solution to the Euler-Lagrange equations describing both free or accelerated motion. It carries explicitly features of a point charge and the de Broglie wave. A system of well separated charges moving with nonrelativistic velocities are represented accurately as wave-corpuscles governed by the Newton motion equations for point charges interacting with the Lorentz forces. In this regime the nonlinearities are "stealthy" and don't show explicitly anywhere, but they provide for binding forces that keep localized every individual charge.

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1 Introduction

We all know from textbooks that if there is a point charge q of the mass m in an external electromagnetic (EM) field its dynamics is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m\mathbf{v}\left(t\right)\right]=q\left[\mathbf{E}\left(\mathbf{r}\left(t\right),t\right)+\frac{1}{\mathrm{c}}\mathbf{v}\left(t\right)\times\mathbf{B}\left(\mathbf{r}\left(t\right),t\right)\right] \tag{1.0.1}$$

where $\mathbf{r}$ and $\mathbf{v}=\dot{\mathbf{r}}=\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ are respectively charge time dependent position and velocity, $\mathbf{E}\left(t,\mathbf{r}\right)$ and $\mathbf{B}\left(t,\mathbf{r}\right)$ are the electric field and the magnetic induction, and the right-hand side of the equation (1.0.1) is the Lorentz force. We also know that if the charge time dependent position and velocity are $\mathbf{r}$ and $\mathbf{v}$ then there is an associated with them EM field described by the equations

$$\frac{1}{\mathrm{c}}\frac{\partial\boldsymbol{B}}{\partial t}+\nabla\times\boldsymbol{E}=\mathbf{0},\ \nabla\cdot\boldsymbol{B}=0, \tag{1.0.2}$$

$$\frac{1}{\mathrm{c}}\frac{\partial\boldsymbol{E}}{\partial t}-\nabla\times\boldsymbol{B}=-\frac{4\pi}{\mathrm{c}}q\delta\left(\mathbf{x}-\mathbf{r}\left(t\right)\right)\mathbf{v}\left(t\right),\ \nabla\cdot\boldsymbol{E}=4\pi q\delta\left(\mathbf{x}-\mathbf{r}\left(t\right)\right),\ \mathbf{v}\left(t\right)=\dot{\mathbf{r}}\left(t\right), \tag{1.0.3}$$

where δ is the Dirac delta-function. But if naturally we would like to consider the equation (1.0.1) and (1.0.2)-(1.0.3) as a closed system "charge-EM field" there is a problem. The origin of the problems is in the divergence of the EM field exactly at the position of the point charge, as, for instance, for the electrostatic field $\mathbf{E}$ with the Coulomb potential $\frac{q}{|\mathbf{x}-\mathbf{r}|}$ with a singularity at $\mathbf{x}=\mathbf{r}$. If (1.0.1) is replaced for a relativistic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma m\mathbf{v}\left(t\right)\right]=q\left[\mathbf{E}\left(\mathbf{r}\left(t\right),t\right)+\frac{1}{\mathrm{c}}\mathbf{v}\left(t\right)\times\mathbf{B}\left(\mathbf{r}\left(t\right),t\right)\right], \tag{1.0.4}$$

where $\gamma=1/\sqrt{1-\mathbf{v}^2\left(t\right)/\mathrm{c}^2}$ is the Lorentz factor, the system constituted by (1.0.4) and (1.0.2)-(1.0.3) becomes Lorentz invariant and has a Lagrangian that yields it via the variational principle [Barut, (4.21)], [Spohn, (2.36)], but the problem still persists. If one wants to stay within the classical theory of the electromagnetism a possible remedy is the introduction of an extended charge which, though very small, is not a point. Recent studies by A. Yaghjian in [Yaghjian1] show, in particular, that "a fully consistent classical equation of motion for a point charge, unlike that of an extended charge, does not exist". There are two most well known models for such an extended charge: the Abraham rigid charge model and the Lorentz relativistically covariant model. These models are considered, studied and advanced in [Jackson, Sections 16], [Pearle1], [Rohrlich, Sections 2, 6], [Schwinger], [Spohn], [Yaghjian]. Importantly for what we do here, Poincaré suggested in 1905-1906, [Poincare] (see also [Jackson, Sections 16.4-16.6], [Rohrlich, Sections 2.3, 6.1- 6.3], [Pauli RT, Section 63], [Schwinger], [Yaghjian, Section 4.2] and references there in), to add to the Lorentz-Abraham model non-electromagnetic cohesive forces which balance the charge internal repulsive electromagnetic forces and remarkably restore also the covariance of the entire model. W. Pauli argues very convincingly based on the relativity principle in [Pauli RT, Section 63] a necessity to introduce for the electron an energy of non-electromagnetic origin.

An alternative approach to deal with the above-mentioned divergences goes back to G. Mie who proposed to modify the Maxwell equations making them nonlinear, [Pauli RT, Section 64], [Weyl STM, Section 26] and a particular example of the Mie approach is the Born-Infeld theory, [Born Infeld 1]. Recently M. Kiessling showed that, [Kiessling 1], "a relativistic Hamilton–Jacobi type law of point charge motion can be consistently coupled with the nonlinear Maxwell–Born–Infeld field equations to obtain a well-defined relativistic classical electrodynamics with point charges".

A substantially different approach to elementary charges was pursued by E. Schrödinger and L. de Broglie who tried to develop a concept of *wavepacket* as a model for spatially localized charge. The Schrödinger wave theory, [Schrodinger ColPap], was inspired by de Broglie ideas on the material wave, [de Broglie 2], [Barut, Section II.1]. The theory was very successful in describing quantum phenomena in the hydrogen atom, but it had great difficulties in treating the elementary charge as the material wave as it moves and interacts with other elementary charges. M. Born commented on this, [Born1, Chapter IV.7]: "To begin with, Schrödinger attempted to interpret corpuscles and particularly electrons, as *wave packets*. Although his formulae are entirely correct, his interpretation cannot be maintained, since on the one hand, as we have already explained above, the wave packets must in course of time become dissipated, and on the other hand the description of the interaction of two electrons as a collision of two wave packets in ordinary three-dimensional space lands us in grave difficulties."

We develop here a concept of wave-corpuscle, which is understood as a spatially localized excitation in a dispersive medium, and which is to substitute for the point charge concept. Our approach to a spatially distributed but localized elementary charge has some features in common with the discussed above concepts of extended charge, but it differs from any of them substantially. In particular, our approach provides for an electromagnetic theory in which (i) a "bare" elementary charge and the EM field described by the Maxwell equations form an inseparable entity; (ii) every elementary "bare" charge interacts directly only with the EM field; (iii) the EM field is a single entity providing for interaction between "bare" elementary charges insuring the maximum speed of interaction not to ever exceed the speed of light. To emphasize the inseparability of the "bare" elementary charge from the EM field we refer to their entity as to *dressed charge*.

The best way to describe our concept of a spatially distributed but localized dressed charge in one word is by the name *wave-corpuscle* since it is a stable localized excitation of a dispersive medium propagating in the three-dimensional space. An instructive example of a wave-corpuscle is furnished by our nonrelativistic charge model. In that model in the simplest case an *ideal wave-corpuscle* is described by a complex-valued wave function ψ of the form

$$\psi = \psi(t,\mathbf{x}) = \exp\left\{\frac{\mathrm{i}}{\hbar}\left[\mathbf{p}(t)\cdot\mathbf{x} - \int_0^t \frac{\mathbf{p}(t')^2}{2m}\,\mathrm{d}t'\right]\right\}\mathring{\psi}(|\mathbf{x}-\mathbf{r}(t)|), \tag{1.0.5}$$

where $\mathring{\psi}(s)$, $s \geq 0$, is a non negative, monotonically decaying function which vanishes at the infinity at a sufficient fast rate. Importantly, for the above wave function ψ to be an exact solution of corresponding field equations, the parameters $\mathbf{r}(t)$ and $\mathbf{p}(t)$ must satisfy Newton equation which in this simplest case has the form

$$m\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = q\mathbf{E}_{\mathrm{ex}},\ \ \mathbf{p}(t) = m\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}, \tag{1.0.6}$$

where m and q are respectively its mass and the charge and $\mathbf{E}_{\mathrm{ex}}(t)$ is an external homogeneous electric field. We underline that Newton equation is not postulated as in (1.0.1) or (1.0.4) but rather is derived from the field equations. *The ideal wave-corpuscle wave function $\psi(t,\mathbf{x})$ defined by (1.0.5), (1.0.6) forms together with the corresponding EM field an exact solution to the relevant Euler-Lagrange field equations describing an accelerating dressed charge.*

The point charge momentum $\mathbf{p}(t)$ *turns out to be exactly equal to the total momentum of the charge as a wave-corpuscle and its electromagnetic field. Remarkably the point charge features appear in the phase and amplitude of the ideal wave-corpuscle in a transparent and direct way without any limit process. The wave-corpuscle is a material wave, the quantity* $q|\psi(t,\mathbf{x})|^2$ *corresponds to the charge density and the density* $|\psi(t,\mathbf{x})|^2$ *is not given a probabilistic interpretation. The wave-corpuscle provides we believe an alternative resolution to the wave-particle duality problem.*

2 Sketch of the wave-corpuscle mechanics

We describe a bare single elementary charge by a complex-valued scalar field $\psi = \psi(x) = \psi(t,\mathbf{x})$, where $x = (t,\mathbf{x}) \in \mathbb{R}^4$ is the space-time variable. The charge is coupled at all times with the classical EM field as described by its potentials $A^\mu = (\varphi, \mathbf{A})$ related to the EM field by the standard formulas

$$\mathbf{E} = -\nabla\varphi - \frac{1}{\mathrm{c}}\partial_t \mathbf{A},\ \mathbf{B} = \nabla \times \mathbf{A}, \tag{2.0.7}$$

where c is the speed of light. The dynamics of the system of a single charge and the EM field is described via its Lagrangian

$$\begin{aligned} L_0(\psi, A^\mu) = {} & \frac{\chi^2}{2m}\left\{\frac{1}{\mathrm{c}^2}\left|\tilde{\partial}_t\psi\right|^2 - \left|\tilde{\nabla}\psi\right|^2 - \kappa_0^2|\psi|^2 - G(\psi^*\psi)\right\} \\ & + \frac{1}{8\pi}\left[\left(\nabla\varphi + \frac{1}{\mathrm{c}}\partial_t\mathbf{A}\right)^2 - (\nabla\times\mathbf{A})^2\right], \end{aligned} \tag{2.0.8}$$

where $\tilde{\partial}_t$ and $\tilde{\nabla}$ are the covariant differentiation operators defined by

$$\tilde{\partial}_t = \partial_t + \frac{\mathrm{i}q\varphi}{\chi},\ \tilde{\nabla} = \nabla - \frac{\mathrm{i}q\mathbf{A}}{\chi\mathrm{c}},\ \tilde{\partial}_t^* = \partial_t - \frac{\mathrm{i}q\varphi}{\chi},\ \tilde{\nabla}^* = \nabla + \frac{\mathrm{i}q\mathbf{A}}{\chi\mathrm{c}}, \tag{2.0.9}$$

$m > 0$ is the *charge mass*, $\chi > 0$ is a constant similar to the Planck constant $\hbar = \frac{h}{2\pi}$ and it might be dependent on the charge; q is the total charge of the particle.

Let us take a closer look at the components of the Lagrangian (2.0.8). It involves constants κ_0, c, χ and m and, acting similarly to the case of the Klein-Gordon equation for a relativistic particle (see [Pauli PWM, Sections 1, 18, 19] and Section 10.10), we introduce a fundamental frequency ω_0 relating it to the above constants by the following formulas

$$\omega_0 = \frac{m\mathrm{c}^2}{\chi},\ \kappa_0 = \frac{\omega_0}{\mathrm{c}} = \frac{m\mathrm{c}}{\chi}. \tag{2.0.10}$$

A key component of the Lagrangian in (2.0.8) is a real-valued nonlinear function $G(s)$, $s \geq 0$, providing for the charge cohesive self-interaction. The second part of the expression (2.0.8) is the standard Lagrangian of the EM field coupled to the charge via the covariant derivatives. Observe that the Lagrangian L_0 defined by (2.0.8)-(2.0.9) is manifestly (i) local; (ii) Lorentz and gauge invariant, and (iii) it has a local nonlinear term providing for a cohesive self-force similar to the Poincaré force for the Lorentz-Poincaré model of an extended charge.

Since a single charge is coupled at all times to the EM field we always deal with the *system "charge-EM field"*, $\{\psi, \psi^*, A^\mu\}$, and call it for short *dressed charge*. The dressed charge motion is governed by the relevant Euler-Lagrange field equations (see (3.0.6), (3.0.7)), and when the charge is at rest in the origin $\mathbf{x} = \mathbf{0}$ it is described by the fields

$$\psi(t, \mathbf{x}) = \mathrm{e}^{-\mathrm{i}\omega_0 t} \mathring{\psi}(|\mathbf{x}|), \ \varphi(t, \mathbf{x}) = \mathring{\varphi}(|\mathbf{x}|), \ \mathbf{A}(t, \mathbf{x}) = \mathbf{0}, \tag{2.0.11}$$

where ψ^* is the complex conjugate to ψ and the real valued functions $\mathring{\psi}$ and $\mathring{\varphi}$ satisfy the following system of equations

$$-\Delta\mathring{\varphi} = 4\pi\mathring{\rho}, \ \mathring{\rho} = q\left(1 - \frac{q\mathring{\varphi}}{mc^2}\right)\mathring{\psi}^2, \tag{2.0.12}$$

$$-\Delta\mathring{\psi} + \frac{m\mathring{\varphi}}{\chi^2} q\left(2 - \frac{q\mathring{\varphi}}{mc^2}\right)\mathring{\psi} + G'\left(|\mathring{\psi}|^2\right)\mathring{\psi} = 0. \tag{2.0.13}$$

where $\Delta = \nabla^2$ is Laplace operator. We refer to the state of the dressed charge of the form (2.0.11) as ω_0-*static*. The functions $\mathring{\psi}$ and $\mathring{\varphi}$ in the above formulas are instrumental for our constructions and we refer to them respectively as the *charge form factor* and *form factor potential*. Using Green's function to solve equation (2.0.12) we see that the charge form factor $\mathring{\psi}$ determines the Coulomb-like potential $\mathring{\varphi} = \mathring{\varphi}_{\mathring{\psi}}$ by the formula

$$\mathring{\varphi} = \mathring{\varphi}_{\mathring{\psi}} = 4\pi q\left(-\Delta + \frac{4\pi q^2}{mc^2}\mathring{\psi}^2\right)^{-1}\mathring{\psi}^2. \tag{2.0.14}$$

Consequently, plugging in the above expression into the equation (2.0.13) we get the following nonlinear equation

$$-\Delta\mathring{\psi} + \frac{m\mathring{\varphi}_{\mathring{\psi}}}{\chi^2} q\left(2 - \frac{q\mathring{\varphi}_{\mathring{\psi}}}{mc^2}\right)\mathring{\psi} + G'\left(|\mathring{\psi}|^2\right)\mathring{\psi} = 0. \tag{2.0.15}$$

The above equation (2.0.15) signifies a complete balance of the three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term $-\Delta\mathring{\psi}$; (ii) charge's electromagnetic self-interaction force associated with the term $\frac{m\mathring{\varphi}_{\mathring{\psi}}}{\chi^2}\left(2q - \frac{q^2\mathring{\varphi}_{\mathring{\psi}}}{mc^2}\right)\mathring{\psi}$; (iii) internal nonlinear self-interaction of the charge associated with the term $G'\left(|\mathring{\psi}|^2\right)\mathring{\psi}$. In what follows we refer to the equation (2.0.15) as *charge equilibrium equation*. Importantly, the static charge equilibrium equation (2.0.15) establishes an explicit relation between the form factor $\mathring{\psi}$ and the self-interaction nonlinearity G. Hence being given the form factor $\mathring{\psi}$ we can find from the equilibrium equation (2.0.15) the self-interaction nonlinearity G which exactly produces this factor under the assumption that $\mathring{\psi}(r)$, $r \geq 0$ is a nonnegative, monotonically decaying and sufficiently smooth function. *The later is a key feature of our approach: it allows to choose the form factor $\mathring{\psi}$ and then to determine matching self-interaction nonlinearity G rather than to deal with solving a nontrivial nonlinear partial differential equation.*

Thus, to summarize an important point of our method: we pick the form factor $\mathring{\psi}$ and then the nonlinear self interaction function G is determined based on a physically sound ground: the charge equilibrium equation (2.0.15). Needless to say that under this approach the nonlinearity G is not expected to be a simple polynomial function but rather a function with properties that ought to be established. Then having fixed the nonlinear self-interaction G based on the charge equilibrium equation (2.0.15) the challenge is to figure out the dynamics of the charge as it interacts with other charges or is acted upon by an external EM field and hence accelerates. The nonlinear self-interaction G evidently brings into the charge model non-electromagnetic forces, the necessity of which for a consistent relativistic electromagnetic theory was argued convincingly by W. Pauli in [Pauli RT, Section 63]. It is worth to point out that the nonlinearity G introduced via the charge equilibrium equation (2.0.15) differs significantly from nonlinearities considered in similar problems in literature including attempts to introduce nonlinearity in the quantum mechanics, [Bialynicki], [Holland], [Weinberg]. Important features of our nonlinearity include: (i) the boundedness of its derivative $G'(s)$ for $s \geq 0$ with consequent boundedness from below of the wave energy; (ii) non analytic behavior for small s that is for small wave amplitudes.

We would like to mention that an idea to use concept of a solitary wave in nonlinear dispersive media for modelling wave-particles was quite popular. Luis de Broglie tried to use it in his pursuit of the material wave mechanics. G. Lochak wrote in his preface to the de Broglie's monograph, [de Broglie 2, page XXXIX]: "...The first idea concerns the solitons, which we would call *ondes à bosses* (humped waves) at the *Institut Henri Poincaré*. This idea of de Broglie's used to be considered as obsolete and too classical, but it is now quite well known, as I mentioned above, and is likely to be developed in the future, but only provided we realize what the obstacle is and has been for twenty-five years: It resides in the lack of a general principle in the name of which we would be able to choose one nonlinear wave equation from among the infinity of possible equations. If we succeed one day in finding such an equation, a new microphysics will arise." G. Lochak raised an interesting point of the necessity of a general principle that would allow to choose one nonlinearity among infinitely many. We agree to G. Lochak to the extend that there has to be an important physical principle that would allow to choose the nonlinearity but whether it has to be unique is different matter. In our approach such a principle is the exact balance of all forces for the resting dressed charge via the static charge equilibrium equation (2.0.15). As to a possibility of spatially localized excitations such as wave-packets to maintain their basic properties when propagate in a dispersive medium with a nonlinearity we refer to our work [Babin Figoin 1]-[Babin Figotin 3].

The gauge invariance of the Lagrangian L_0 allows us to introduce in a standard fashion the *microcharge density* ρ and the *microcurrent density* $\mathbf{J}$ by

$$\rho = -\frac{\chi q}{2mc}\mathrm{i}\left(\tilde{\partial}_t^* \psi^* \psi - \psi^* \tilde{\partial}_t \psi\right), \ \mathbf{J} = \frac{\chi q}{2m}\mathrm{i}\left(\tilde{\nabla}^* \psi^* \psi - \psi^* \tilde{\nabla} \psi\right). \tag{2.0.16}$$

They satisfy the conservation (continuity) equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0, \tag{2.0.17}$$

and, consequently, the total charge is conserved:

$$\int_{\mathbb{R}^3} \rho(t, \mathbf{x})\, \mathrm{d}\mathbf{x} = const. \tag{2.0.18}$$

For the fundamental pair $\left\{\mathring{\psi}, \mathring{\varphi}\right\}$ the corresponding microcharge density defined by (2.0.16) turns into

$$\rho = \rho(|\mathbf{x}|) = q\left(1 - \frac{q\mathring{\varphi}(|\mathbf{x}|)}{mc^2}\right)\mathring{\psi}^2(|\mathbf{x}|). \tag{2.0.19}$$

Note that equation (2.0.12) turns into the classical equation for the Coulomb potential if ρ is replaced by $q\delta(x)$ where delta function has standard property $\int \delta(\mathbf{x})\, \mathrm{d}\mathbf{x} = 1$. Since we want

$\mathring{\varphi}$ to behave as Coulomb electrostatic potential at large distances and q to be the charge, we introduce the following *charge normalization condition* imposed on the form factor $\mathring{\psi}$

$$\int_{\mathbb{R}^3}\left(1-\frac{q\mathring{\varphi}\left(|\mathbf{x}|\right)}{mc^2}\right)\mathring{\psi}^2\left(|\mathbf{x}|\right)\mathrm{d}\mathbf{x}=1. \tag{2.0.20}$$

Notice that we introduced above terms *microcharge* and *microcurrent* densities to emphasize their relation to the internal structure of elementary charges and difference from commonly used charge and the current densities as macroscopic quantities. It is worth noticing though that if it comes to the interaction with the electromagnetic field the "micro" charges and microcurrents densities behave exactly the same way as the macroscopic charges and densities, but microcharges are also subjects to the internal elastic and nonlinear self-interaction forces of non-electromagnetic nature.

2.1 Energy considerations

Let us denote by $\mathcal{E}_0\left(\psi, A^\mu\right)$ the energy of the dressed charge $\{\psi, A^\mu\}$ derived from the Lagrangian L_0. We found that for the fundamental pair $\left\{\mathring{\psi},\mathring{\varphi}\right\}$ the energy $\mathcal{E}_0$ can be written in the following form

$$\mathcal{E}_0\left(\mathring{\psi},\mathring{\varphi}\right)=\mathcal{E}_0\left(\mathring{\psi}\right)=mc^2+\mathcal{E}_0'\left(\mathring{\psi}\right), \tag{2.1.1}$$
$$\mathcal{E}_0'\left(\mathring{\psi}\right)=\frac{2}{3}\int_{\mathbb{R}^3}\left[\frac{\chi^2}{2m}\nabla\mathring{\psi}^*\cdot\nabla\mathring{\psi}-\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi}\right]\mathrm{d}x,$$

where we use the relation $\mathring{\varphi}=\mathring{\varphi}_{\mathring{\psi}}$ from (2.0.14) to emphasize an important fact that the above energy $\mathcal{E}_0$ is a functional of $\mathring{\psi}$ and the model constants only. We refer to the energy $\mathcal{E}_0'\left(\mathring{\psi}\right)$ defined in (2.1.1) as the relative energy. The significance of the representation (2.1.1) for the energy $\mathcal{E}_0$ is in the fact that it does not explicitly involve the nonlinear self-interaction G.

Applying to the energy $\mathcal{E}_0$ the Einstein principle of equivalence of mass and energy, namely $E=mc^2$, [Pauli RT, Section 41], we define the *dressed charge mass* $\tilde{m}=\tilde{m}\left(\mathring{\psi}\right)$ by the equality

$$\mathcal{E}_0\left(\mathring{\psi}\right)=\tilde{m}\mathrm{c}^2=\tilde{m}\left(\mathring{\psi}\right)\mathrm{c}^2. \tag{2.1.2}$$

Combining the relation (2.1.2) with (2.1.1) we readily obtain

$$\left(\tilde{m}-m\right)\mathrm{c}^2=\mathcal{E}_0'\left(\mathring{\psi}\right)=\frac{2}{3}\int_{\mathbb{R}^3}\left[\frac{\chi^2}{2m}\nabla\mathring{\psi}^*\cdot\nabla\mathring{\psi}-\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi}\right]\mathrm{d}x. \tag{2.1.3}$$

We also want the fundamental frequency ω_0 to satisfy the Einstein relation $\mathcal{E}_0=\hbar\omega_0$, which would determine ω_0 as a function of $\mathring{\psi}$, constants c, m, q and, importantly, χ, namely

$$\mathcal{E}_0\left(\mathring{\psi}\right)=\hbar\omega_0\left(\mathring{\psi},\chi\right),\ \text{or} \tag{2.1.4}$$
$$\omega_0=\omega_0\left(\mathring{\psi},\chi\right)=\frac{1}{\hbar}\left\{mc^2+\frac{2}{3}\int_{\mathbb{R}^3}\left[\frac{\chi^2}{2m}\nabla\mathring{\psi}^*\cdot\nabla\mathring{\psi}-\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi}\right]\mathrm{d}x\right\}.$$

Then to be consistent with the earlier relation (2.0.10) we have to set the value of χ so that

$$\chi\omega_0\left(\mathring{\psi},\chi\right)=mc^2, \tag{2.1.5}$$

which, in view of the representation (2.1.4) for $\omega_0\left(\mathring{\psi},\chi\right)$, is equivalent to the requirement for $\chi=\chi\left(\mathring{\psi}\right)$ to be a positive solution to the following cubic equation

$$\chi\left\{c_2\chi^2+c_1\right\}=\hbar,\text{ where} \tag{2.1.6}$$

$$c_2=\frac{\chi^2}{3m^2c^2}\int_{\mathbb{R}^3}\nabla\mathring{\psi}^*\cdot\nabla\mathring{\psi}\,\mathrm{d}x,\;\; c_1=1-\frac{2}{3mc^2}\int_{\mathbb{R}^3}\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi}\,\mathrm{d}x,$$

where $\mathring{\varphi}$, in view of defining it equation (2.0.12), depends only on $\mathring{\psi}$ and constants c, m, q. Notice that cubic equation (2.1.6) always has a positive solution, and if, in addition to that, we know that $c_1\geq 0$ then the left-hand side of the equation (2.1.6) is a monotonically increasing function for $\chi\geq 0$ implying that the solution is unique.

In the case of a generic form factor $\mathring{\psi}$ the relative energy $\mathcal{E}_0'\left(\mathring{\psi}\right)$ does not necessarily have to vanish and, in view of the formula (2.1.3), the mass $\tilde{m}$ may be different from m. Then, as it follows from the relation (2.1.2)-(2.1.6) $\chi\neq\hbar$. The very same relations also readily imply that

$$\chi=\hbar\text{ if }\tilde{m}=m. \tag{2.1.7}$$

For a number of reasons, mainly for the perfect agreement between the relativistic energy-momentum and its nonrelativistic approximation constructed below, it is attractive to have $\tilde{m}=m$ implying also, in view of (2.1.7), $\chi=\hbar$. The question though is if that is possible. The answer is affirmative and the equality $\tilde{m}=m$ of the two masses can be achieved as follows. We introduce for the form factor its *size representation* involving size parameter a and normalization constant $\mathring{C}$:

$$\mathring{\psi}\left(s\right)=\mathring{\psi}_a\left(s\right)=\frac{\mathring{C}}{a^{3/2}}\mathring{\psi}_1\left(\frac{s}{a}\right),\;\mathring{\varphi}\left(s\right)=\mathring{\varphi}_a\left(s\right)=\frac{q}{a}\mathring{\varphi}_1\left(\frac{s}{a}\right),\;a>0,\;s\geq 0, \tag{2.1.8}$$

where the function $\mathring{\psi}_1\left(|\mathbf{x}|\right)$ satisfies the normalization condition:

$$\int_{\mathbb{R}^3}\mathring{\psi}_1^2\left(|\mathbf{x}|\right)\,\mathrm{d}\mathbf{x}=1. \tag{2.1.9}$$

We consider then values of $\mathring{C}$ and a in the representation (2.1.8), (2.1.9) that satisfy two conditions: (i) the charge normalization condition (2.0.20), namely

$$\mathring{C}^2\int_{\mathbb{R}^3}\left(1-\frac{q\mathring{\varphi}_a\left(|\mathbf{x}|\right)}{mc^2}\right)\mathring{\psi}_a^2\left(|\mathbf{x}|\right)\,\mathrm{d}\mathbf{x}=1, \tag{2.1.10}$$

and the energy normalization condition

$$\mathcal{E}_0'\left(\mathring{\psi}_{a_0}\right)=0. \tag{2.1.11}$$

We provide arguments in Subsection 7.2 based on the smallness of the Sommerfeld fine structure constant showing that there exist such $\mathring{C}$, $a=a_0$ for which the both normalization conditions (2.1.10) and (2.1.11) can hold . In view of (2.1.1) and (2.1.7) the above equality implies

$$\mathcal{E}_0\left(\mathring{\psi}_{a_0}\right) = m\mathrm{c}^2 \text{ and } \tilde{m} = m,\ \chi = \hbar. \tag{2.1.12}$$

In other words, the requirement $\tilde{m} = m$ fixes the charge size $a = a_0$ as well as the constant $\chi = \hbar$, the magnitude of a_0 is of the same order as Bohr's radius. One might ask if it is necessary to require the exact equality $\tilde{m} = m$? For a good agreement between the relativistic energy-momentum and its certain constructed below nonrelativistic approximation it would be sufficient for $\tilde{m} - m$ to be small enough rather than zero. For this reason and because of the scope of this paper we decided not to impose here the exact mass equality $\tilde{m} = m$ leaving this decision for the future work. *So from now on we assume that the value of the constant χ to be set by equations (2.1.5), (2.1.6).*

2.2 Moving free charge

Using the Lorentz invariance of the system we can obtain as it is often done a representation for the dressed charge moving with a constant velocity $\mathbf{v}$ simply by applying to the rest solution (2.0.11) the Lorentz transformation from the original "rest frame" to the frame in which the "rest frame" moves with the constant velocity $\mathbf{v}$ as described by the formulas (10.1.6), (10.4.12) (so x' and x correspond respectively to the "rest" and "moving" frames). Namely, introducing

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{\mathrm{c}},\ \beta = |\boldsymbol{\beta}|\,,\ \gamma = \left(1 - \left(\frac{v}{\mathrm{c}}\right)^2\right)^{-1/2}, \tag{2.2.1}$$

we obtain the following representation for the dressed charge moving with the velocity $\mathbf{v}$

$$\psi(t, \mathbf{x}) = \mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{x})}\mathring{\psi}(\mathbf{x}')\,,\ \varphi(t, \mathbf{x}) = \gamma\mathring{\varphi}(|\mathbf{x}'|)\,,\ \mathbf{A}(t, \mathbf{x}) = \gamma\boldsymbol{\beta}\mathring{\varphi}(|\mathbf{x}'|)\,, \tag{2.2.2}$$

$$\mathbf{x}' = \mathbf{x} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\mathbf{v}t, \text{ or } \mathbf{x}'_{\parallel} = \gamma\left(\mathbf{x}_{\parallel} - \mathbf{v}t\right),\ \mathbf{x}'_{\perp} = \mathbf{x}_{\perp}, \tag{2.2.3}$$

$$\omega = \gamma\omega_0,\ \mathbf{k} = \gamma\boldsymbol{\beta}\frac{\omega_0}{\mathrm{c}}, \tag{2.2.4}$$

where $\mathbf{x}_{\parallel}$ and $\mathbf{x}_{\perp}$ refer respectively to the components of $\mathbf{x}$ parallel and perpendicular to the velocity $\mathbf{v}$, with the fields

$$\mathbf{E}(t, \mathbf{x}) = -\gamma\nabla\mathring{\varphi}(|\mathbf{x}'|) + \frac{\gamma^2}{\gamma + 1}(\boldsymbol{\beta}\cdot\nabla\mathring{\varphi}(|\mathbf{x}'|))\boldsymbol{\beta},\ \mathbf{B}(t, \mathbf{x}) = \gamma\boldsymbol{\beta}\times\nabla\mathring{\varphi}(|\mathbf{x}'|)\,. \tag{2.2.5}$$

The above formulas (which provide a solution to field equations (3.0.6), (3.0.7)) indicate that the fields of the dressed charge contract by the factor γ as it moves with the velocity $\mathbf{v}$ compare to their rest state. The first oscillatory exponential factor in (2.2.1) is the de Broglie plane wave of a frequency $\omega = \omega(\mathbf{k})$ and a wave-vector $\mathbf{k}$ satisfying

$$\omega^2 - \mathrm{c}^2\mathbf{k}^2 = \omega_0^2,\ \chi\omega_0 = m\mathrm{c}^2. \tag{2.2.6}$$

Based on the Lagrangian L_0 we found the symmetric energy-momentum tensor which shows that the dressed charge moving with a constant velocity $\mathbf{v}$ and described by (2.2.1)-(2.2.3) has the energy $\mathcal{E}$ and the momentum $\mathbf{p}$ which satisfy the *Einstein-de Broglie relations*

$$\mathcal{E} = \hbar\omega,\ \mathbf{P} = \hbar\mathbf{k}. \tag{2.2.7}$$

In addition to that, the charge velocity $\mathbf{v}$ and its de Broglie wave vector $\mathbf{k}$ satisfy the following relation

$$\mathbf{v} = \nabla_{\mathbf{k}}\omega(\mathbf{k}) \tag{2.2.8}$$

signifying that the velocity $\mathbf{v}$ of the dressed charge is the group velocity of the linear medium with the dispersion relation (2.2.6).

The second factor in the formula (2.2.2) for ψ involves the form factor $\mathring{\psi}(r)$, $r \geq 0$, which is a monotonically decreasing decaying at infinity function of r. For such a form factor the form factor potential $\mathring{\varphi}(r)$ decays at infinity as the Coulomb potential as it follows from (2.0.14), i.e. $\mathring{\varphi}(r) \sim qr^{-1}$ for large r. Consequently, the dressed charge moving with constant velocity $\mathbf{v}$ as described by equations (2.2.2)-(2.2.3) remains well localized and does not disperse in the space at all times justifying its characterization as a wave-corpuscle.

2.3 Nonrelativistic approximation for the charge in an external EM field

Our nonrelativistic Lagrangian for a single charge in external EM field with potentials $\varphi_{\mathrm{ex}}, \mathbf{A}_{\mathrm{ex}}$ has the form

$$\hat{L}_0(\psi,\psi^*,\varphi) = \frac{\chi}{2}\mathrm{i}\left[\psi^*\tilde{\partial}_t\psi - \psi\tilde{\partial}_t^*\psi^*\right] - \frac{\chi^2}{2m}\left\{\tilde{\nabla}\psi\tilde{\nabla}^*\psi^* + G(\psi^*\psi)\right\} - \frac{|\nabla\varphi|^2}{8\pi}, \tag{2.3.1}$$

$$\tilde{\partial}_t = \partial_t + \frac{\mathrm{i}q\bar{\varphi}}{\chi},\ \tilde{\nabla} = \nabla - \frac{\mathrm{i}q\mathbf{A}_{\mathrm{ex}}}{\chi c},\ \tilde{\partial}_t^* = \partial_t - \frac{\mathrm{i}q\bar{\varphi}}{\chi},\ \tilde{\nabla}^* = \nabla + \frac{\mathrm{i}q\mathbf{A}_{\mathrm{ex}}}{\chi c}, \bar{\varphi} = \varphi + \varphi_{\mathrm{ex}}. \tag{2.3.2}$$

For simplicity, we consider at first the case where external magnetic field is absent, $\mathbf{A}_{\mathrm{ex}} = 0$. The field equations for this Lagrangian take the form

$$\chi\mathrm{i}\partial_t\psi - q(\varphi+\varphi_{\mathrm{ex}})\psi = -\frac{\chi^2}{2m}\left[\Delta\psi - G'(\psi^*\psi)\psi\right], \tag{2.3.3}$$

$$-\Delta\varphi = 4\pi q\psi\psi^*,\ \text{where } G'(s) = \partial_s G, \tag{2.3.4}$$

and we refer to the pair $\{\psi,\varphi\}$ as *dressed charge.* Recall that ψ^* is complex conjugate to ψ.

Let us consider now the case of resting charge with no external EM field described by a static time independent solution to the equations (2.3.3)-(2.3.4). These equations under assumption $\mathbf{E}_{\mathrm{ex}}(t) = 0$ turn into the following *rest charge equations* for a static state as described by time independent real-valued radial functions $\mathring{\psi} = \mathring{\psi}(|\mathbf{x}|)$ and $\mathring{\varphi} = \mathring{\varphi}(|\mathbf{x}|)$:

$$-\Delta\mathring{\varphi} = 4\pi q\left|\mathring{\psi}\right|^2, \tag{2.3.5}$$

$$-\Delta\mathring{\psi} + \frac{2m}{\chi^2}q\mathring{\varphi}\mathring{\psi} + G'\left(\left|\mathring{\psi}\right|^2\right)\mathring{\psi} = 0. \tag{2.3.6}$$

The quantities $\mathring{\psi}$ and $\mathring{\varphi}$ are fundamental for our theory and we refer to them respectively as *form factor* and form *factor potential.* In view of the equation (2.3.5) the charge form factor $\mathring{\psi}$ determines the form factor potential $\mathring{\varphi}$ by the formula

$$\mathring{\varphi}(|\mathbf{x}|) = \mathring{\varphi}_{\mathring{\psi}}(|\mathbf{x}|) = 4\pi q\int_{\mathbb{R}^3}\frac{\mathring{\psi}^2(|\mathbf{y}|)}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y}, \tag{2.3.7}$$

and if we plug in the above expression into the equation (2.3.6) we get the following nonlinear equation

$$-\Delta\mathring{\psi}+\frac{2mq}{\chi^{2}}\mathring{\varphi}_{\mathring{\psi}}\mathring{\psi}+G'\left(\left|\mathring{\psi}\right|^{2}\right)\mathring{\psi}=0. \tag{2.3.8}$$

The equation (2.3.8) signifies a complete balance of the three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term $-\Delta\mathring{\psi}$; (ii) charge's electromagnetic self-interaction force associated with the term $\frac{2mq}{\chi^{2}}\mathring{\varphi}_{\mathring{\psi}}\mathring{\psi}$; (iii) internal nonlinear self-interaction of the charge associated with the term $G'\left(\left|\mathring{\psi}\right|^{2}\right)\mathring{\psi}$. We refer to the equation (2.3.8) establishing an explicit relation between the form factor $\mathring{\psi}$ and the self-interaction nonlinearity G as the *charge equilibrium equation.* Hence being given the form factor $\mathring{\psi}$ we can find from the equilibrium equation (2.3.8) the self-interaction nonlinearity G which exactly produces this factor under the assumption that $\mathring{\psi}(r)$, $r\geq 0$ is a nonnegative, monotonically decaying and sufficiently smooth function. *Thus, we pick the form factor $\mathring{\psi}$ and then the nonlinear self interaction function G is determined based on the charge equilibrium equation (2.3.8).* One of the advantages of determining G in terms of $\mathring{\psi}$ is that we more often use properties of $\mathring{\psi}$ in our analysis rather than properties of G. Note that after the nonlinearity G is determined, it is fixed forever and solutions of equations (2.3.3)-(2.3.4) may evolve in time, they do not need to coincide with $\left\{\mathring{\psi},\mathring{\varphi}\right\}$. Details and examples of the construction of the nonlinear self-interaction function G based on the form factor are provided in the following sections.

The 4-microcurrent density J^{μ} related to the Lagrangian $\hat{L}_0$ is

$$J^{\mu}=(\mathrm{c}\rho,\mathbf{J}),\ \rho=q\psi\psi^{*},\ \mathbf{J}=\frac{\mathrm{i}\chi q}{2m}\left[\psi\nabla^{*}\psi^{*}-\psi^{*}\nabla\psi\right]=\frac{\chi q}{m}\operatorname{Im}\frac{\nabla\psi}{\psi}\left|\psi\right|^{2}, \tag{2.3.9}$$

and in the absence of external fields it satisfies the conservation/continuity equations

$$\partial_{\nu}J^{\nu}=0,\ \partial_{t}\rho+\nabla\cdot\mathbf{J}=0,\ J^{\nu}=(\rho\mathrm{c},\mathbf{J}). \tag{2.3.10}$$

For the fundamental pair $\left\{\mathring{\psi},\mathring{\varphi}\right\}$ the corresponding microcharge density defined by (2.3.9) turns into

$$\rho=\rho\left(|\mathbf{x}|\right)=q\left|\mathring{\psi}\right|^{2}\left(|\mathbf{x}|\right). \tag{2.3.11}$$

The *charge normalization condition* (2.0.20) which ensures that $\mathring{\varphi}\left(|\mathbf{x}|\right)$ is close to Coulomb's potential with proper charge q for large $|\mathbf{x}|$ now takes simpler form

$$\int_{\mathbb{R}^{3}}\mathring{\psi}^{2}\left(|\mathbf{x}|\right)\mathrm{d}\mathbf{x}=1. \tag{2.3.12}$$

Interestingly the momentum and the current densities of the dressed charge derived from $\hat{L}_0$ are identical up to a factor $\frac{m}{q}$, namely

$$\mathbf{P}=\frac{m}{q}\mathbf{J}=\frac{\mathrm{i}\chi}{2}\left[\psi\nabla^{*}\psi^{*}-\psi^{*}\nabla\psi\right]=\chi\operatorname{Im}\frac{\nabla\psi}{\psi}\left|\psi\right|^{2}. \tag{2.3.13}$$

It turns out the field equations (2.3.3)-(2.3.4) have a closed form solution in terms of (i) the static dressed charge state $\left\{\mathring{\psi},\mathring{\varphi}\right\}$ satisfying (2.3.5)-(2.3.6) and (ii) the basic quantities related to the point charge of the mass m moving in the external homogeneous electric field $\mathbf{E}_{\mathrm{ex}}(t)$ with the electric potential $\varphi_{\mathrm{ex}}(t,\mathbf{x})=\varphi_{\mathrm{ex}}^{0}(t)-\mathbf{E}_{\mathrm{ex}}(t)\cdot\mathbf{x}$. Indeed, let $\mathbf{r}$, $\mathbf{v}$, $\mathbf{p}$, $L_{\mathrm{p}}(\mathbf{v},\mathbf{r})$, s_{p} and $H_{\mathrm{p}}(\mathbf{p},\mathbf{r},t)$ be respectively the point charge position, velocity, momentum, Lagrangian, action and Hamiltonian satisfying the following familiar relations

$$L_{\mathrm{p}}(\mathbf{v},\mathbf{r},t)=\frac{m\mathbf{v}^{2}}{2}+q\mathbf{E}_{\mathrm{ex}}(t)\cdot\mathbf{r},\ \frac{\mathrm{d}s_{\mathrm{p}}}{\mathrm{d}t}=L_{\mathrm{p}}(\mathbf{v},\mathbf{r})=\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\cdot\mathbf{r})-\frac{\mathbf{p}^{2}}{2m},\tag{2.3.14}$$

$$H_{\mathrm{p}}(\mathbf{p},\mathbf{r},t)=\frac{\mathbf{p}^{2}}{2m}-q\mathbf{E}_{\mathrm{ex}}(t)\cdot\mathbf{r},\ \mathbf{p}=m\mathbf{v},$$

$$\mathbf{v}=\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t},\ \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}=q\mathbf{E}_{\mathrm{ex}}(t)\ \text{ or equivalently } m\frac{\mathrm{d}^{2}\mathbf{r}}{\mathrm{d}t^{2}}=q\mathbf{E}_{\mathrm{ex}}.\tag{2.3.15}$$

We recognize in the expression $q\mathbf{E}_{\mathrm{ex}}(t)$ in (2.3.15) the Lorentz force due the external electric field $\mathbf{E}_{\mathrm{ex}}(t)$. We refer to $L_{\mathrm{p}}(\mathbf{v},\mathbf{r})$ and the equations (2.3.15) respectively as *complimentary point charge Lagrangian and motion equations*.

Then the field equations (2.3.3)-(2.3.4) have the following closed form solution

$$\psi=\psi(t,\mathbf{x})=\mathrm{e}^{\frac{\mathrm{i}S}{\chi}}\mathring{\psi}(|\mathbf{x}-\mathbf{r}|),\ S=\mathbf{p}\cdot(\mathbf{x}-\mathbf{r})+s_{\mathrm{p}}-q\int_{0}^{t}\varphi_{\mathrm{ex}}^{0}(t')\,\mathrm{d}t'\tag{2.3.16}$$

$$\varphi(t,\mathbf{x})=\mathring{\varphi}_{0}(|\mathbf{x}-\mathbf{r}|),$$

where ψ in view of relations (2.3.14) can be also represented as

$$\psi=\psi(t,\mathbf{x})=\mathrm{e}^{\frac{\mathrm{i}S}{\chi}}\mathring{\psi}(|\mathbf{x}-\mathbf{r}|),\ S=\mathbf{p}\cdot\mathbf{x}-\int_{0}^{t}\frac{\mathbf{p}^{2}}{2m}\,\mathrm{d}t'-q\int_{0}^{t}\varphi_{\mathrm{ex}}^{0}(t')\,\mathrm{d}t'.\tag{2.3.17}$$

A similar exact solution is given for a class of EM fields with non-zero, spatially constant magnetic field $\mathbf{B}_{\mathrm{ex}}(t)$. Of course, in latter case the Lorentz force involves $\mathbf{B}_{\mathrm{ex}}$ as in (1.0.1) (see Section 5.5 for details). Looking at the exact solution (2.3.16), (2.3.17) to the field equations that describes the *accelerating charge* we would like to acknowledge the *truly remarkable simplicity and transparency of relations between two concepts of the charge*: *charge as a field* $\{\psi,\varphi\}$ in (2.3.16), (2.3.17) and *charge as a point* described by equations (2.3.14) and (2.3.15). Indeed, the wave amplitude $\mathring{\psi}(|\mathbf{x}-\mathbf{r}(t)|)$ in (2.3.16) is a soliton-like field moving exactly as the point charge described by its position $\mathbf{r}(t)$. *The exponential factor* $\mathrm{e}^{\frac{\mathrm{i}S}{\chi}}$ *is just a plane wave with the phase* S *that depends only on the point charge position* $\mathbf{r}$ *and momentum* $\mathbf{p}$ *and a time dependent gauge term, and it does not depend on the nonlinear self-interaction.* The phase S has a term in which we readily recognize the de Broglie wave-vector $\mathbf{k}(t)$ described exactly in terms of the point charge quantities, namely

$$\mathbf{k}(t)=\frac{\mathbf{p}(t)}{\chi}=\frac{m}{\chi}\mathbf{v}(t).\tag{2.3.18}$$

Notice that the dispersion relation $\omega=\omega(\mathbf{k})$ of the linear kinetic part of the field equations (2.3.3) for ψ is

$$\omega(\mathbf{k})=\frac{\chi\mathbf{k}^{2}}{2m},\ \text{implying that the group velocity } \nabla_{\mathbf{k}}\omega(\mathbf{k})=\frac{\chi\mathbf{k}}{m}.\tag{2.3.19}$$

Combining the expression (2.3.19) for the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$ with the expression (2.3.18) for wave vector $\mathbf{k}(t)$ we establish another exact relation

$$\mathbf{v}(t)=\nabla_{\mathbf{k}}\omega(\mathbf{k}(t)),\tag{2.3.20}$$

signifying the equality between the point charge velocity $\mathbf{v}(t)$ and the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k}(t))$ at the de Broglie wave vector $\mathbf{k}(t)$. Using the relations (2.3.9) and (2.3.13) we readily obtain the following representations for the micro-charge, the micro-current and momentum densities

$$\rho(t,\mathbf{x}) = q\mathring{\psi}^2(|\mathbf{x}-\mathbf{r}(t)|), \ \mathbf{J}(t,\mathbf{x}) = q\mathbf{v}(t)\mathring{\psi}^2(|\mathbf{x}-\mathbf{r}(t)|), \tag{2.3.21}$$

$$\mathbf{P}(t,\mathbf{x}) = \frac{m}{q}\mathbf{J}(t,\mathbf{x}) = \mathbf{p}(t)\mathring{\psi}^2(|\mathbf{x}-\mathbf{r}(t)|). \tag{2.3.22}$$

We also found that the total dressed charge field momentum $\mathsf{P}(t)$ and the total current $\mathsf{J}(t)$ for the solution (2.3.16) are expressed exactly in terms of point charge quantities, namely

$$\mathsf{P}(t) = \frac{m}{q}\mathsf{J}(t) = \int_{\mathbb{R}^3}\frac{\chi q}{m}\operatorname{Im}\frac{\nabla\psi}{\psi}|\psi|^2\,\mathrm{d}\mathbf{x} = \mathsf{p}(t) = m\mathbf{v}(t). \tag{2.3.23}$$

To summarize the above analysis we may state that even when the charge accelerates it perfectly combines the properties of a wave and a corpuscle, justifying the name wave-corpuscle mechanics. Its wave nature shows, in particular, in the de Broglie exponential factor and the equality (2.3.20) indicating the wave origin the charge motion. The corpuscle properties are manifested in the factor $\mathring{\psi}(|\mathbf{x}-\mathbf{r}(t)|)$ and soliton like propagation with $\mathbf{r}(t)$ satisfying the classical point charge evolution equation (2.3.15). Importantly, the introduced nonlinearities are stealthy in the sense that they don't show in the dynamics and kinematics of what appears to be soliton-like waves propagating as classical point charges.

2.4 Many interacting charges

A qualitatively new physical phenomenon in the theory of two or more charges compared with the theory of a single charge is obviously interaction between them. In our approach *any individual "bare" charge interacts directly only with the EM* field and consequently different charges interact with each other only indirectly through the EM field. We develop the theory for many interacting charges for the both relativistic and nonrelativistic cases and in this section we provide key elements of the nonrelativistic theory. The primary focus of our studies on many charges is correspondence between our wave theory and the point charge mechanics in the regime of remote interaction when the charges are separated by large distances compared to their sizes.

Studies of charge interactions at short distances are not in the scope of this paper though our approach allows to study short distance interactions and we provide as an example the Hydrogen atom model in Section 8. The main purpose of that exercise is to show a similarity between our and Schrodinger's Hydrogen atom models and to contrast it to any kind of Kepler's model. Another point we can make based on our Hydrogen atom model is that our theory does provide a basis for a regime of close interaction between two charges which differs significantly from the regime of remote interaction which is the primary focus of this paper.

Let us consider a system of N charges described by complex-valued fields ψ^ℓ, $\ell = 1,\ldots,N$. The system nonrelativistic Lagrangian $\hat{\mathcal{L}}$ is constructed based on individual charge Lagrangians $\hat{L}^\ell$ of the form (2.3.1)-(2.3.2) and an assumption that every charge interacts directly only with the EM field, including the external one with potentials $\varphi_{\mathrm{ex}}(t,\mathbf{x})$, $\mathbf{A}_{\mathrm{ex}}(t,\mathbf{x})$, namely

$$\hat{\mathcal{L}} = \sum_{\ell} \hat{L}^{\ell} + \frac{|\nabla\varphi|^2}{8\pi}, \text{ where} \tag{2.4.1}$$

$$\hat{L}^{\ell} = \frac{\chi}{2}\mathrm{i}\left[\psi^{\ell*}\tilde{\partial}_t^{\ell}\psi^{\ell} - \psi^{\ell}\tilde{\partial}_t^{\ell*}\psi^{\ell*}\right] - \frac{\chi^2}{2m}\left\{\tilde{\nabla}\psi^{\ell}\tilde{\nabla}^*\psi^{\ell*} + G^{\ell}\left(\psi^{\ell*}\psi^{\ell}\right)\right\},$$

$$\tilde{\partial}_t^{\ell} = \partial_t + \frac{\mathrm{i}q^{\ell}\left(\varphi+\varphi_{\mathrm{ex}}\right)}{\chi}, \ \tilde{\nabla} = \nabla - \frac{\mathrm{i}q\mathbf{A}_{\mathrm{ex}}}{\chi c}.$$

The nonlinear self-interaction terms G^{ℓ} in (2.4.1) are determined through the charge equilibrium equation (2.3.8) and they can be the same for different ℓ or there may be several types of charges, for instance, protons and electrons. The field equations for this Lagrangian are

$$\mathrm{i}\chi\partial_t\psi^{\ell} = -\frac{\chi^2\nabla^2\psi^{\ell}}{2m^{\ell}} - \frac{\chi q^{\ell}\mathbf{A}_{\mathrm{ex}}\cdot\nabla\psi^{\ell}}{m^{\ell}c i} + q^{\ell}\left(\varphi+\varphi_{\mathrm{ex}}\right)\psi^{\ell} + \frac{q^{\ell 2}\mathbf{A}_{\mathrm{ex}}^2\psi^{\ell}}{2m^{\ell}\mathrm{c}^2} + \left[G_a^{\ell}\right]'\left(\left|\psi^{\ell}\right|^2\right)\psi^{\ell}, \tag{2.4.2}$$

$$-\nabla^2\varphi = 4\pi\sum_{\ell=1}^{N} q^{\ell}\left|\psi^{\ell}\right|^2, \text{ and } \psi^{\ell*} = \text{complex conjugate of } \psi^{\ell}, \ \ell = 1,...,N.$$

Obviously, the equations with different ℓ are coupled only through the potential φ which is responsible for the charge interaction. The Lagrangian $\hat{\mathcal{L}}$ is gauge invariant with respect to the first and the second gauge transformations (10.7.6), (10.7.7) and consequently every ℓ-th charge has a conserved current

$$J^{\ell\mu} = \left(\mathrm{c}\rho^{\ell}, \mathbf{J}^{\ell}\right), \ \rho^{\ell} = q\left|\psi^{\ell}\right|^2, \ \mathbf{J}^{\ell} = \left(\frac{\chi q^{\ell}}{m^{\ell}}\operatorname{Im}\frac{\nabla\psi^{\ell}}{\psi^{\ell}} - \frac{q^{\ell 2}\mathbf{A}_{\mathrm{ex}}}{m^{\ell}\mathrm{c}}\right)\left|\psi^{\ell}\right|^2 \tag{2.4.3}$$

satisfying the conservation/continuity equations

$$\partial_t\rho^{\ell} + \nabla\cdot\mathbf{J}^{\ell} = 0 \text{ or } m^{\ell}\partial_t\left|\psi^{\ell}\right|^2 + \nabla\cdot\left(\chi\operatorname{Im}\frac{\nabla\psi^{\ell}}{\psi^{\ell}}\left|\psi^{\ell}\right|^2 - \frac{q^{\ell}}{\mathrm{c}}\mathbf{A}_{\mathrm{ex}}\left|\psi^{\ell}\right|^2\right) = 0. \tag{2.4.4}$$

The differential form (2.4.4) of the charge conservation imply the conservation of every total ℓ-th charge and we require every total ℓ-th conserved charge to be exactly q^{ℓ}, rather than just any constant, and that is reduced to the following *charge normalization* conditions

$$\int_{\mathbb{R}^3}\left|\psi^{\ell}\right|^2 \mathrm{d}\mathbf{x} = 1, \ t \geq 0, \ \ell = 1,...,N. \tag{2.4.5}$$

We attribute to every ℓ-th charge its potential φ^{ℓ} by the formula

$$\varphi^{\ell}(t,\mathbf{x}) = q^{\ell}\int_{\mathbb{R}^3}\frac{\left|\psi^{\ell}\right|^2(t,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y}. \tag{2.4.6}$$

Hence we can write equation for φ in (2.4.2) in the form

$$\nabla^2\varphi^{\ell} = -4\pi q^{\ell}\left|\psi^{\ell}\right|^2, \ \ \varphi = \sum_{\ell=1}^{N}\varphi^{\ell}. \tag{2.4.7}$$

An analysis of the system of charges energy-momentum and its partition between individual charges shows another important property of the Lagrangian: *the charge gauge invariant momentum density* $\tilde{\mathbf{p}}^{\ell}$ *equals exactly the microcurrent density* $\mathbf{J}^{\ell}$ *multiplied by the constant* m^{ℓ}/q^{ℓ}, namely:

$$\tilde{\mathbf{p}}^{\ell}=\frac{m^{\ell}}{q^{\ell}}\mathbf{J}^{\ell}=\frac{\mathrm{i}\chi}{2}\left[\psi^{\ell}\tilde{\nabla}^{\ell*}\psi^{\ell*}-\psi^{\ell*}\tilde{\nabla}^{\ell}\psi^{\ell}\right]=\left(\chi\,\mathrm{Im}\,\frac{\nabla\psi^{\ell}}{\psi^{\ell}}-\frac{q^{\ell}\mathbf{A}_{\mathrm{ex}}}{\mathrm{c}}\right)\left|\psi^{\ell}\right|^{2},\tag{2.4.8}$$

that can be viewed as the momentum density kinematic representation:

$$\tilde{\mathbf{p}}^{\ell}(t,\mathbf{x})=m\mathbf{v}^{\ell}(t,\mathbf{x}),\ \text{where }\mathbf{v}^{\ell}(t,\mathbf{x})=\mathbf{J}^{\ell}(t,\mathbf{x})/q^{\ell}\text{ is the velocity density.}\tag{2.4.9}$$

To quantify the conditions of the remote interaction we make use of explicit dependence on the size parameter a of the nonlinearity $G^{\ell}=G_{a}^{\ell}$ as in (4.4.12) and take the size parameter as a spatial scale characterizing sizes of the fields ψ_{a}^{ℓ} and φ_{a}^{ℓ}. The charges separation is measured roughly by a minimal distance $R_{\min}$ between any two charges. Another relevant spatial scale R_{ex} measures a typical scale of spatial inhomogeneity of external fields near charges. Consequently, conditions of remote interaction are measured by the dimensionless ratio a/R where characteristic spatial scale $R=\min(R_{\min},R_{\mathrm{ex}})$ with the *condition* $a/R\ll 1$ *to define the regime of remote interaction.*

Next we would like to take a look on how the point charge mechanics is manifested in our wave mechanics governed by the Lagrangian $\hat{\mathcal{L}}$. There are two distinct ways to correspond our field theory to the classical point charge mechanics in the case when all charges are well separated satisfying the condition $a/R\ll 1$. The first way is via averaged quantities in spirit of the well known in quantum mechanics *Ehrenfest Theorem,* [Schiff, Sections 7, 23], and the second one via a construction of approximate solutions to the field equations (2.4.2) when every charge is represented as wave-corpuscle similar to one from (2.3.16).

We construct the correspondence via averaged quantities beginning with defining ℓ-th charge location $\mathbf{r}_{a}^{\ell}(t)$ and its potential φ_{a}^{ℓ} by the relations

$$\mathbf{r}^{\ell}(t)=\mathbf{r}_{a}^{\ell}(t)=\int_{\mathbb{R}^{3}}\mathbf{x}\left|\psi_{a}^{\ell}(t,\mathbf{x})\right|^{2}\,\mathrm{d}\mathbf{x},\ \varphi_{a}^{\ell}(t,\mathbf{x})=q^{\ell}\int_{\mathbb{R}^{3}}\frac{\left|\psi_{a}^{\ell}\right|^{2}(t,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y}.\tag{2.4.10}$$

We consider first a simpler case when the external magnetic field vanishes, i.e. $\mathbf{A}_{\mathrm{ex}}=0$, and introduce potential $\varphi_{\mathrm{ex},a}^{\ell}$ describing the interaction of ℓ-th charge with all other charges:

$$\varphi_{\mathrm{ex},a}^{\ell}=\varphi+\varphi_{\mathrm{ex}}-\varphi_{a}^{\ell}=\varphi_{\mathrm{ex}}+\sum_{\ell'\neq\ell}\varphi_{a}^{\ell'}.\tag{2.4.11}$$

In this case based on the momentum balance equation for every ℓ-th charge combined with the momentum density kinematic representation (2.4.7), (2.4.8) and the representation (2.4.11) we obtain the following exact motion equations

$$m^{\ell}\frac{\mathrm{d}^{2}\mathbf{r}^{\ell}}{\mathrm{d}t^{2}}=-\int_{\mathbb{R}^{3}}q^{\ell}\left|\psi_{a}^{\ell}\right|^{2}\nabla\varphi_{\mathrm{ex},a}^{\ell}\,\mathrm{d}\mathbf{x},\ \ell=1,...,N,\tag{2.4.12}$$

where we recognize in the integrand the Lorentz force density exerted on ℓ-th charge by other charges and the external field. Suppose now that fields $\psi_{a}^{\ell}(t,\mathbf{x})$ are localized about point $\mathbf{r}_a^\ell(t)$ with the localization radius of the order a and observe that normalization (2.4.5) combined with the localization imply that

$$\left|\psi_a^\ell(t,\mathbf{x})\right|^2 \to \delta\left(\mathbf{x}-\mathbf{r}_0^\ell(t)\right),\ a\to 0,\ \text{implying } \mathbf{r}_0^\ell(t)=\lim_{a\to 0}\mathbf{r}_a^\ell(t). \tag{2.4.13}$$

The relations (2.4.10) and (2.4.13) imply consequently

$$\varphi_a^\ell(t,\mathbf{x})\to\varphi_0^\ell=\frac{q^\ell}{\left|\mathbf{x}-\mathbf{r}_0^\ell\right|}\ \text{as } a\to 0,\ \text{and } \varphi_{\text{ex},0}^\ell(t,\mathbf{x})=\varphi_{\text{ex}}(t,\mathbf{x})+\sum_{\ell'\neq\ell}\frac{q^{\ell'}}{\left|\mathbf{x}-\mathbf{r}_0^{\ell'}(t)\right|}. \tag{2.4.14}$$

Then combining relations (2.4.10), (2.4.13) we can pass in (2.4.12) to the limit $a\to 0$ obtaining Newton's motion equations for the system of N point charges

$$m^\ell\frac{\mathrm{d}^2\mathbf{r}_0^\ell}{\mathrm{d}t^2}(t)=-q^\ell\nabla\varphi_{\text{ex},a}^\ell\left(\mathbf{r}_0^\ell(t),t\right),\ \ell=1,...,N, \tag{2.4.15}$$

where in the right-hand side of equation (2.4.15) we see the Lorentz force acting on ℓ-th point charge interacting with other charges via point charges Coulomb potentials (2.4.14).

In the case when the external magnetic field potential $\mathbf{A}_{\text{ex}}(t,\mathbf{x})$ does not vanish we establish based on the exact motion equations similar to (2.4.12) that in the limit as $a\to 0$ the following point charges motion equation hold:

$$m^\ell\frac{\mathrm{d}^2\mathbf{r}_0^\ell}{\mathrm{d}t^2}=\mathsf{f}^\ell,\ \mathsf{f}^\ell=q^\ell\mathbf{E}^\ell+\frac{q^\ell}{\mathrm{c}}\frac{\mathrm{d}\mathbf{r}_0^\ell}{\mathrm{d}t}\times\mathbf{B}^\ell,\ \ell=1,...N, \tag{2.4.16}$$

with the Lorentz force f^ℓ involving electric and magnetic fields defined by classical formulas:

$$\mathbf{E}^\ell=-\left[\nabla\varphi_{\text{ex},0}^\ell\left(\mathbf{r}_0^\ell\right)+\frac{1}{\mathrm{c}}\partial_t\mathbf{A}_{\text{ex}}\left(\mathbf{r}_0^\ell\right)\right],\ \mathbf{B}^\ell=\nabla\times\mathbf{A}_{\text{ex}}\left(t,\mathbf{r}_0^\ell\right). \tag{2.4.17}$$

Obviously, the above force f^ℓ coincides with the Lorentz force acting on a point charge as in (1.0.1). To summarize our first way of correspondence, we observe that the exact motion equations (2.4.12) form a basis for relating the field and point mechanics under an assumption that charge fields remain localized during time interval of interest. *Notice the motion equations (2.4.12) do not involve the nonlinear interaction terms G^ℓ in an explicit way justifying their characterization as "stealthy" in the regime of remote interactions.* As to the assumption that the charge fields remain localized, it has to be verified based on the field equations (2.4.2) where the nonlinear interaction terms G^ℓ provide for cohesive forces for individual charges. The fact that they can do just that is demonstrated for a single charge represented as wave-corpuscle (2.3.16)-(2.3.17) as it accelerates in an external EM field.

The last point we made naturally brings us to the second way of correspondence between the charges as fields and points when they interact. This way of correspondence is based on an established by us fact that all charges fields ψ_a^ℓ can be represented as wave-corpuscles (2.3.16)-(2.3.17) which though do not satisfy the field equations (2.4.2) exactly but rather they satisfy them with small discrepancies in the regime of remote interaction when $a/R_{\min}\ll 1$. More detailed presentation of that idea is as follows. Consider for simplicity a simpler case when $\mathbf{A}_{\text{ex}}=0$ and introduce the following wave-corpuscle representation similar to (2.3.16)-(2.3.17)

$$\psi_a^\ell(t,\mathbf{x})=\mathrm{e}^{\frac{\mathrm{i}S^\ell}{\chi}}\mathring{\psi}\left(\left|\mathbf{x}-\mathbf{r}_0^\ell\right|\right)\mathring{\psi}^\ell\left(\left|\mathbf{x}-\mathbf{r}_0^\ell\right|\right),\ \varphi_a^\ell(t,\mathbf{x})=q^\ell\mathring{\varphi}^\ell\left(\mathbf{x}-\mathbf{r}_0^\ell\right),\ \text{where} \tag{2.4.18}$$

$$S^\ell(t,\mathbf{x})=\mathbf{p}_0^\ell\cdot\mathbf{x}-\int_0^t\frac{\mathbf{p}_0^{\ell 2}}{2m}\,\mathrm{d}t'-q\int_0^t\varphi_{\text{ex}}^0\left(t',\mathbf{r}_0^\ell\right)\,\mathrm{d}t',\ \mathbf{p}_0^\ell=m^\ell\frac{\mathrm{d}\mathbf{r}_0^\ell}{\mathrm{d}t} \tag{2.4.19}$$

and the position functions $\mathbf{r}_0^\ell(t)$ satisfy the Newton motion equations (2.4.16). Its turns out that wave-corpuscles $\{\psi_a^\ell, \varphi_a^\ell\}$ defined by (2.4.18), (2.4.19) and the complimentary point charge Newton motion equations (2.4.16) solve the field equations (2.4.2) with a small discrepancy which approaches zero as a/R approaches zero. The point charge mechanics features are transparently integrated into the fields $\{\psi_a^\ell, \varphi_a^\ell\}$ in (2.4.18), (2.4.19) via the de Broglie factor phases S^ℓ and spatial shifts $\mathbf{r}_0^\ell$. Comparing with motion of a single charge in external field we observe that now the acceleration of the corpuscle center $\mathbf{r}_0^\ell(t)$ is determined not only by Lorentz force due to the external field but also by electric interaction with remaining charges $\mathring{\psi}^{\ell'}$, $\ell' \neq \ell$ according to the Coulomb law (2.4.14), (2.4.16).

We would like to point out that when analyzing the system of charges in the regime of remote interaction we do not use any specific form of the nonlinearities, but the nonlinearity is necessary to provide cohesion and to ensure small scale equilibrium of every corpuscle in the dynamics. Note that solutions of field equations (2.4.2) depend on the size parameter a which is proportional to the radius of the corpuscle and which enters through the nonlinearity G_a^ℓ, but integral equations (2.4.12) do not involve explicit dependence on a. Equation (2.4.18) which describes the structure of the corpuscle involves a only through radial shape factors $\mathring{\psi}^\ell = \mathring{\psi}_a^\ell$ and through the electric potential $\mathring{\varphi}^\ell = \mathring{\varphi}_a^\ell$. The dependence of $\mathring{\psi}_a^\ell$ on a is explicitly singular at zero as should be expected since in the singular limit $a \to 0$ the wave-corpuscle should turn into the point charge with the square of amplitude described by a delta function as in (1.0.3). Nevertheless, for *arbitrary small* $a > 0$ the wave-corpuscle structure of every charge is preserved including its principal wave-vector. The dependence of $\mathring{\varphi}_a^\ell$ on small a can be described as a regular convergence to the classical singular Coulomb potential, see (4.6.9) for details. That allows to apply representation (2.4.18) to arbitrary small charges with radius proportional to a without compromising the accuracy of the description and, in fact, increasing the accuracy as $a \to 0$.

2.5 Comparative summary with the Schrödinger wave mechanics

The nonrelativistic version of our wave mechanics has many features in common with the Schrödinger wave mechanics. In particular, the charges wave functions are complex valued, they satisfy equations resembling the Schrödinger equation, the charge normalization condition is the same as in the Schrödinger wave mechanics. Our theory provides for a Hydrogen atom model which has a lot in common with that of Schrödinger but its detailed study is outside of the scope of this article. There are though features of our wave theory that distinguish it significantly from the Schrödinger wave mechanics and they are listed below.

- Charges are always coupled with and inseparable from the EM field.
- Every charge has a nonlinear self-interaction term in its Lagrangian providing for a cohesive force holding it together as it moves freely or accelerates.
- A single charge either free or in external EM field is described by a soliton-like wave function parametrized by the position and the momentum related to the corresponding point mechanics. It propagates in the space without dispersion even when accelerates, and that addresses one of the above mentioned "grave difficulties" with the Schrödinger's interpretation of the wave function expressed by M. Born.

- When dressed charges are separated by distances considerably larger than their sizes their wave functions and the corresponding EM fields maintain soliton-like representation.

- The correspondence between the wave mechanics and a point mechanics comes through the closed form soliton-like representation of wave functions in which point mechanics positions and momenta enter as parameters. In particular, the wave function representation includes the de Broglie wave vector as an exact parameter, it equals up the Planck constant to the point mechanics momentum. In addition to that, the corresponding group velocity matches exactly the velocity of soliton-like solution and the point mechanics velocity.

- In the case of many interacting charges every charge is described by its wave function over the same three dimensional space in contrast to the Schrödinger wave mechanics for many charges requiring multidimensional configuration space.

- Our theory has a relativistic version based on a local, gauge and Lorentz invariant Lagrangian with most of listed above features.

3 Single free relativistic charge

A single free charge is described by a complex scalar field $\psi = \psi(t, \mathbf{x})$ and it is coupled to the EM field described by its 4-potential $A^\mu = (\varphi, \mathbf{A})$. To emphasize the fact that our charge is always coupled with the EM field we name the pair $\{\psi, A^\mu\}$ *dressed charge*. So whenever we use the term dressed charge we mean the charge and the EM field as an inseparable entity. The dressed charge is called free if there no any external forces acting upon it. The free charge Lagrangian is defined by the following formula

$$L_0(\psi, A^\mu) = \frac{\chi^2}{2m}\left\{\psi^*_{;\mu}\psi^{;\mu} - \kappa_0^2\psi^*\psi - G(\psi^*\psi)\right\} - \frac{F^{\mu\nu}F_{\mu\nu}}{16\pi}, \text{ where} \tag{3.0.1}$$

$$\kappa_0 = \frac{\omega_0}{\mathrm{c}} = \frac{m\mathrm{c}}{\chi},\ \omega_0 = \frac{m\mathrm{c}^2}{\chi},\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.0.2}$$

$$\psi^{;\mu} = \tilde{\partial}^\mu\psi,\ \psi^{;\mu*} = \tilde{\partial}^{*\mu}\psi^*,\ \tilde{\partial}^\mu = \partial^\mu + \frac{\mathrm{i}q}{\chi\mathrm{c}}A^\mu,\ \tilde{\partial}^{*\mu} = \partial^\mu - \frac{\mathrm{i}q}{\chi\mathrm{c}}A^\mu. \tag{3.0.3}$$

In the above relations $m > 0$ is the classical mass of the charge, and $\chi > 0$ is a parameter similar to the Planck constant $\hbar$ value of which will be set latter on to satisfy the Einstein relation $\mathcal{E} = \hbar\omega_0$. The term $G(\psi^*\psi)$ corresponds to the nonlinear self-interaction and is to be determined later, ψ^* is complex conjugate to ψ. The above Lagrangian involves the so-called *covariant differentiation operators* $\tilde{\partial}^\mu$ and $\tilde{\partial}^{*\mu}$ with abbreviated notations $\psi^{;\mu}$ and $\psi^{;\mu*}$ for the corresponding *covariant derivatives*. In what follows we use also the following abbreviations

$$\partial^\mu\psi = \psi^{,\mu},\ \partial^\mu\psi^* = \psi^{,\mu*}. \tag{3.0.4}$$

We remind also that

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{\mathrm{c}}\partial_t, \nabla\right),\ \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{\mathrm{c}}\partial_t, -\nabla\right), \tag{3.0.5}$$

$$A^\mu = (\varphi, \mathbf{A}),\ A_\mu = (\varphi, -\mathbf{A}),\ \mathbf{E} = -\nabla\varphi - \frac{1}{\mathrm{c}}\partial_t\mathbf{A},\ \mathbf{B} = \nabla\times\mathbf{A}.$$

Evidently the Lagrangian L_0 defined by the formulas (3.0.1)-(3.0.3) is obtained from the Klein-Gordon Lagrangian, [Griffiths, Section 7.1, 11.2], [Barut, III.3], by adding to it the nonlinear term $G(\psi^*\psi)$. The Lagrangian expression indicates that the charge is coupled to the EM field through the covariant derivatives, and such a coupling is well known and called *minimal*. The Klein-Gordon Lagrangian is a commonly used model for a *relativistic spinless charge*, and the introduced nonlinearity $G(\psi^*\psi)$ can provide for a binding self-force. Nonlinear alterations of the Klein-Gordon Lagrangian were considered in the literature, see, for instance, [Griffiths, Section 11.7, 11.8] and [Benci Fortunato], for rigorous mathematical studies, but our way to choose of the nonlinearity $G(\psi^*\psi)$ differs from those.

Observe that the Lagrangian L_0 defined by (3.0.1)-(3.0.3) is manifestly Lorentz and gauge invariant, and it is a special case of a general one charge Lagrangian studied in Section 10.6. This allows us to apply to the Lagrangian L_0 formulas from there to get the field equations, the 4-microcurrent and the energy-momenta tensors. Consequently, the Euler-Lagrange field equations (10.6.3) take here the form

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{\mathrm{c}} J^\mu,\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{3.0.6}$$

$$\left[\tilde{\partial}_\mu \tilde{\partial}^\mu + \kappa_0^2 + G'(\psi^*\psi)\right]\psi = 0, \quad \text{where } \tilde{\partial}^\mu = \partial^\mu + \frac{\mathrm{i}q}{\chi \mathrm{c}} A^\mu. \tag{3.0.7}$$

The formula (10.6.5) for the 4-microcurrent density J^μ turns into

$$J^\mu = -\frac{\chi q}{2m}\mathrm{i}\left(\tilde{\partial}^{\mu*}\psi^*\psi - \psi^*\tilde{\partial}^\mu\psi\right) = -\left(\frac{\chi q}{m}\operatorname{Im}\frac{\partial^\mu\psi}{\psi} + \frac{q^2}{mc}A^\mu\right)|\psi|^2, \tag{3.0.8}$$

or, in the time-space variables,

$$\begin{gathered}\rho = -\frac{\chi q}{2mc}\mathrm{i}\left(\tilde{\partial}_t^*\psi^*\psi - \psi^*\tilde{\partial}_t\psi\right) = -\left(\frac{\chi q}{mc^2}\operatorname{Im}\frac{\partial_t\psi}{\psi} + \frac{q^2\varphi}{mc^2}\right)|\psi|^2, \\ \mathbf{J} = \frac{\chi q}{2m}\mathrm{i}\left(\tilde{\nabla}^*\psi^*\psi - \psi^*\tilde{\nabla}\psi\right) = \left(\frac{\chi q}{m}\operatorname{Im}\frac{\nabla\psi}{\psi} - \frac{q^2\mathbf{A}}{mc}\right)|\psi|^2.\end{gathered} \tag{3.0.9}$$

The above formulas for 4-microcurrent density J^μ are well known in the literature, see for instance, [Wentzel, (11.3)], [Morse Feshbach 1, Section 3.3, (3.3.27), (3.3.34), (3.3.35)]. It satisfies the conservation/continuity equations

$$\partial_\nu J^\nu = 0,\ \partial_t\rho + \nabla\cdot\mathbf{J} = 0,\ J^\nu = (\rho\mathrm{c}, \mathbf{J}). \tag{3.0.10}$$

Consequently, the total charge $\int \rho(\mathbf{x})\,\mathrm{d}\mathbf{x}$ of the elementary charge remains constant in the course of evolution, and we impose *charge normalization* condition which extends (2.0.20), namely

$$\int_{\mathbb{R}^3}\frac{\rho(\mathbf{x})}{q}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3}\left[-\left(\frac{\chi}{mc^2}\operatorname{Im}\frac{\partial_t\psi}{\psi} + \frac{q\varphi}{mc^2}\right)|\psi|^2\right](\mathbf{x})\,\mathrm{d}\mathbf{x} = 1. \tag{3.0.11}$$

We would like to stress that the equation (3.0.11) is perfectly consistent with the field equations and the conservation laws (3.0.10), and *it constitutes an independent and physically significant constraint for the total charge to be exactly q as in the Coulomb potential rather than an arbitrary constant.*

Applying general formulas (10.6.8)-(10.6.9) to the Lagrangian L_0 defined by (3.0.1)-(3.0.3) we obtain the following representations for the symmetric and gauge invariant energy-momenta tensors $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ for respectively the charge and the EM field the following formulas

$$T^{\mu\nu} = \frac{\chi^2}{2m}\left\{\left[\psi^{;\mu *}\psi^{;\nu} + \psi^{;\mu}\psi^{;\nu *}\right] - \left[\psi^*_{;\mu}\psi^{;\mu} - \kappa_0^2\psi^*\psi - G\left(\psi^*\psi\right)\right]g^{\mu\nu}\right\}, \tag{3.0.12}$$

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(g^{\mu\gamma}F_{\gamma\xi}F^{\xi\nu} + \frac{1}{4}g^{\mu\nu}F_{\gamma\xi}F^{\gamma\xi}\right), \; F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{3.0.13}$$

Energy conservation equations which we derive later in (10.5.29)-(10.5.30) turn here into

$$\partial_\mu T^{\mu\nu} = f^\nu, \; \partial_\mu \Theta^{\mu\nu} = -f^\nu, \tag{3.0.14}$$

where

$$f^\nu = \frac{1}{\mathrm{c}}J_\mu F^{\nu\mu} = \left(\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}, \rho\mathbf{E} + \frac{1}{\mathrm{c}}\mathbf{J}\times\mathbf{B}\right) \text{ is the Lorentz force density.} \tag{3.0.15}$$

3.1 Charge at rest

We say the dressed charge to be at rest at the origin $\mathbf{x} = \mathbf{0}$ if it is a radial solution to the field equations (3.0.6)-(3.0.7) of the following special form

$$\psi\left(t,\mathbf{x}\right) = \mathrm{e}^{-\mathrm{i}\omega_0 t}\mathring{\psi}\left(|\mathbf{x}|\right), \; \varphi\left(t,\mathbf{x}\right) = \mathring{\varphi}\left(|\mathbf{x}|\right), \; \mathbf{A}\left(t,\mathbf{x}\right) = \mathbf{0}, \; \omega_0 = \frac{m\mathrm{c}^2}{\chi}, \tag{3.1.1}$$

and we refer to such a solution as ω_0-*static*. Observe that as it follows from (3.0.8), (3.0.9) the micro-density ρ, micro-current $\mathbf{J}$ and 4-microcurrent J^ν for the ω_0-static solution (3.1.1) are

$$\rho = q\left(1 - \frac{q\mathring{\varphi}}{m\mathrm{c}^2}\right)\mathring{\psi}^2, \; \mathbf{J} = \mathbf{0}, \text{ and } \; J^\nu = \left(\rho\mathrm{c}, \mathbf{0}\right). \tag{3.1.2}$$

The charge normalization condition (3.0.11) then turns then into (2.0.20).

For the charge at rest as described by relations (3.1.1) the field equations (3.0.6)-(3.0.7) turn into the following system of two equations for the real-valued functions $\mathring{\psi}$ and $\mathring{\varphi}$ which we call *rest charge equations*:

$$-\Delta\mathring{\varphi} = 4\pi\mathring{\rho}, \; \mathring{\rho} = q\left(1 - \frac{q\mathring{\varphi}_a}{m\mathrm{c}^2}\right)\mathring{\psi}^2, \tag{3.1.3}$$

$$-\Delta\mathring{\psi} + \frac{mq\mathring{\varphi}}{\chi^2}\left(2 - \frac{q\mathring{\varphi}}{m\mathrm{c}^2}\right)\mathring{\psi} + G'\left(|\mathring{\psi}|^2\right)\mathring{\psi} = 0. \tag{3.1.4}$$

The radial functions $\mathring{\psi}$ and $\mathring{\varphi}$ play instrumental role in our constructions and we name them respectively *charge form factor* and *form factor potential*. As it follows from the equation (3.1.3) the charge form factor $\mathring{\psi} = \mathring{\varphi}_{\mathring{\psi}}$ determines the form factor potential $\mathring{\varphi}$ by the formula (2.0.14). Consequently, plugging in the above expression into the equation (2.0.13) we get the nonlinear equation (2.0.15) as follows:

$$-\Delta\mathring{\psi} + \frac{m\mathring{\varphi}_{\mathring{\psi}}}{\chi^2}q\left(2 - \frac{q\mathring{\varphi}_{\mathring{\psi}}}{m\mathrm{c}^2}\right)\mathring{\psi} + G'\left(|\mathring{\psi}|^2\right)\mathring{\psi} = 0. \tag{3.1.5}$$

As it is shown in next section the equation (3.1.5) signifies a complete balance (equilibrium) of the three forces acting upon the resting charge: (i) internal elastic deformation force associated with the term $-\Delta\mathring{\psi}$; (ii) charge's electromagnetic self-interaction force associated with the term $\frac{m\mathring{\varphi}_{\mathring{\psi}}}{\chi^2}\left(2q-\frac{q^2\mathring{\varphi}_{\mathring{\psi}}}{mc^2}\right)\mathring{\psi}$; (iii) internal nonlinear self-interaction of the charge associated with the term $G'\left(|\mathring{\psi}|^2\right)\mathring{\psi}$. We refer to the equation (3.1.5) as *charge equilibrium equation* or just the *equilibrium equation*. Importantly, the charge equilibrium equation (3.1.5) establishes an explicit relation between the form factor $\mathring{\psi}$ and the self-interaction nonlinearity G.

Now we come to a key point of our construction: determination of the nonlinearity G from the equilibrium equation (3.1.5). First we pick and fix a form factor $\mathring{\psi}(r)$, $r \geq 0$, which is assumed to be a nonnegative, monotonically decaying and sufficiently smooth function. Then we determine consequently G' and G from the equilibrium equation (3.1.5). This gives us at once the desired state of resting charge $\left\{\mathring{\psi},\mathring{\varphi}\right\}$ without solving any nontrivial nonlinear partial differential equation which is a stumbling block in most of theories involving nonlinearities. Of course such a benefit of our approach comes at a cost of dealing with a nontrivial nonlinearity G at all further steps, but it turns out that the definition of the nonlinearity via the equilibrium equation (3.1.5) is constructive enough for representing many important physical quantities in terms of $\mathring{\psi}$, $\mathring{\varphi}$ and G without explicit formulas for them. Curiously, for certain choices of $\mathring{\psi}$ one can find explicit formulas for $\mathring{\varphi}$, G and other important physical quantities as we show in Section 4.5.

3.2 Energy-momentum tensor, forces and equilibrium

In any classical field theory over the four dimensional continuum of space and time the energy-momentum tensor is of a fundamental importance. It provides for the density of the energy, the momentum and the surface forces as well as for the conservations laws that govern the energy and momentum transport in the space and time. It is worth to point out that it is the differential form of the energy-momentum conservation rather than the original field equations involve the densities of energy, momentum and forces and consequently they are more directly related to corpuscular properties of the fields. In particular, for charge model we study here the Lorentz force density arises in the differential form of the energy-momentum conservation equations and not in the original field equations. For detailed considerations of the structure and properties of the energy-momentum tensor including its symmetry, gauge invariance and conservation laws we refer the reader to Section 10. Here using the results of that section we compute and analyze the energy-momentum tensor for the Lagrangian L_0 defined by the formulas (3.0.1)-(3.0.3) and for the ω_0-static state defined by (3.1.1).

Using the interpretation form (10.2.19)-(10.2.20) of the energy-momentum $T^{\mu\nu}$ and formulas (3.0.12)-(3.0.13) we find that the energy-momentum tensor takes the following form

$$T^{\mu\nu}=\begin{bmatrix} u & \mathrm{c}p_1 & \mathrm{c}p_2 & \mathrm{c}p_3 \\ \mathrm{c}^{-1}s_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ \mathrm{c}^{-1}s_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ \mathrm{c}^{-1}s_3 & -\sigma_{31} & -\sigma^{32} & -\sigma_{33} \end{bmatrix}, \tag{3.2.1}$$

where the energy density u, the momentum and the energy flux components p_j and s_j are as follows

$$u=\frac{\chi^2}{2m}\left[\left(\nabla\mathring{\psi}\right)^2+G\left(\mathring{\psi}^2\right)\right]+\left(mc^2-q\mathring{\varphi}+\frac{q\mathring{\varphi}^2}{2mc^2}\right)\mathring{\psi}^2, \tag{3.2.2}$$

$$p^j=0,\ s^j=0,\ j=1,2,3, \tag{3.2.3}$$

and the stress tensor components σ_{ij} are represented by the formulas

$$\sigma_{ij}=-\frac{\chi^2}{m}\left[\partial_i\mathring{\psi}\partial_j\mathring{\psi}-\frac{1}{2}\left(\nabla\mathring{\psi}\right)^2\delta_{ij}\right]+ \tag{3.2.4}$$
$$\left[q\left(\mathring{\varphi}-\frac{q}{2mc^2}\mathring{\varphi}^2\right)\mathring{\psi}^2+\frac{\chi^2}{2m}G\left(\mathring{\psi}^2\right)\right]\delta_{ij},\ i,j=1,2,3.$$

Notice the vanishing of the momentum $\mathbf{p}$ and the energy flux $\mathbf{s}$ in (3.2.3) is yet another justification for the name ω_0-static solution. Observe also that for the ω_0-static state defined by (3.1.1) the EM field is

$$\mathbf{E}=-\nabla\mathring{\varphi},\ \mathbf{B}=\mathbf{0}. \tag{3.2.5}$$

Using the representation (10.4.21)-(10.4.22) for EM energy-momentum $\Theta^{\mu\nu}$ combined with the formulas (3.2.5) for the EM field we obtain the following representation of $\Theta^{\mu\nu}$ for the ω_0-static solution (3.1.1):

$$\Theta^{\mu\nu}=\begin{bmatrix} w & \mathrm{c}\mathbf{g}\\ \mathrm{c}\mathbf{g} & -\tau_{ij}\end{bmatrix},\ \text{where } w=\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi},\ g_i=0,\ i=1,2,3, \tag{3.2.6}$$

$$\Theta^{ij}=-\tau_{ij}=-\frac{1}{4\pi}\left[\partial_i\mathring{\varphi}\partial_j\mathring{\varphi}-\frac{\left(\nabla\mathring{\varphi}\right)^2}{2}\delta_{ij}\right],\ i,j=1,2,3. \tag{3.2.7}$$

Combining the conservation law (3.0.14) with general representation (3.2.1) of the charge energy-momentum tensor $T^{\mu\nu}$ we obtain

$$\partial_t p_i=\sum_{j=1,2,3}\partial_j\sigma_{ji}+\left[\rho E_i+\frac{1}{\mathrm{c}}\left(\mathbf{J}\times\mathbf{B}\right)_i\right]=0,\ i=1,2,3. \tag{3.2.8}$$

Notice that for the ω_0-static solution (3.1.1), in view of (3.2.5), (3.2.6), (3.2.7), the equation (3.2.8) turns into the equilibrium equations

$$\sum_{j=1,2,3}\partial_j\sigma_{ji}-\rho\partial_i\mathring{\varphi}=0,\ i=1,2,3. \tag{3.2.9}$$

Observe now that the stress tensor (str.t.) σ_{ij} defined (3.2.4) can be naturally decomposed in three components which we name as follows

$$\sigma_{ij}=\sigma_{ij}^{\mathrm{el}}+\sigma_{ij}^{\mathrm{em}}+\sigma_{ij}^{\mathrm{nl}},\ i,j=1,2,3,\ \text{where} \tag{3.2.10}$$

$$\sigma_{ij}^{\mathrm{el}}=-\frac{\chi^2}{m}\left[\partial_i\mathring{\psi}\partial_j\mathring{\psi}-\frac{1}{2}\left(\nabla\mathring{\psi}\right)^2\delta_{ij}\right]\ \text{is elastic deformation str. t.,} \tag{3.2.11}$$

$$\sigma_{ij}^{\mathrm{em}}=-p^{\mathrm{em}}\delta_{ij},\ p^{\mathrm{em}}=-q\left(\mathring{\varphi}-\frac{q\mathring{\varphi}^2}{2mc^2}\right)\mathring{\psi}^2\ \text{is EM interaction str. t.,} \tag{3.2.12}$$

$$\sigma_{ij}^{\mathrm{nl}}=-p^{\mathrm{nl}}\delta_{ij},\ p^{\mathrm{nl}}=-\frac{\chi^2 G\left(\mathring{\psi}^2\right)}{2m}\ \text{is nonlinear self-interaction str. t.,} \tag{3.2.13}$$

and consequently the respective volume force densities are

$$\sum_{j=1,2,3} \partial_j \sigma_{ij}^{\mathrm{el}} = f_i^{\mathrm{el}} = -\frac{\chi^2}{m} \Delta \mathring{\psi} \partial_i \mathring{\psi},\ i = 1,2,3 \tag{3.2.14}$$

$$\sum_{j=1,2,3} \partial_j \sigma_{ij}^{\mathrm{em}} = f_i^{\mathrm{em}} + \rho \partial_i \mathring{\varphi},\ f_i^{\mathrm{em}} = q \left(2\mathring{\varphi} - \frac{q}{mc^2} \mathring{\varphi}^2\right) \mathring{\psi} \partial_i \mathring{\psi}, \tag{3.2.15}$$

$$\sum_{j=1,2,3} \partial_j \sigma_{ij}^{\mathrm{nl}} = f_i^{\mathrm{nl}} = \frac{\chi^2}{m} G'\left(\mathring{\psi}^2\right) \mathring{\psi} \partial_i \mathring{\psi}. \tag{3.2.16}$$

Notice that the volume force density for the electromagnetic interaction stress in (3.2.15) has two parts: f_i^{em}, which we call *internal electromagnetic force*, and $\rho \partial_i \mathring{\varphi}$ which is the minus Lorentz force. Observe that the stress tensor $\sigma_{ij}^{\mathrm{el}}$ has the structure similar to the one for compressional waves, see Section 10.9 and (10.9.6), whereas the both stress tensors $\sigma_{ij}^{\mathrm{em}}$ and $\sigma_{ij}^{\mathrm{em}}$ have the structure typical for perfect fluids, [Moller, Section 6.6], with respective hydrostatic pressures p^{em} and p^{nl} defined by the relations (3.2.12)-(3.2.13). *Notice that the formula (3.2.13) provides an interpretation of the nonlinearity $G\left(\psi^2\right)$: $p^{\mathrm{nl}} = -\chi^2 G\left(\psi^2\right)/(2m)$ is the hydrostatic pressure when the charge as at rest.*

Based on the equalities (3.2.14)-(3.2.16) we can recast the equilibrium equation (3.2.9) as

$$f_i^{\mathrm{el}} + f_i^{\mathrm{em}} + f_i^{\mathrm{nl}} = 0,\ i = 1,2,3,\ \text{or} \tag{3.2.17}$$

$$\left[-\frac{\chi^2}{m} \Delta \mathring{\psi} + q\left(2 - \frac{q}{mc^2} \mathring{\varphi}\right) \mathring{\varphi} \mathring{\psi} + \frac{\chi^2}{m} G'\left(\mathring{\psi}^2\right) \mathring{\psi}\right] \partial_i \mathring{\psi} = 0.$$

The equation (3.2.17) signifies the ultimate equilibrium for the ω_0-static charge. It is evident from equation (3.2.17) that the scalar expression in the brackets before $\nabla \mathring{\psi}$ up to the factor $\frac{m}{\chi^2}$ is exactly the left-hand side of the equilibrium equation (3.1.5). In fact if $\nabla \mathring{\psi} \neq 0$ then the equilibrium equation (3.2.17) is equivalent to the scalar equilibrium equation (3.1.5).

Notice that since the $\mathring{\psi}(|\mathbf{x}|)$ and $\mathring{\varphi}(|\mathbf{x}|)$ are radial and monotonically decaying functions of $|\mathbf{x}|$ we readily have

$$\nabla \mathring{\psi}(|\mathbf{x}|) = -\left|\nabla \mathring{\psi}\right| \hat{\mathbf{x}},\ \nabla \mathring{\varphi}(|\mathbf{x}|) = -\left|\nabla \mathring{\varphi}\right| \hat{\mathbf{x}},\ \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|} = (\hat{x}_1, \hat{x}_2, \hat{x}_3). \tag{3.2.18}$$

The relations (3.2.18) combined with (3.2.14)-(3.2.16) imply that for the resting charge all the forces $\mathbf{f}^{\mathrm{el}}$, $\mathbf{f}^{\mathrm{em}}$ and $\mathbf{f}^{\mathrm{nl}}$ and radial, i.e. they are functions of $|\mathbf{x}|$ and point toward or outward the origin.

Notice that it follows from the relations (3.2.3) and (3.2.6) that *the total momentum P and the energy flux S of the resting dressed charge, i.e. the charge and the EM field together, vanish*, and hence we have

$$\mathbf{P} = \mathbf{0},\ \mathbf{S} = \mathbf{0},\ \text{and } P^{\nu} = (u, \mathbf{0}), \tag{3.2.19}$$

where the energy density u is represented by the formula (3.2.2). We would like to point out that the vanishing for the resting charge of the micro-current $\mathbf{J}$ in (3.1.2) as well as the momentum $\mathbf{P}$ and the energy flux $\mathbf{S}$ in (3.2.19) justifies the name ω_0-static solution.

Using the general formulas (10.2.16) for the angular momentum density $M^{\mu\nu\gamma}$ and combining them with the relations (3.2.1)-(3.2.3) for the energy-momentum tensor $T^{\mu\nu}$ we readily obtain that the *total angular momentum $J^{\nu\gamma}$ vanishes*, namely

$$M^{0\nu\gamma} = 0 \text{ implying } J^{\nu\gamma} = \int_{\mathbb{R}^3} M^{0\nu\gamma}(x)\, \mathrm{d}\mathbf{x} = 0. \tag{3.2.20}$$

3.3 Frequency shifted Lagrangian and the reduced energy

Time harmonic factor $\mathrm{e}^{-\mathrm{i}\omega_0 t}$ which appears in ω_0-static states as in (3.1.1) plays a very important role in this theory including the nonrelativistic case. To reflect that we introduce a change of variables

$$\psi(t,\mathbf{x}) \to \mathrm{e}^{-\mathrm{i}\omega_0 t}\psi(t,\mathbf{x}) \tag{3.3.1}$$

and substitute it in the Lagrangian L_0 defined by (3.0.1) to obtain Lagrangian L_{ω_0}, which we call *frequency shifted*, namely

$$\begin{gathered} L_{\omega_0}(\psi, A^\mu) = \frac{\chi}{2}\mathrm{i}\left(\psi^*\tilde{\partial}_t\psi - \psi\tilde{\partial}_t^*\psi^*\right) + \\ +\frac{\chi^2}{2m}\left\{\frac{1}{\mathrm{c}^2}\tilde{\partial}_t\psi\tilde{\partial}_t^*\psi^* - \tilde{\nabla}\psi\tilde{\nabla}^*\psi^* - G(\psi^*\psi)\right\} - \frac{F^{\mu\nu}F_{\mu\nu}}{16\pi}, \\ \tilde{\partial}_t = \partial_t + \frac{\mathrm{i}q\varphi}{\chi},\ \tilde{\nabla} = \nabla - \frac{\mathrm{i}q\mathbf{A}}{\chi\mathrm{c}},\ \tilde{\partial}_t^* = \partial_t - \frac{\mathrm{i}q\varphi}{\chi},\ \tilde{\nabla}^* = \nabla + \frac{\mathrm{i}q\mathbf{A}}{\chi\mathrm{c}}. \end{gathered} \tag{3.3.2}$$

If we use the relation (10.4.16) we can rewrite it in the form

$$\begin{gathered} L_{\omega_0}(\psi, \psi^*, A^\mu) = \frac{\chi}{2}\mathrm{i}\left(\psi^*\tilde{\partial}_t\psi - \psi\tilde{\partial}_t^*\psi^*\right) + \\ \frac{\chi^2}{2m}\left\{\frac{1}{\mathrm{c}^2}\tilde{\partial}_t\psi\tilde{\partial}_t^*\psi^* - \tilde{\nabla}\psi\tilde{\nabla}^*\psi^* - G(\psi^*\psi)\right\} + \frac{1}{8\pi}\left[\left(\nabla\varphi + \frac{1}{\mathrm{c}}\partial_t\mathbf{A}\right)^2 - (\nabla\times\mathbf{A})^2\right] \end{gathered} \tag{3.3.3}$$

The Lagrangian L_{ω_0} defined by the formula (3.3.2) is manifestly gauge and space-time translation invariant, it also invariant with respect to space rotations but it is not invariant with respect to the entire group of Lorentz transformations. Notice also ω_0-static states for the original Lagrangian defined by (3.0.1) turns into regular static states for the Lagrangian L_{ω_0}, and that was one of the reasons to introduce it.

For a ω_0-static state $\left\{\mathrm{e}^{-\mathrm{i}\omega_0 t}\mathring{\psi}, \mathrm{e}^{\mathrm{i}\omega_0 t}\mathring{\psi}^*, \mathring{\varphi}\right\}$ satisfying the field equations (3.1.3)-(3.1.4) its canonical density of energy $\mathring{u}_{L_0}\left(\mathring{\psi},\ \mathring{\varphi}\right)$ as defined by (10.2.5) can be simply related to the canonical energy $\mathring{u}_{L_{\omega_0}}\left(\mathring{\psi},\ \mathring{\varphi}\right)$ of the frequency shifted Lagrangian L_{ω_0}. Indeed applying the arguments provided in Section 10.8.1, particularly relations (10.8.37)-(10.8.40), and combined with the representation (3.3.3) we find that

$$\begin{gathered} \mathring{u}_{L_0}\left(\mathring{\psi},\ \mathring{\varphi}\right) = \frac{m\mathrm{c}^2}{q}\rho - L_{\omega_0}\left(\mathring{\psi}, \mathring{\psi}^*, \mathring{\varphi}\right) = \\ = \frac{m\mathrm{c}^2}{q}\rho + \frac{\chi^2}{2m}\left[\left(\nabla\mathring{\psi}\right)^2 + \left(\frac{\omega_0}{\mathrm{c}} - \frac{q}{\chi\mathrm{c}}\mathring{\varphi}\right)^2\mathring{\psi}^2 + G\left(\mathring{\psi}^2\right)\right] - \frac{(\nabla\mathring{\varphi})^2}{8\pi}, \end{gathered} \tag{3.3.4}$$

and that the total energy in this state can be represented in the form (2.1.1) using results of Section 10.8. The energy representation (2.1.1) is important to us since it does not involve explicitly the nonlinear self-interaction G.

3.4 Moving charge

As it is often done in the literature we use the Lorentz invariance of the system to obtain the state of the dressed charge moving with a constant velocity $\mathbf{v}$. Namely, we apply to the rest solution described by (3.1.1)-(3.1.4) the Lorentz transformation from the original "rest frame" to the frame in which the "rest frame" moves with the constant velocity $\mathbf{v}$ as described by the formulas (10.1.6), (10.4.12) (so $\mathbf{x}'$ and $\mathbf{x}$ correspond respectively to the "rest" and "moving" frames) yielding

$$\psi(t,\mathbf{x}) = \mathrm{e}^{-\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{x})}\mathring{\psi}(\mathbf{x}'),\ \varphi(t,\mathbf{x}) = \gamma\mathring{\varphi}(|\mathbf{x}'|),\ \mathbf{A}(t,\mathbf{x}) = \gamma\boldsymbol{\beta}\mathring{\varphi}(|\mathbf{x}'|), \tag{3.4.1}$$

$$\mathbf{E}(t,\mathbf{x}) = -\gamma\nabla\mathring{\varphi}(|\mathbf{x}'|) + \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\nabla\mathring{\varphi}(|\mathbf{x}'|))\boldsymbol{\beta},\ \mathbf{B}(t,\mathbf{x}) = \gamma\boldsymbol{\beta}\times\nabla\mathring{\varphi}(|\mathbf{x}'|), \tag{3.4.2}$$

where

$$\omega = \gamma\omega_0,\ \mathbf{k} = \gamma\boldsymbol{\beta}\frac{\omega_0}{\mathrm{c}},\ \boldsymbol{\beta} = \frac{\mathbf{v}}{\mathrm{c}},\ \beta = |\boldsymbol{\beta}|,\ \gamma = \left(1-\left(\frac{v}{\mathrm{c}}\right)^2\right)^{-1/2}, \tag{3.4.3}$$

$$\mathbf{x}' = \mathbf{x} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\mathbf{v}t,\ \text{or}\ \mathbf{x}'_\parallel = \gamma\left(\mathbf{x}_\parallel - \mathbf{v}t\right),\ \mathbf{x}'_\perp = \mathbf{x}_\perp, \tag{3.4.4}$$

where $\mathbf{x}_\parallel$ and $\mathbf{x}_\perp$ refer respectively to the components of $\mathbf{x}$ parallel and perpendicular to the velocity $\mathbf{v}$, ψ^* is complex conjugate to ψ. The above formulas provide a solution to field equations (3.0.6), (3.0.7) and indicate that the fields of the dressed charge contract by the factor γ as it moves with the velocity $\mathbf{v}$ compare to their rest state. The first oscillatory exponential factor in (3.4.1) is the *de Broglie plane wave* of the frequency ω and the *de Broglie wave-vector* $\mathbf{k}$. Notice that the equalities (3.4.3) readily imply the following relations between ω, $\mathbf{k}$ and $\mathbf{v}$

$$\omega = \omega(\mathbf{k}) = \sqrt{\omega_0^2 + \mathrm{c}^2\mathbf{k}^2},\ \mathbf{v} = \nabla_{\mathbf{k}}\omega(\mathbf{k}),\ \text{where}\ \omega_0 = \frac{\mathcal{E}_0\left(\mathring{\psi}\right)}{\hbar} = \frac{m\mathrm{c}^2}{\chi}, \tag{3.4.5}$$

and we refer to Section 2, formulas (2.1.4)-(2.1.6), for the values of the frequency ω_0 and the constant χ.

Notice that the above relations show, in particular, that for the freely moving dressed charge defined by equalities (3.4.1)-(3.4.4) its velocity $\mathbf{v}$ equals exactly the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$ computed for the de Broglie wave vector $\mathbf{k}$. This fact clearly points to the wave origin of the charge kinematics as it moves in the three dimensional space continuum with the dispersion relation $\omega = \sqrt{\omega_0^2 + \mathrm{c}^2\mathbf{k}^2}$. Notice that this dispersion relation is identical to the dispersion relation of the Klein-Gordon equation as a model for a free charge, [Pauli PWM, Section 18].

Now we consider total 4-momentum P obtained from its density by integration over the space $\mathbb{R}^3$. Since the dressed charge is a closed system its total 4-momentum $\mathsf{P}^\nu = (\mathsf{E}, \mathrm{c}\mathsf{P})$ is 4-vector, see the end of Section 10.2. Using this vector property and the value $\mathsf{P}^\nu = \left(\mathcal{E}_0\left(\mathring{\psi}\right), \mathbf{0}\right)$ for the resting dressed charge we find, by applying the relevant Lorentz transformation, that the dressed charge 4-momentum P^ν satisfies

$$\mathsf{P}^\nu = (\mathsf{E}, \mathrm{c}\mathsf{P}),\ \mathsf{E} = \hbar\omega,\ \mathsf{P} = \hbar\mathbf{k}, \tag{3.4.6}$$

showing that *the Einstein-de Broglie relations hold for the moving charge.* We would like to point out that, though the above argument used to obtain the relations (3.4.6) is rather standard, in our case relations (3.4.6) are deduced rather than rationally imposed.

Observe that our relations (3.4.5) under the assumption that $\chi = \hbar$ are identical to those of a free charge as described by the Klein-Gordon equation, [Pauli PWM, Sections 1, 18], (see also Section 10.10) but there are several significant differences between the two models which are as follows. First of all, our charge is a dressed charge described by the pair $\{\psi, A^{\mu}\}$. From the very outset it includes the EM field as its inseparable part whereas the Klein-Gordon model describes a free charge by a complex-valued wave function ψ which is not coupled to its own EM field (not to be confused to with an external EM field). Second of all, our free dressed charge as it moves evidently preserves its shape up the natural Lorentz construction whereas any wavepacket satisfying Klein-Gordon equation spreads out in the course of time.

3.5 Correspondence with the point charge mechanics

The free dressed charge as described by equalities (3.4.1)-(3.4.4) allows for a certain reduction to the model of point charge (mass). Notice that combining the relations (3.4.6) with (3.4.3) we obtain the well known point mass kinematic representations (10.1.13) for the total energy E and the momentum P of the dressed charge, namely

$$\mathsf{P} = \hbar\mathbf{k} = \gamma\boldsymbol{\beta}\frac{\hbar\omega_0}{\mathrm{c}} = \gamma\tilde{m}\mathbf{v}, \gamma = \left(1 - \left(\frac{v}{\mathrm{c}}\right)^2\right)^{-1/2}, \tag{3.5.1}$$

$$\mathsf{E} = \hbar\omega = \hbar\gamma\omega_0 = \gamma\tilde{m}\mathrm{c}^2 = \mathrm{c}\sqrt{\mathsf{P}^2 + \tilde{m}^2\mathrm{c}^2}, \tag{3.5.2}$$

where $\tilde{m}$ is the dressed charge mass defined by (2.1.2). We can also reasonably assign to the dressed charge described by equalities (3.4.1)-(3.4.4) a location $\mathbf{r}(t)$ at any instant t of time which is obtained from the (3.4.4) by setting there $\mathbf{x}'$ and solving it for $\mathbf{x}$, $\mathbf{r}(t) = \mathbf{x}(\mathbf{x}', t)$. Not surprisingly, its solution is

$$\mathbf{r}(t) = \mathbf{v}t. \tag{3.5.3}$$

An elementary examination confirms that $(\mathrm{c}t, \mathbf{v}t)$ transforms as a 4-vector implying that the definition (3.5.3) is both natural and relativistically consistent. From (3.5.3) we readily obtain another fundamental relation for the point charge

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}. \tag{3.5.4}$$

4 Single nonrelativistic free and resting charge

The nonrelativistic case, i.e. the case when a charge moves with a velocity much smaller than the velocity of light, is important for our studies for at least two reasons. First of all, we need it to relate the wave-corpuscle mechanics to the Newtonian mechanics for point charges in EM field. Second, in the nonrelativistic case we can carry out rather detailed analytical studies of many of physical quantities in a closed form. With that in mind, we would like to treat the nonrelativistic case not just as an approximation to the relativistic theory but rather as a case on its own, and we do it by constructing a certain *nonrelativistic Lagrangian* $\hat{L}_0$ intimately related to the relativistic Lagrangian defined in (3.0.1)-(3.0.3). This nonrelativistic Lagrangian constitutes a fundamental basis for our nonrelativistic studies including the construction of the nonlinear self-interaction. The relation between the relativistic and nonrelativistic Lagrangians is considered in Section 7.

The nonrelativistic Lagrangian $\hat{L}_0$ is constructed as a certain nonrelativistic modification of the frequency shifted Lagrangian $L_{\omega_0}(\psi, A^{\mu})$ introduced in Section 3.3. The first step in this modification is the change of variables (3.3.1), namely

$$\psi(t, \mathbf{x}) \rightarrow \mathrm{e}^{-\mathrm{i}\omega_0 t}\psi(t, \mathbf{x}) \tag{4.0.5}$$

which was the initial step in the construction of the frequency shifted Lagrangian L_{ω_0} defined by (3.3.2)-(3.3.3). Then a gauge invariant and nonrelativistic Lagrangian $\hat{L}_0$ is obtained from from the Lagrangian L_{ω_0} by omitting in (3.3.3) the term $\frac{\chi^2}{2mc^2}\tilde{\partial}_t\psi\tilde{\partial}_t^*\psi^*$ and setting $\mathbf{A}=0$, namely

$$\hat{L}_0\left(\psi,\psi^*,\varphi\right)=\frac{\chi\mathrm{i}}{2}\left[\psi^*\tilde{\partial}_t\psi-\psi\tilde{\partial}_t^*\psi^*\right]-\frac{\chi^2}{2m}\left[\nabla\psi\nabla\psi^*+G\left(\psi^*\psi\right)\right]+\frac{\left|\nabla\varphi\right|^2}{8\pi}, \tag{4.0.6}$$
$$\tilde{\partial}_t=\partial_t+\frac{\mathrm{i}q\varphi}{\chi},\ \tilde{\partial}_t^*=\partial_t-\frac{\mathrm{i}q\varphi}{\chi}$$

where, we remind, the term $G\left(\psi^*\psi\right)$ corresponds to the charge nonlinear self-interaction. Observe that the assumption $\mathbf{A}=0$ in view of (10.4.6) readily implies

$$\mathbf{E}=-\nabla\varphi-\frac{1}{\mathrm{c}}\partial_t\mathbf{A}=-\nabla\varphi,\ \mathbf{B}=\nabla\times\mathbf{A}=\mathbf{0}. \tag{4.0.7}$$

Hence, the EM field tensor $F^{\mu\nu}$ defined by (10.4.5) takes here a simpler form

$$F^{\mu\nu}=\begin{bmatrix}0&-E_1&-E_2&-E_3\\E_1&0&0&0\\E_2&0&0&0\\E_3&0&0&0\end{bmatrix}. \tag{4.0.8}$$

The mentioned gauge invariance is understood with respect first to the gauge transformation of the first kind (global) as in (10.5.7) and of the second (local) types as in (10.5.7)-(10.5.8), namely

$$\psi\rightarrow\mathrm{e}^{\mathrm{i}\gamma}\psi,\ \psi^*\rightarrow\mathrm{e}^{-\mathrm{i}\gamma}\psi^*,\ \text{where }\gamma\text{ is any real constant}, \tag{4.0.9}$$

and with respect to a reduced version of the second type gauge transformation

$$\psi\rightarrow\mathrm{e}^{-\frac{\mathrm{i}q\lambda(t)}{\chi}}\psi,\ \psi^*\rightarrow\mathrm{e}^{\frac{\mathrm{i}q\lambda(t)}{\chi}}\psi^*,\ \varphi\rightarrow\varphi+\partial_t\lambda\left(t\right), \tag{4.0.10}$$

which is similar to (10.5.8) but the function $\lambda\left(t\right)$ may depend only on time.

Evidently the EM field of the charge is represented in the above Lagrangian $\hat{L}_0$ only by its scalar potential φ and the corresponding electric field $\mathbf{E}=-\nabla\varphi$ since $\mathbf{A}=0$. The charge magnetic field is identically zero in view of the equalities (4.0.7), and, consequently, any radiation phenomena are excluded in this model. The Lagrangian $\hat{L}_0$ can be viewed as a field version of point charges model (6.1.30) that neglects all retardation effects in the static limit (zeroth order in $\frac{v}{\mathrm{c}}$) with the "instantaneous" interaction Lagrangian $-\frac{q_1q_2}{|\mathbf{r}_1-\mathbf{r}_2|}$ between two charges, [Jackson, Section 12.6]. More detailed discussion on the relations between relativistic and nonrelativistic Lagrangians and the corresponding Euler-Lagrange equations is provided in Section 7.

The Euler-Lagrange field equations for this Lagrangian are

$$\chi\mathrm{i}\tilde{\partial}_t\psi=\frac{\chi^2}{2m}\left[-\Delta\psi+G'\left(\psi^*\psi\right)\psi\right], \tag{4.0.11}$$

$$-\Delta\varphi=4\pi q\psi\psi^*,\ \text{where } G'\left(s\right)=\partial_sG\left(s\right),\tilde{\partial}_t=\partial_t+\frac{\mathrm{i}q\varphi}{\chi} \tag{4.0.12}$$

and we refer to the pair $\{\psi,\varphi\}$ as *dressed charge*. Notice that we always take ψ^* to be complex conjugate to ψ. Hence, taking into account the form of the covariant time derivative from (4.0.6) we can recast the field equations (4.0.11)-(4.0.12) for the dressed charge as

$$\chi \mathrm{i}\partial_t\psi=\frac{\chi^2}{2m}\left[-\Delta+\frac{2mq}{\chi^2}\varphi+G'\left(|\psi|^2\right)\right]\psi,\ -\Delta\varphi=4\pi q\,|\psi|^2\,. \tag{4.0.13}$$

which imply (2.3.3), (2.3.4).

Applying the general formulas (10.7.12)-(10.7.15) for the charge and current densities to the Lagrangian $\hat{L}_0$ we obtain expressions (2.3.9) for the densities and since external fields are absent, the current J^μ satisfies the conservation/continuity equations (2.3.10). Consequently, the total charge remains constant in the course of evolution, and as always we set this constant charge to be exactly q, namely we impose *charge normalization* condition (2.3.12)

$$\int_{\mathbb{R}^3}\rho\left(x\right)\,\mathrm{d}\mathbf{x}=q\int_{\mathbb{R}^3}\psi\psi^*\mathrm{d}\mathbf{x}=q\text{ or }\int_{\mathbb{R}^3}|\psi|^2\,\mathrm{d}\mathbf{x}=1. \tag{4.0.14}$$

As in the relativistic case the equation (2.3.10) follows from the field equations, therefore (4.0.14) is preserved for all times.

4.1 Symmetries and conservation laws

To carry out a systematic analysis of conservation laws associated with the Lagrangian $\hat{L}_0$ defined by (4.0.6) via Noether theorem, see Section 10.3, we need to find a Lie group of transformations which preserve it. The Lagrangian $\hat{L}_0$ is not invariant with respect to either the Lorentz or the Galilean groups of transformations. But a straightforward examination shows that $\hat{L}_0$ is invariant with respect to the following *Galilean-gauge group of transformations*

$$\begin{aligned} t' &= t,\ \mathbf{x}'=\mathbf{x}-\mathbf{v}t\text{ or }t=t',\ \mathbf{x}=\mathbf{x}'+\mathbf{v}t', \\ x^{0\prime} &= x^0,\ \mathbf{x}'=\mathbf{x}-\frac{\mathbf{v}}{\mathrm{c}}x^0\text{ or }x^0=x^{0\prime},\ \mathbf{x}=\mathbf{x}'+\frac{\mathbf{v}}{\mathrm{c}}x^{0\prime}, \end{aligned} \tag{4.1.1}$$

$$\begin{gathered} \psi\left(t,\mathbf{x}\right)=\mathrm{e}^{\mathrm{i}\frac{m}{2\chi}\left(\mathbf{v}^2t'+2\mathbf{v}\cdot\mathbf{x}'\right)}\psi'\left(t',\mathbf{x}'\right),\ \psi^*\left(t,\mathbf{x}\right)=\mathrm{e}^{-\mathrm{i}\frac{m}{2\chi}\left(\mathbf{v}^2t'+2\mathbf{v}\cdot\mathbf{x}'\right)}\psi^{*\prime}\left(t',\mathbf{x}'\right)\text{ or} \\ \psi'\left(t',\mathbf{x}'\right)=\mathrm{e}^{\mathrm{i}\frac{m}{2\chi}\left(\mathbf{v}^2t-2\mathbf{v}\cdot\mathbf{x}\right)}\psi\left(t,\mathbf{x}\right),\ \psi^{*\prime}\left(t',\mathbf{x}'\right)=\mathrm{e}^{-\mathrm{i}\frac{m}{2\chi}\left(\mathbf{v}^2t-2\mathbf{v}\cdot\mathbf{x}\right)}\psi^*\left(t,\mathbf{x}\right),\text{ and} \\ \varphi\left(t,\mathbf{x}\right)=\varphi'\left(t',\mathbf{x}'\right). \end{gathered} \tag{4.1.2}$$

One can also verify that the above transformations form an Abelian (commutative) group of transformation parametrized by the velocity parameter $\mathbf{v}$. It is curious to observe that according to the Galilean-gauge transformations (4.1.1), (4.1.2) the charge wave function does not transform as a scalar as in the relativistic case. These transformations were used in studies of nonlinear Schrödinger equations, [Sulem, Section 2.3].

The defined above Galilean-gauge group is naturally extended to the *general inhomogeneous Galilean-gauge group* by adding to it the group of spacial rotations O and space-time translations a^μ, namely

$$t'=t+\tau,\ \mathbf{x}'=O\mathbf{x}-\mathbf{v}t+\mathbf{a}\text{ or }t=t'-\tau,\ \mathbf{x}=O^{-1}\left(\mathbf{x}'+\mathbf{v}t'-\mathbf{a}\right), \tag{4.1.3}$$

$$\psi'(t',\mathbf{x}') = \mathrm{e}^{\mathrm{i}\frac{m}{2\chi}\left[\mathbf{v}^2(t+\tau)-2\mathbf{v}\cdot(O\mathbf{x}+\mathbf{a})\right]}\psi(t,\mathbf{x}), \tag{4.1.4}$$
$$\psi^{*\prime}(t',\mathbf{x}') = \mathrm{e}^{-\mathrm{i}\frac{m}{2\chi}\left[\mathbf{v}^2(t+\tau)-2\mathbf{v}\cdot(O\mathbf{x}+\mathbf{a})\right]}\psi^*(t,\mathbf{x}), \text{ and } \varphi'(t',\mathbf{x}') = \varphi(t,\mathbf{x}).$$

The *infinitesimal form* of the above group of transformations is as follows

$$t' = t+\tau,\ \mathbf{x}' = \mathbf{x}+\boldsymbol{\xi}\times\mathbf{x}-\mathbf{v}t+\mathbf{a},\ a^\mu = (\mathrm{c}\tau,\mathbf{a}),\ a^0 = \mathrm{c}\tau,\ x^0 = \mathrm{c}t, \tag{4.1.5}$$

or, equivalently,

$$x'^\mu = x^\mu + \xi^{\mu\nu}x_\nu + a^\mu,\ \xi^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{v^1}{\mathrm{c}} & 0 & -\xi^3 & \xi^2 \\ -\frac{v^2}{\mathrm{c}} & \xi^3 & 0 & -\xi^1 \\ -\frac{v^3}{\mathrm{c}} & -\xi^2 & \xi^1 & 0 \end{bmatrix}, \tag{4.1.6}$$

where the real number τ and the coordinates of the tree three-dimensional vectors $\mathbf{v}$, $\boldsymbol{\xi}$, $\mathbf{a}$ provide for the total of ten real parameters as in the case of the infinitesimal inhomogeneous Lorentz group defined by (10.1.10). The infinitesimal form of the transformations (4.1.4) is

$$\delta\psi = -\mathrm{i}\frac{m}{\chi}\left(\sum_{j=1,2,3} x^j\cdot\delta v^j\right)\psi,\ \delta\psi^* = \mathrm{i}\frac{m}{\chi}\left(\sum_{j=1,2,3} x^j\cdot\delta v^j\right)\psi^*,\ \delta\varphi = 0. \tag{4.1.7}$$

Coming back to the analysis of basic features of our model we acknowledge the use in this section of the relativistic conventions for upper and lower indices and the summation as in Section 10.1 including

$$x^\mu = \left(x^0,\mathbf{x}\right),\ x_\mu = \left(x^0,-\mathbf{x}\right),\ x^0 = \mathrm{c}t. \tag{4.1.8}$$

Carrying out the Noether currents analysis as in Section 10.3 for the Lagrangian $\hat{L}_0$ we obtain 10 conservation laws which, as it turns out, can be formulated in terms of the canonical energy-momentum tensor $\mathring{\mathcal{T}}^{\mu\nu}$, which in turn is obtained from the general formula (10.2.5):

$$\mathring{\mathcal{T}}^{\mu\nu} = \frac{\partial\hat{L}_0}{\partial\psi_{,\mu}}\psi_{,\nu} + \frac{\partial\hat{L}_0}{\partial\psi^*_{,\mu}}\psi^*_{,\nu} + \frac{\partial\hat{L}_0}{\partial\varphi_{,\mu}}\varphi_{,\nu} - \hat{L}_0 g^{\mu\nu}. \tag{4.1.9}$$

Namely, we get the total of ten conservation laws:

$$\partial_\mu\mathring{\mathcal{T}}^{\mu\nu} = 0 \text{ - energy-momentum conserv.,} \tag{4.1.10}$$
$$\mathring{\mathcal{T}}^{ij} = \mathring{\mathcal{T}}^{ji},\ i,j = 1,2,3 \text{ - space angular momentum conserv.,} \tag{4.1.11}$$
$$P^i = \mathring{\mathcal{T}}^{0i} = \frac{m}{q}J^i,\ i = 1,2,3 \text{ - time-space angular momentum conserv.} \tag{4.1.12}$$

The the first four standard conservation laws (4.1.10) are associated with the Noether's currents with respect to space-time translations a^μ. The second three conservations laws in (4.1.11) are associated with space rotations parameters $\boldsymbol{\xi}$, and they turn into the symmetry of the energy-momentum tensor $\mathring{\mathcal{T}}^{\mu\nu}$ for the spatial indices similarly to relations (10.2.16)-(10.2.17). The form of the *last three conservation laws (4.1.12) is special to the nonrelativistic Lagrangian $\hat{L}_0$, and it is due the Galilean-gauge invariance (4.1.2), (4.1.4).* These relations indicate that the *total momentum density P^i is identically equal up to the factor $\frac{m}{q}$ to the microcurrent density J^i* defined by (2.3.10). This important identity is analogous to the kinematic representation $\mathbf{p} = m\mathbf{v}$ of the momentum $\mathbf{p}$ of a point charge. It is related to the velocity components $\mathbf{v}$ in the Galilean-gauge transformations (4.1.2), (4.1.4), and can be traced to the infinitesimal transformation (4.1.7) and the phases in (4.1.4). The proportionality of the momentum and the current is known to occur for systems governed by the nonlinear Schrödinger equations, [Sulem, Section 2.3].

The issue of fundamental importance of studies of the energy-momentum tensor has been already addressed in the beginning of Section 3.2. To find the energy-momentum tensor for the Lagrangian $\hat{L}_0$ we apply to it the general formulas from Section 10.6. The canonical energy-momentum $\mathring{\Theta}^{\mu\nu}$ for the EM field is obtained by applying the general formula (10.2.5) to the Lagrangian $\hat{L}_0$ yielding

$$\mathring{\Theta}^{\mu\nu} = \begin{bmatrix} \mathring{w} & \mathrm{c}\mathring{g}_1 & \mathrm{c}\mathring{g}_2 & \mathrm{c}\mathring{g}_3 \\ \mathrm{c}^{-1}\mathring{s}_1 & -\mathring{\tau}_{11} & -\mathring{\tau}_{12} & -\mathring{\tau}_{13} \\ \mathrm{c}^{-1}\mathring{s}_2 & -\mathring{\tau}_{21} & -\mathring{\tau}_{22} & -\mathring{\tau}_{23} \\ \mathrm{c}^{-1}\mathring{s}_3 & -\mathring{\tau}_{31} & -\mathring{\tau}_{32} & -\mathring{\tau}_{33} \end{bmatrix} = \begin{bmatrix} \mathring{w} & 0 & 0 & 0 \\ \mathrm{c}^{-1}\mathring{s}_1 & -\mathring{\tau}_{11} & -\mathring{\tau}_{12} & -\mathring{\tau}_{13} \\ \mathrm{c}^{-1}\mathring{s}_2 & -\mathring{\tau}_{21} & -\mathring{\tau}_{22} & -\mathring{\tau}_{23} \\ \mathrm{c}^{-1}\mathring{s}_3 & -\mathring{\tau}_{31} & -\mathring{\tau}_{32} & -\mathring{\tau}_{33} \end{bmatrix}, \tag{4.1.13}$$

$$\begin{gathered} \mathring{w} = -\frac{|\nabla\varphi|^2}{8\pi},\ \mathring{g}_j = 0,\ \mathring{s}_j = \mathrm{c}\frac{\partial_j\varphi\partial_0\varphi}{4\pi}, \\ \mathring{\tau}_{jj} = \frac{\partial_j^2\varphi}{4\pi} - \frac{|\nabla\varphi|^2}{8\pi} = \frac{\partial_j^2\varphi}{4\pi} + \mathring{w},\ \mathring{\tau}_{ij} = \frac{\partial_i\varphi\partial_j\varphi}{4\pi}. \end{gathered} \tag{4.1.14}$$

The gauge invariant energy-momentum of the EM field takes the form

$$\Theta^{\mu\nu} = \begin{bmatrix} w & \mathrm{c}g_1 & \mathrm{c}g_2 & \mathrm{c}g_3 \\ \mathrm{c}^{-1}g_1 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ \mathrm{c}^{-1}g_2 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ \mathrm{c}^{-1}g_3 & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix} = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ 0 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ 0 & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix}, \tag{4.1.15}$$

with matrix entries

$$\begin{gathered} \partial_0 w = \frac{\mathbf{J}\cdot\nabla\varphi}{\mathrm{c}} = -\frac{\mathbf{J}\cdot\mathbf{E}}{\mathrm{c}},\ g_j = 0,\ s_j = 0, \\ \tau_{jj} = \frac{\partial_j^2\varphi}{4\pi} - \frac{|\nabla\varphi|^2}{8\pi},\ \tau_{ij} = \frac{\partial_i\varphi\partial_j\varphi}{4\pi}. \end{gathered} \tag{4.1.16}$$

As we can see the relation (4.1.16) involves the time derivative $\partial_0 w$ for the energy density w rather then density itself, and it follows from it that

$$\begin{aligned} w = w(t,\mathbf{x}) &= w_0(\mathbf{x}) + \int_{-\infty}^{t} \frac{\mathbf{J}(t',\mathbf{x})\cdot\nabla\varphi(t',\mathbf{x})}{\mathrm{c}}\,\mathrm{d}t' \\ &= w_0(\mathbf{x}) + \int_{-\infty}^{t} -\frac{\mathbf{J}(t',\mathbf{x})\cdot\mathbf{E}(t',\mathbf{x})}{\mathrm{c}}\,\mathrm{d}t', \end{aligned} \tag{4.1.17}$$

where $w_0(\mathbf{x})$ is a time independent energy density. Notice also that combining the relations (2.3.9), (2.3.10), (4.0.12) and (4.1.17) we obtain the following identity

$$\begin{gathered} \partial_0\int_{\mathbb{R}^3} w\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3}\frac{\mathbf{J}\cdot\nabla\varphi}{\mathrm{c}}\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^3}\frac{\varphi(\nabla\cdot\mathbf{J})}{\mathrm{c}}\,\mathrm{d}\mathbf{x} = \frac{1}{4\pi\mathrm{c}}\int_{\mathbb{R}^3}\varphi\partial_t\rho\,\mathrm{d}\mathbf{x} = \\ -\frac{1}{4\pi\mathrm{c}}\int_{\mathbb{R}^3}\varphi\partial_t\nabla^2\varphi\,\mathrm{d}\mathbf{x} = \frac{1}{4\pi\mathrm{c}}\int_{\mathbb{R}^3}\nabla\varphi\cdot\partial_t\nabla\varphi\,\mathrm{d}\mathbf{x} = \partial_0\int_{\mathbb{R}^3}\frac{(\nabla\varphi)^2}{8\pi}\,\mathrm{d}\mathbf{x}. \end{gathered} \tag{4.1.18}$$

A consistent with the canonical energy-momentum choice for $w_0(\mathbf{x})$ in (4.1.17) is $w_0(\mathbf{x}) = \frac{(\nabla\varphi)^2}{8\pi}$.

The canonical energy-momentum tensor $\mathring{T}^{\mu\nu}$ is not gauge invariant, but the following decomposition holds for it

$$\mathring{T}^{\mu\nu} = \tilde{T}^{\mu\nu} + \frac{1}{\mathrm{c}} J^\mu A^\nu, \; A^\nu = (\varphi, 0) \tag{4.1.19}$$

where $\tilde{T}^{\mu\nu}$ is a gauge invariant energy-momentum obtained by applying formula (10.6.8) to the Lagrangian $\hat{L}_0$, namely

$$\tilde{T}^{\mu\nu} = \begin{bmatrix} \tilde{u} & \mathrm{c}\tilde{p}_1 & \mathrm{c}\tilde{p}_2 & \mathrm{c}\tilde{p}_3 \\ \mathrm{c}^{-1}\tilde{s}_1 & -\tilde{\sigma}_{11} & -\tilde{\sigma}_{12} & -\tilde{\sigma}_{13} \\ \mathrm{c}^{-1}\tilde{s}_2 & -\tilde{\sigma}_{21} & -\tilde{\sigma}_{22} & -\tilde{\sigma}_{23} \\ \mathrm{c}^{-1}\tilde{s}_3 & -\tilde{\sigma}_{31} & -\tilde{\sigma}_{32} & -\tilde{\sigma}_{33} \end{bmatrix}, \text{ where} \tag{4.1.20}$$

$$\tilde{u} = \frac{\chi^2}{2m}\left[|\nabla\psi|^2 + G\left(|\psi|^2\right)\right], \tag{4.1.21}$$

$$\tilde{p}_j = \frac{\chi \mathrm{i}}{2}\left(\psi\partial_j\psi^* - \psi^*\partial_j\psi\right), \; \tilde{s}_j = -\frac{\chi^2 \mathrm{i}}{2m}\left(\tilde{\partial}_t\psi\partial_j\psi^* + \tilde{\partial}_t^*\psi^*\partial_j\psi\right), \; j = 1,2,3, \tag{4.1.22}$$

and the stress tensor components σ_{ij} are represented by the formulas

$$\tilde{\sigma}_{ii} = \tilde{u} - \frac{\chi^2}{m}\partial_i\psi\partial_i\psi^* + \frac{\chi \mathrm{i}}{2}\left(\psi\tilde{\partial}_t\psi^* - \psi^*\tilde{\partial}_t\psi\right), \tag{4.1.23}$$

$$\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji} = -\frac{\chi^2}{2m}\left(\partial_i\psi\partial_j\psi^* + \partial_j\psi\partial_i\psi^*\right) \text{ for } i \neq j, \; i,j = 1,2,3.$$

One can verify using the field equations (4.0.11), (4.0.12) and the current conservation law (2.3.10) that the canonical and gauge invariant energy-momentum tensors satisfy the the following relations

$$\mathring{\Theta}^{\mu j} = \Theta^{\mu j}, \; \mathring{T}^{\mu j} = \tilde{T}^{\mu j}, \; j = 1,2,3 \text{ and } \partial_\mu\left[\left(\Theta^{\mu 0} + \tilde{T}^{\mu 0}\right) - \left(\mathring{\Theta}^{\mu 0} + \mathring{T}^{\mu 0}\right)\right] = 0, \tag{4.1.24}$$

and that the conservation laws in view of the representation (4.0.8) take the following form

$$\partial_\mu\tilde{T}^{\mu\nu} = f^\nu, \; \partial_\mu\Theta^{\mu\nu} = -f^\nu, \; f^\nu = \frac{1}{\mathrm{c}}J_\mu F^{\nu\mu} = \left(\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}, \rho\mathbf{E}\right), \tag{4.1.25}$$

where we recognize in f^ν the Lorentz force density.

4.2 Resting charge

For resting charge the representations (4.1.15)-(4.1.17) and (4.1.20)-(4.1.23) for the energy-momentum tensors $\Theta^{\mu\nu}$ and $\tilde{T}^{\mu\nu}$ turns into

$$\Theta^{\mu\nu} = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & -\tau_{11} & -\tau_{12} & -\tau_{13} \\ 0 & -\tau_{21} & -\tau_{22} & -\tau_{23} \\ 0 & -\tau_{31} & -\tau_{32} & -\tau_{33} \end{bmatrix}, \; \tilde{T}^{\mu\nu} = \begin{bmatrix} \tilde{u} & 0 & 0 & 0 \\ 0 & -\tilde{\sigma}_{11} & -\tilde{\sigma}_{12} & -\tilde{\sigma}_{13} \\ 0 & -\tilde{\sigma}_{21} & -\tilde{\sigma}_{22} & -\tilde{\sigma}_{23} \\ 0 & -\tilde{\sigma}_{31} & -\tilde{\sigma}_{32} & -\tilde{\sigma}_{33} \end{bmatrix}. \tag{4.2.1}$$

showing, in particular, that the momentum and flux densities for the charge and for the EM field are all identically zero. Consequently, the total momentum P and the energy flux S of the resting dressed charge vanish and we have

$$\mathsf{P}=\mathbf{0},\ \mathsf{S}=\mathbf{0},\ \text{and}\ \mathsf{P}^{\nu}=(\tilde{u},\mathbf{0}),\ \tilde{u}=\frac{\chi^2}{2m}\left[\left(\nabla\mathring{\psi}\right)^2+G\left(\mathring{\psi}^2\right)\right]. \tag{4.2.2}$$

Observe now that the stress tensor (str. t.) σ_{ij} defined by relations (4.1.23) in the case of resting charge can be naturally decomposed into three components which we name as follows

$$\begin{aligned}
\sigma_{ij} &= \sigma_{ij}^{\mathrm{el}}+\sigma_{ij}^{\mathrm{em}}+\sigma_{ij}^{\mathrm{nl}},\ i,j=1,2,3,\ \text{where} && (4.2.3)\\
\sigma_{ij}^{\mathrm{el}} &= -\frac{\chi^2}{m}\left[\partial_i\mathring{\psi}\partial_j\mathring{\psi}-\frac{1}{2}\left(\nabla\mathring{\psi}\right)^2\delta_{ij}\right]\ \text{is elastic deformation str. t.,} && (4.2.4)\\
\sigma_{ij}^{\mathrm{em}} &= -p^{\mathrm{em}}\delta_{ij},\ p^{\mathrm{em}}=-q\mathring{\varphi}\mathring{\psi}^2\ \text{is EM interaction str. t.,} && (4.2.5)\\
\sigma_{ij}^{\mathrm{nl}} &= -p^{\mathrm{nl}}\delta_{ij},\ p^{\mathrm{nl}}=-\frac{\chi^2G\left(\mathring{\psi}^2\right)}{2m}\ \text{is nonlinear self-interaction str. t,} && (4.2.6)
\end{aligned}$$

and consequently the respective volume force densities are

$$\begin{aligned}
\sum_{j=1,2,3}\partial_j\sigma_{ij}^{\mathrm{el}} &= f_i^{\mathrm{el}}=-\frac{\chi^2}{m}\Delta\mathring{\psi}\partial_i\mathring{\psi},\ i=1,2,3, && (4.2.7)\\
\sum_{j=1,2,3}\partial_j\sigma_{ij}^{\mathrm{em}} &= f_i^{\mathrm{em}}+\rho\partial_i\mathring{\varphi},\ f_i^{\mathrm{em}}=2q\mathring{\varphi}\mathring{\psi}\partial_i\mathring{\psi}, && (4.2.8)\\
\sum_{j=1,2,3}\partial_j\sigma_{ij}^{\mathrm{nl}} &= f_i^{\mathrm{nl}}=\frac{\chi^2}{m}G'\left(\mathring{\psi}^2\right)\mathring{\psi}\partial_i\mathring{\psi}. && (4.2.9)
\end{aligned}$$

Notice that the volume force density for the electromagnetic interaction stress in (4.2.8) has two parts: f_i^{em}, which we call *internal electromagnetic force*, and $\rho\partial_i\mathring{\varphi}$ which is the minus Lorentz force. Observe that the stress tensor $\sigma_{ij}^{\mathrm{el}}$ has the structure similar to the one for compressional waves, see Section 10.9 and (10.9.6), whereas the both stress tensors $\sigma_{ij}^{\mathrm{em}}$ and $\sigma_{ij}^{\mathrm{em}}$ have the structure typical for perfect fluids, [Moller, Section 6.6], with respective hydrostatic pressures p^{em} and p^{nl} defined by the relations (4.2.5)-(4.2.6).

Based on the equalities (4.2.7)-(4.2.9) we can recast the equilibrium equation (3.2.9) as

$$\begin{gathered}
f_i^{\mathrm{el}}+f_i^{\mathrm{em}}+f_i^{\mathrm{nl}}=0,\ i=1,2,3,\ \text{or} \\
\left[-\frac{\chi^2}{m}\Delta\mathring{\psi}+2q\mathring{\varphi}\mathring{\psi}+\frac{\chi^2}{m}G'\left(\mathring{\psi}^2\right)\mathring{\psi}\right]\partial_i\mathring{\psi}=0.
\end{gathered} \tag{4.2.10}$$

The equation (4.2.10) signifies the ultimate equilibrium for the static charge. It is evident from equation (4.2.10) that the scalar expression in the brackets before $\nabla\mathring{\psi}$ up to the factor $\frac{m}{\chi^2}$ is exactly the left-hand side of the equilibrium equation (2.3.6). In fact if $\nabla\mathring{\psi}\neq 0$ then the equilibrium equation (4.2.10) is equivalent to the scalar equilibrium equation (2.3.6).

Notice that since the $\mathring{\psi}(|\mathbf{x}|)$ and $\mathring{\varphi}(|\mathbf{x}|)$ are radial and monotonically decaying functions of $|\mathbf{x}|$ we readily have

$$\nabla\mathring{\psi}(|\mathbf{x}|)=\hat{\mathbf{x}}-\left|\nabla\mathring{\psi}\right|\hat{\mathbf{r}},\ \nabla\mathring{\varphi}(|\mathbf{r}|)=-\left|\nabla\mathring{\varphi}\right|\hat{\mathbf{x}},\ \hat{\mathbf{x}}=\frac{\mathbf{x}}{|\mathbf{x}|}=(\hat{x}_1,\hat{x}_2,\hat{x}_3). \tag{4.2.11}$$

The relations (4.2.11) combined with (4.2.7)-(4.2.9) imply that for the resting charge all the forces $\mathbf{f}^{\mathrm{el}}$, $\mathbf{f}^{\mathrm{em}}$ and $\mathbf{f}^{\mathrm{nl}}$ and radial, i.e. they are functions of $|\mathbf{x}|$ and point toward or outward the origin.

The total energy of the resting dressed charge $\mathcal{E}\left(\mathring{\psi}\right)$ can be estimated based on either canonical energy-momentum or the gauge invariant one. If we use the canonical energy-momentum tensors defined by (4.1.13), (4.1.14) and (4.1.19)-(4.1.21) we find the following expressions for the respectively the charge energy density $\mathring{u}$, the EM field energy density $\mathring{w}$ and the total energy density of the dressed charge $\mathring{u}+\mathring{w}$:

$$\mathring{u}+\mathring{w}=\left(\tilde{u}+\mathring{\psi}^2\mathring{\varphi}\right)+\mathring{w}=\frac{\chi^2}{2m}\left[\left(\nabla\mathring{\psi}\right)^2+G\left(\mathring{\psi}^2\right)\right]+\mathring{\psi}^2\mathring{\varphi}-\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi}. \qquad (4.2.12)$$

Using now the results of Section 10.8 including the relation (10.8.15) we obtain the following representation for the total energy of the resting dressed charge

$$\mathcal{E}\left(\mathring{\psi}\right)=\int_{\mathbb{R}^3}\left(\mathring{u}+\mathring{w}\right)\,\mathrm{d}\mathbf{x}=\frac{2}{3}\int_{\mathbb{R}^3}\left[\frac{\chi^2\left(\nabla\mathring{\psi}\right)^2}{2m}-\frac{\left(\nabla\mathring{\varphi}\right)^2}{8\pi}\right]\,\mathrm{d}\mathbf{x}. \qquad (4.2.13)$$

If we wanted to use the gauge invariant energy-momentum tensors (4.1.15)-(4.1.23) for the same evaluation, a consistent with the canonical energy-momentum choice for $w_0(\mathbf{x})$ in (4.1.17) would be $w_0(\mathbf{x})=\mathring{w}=-\frac{(\nabla\varphi)^2}{8\pi}$.

4.3 Freely moving charge

We can use the invariance of the Lagrangian $\hat{L}_0$ with respect to Galilean-gauge transformations (4.1.1)-(4.1.2) to obtain freely moving charge solution to the field equations (4.0.11)-(4.0.12) based on the resting charge solution (2.3.5)-(2.3.6) similarly to what is done in the relativistic case where we obtain freely moving charge solution applying Lorentz transformation to the resting one. Namely, the field equations (2.3.3)-(2.3.4) have the following closed form solution

$$\psi=\psi(t,\mathbf{x})=\mathrm{e}^{\frac{\mathrm{i}S}{\chi}}\mathring{\psi}\left(|\mathbf{x}-\mathbf{v}t|\right),\ S=\frac{m}{2}\left[\mathbf{v}^2t+2\mathbf{v}\cdot(\mathbf{x}-\mathbf{v}t)\right],\ \varphi(t,\mathbf{x})=\mathring{\varphi}_0\left(|\mathbf{x}-\mathbf{v}t|\right), \qquad (4.3.1)$$

where ψ in view of relations (2.3.14) can be also represented as

$$\psi=\psi(t,\mathbf{x})=\mathrm{e}^{\frac{\mathrm{i}S}{\chi}}\mathring{\psi}\left(|\mathbf{x}-\mathbf{v}t|\right),\ S=\mathbf{p}\cdot\mathbf{x}-\frac{\mathbf{p}^2t}{2m},\ \mathbf{p}=m\mathbf{v}. \qquad (4.3.2)$$

Solutions of a similar form propagating with a constant speed are well-known in theory of Nonlinear Schrodinger equations, see [Sulem] and references therein. In what follows we refer to a wave function represented by the formulas (4.3.1), (4.3.2) as a *wave-corpuscle*. Looking at the exact solution (4.3.1), (4.3.2) to the field equations describing the *freely moving charge* we observe that it harmoniously integrates the features of the point charge. Indeed, the wave amplitude $\mathring{\psi}\left(|\mathbf{x}-\mathbf{v}t|\right)$ in (4.3.1) is a soliton-like field moving exactly as a free point charge described by its position $\mathbf{r}=\mathbf{v}t$. *The exponential factor* $\mathrm{e}^{\frac{\mathrm{i}S}{\chi}}$ *is a plane wave with the phase* S *that depends only on the point charge position* $\mathbf{v}t$ *and momentum* $\mathbf{p}=m\mathbf{v}$*, and it does not depend on the nonlinear self-interaction.* The phase S has a term in which we readily recognize the de Broglie wave-vector $\mathbf{k}$ described exactly in terms of the point charge quantities, namely

$$\mathbf{k}=\frac{\mathbf{p}}{\chi}=\frac{m}{\chi}\mathbf{v}. \tag{4.3.3}$$

Notice that the dispersion relation $\omega=\omega(\mathbf{k})$ of the linear part of the field equations (4.0.11) for ψ is

$$\omega(\mathbf{k})=\frac{\chi\mathbf{k}^2}{2m}, \text{ implying that the group velocity } \nabla_{\mathbf{k}}\omega(\mathbf{k})=\frac{\chi\mathbf{k}}{m}. \tag{4.3.4}$$

Combining the expression (4.3.4) for the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$ with the expression (4.3.3) for wave vector $\mathbf{k}$ we establish another exact relation

$$\mathbf{v}=\nabla_{\mathbf{k}}\omega(\mathbf{k}), \tag{4.3.5}$$

signifying the equality between the point charge velocity $\mathbf{v}$ and the group velocity $\nabla_{\mathbf{k}}\omega(\mathbf{k})$ at the de Broglie wave vector $\mathbf{k}$. Using the relations (2.3.9) and (4.1.12) we readily obtain the following representations for the micro-charge, the micro-current and momentum densities

$$\rho(t,\mathbf{x})=q\mathring{\psi}^2(|\mathbf{x}-\mathbf{v}t|), \ \mathbf{J}(t,\mathbf{x})=q\mathbf{v}\mathring{\psi}^2(|\mathbf{x}-\mathbf{v}t|), \tag{4.3.6}$$

$$\mathbf{P}(t,\mathbf{x})=\frac{m}{q}\mathbf{J}(t,\mathbf{x})=\mathbf{p}\mathring{\psi}^2(|\mathbf{x}-\mathbf{v}t|). \tag{4.3.7}$$

The above expressions and charge normalization condition (4.0.14) readily imply the following representations for the total dressed charge field momentum $\mathbf{P}$ and the total current $\mathbf{J}$ for the solution (2.3.16) in terms of point charge quantities, namely

$$\mathbf{P}=\frac{m}{q}\mathbf{J}=\int_{\mathbb{R}^3}\frac{\chi q}{m}\operatorname{Im}\frac{\nabla\mathring{\psi}}{\mathring{\psi}}\mathring{\psi}^2\,\mathrm{d}\mathbf{x}=\mathbf{p}=m\mathbf{v}. \tag{4.3.8}$$

4.4 Nonlinear self-interaction and its basic properties

As we have already explained in the beginning of Section 4 the *nonlinear self interaction function G is determined from the charge equilibrium equation* (2.3.8) based on the form factor $\mathring{\psi}$ and the form factor potential $\mathring{\varphi}$. It is worth to point out that such a nonlinearity differs significantly from nonlinearities considered in similar problems in literature. Important features of our nonlinearity include: (i) the boundedness of its derivative $G'(s)$ for $s\geq 0$ with consequent boundedness from below of the wave energy; (ii) non analytic behavior for small s that is for small wave amplitudes.

In this section we consider the construction of the function G, study its properties and provide examples for which the construction of G is carried out explicitly. Throughout this section we have

$$\psi,\mathring{\psi}\geq 0 \text{ and hence } |\psi|=\psi.$$

We introduce explicitly the *size parameter a* through the following representation of the fundamental functions $\mathring{\psi}(r)$ and $\mathring{\varphi}(r)$

$$\mathring{\psi}(r)=\mathring{\psi}_a(r)=a^{-3/2}\mathring{\psi}_1\left(a^{-1}r\right), \ \mathring{\varphi}(r)=\mathring{\varphi}_a(r)=a^{-1}\mathring{\varphi}_1\left(a^{-1}r\right), \ a>0, \tag{4.4.1}$$

where $\mathring{\psi}_1(\mathsf{r})$ and $\mathring{\varphi}_1(\mathsf{r})$ are functions of the dimensionless parameter r, and, as a consequence of (4.0.14), the function $\mathring{\psi}_a(r)$ satisfies the charge normalization condition

$$\int_{\mathbb{R}^3}\mathring{\psi}_a^2(|\mathbf{x}|)\,\mathrm{d}\mathbf{x}=1 \text{ for all } a>0. \tag{4.4.2}$$

The *size parameter* a naturally has the dimension of the length, but yet we do not identify it with the size. Indeed, any properly defined spatial size of $\mathring{\psi}_a$, based, for instance, on the variance or on an energy-based scale as in (4.7.3), is proportional to a, with a coefficient depending on $\mathring{\psi}_1$. The charge equilibrium equation (2.3.8) can be written in the following form

$$\frac{\chi^2}{2m} G_a'\left(\mathring{\psi}_a^2\right)\mathring{\psi}_a = \frac{\chi^2}{2m}\nabla^2\mathring{\psi}_a - q\mathring{\varphi}_a\mathring{\psi}_a, \tag{4.4.3}$$

$$\nabla^2\mathring{\varphi}_a = -4\pi q\left|\mathring{\psi}_a\right|^2. \tag{4.4.4}$$

The function $\mathring{\psi}_a(r)$, $r \geq 0$ is assumed to be positive, monotonically decreasing function of r, and to satisfy the charge normalization condition (4.4.2). Recall that $\mathring{\psi}_a(|\mathbf{x}|)$ and $\mathring{\varphi}_a(|\mathbf{x}|)$ are radial functions and consequently when solving the equation (4.4.4) for $\mathring{\varphi}_a$ we obtain (see Section 4.6 and (4.6.7) for details) the formula

$$\mathring{\varphi}_a(r) = \frac{q}{r}\left[1 - \frac{4\pi}{a}\int_{r/a}^{\infty}\left(r_1 - \frac{r}{a}\right) r_1 \mathring{\psi}_1^2(r_1)\, \mathrm{d}r_1\right]. \tag{4.4.5}$$

Obviously, if $\mathring{\psi}_1^2(r)$ decays sufficiently fast as $r \to \infty$ and a sufficiently small then the potential $\mathring{\varphi}_a(r)$ is very close to the Coulomb potential q/r as we show in Section 4.6.

Let us look first at the case $a = 1$, $\mathring{\psi}_a = \mathring{\psi}_1$, $\mathring{\varphi}_a = \mathring{\varphi}_1$, for which the equation (4.4.3) yields the following representation for $G'\left(\mathring{\psi}_1^2\right)$ from (4.4.3)

$$G'\left(\mathring{\psi}_1^2(r)\right) = \frac{\left(\nabla^2\mathring{\psi}_1\right)(r)}{\mathring{\psi}_1(r)} - \frac{2m}{\chi^2} q\mathring{\varphi}_1(r). \tag{4.4.6}$$

Since $\mathring{\psi}_1^2(r)$ is a monotonic function we can find its inverse $r = r\left(\psi^2\right)$ yielding

$$G'(s) = \left[\frac{\nabla^2\mathring{\psi}_1}{\mathring{\psi}_1} - \frac{2m}{\chi^2} q\mathring{\varphi}_1\right](r(s)),\ 0 = \mathring{\psi}_1^2(\infty) \leq s \leq \mathring{\psi}_1^2(0). \tag{4.4.7}$$

We extend then $G'(s)$ for $s \geq \mathring{\psi}_1^2(0)$ to be a constant, namely

$$G'(s) = G'\left(\mathring{\psi}_1^2(\infty)\right) \text{ if } s \geq \mathring{\psi}_1^2(\infty). \tag{4.4.8}$$

Observe that the positivity and the monotonicity of the form factor $\mathring{\psi}_1$ *was instrumental for recovering the function* $G'(s)$ *from the charge balance equation (4.4.3).*

Using the representation (4.4.7) for the function $G'(s)$ we decompose it naturally into two components:

$$G'(s) = G_\nabla'(s) - \frac{2}{a_\chi} G_\varphi'(s),\ \text{where } a_\chi = \frac{\chi^2}{mq^2} \tag{4.4.9}$$

$$G_\nabla'\left(\mathring{\psi}_1^2\right) = \frac{\left(\nabla^2\mathring{\psi}_1\right)}{\mathring{\psi}_1},\ G_\varphi'\left(\mathring{\psi}_1^2\right) = \frac{\mathring{\varphi}_1}{q} = \mathring{\phi}_1. \tag{4.4.10}$$

We refer to $G'_{\nabla}(s)$ and $G'_{\varphi}(s)$ respectively as *elastic and EM components*. In the case of arbitrary size parameter a we find first that

$$G'_{\nabla,a}(s) = a^{-2} G'_{\nabla,1}\left(a^3 s\right),\ G'_{\varphi,a}(s) = a^{-1} G'_{\varphi,1}\left(a^3 s\right),\ a > 0, \tag{4.4.11}$$

and then combining (4.4.11) with (4.4.9) and (4.4.10) we obtain the following representation for the function $G'_a(s)$

$$G'_a(s) = \frac{G'_{\nabla,1}(a^3 s)}{a^2} - \frac{2G'_{\varphi,1}(a^3 s)}{a a_\chi},\ a_\chi = \frac{\chi^2}{mq^2}. \tag{4.4.12}$$

Let us take a look at general properties of $G'(s)$ and its components $G'_{\nabla}(s)$ and $G'_{\varphi}(s)$ as they follow from defining them relations (4.4.7)-(4.4.12). Starting with EM component $G'_{\varphi}(s)$ we notice that $\mathring{\varphi}_1(|\mathbf{x}|)$ is a radial solution to the equation (4.4.4). Combining that with $\psi^2 \geq 0$ and using the Maximum principle we conclude that $\mathring{\varphi}_1(|\mathbf{x}|)/q$ *is a positive function without local minima, implying that it is a monotonically decreasing function of* $|\mathbf{x}|$. Consequently, $G'_{\varphi}(s)$ defined by (4.4.10) is a monotonically increasing function of s, and hence

$$G'_{\varphi}(s) > 0 \text{ for all } s > 0 \text{ and } G'_{\varphi}(0) = 0. \tag{4.4.13}$$

Note that $G'_{\varphi}(s)$ *is not differentiable at zero*, and that can be seen by comparing the behavior of $\mathring{\varphi}_1(r)$ and $\mathring{\psi}_1(r)$ at infinity. Indeed, $\varphi_1(r)/q \sim r^{-1}$ as $r \to \infty$ and since $\mathring{\psi}^2(|\mathbf{x}|)$ is integrable, it has to decay faster than $|\mathbf{x}|^{-3}$ as $|\mathbf{x}| \to \infty$. Consequently, $\left|G'_{\varphi}(s)\right|$ for small s has to be greater than $s^{1/3}$ that prohibits its the differentiability at zero. One has to notice though that the nonlinearity $G'\left(|\psi|^2\right)\psi$ as it enters field equation (4.0.13) is differentiable for all ψ including zero, hence it satisfies Lipschitz condition required for uniqueness of solutions of initial value problem for (4.0.13).

Let us look at the elastic component $G'_{\nabla}(s)$ defined by the relations (4.4.10). Since $\mathring{\psi}(|\mathbf{x}|) > 0$ the sign of $G'_{\nabla}\left(|\psi|^2\right)$ coincides with the sign of $\nabla^2 \mathring{\psi}_1(|\mathbf{x}|)$. At the origin $\mathbf{x} = \mathbf{0}$ the function $\mathring{\psi}_1(|\mathbf{x}|)$ has its maximum and consequently $G'_{\nabla}(s) \leq 0$ for all s close to $s = \mathring{\psi}_1^2(\infty)$, implying

$$G'_{\nabla}(s) \leq 0 \text{ for } s \gg 1. \tag{4.4.14}$$

The Laplacian applied to radial functions $\mathring{\psi}_1$ takes the form $\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\mathring{\psi}_1 |\mathbf{x}|\right)$. Consequently, if $r\mathring{\psi}_1(r)$ is convex at $r = |\mathbf{x}|$ we have $\nabla^2 \mathring{\psi}_1(|\mathbf{x}|) \geq 0$. Since $r^2 \mathring{\psi}(r)$ is integrable we can naturally assume that $|\mathbf{x}|\,\mathring{\psi}_1(|\mathbf{x}|) \to 0$ as $|\mathbf{x}| \to \infty$. Then if the second derivative of $r\mathring{\psi}_1(r)$ has a constant sign near infinity, it must be non-negative. For an exponentially decaying $\mathring{\psi}_1(r)$ the second derivative of $r\mathring{\psi}_a(r)$ is positive implying

$$G'_{\nabla}(s) > 0 \text{ for } s \ll 1. \tag{4.4.15}$$

Combining that with the equality $G'_{\varphi}(0) = 0$ from (4.4.13) we readily obtain

$$G'(s) > 0 \text{ for } s \ll 1. \tag{4.4.16}$$

¿From the relations (4.4.9), (4.4.13), (4.4.14) we also obtain

$$G'(s) < 0 \text{ if } s \gg 1. \tag{4.4.17}$$

We remind that the sign of the $G'(s)$ according to the representation (4.2.9) for nonlinear self-interaction force density f_i^{nl} controls its direction.

4.5 Examples of nonlinearities

In this section we provide two examples of the form factor $\mathring{\psi}$ for which the form factor potential $\mathring{\varphi}_0$ and the corresponding nonlinear self-interaction function G can be constructed explicitly. The first example is for the form factor $\mathring{\psi}(r)$ decaying as a power law as $r \to \infty$. In this the case the both $\mathring{\varphi}_0$ and G are represented by rather simple, explicit formulas, but some properties of these functions are not as appealing. Namely, the variance of the function $\mathring{\psi}$ is infinite and the rate of approximation of the exact Coulomb potential by $\mathring{\varphi}_a(\mathbf{x})$ for small a is not as fast. The second example is for the form factor $\mathring{\psi}(r)$ decaying exponentially as $r \to \infty$. In this case representations for $\mathring{\varphi}_0$ and G are more involved compared with the power law form factor but all the properties of $\mathring{\psi}$ and $\mathring{\varphi}$ are satisfactory in any regard.

4.5.1 Nonlinearity for the form factor decaying as a power law

We introduce here a form factor $\mathring{\psi}_1(r)$ decaying as a power law of the form

$$\mathring{\psi}_1(r) = \frac{c_{\mathrm{pw}}}{\left(1+r^2\right)^{5/4}},\ c_{\mathrm{pw}} = \frac{3^{1/2}}{(4\pi)^{1/2}} \text{ implying } 0 < \mathring{\psi}_1(r) \leq c_{\mathrm{pw}}. \tag{4.5.1}$$

This function evidently is positive and monotonically decreasing as required. Let us find now $G'_{\nabla}(s)$ and G'_{φ} based on the relations (4.4.10). An elementary computation shows that

$$\nabla^2 \mathring{\psi}_1 = \frac{15}{4c_{\mathrm{pw}}^{4/5}}\left(1 - \frac{3}{c_{\mathrm{pw}}^{4/5}}\mathring{\psi}_1^{4/5}\right)\mathring{\psi}_1^{1+4/5}, \text{ implying}$$

$$G'_{\nabla}(s) = \frac{15 s^{2/5}}{4c_{\mathrm{pw}}^{4/5}} - \frac{45 s^{4/5}}{4c_{\mathrm{pw}}^{8/5}},\ G_{\nabla}(s) = \frac{75 s^{7/5}}{28 c_{\mathrm{pw}}^{4/5}} - \frac{25 s^{9/5}}{4c_{\mathrm{pw}}^{8/5}},\ \text{for } 0 \leq s \leq c_{\mathrm{pw}}^2. \tag{4.5.2}$$

To determine G'_{φ} we find by a straightforward examination that function

$$\mathring{\varphi}_1 = q\mathring{\phi}_1 = \frac{q}{\left(1+r^2\right)^{1/2}} = \frac{q}{c_{\mathrm{pw}}^{2/5}}\mathring{\psi}_1^{2/5} \text{ solves } \nabla^2\mathring{\varphi}_1 = -4\pi q \mathring{\psi}_1^2 \tag{4.5.3}$$

that together with (4.4.10) yields

$$G'_{\varphi}(s) = \frac{s^{1/5}}{c_{\mathrm{pw}}^{2/5}},\ G_{\varphi}(s) = \frac{5 s^{6/5}}{6 c_{\mathrm{pw}}^{2/5}},\ \text{for } 0 \leq s \leq c_{\mathrm{pw}}^2. \tag{4.5.4}$$

Observe that both components $G'_{\nabla}(s)$ and $G'_{\varphi}(s)$ in (4.5.2), (4.5.4) of the total nonlinearity $G'_{\nabla}(s)$ defined by (4.4.9) *are not differentiable* at $s = 0$.

If we explicitly introduce size parameter a into the form factor, namely

$$\psi_a(r) = \frac{a^{-3/2} c_{\mathrm{pw}}}{\left(1+r^2/a^2\right)^{5/4}} = \frac{a c_{\mathrm{pw}}}{\left(a^2+r^2\right)^{5/4}}, \tag{4.5.5}$$

then combining (4.5.2), (4.5.4) with (4.4.11) we obtain the following representation for the nonlinearity components

$$G'_{\nabla,a}(s) = \frac{15 s^{2/5}}{4a^{4/5}c_{\mathrm{pw}}^{4/5}} - \frac{45 a^{2/5} s^{4/5}}{4c_{\mathrm{pw}}^{8/5}}, \tag{4.5.6}$$

$$G_{\nabla,a}(s) = \frac{75 s^{7/5}}{28 a^{4/5} c_{\mathrm{pw}}^{4/5}} - \frac{25 a^{2/5} s^{9/5}}{4c_{\mathrm{pw}}^{8/5}} \text{ for } 0 \leq s \leq c_{\mathrm{pw}}^2 a^{-3},$$

$$G'_{\varphi,a}(s) = \frac{s^{1/5}}{a^{2/5} c_{\mathrm{pw}}^{2/5}}, \; G_{\varphi}(s) = \frac{5 s^{6/5}}{6 a^{2/5} c_{\mathrm{pw}}^{2/5}} \text{ for } 0 \leq s \leq c_{\mathrm{pw}}^{2} a^{-3}. \tag{4.5.7}$$

Notice that the variance of the form factor $\mathring{\psi}_1^2(|\mathbf{x}|)$ decaying as a power law (4.5.1) is infinite, i.e.

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \mathring{\psi}_1^2(|\mathbf{x}|)\, \mathrm{d}\mathbf{x} = 4\pi \int_0^\infty \frac{c_{\mathrm{pw}}^2}{(1+r^2)^{5/2}} r^4 \,\mathrm{d}r = 3 \int_0^\infty \frac{1}{(1+r^2)^{5/2}} r^4 \,\mathrm{d}r = \infty. \tag{4.5.8}$$

4.5.2 Nonlinearity for the form factor decaying exponentially

We introduce here an exponentially decaying form factor $\mathring{\psi}_1$ of the form

$$\mathring{\psi}_1(r) = c_{\mathrm{e}} \mathrm{e}^{-\left(r^2+1\right)^{1/2}} \text{ where } c_{\mathrm{e}} = \left(4\pi \int_0^\infty r^2 \mathrm{e}^{-2\left(r^2+1\right)^{1/2}} \,\mathrm{d}r\right)^{-1/2} \simeq 0.79195, \tag{4.5.9}$$

which is evidently positive and monotonically decreasing as required. The dependence $r(s)$ defined by relation (4.5.9) is as follows for the form factor $\mathring{\psi}_1(r)$ as in (4.5.9)

$$r = \left[\ln^2\left(c_{\mathrm{e}}/\sqrt{s}\right) - 1\right]^{1/2}, \text{ if } \sqrt{s} \leq \mathring{\psi}_1(0) = c_{\mathrm{e}} \mathrm{e}^{-1}. \tag{4.5.10}$$

An elementary computation shows that

$$\nabla^2 \mathring{\psi}_1 = -W \mathring{\psi}_1, \text{ where } W = \frac{2}{(r^2+1)^{\frac{1}{2}}} + \frac{1}{(r^2+1)} + \frac{1}{(r^2+1)^{\frac{3}{2}}} - 1, \tag{4.5.11}$$

implying

$$G'_{\nabla}\left(\psi_1^2(r)\right) = -W(r) = 1 - \frac{2}{(r^2+1)^{\frac{1}{2}}} - \frac{1}{(r^2+1)} - \frac{1}{(r^2+1)^{\frac{3}{2}}}. \tag{4.5.12}$$

Combining (4.5.10) with (4.5.12) we readily obtain the following function

$$G'_{\nabla,1}(s) = \left[1 - \frac{4}{\ln\left(c_{\mathrm{e}}^2/s\right)} - \frac{4}{\ln^2\left(c_{\mathrm{e}}^2/s\right)} - \frac{8}{\ln^3\left(c_{\mathrm{e}}^2/s\right)}\right], \text{ for } \sqrt{s} \leq c_{\mathrm{e}} \mathrm{e}^{-1} \simeq 0.29134 \tag{4.5.13}$$

which is evidently a monotonically decreasing one. We extend for larger s as follows:

$$G'_{\nabla,1}(s) = G'_{\nabla,1}\left(c_{\mathrm{e}}^2 \mathrm{e}^{-2}\right) = -3 \text{ if } \sqrt{s} \geq c_{\mathrm{e}} \mathrm{e}^{-1}. \tag{4.5.14}$$

The relations (4.5.13) and (4.5.14) imply $G'_{\nabla,1}(s)$ takes values in the interval $[1, -3]$. It also follows from (4.5.13) that

$$G'_{\nabla,1}(s) \cong 1 - \frac{4}{\ln 1/s} \text{ as } s \to 0, \tag{4.5.15}$$

implying that the function $G'_{\nabla,1}(s)$ *is not differentiable* at $s = 0$ and consequently is not analytic.

To determine the second component G'_{φ} we need to solve (4.4.10). Using the fact that $\mathring{\varphi}_1$, $\mathring{\psi}_1$ are radial functions we obtain the following equation for $\mathring{\varphi}_1(r)$

$$-\frac{1}{4\pi r} \partial_r^2 \left(r \mathring{\varphi}_1\right) = q c_{\mathrm{e}}^2 \mathrm{e}^{-2\sqrt{r^2+1}}. \tag{4.5.16}$$

We seek such a solution of equation (4.5.16) that is regular at zero and behaves as the Coulomb potential $\frac{q}{r}$ for large r. Taking that into account we obtain after the first integration of (4.5.16)

$$\partial_r \left(r\mathring{\varphi}_1\right) = \pi q c_{\mathrm{e}}^2 \left[1 + 2\left(r^2+1\right)^{1/2}\right] \mathrm{e}^{-2\left(r^2+1\right)^{1/2}}, \tag{4.5.17}$$

and integrating (4.5.17) yields the ultimate formula for the form factor potential

$$\begin{aligned} \mathring{\varphi}_1(r) &= \frac{\pi q c_{\mathrm{e}}^2}{r} \int_0^r \left[1 + 2\left(r_1^2+1\right)^{1/2}\right] \mathrm{e}^{-2\left(r_1^2+1\right)^{1/2}}\, \mathrm{d}r_1 = \\ &= \frac{q}{r} - \frac{\pi q c_{\mathrm{e}}^2}{r} \int_r^\infty \left[1 + 2\left(r_1^2+1\right)^{1/2}\right] \mathrm{e}^{-2\left(r_1^2+1\right)^{1/2}}\, \mathrm{d}r_1. \end{aligned} \tag{4.5.18}$$

The above formula shows that the form factor potential $\mathring{\varphi}_1(r)$ is exponentially close to the Coulomb potential q/r for large r. But if we use the substitution $\left(r_1^2+1\right)^{1/2} = u$ in the second integral in (4.5.18) we can recast $\mathring{\varphi}_1(r)$ in even more convenient form for estimations of its proximity Coulomb potential q/r, namely

$$\begin{aligned} \mathring{\varphi}_1(r) = \frac{q}{r} - \frac{\pi q c_{\mathrm{e}}^2 \left[1 + 2\left(r^2+1\right)^{1/2}\right]}{2r} \mathrm{e}^{-2\left(r^2+1\right)^{1/2}} - \\ - \frac{\pi q c_{\mathrm{e}}^2}{r} \int_{\left(r^2+1\right)^{1/2}}^\infty \left[\frac{(2u+1)}{\left(1-u^{-2}\right)^{1/2}} - 2u\right] \mathrm{e}^{-2u}\, \mathrm{d}u. \end{aligned} \tag{4.5.19}$$

Then based on the relation (4.4.10) and (4.5.18) we find consequently

$$G'_{\varphi,1}\left(\psi^2\right) = \frac{1}{q}\mathring{\varphi}_1\left[\left(\ln^2\left(c_{\mathrm{e}}/\psi\right) - 1\right)^{1/2}\right], \text{ for } \psi \leq c_{\mathrm{e}}\mathrm{e}^{-1}, \tag{4.5.20}$$

$$\begin{gathered} G'_{\varphi,1}\left(\psi^2\right) = \frac{\pi c_{\mathrm{e}}^2}{r(\psi)} \int_0^{r(\psi)} \left[1 + 2\left(r_1^2+1\right)^{1/2}\right] \mathrm{e}^{-2\left(r_1^2+1\right)^{1/2}}\, \mathrm{d}r_1, \text{ where} \\ r(\psi) = \left[\ln^2\left(c_{\mathrm{e}}/\psi\right) - 1\right]^{1/2} \text{ for } \psi \leq c_{\mathrm{e}}\mathrm{e}^{-1}, \end{gathered} \tag{4.5.21}$$

extending $G'_{\varphi,1}\left(\psi^2\right)$ for larger values of ψ as a constant:

$$G'_{\varphi,1}\left(\psi^2\right) = \lim_{r\to 0} \frac{\mathring{\varphi}(r)}{q} = 3\pi c_{\mathrm{e}}^2 \mathrm{e}^{-2} \simeq 0.79998, \text{ for } \psi \geq c_{\mathrm{e}}\mathrm{e}^{-1}. \tag{4.5.22}$$

Using the representation (4.5.19) we obtain the following formula for $G'_{\varphi,1}$

$$\begin{aligned} G'_{\varphi,1}\left(\psi^2\right) = \frac{1}{\left[\ln^2\left(c_{\mathrm{e}}/\psi\right) - 1\right]^{1/2}} - \frac{\pi\left[1 + 2\ln\left(c_{\mathrm{e}}/\psi\right)\right]\psi^2}{2\left[\ln^2\left(c_{\mathrm{e}}/\psi\right) - 1\right]^{1/2}} - \\ - \frac{\pi c_{\mathrm{e}}^2}{\left[\ln^2\left(c_{\mathrm{e}}/\psi\right) - 1\right]^{1/2}} \int_{\ln\left(c_{\mathrm{e}}/\psi\right)}^\infty \left[\frac{(2u+1)}{\left(1-u^{-2}\right)^{1/2}} - 2u\right] \mathrm{e}^{-2u}\, \mathrm{d}u. \end{aligned} \tag{4.5.23}$$

To find $G'_a(s)$ for arbitrary a we use its representation (4.4.12), i.e.

$$G'_a(s) = \frac{G'_{\nabla,1}\left(a^3 s\right)}{a^2} - \frac{2G'_{\varphi,1}\left(a^3 s\right)}{a a_\chi}, \quad a_\chi = \frac{\chi^2}{mq^2}, \tag{4.5.24}$$

and combine with the formulas (4.5.13) and (4.5.23). We don't write the final formula since it is quite long but it is clear that from formulas (4.5.13) and (4.5.23) that $G'_a(s)$ *does not depend analytically on* s at $s=0$, and that the following asymptotic formula holds

$$G'_a(s) = \frac{1}{a^2} - \left(\frac{1}{a^2} + \frac{1}{aa_\chi}\right)\frac{4}{\ln\left(c_\mathrm{e}^2/\left(a^3 s\right)\right)} \text{ for } s \to 0. \tag{4.5.25}$$

The variance of the exponential form factor $\mathring{\psi}_1(r)$ is

$$\int_{\mathbb{R}^3} |\mathbf{x}|^2 \mathring{\psi}_1^2(|\mathbf{x}|)\,\mathrm{d}\mathbf{x} = 4\pi c_\mathrm{e}^2 \int_0^\infty r^4 \mathrm{e}^{-2\left(r^2+1\right)^{1/2}}\,\mathrm{d}r \simeq 3.826\,8. \tag{4.5.26}$$

4.6 Form factor potential proximity to the Coulomb potential

In this subsection we study the proximity of the potential form factor $\mathring{\varphi}_a(|\mathbf{x}|)$ to the Coulomb potential $q/|\mathbf{x}|$ for small a. This is an important issue since it is a well known experimental fact that the Coulomb potential $q/|\mathbf{x}|$ represents the electrostatic field of the charge very accurately even for very small values of $|\mathbf{x}|$.

According to the rest charge equation (2.3.8) and the equation (2.3.7) the potential $\mathring{\phi}_a(|\mathbf{x}|) = \mathring{\varphi}_a(|\mathbf{x}|)/q$ satisfies the following relations

$$\Delta\mathring{\phi}_a(|\mathbf{x}|) = -4\pi\mathring{\psi}_a^2(|\mathbf{x}|) \text{ and hence } \mathring{\phi}_a(|\mathbf{x}|) = \int_{\mathbb{R}^3} \frac{\mathring{\psi}_a^2(|\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} > 0. \tag{4.6.1}$$

where, as always, $\Delta = \nabla^2$. In view of the relations (4.4.1) the dependence of the potential $\mathring{\phi}_a(r)$ on the size parameter a is of the form

$$\mathring{\phi}_a(r) = a^{-1}\phi_1\left(a^{-1}r\right), \tag{4.6.2}$$

and consequently its behavior for small a is determined by the behavior of $\phi_1(r)$ for large r. To find the latter consider the radial solution $z(r)$ to the Poisson equation

$$\frac{1}{r}\left(\frac{d}{dr}\right)^2 z(r) = -4\pi\mathring{\psi}_1^2(r),\ z(r) = r\phi_1(r),\ r \geq 0. \tag{4.6.3}$$

We seek such a solution $z(r)$ to the above equation that is close to the Coulomb potential $1/r$ and hence satisfies the following condition

$$z(r) = r\phi_1(r) \to 1 \text{ as } r \to \infty. \tag{4.6.4}$$

Taking into account (4.6.4) when integrating two times of equation (4.6.3) yields

$$z(r) = 1 - 4\pi\int_r^\infty \int_{r_2}^\infty r_1\mathring{\psi}_1^2(r_1)\,\mathrm{d}r_1\mathrm{d}r_2 = 1 - 4\pi\int_r^\infty (r_1 - r)\,r_1\mathring{\psi}_1^2(r_1)\,\mathrm{d}r_1, \tag{4.6.5}$$

where the second equality in (4.6.5) is obtained by rewriting the preceding repeated integral as as a double integral and changing the order of integration, namely

$$\int_r^\infty \int_{r_2}^\infty r_1\mathring{\psi}_1^2(r_1)\,\mathrm{d}r_1\mathrm{d}r_2 = \int_r^\infty \int_r^{r_1} r_1\mathring{\psi}_1^2(r_1)\,\mathrm{d}r_2\mathrm{d}r_1 = \int_r^\infty (r_1 - r)\,r_1\mathring{\psi}_1^2(r_1)\,\mathrm{d}r_1.$$

In view of the charge normalization condition (4.4.2) we readily obtain from (4.6.5)

$$z(0) = 1 - 4\pi \int_0^\infty r_1^2 \mathring{\psi}_1^2(r_1)\, \mathrm{d}r_1 = 1 - \int_{\mathbb{R}^3} \mathring{\psi}_1^2(|\mathbf{x}|)\, d\mathbf{x} = 0. \tag{4.6.6}$$

The representation (4.6.5) for $z(r) = r\phi_1(r)$ readily implies the following representation that the potential $\phi_1(r)$:

$$\phi_1(r) = \frac{1}{r}\left[1 - 4\pi \int_r^\infty (r_1 - r)\, r_1 \mathring{\psi}_1^2(r_1)\, \mathrm{d}r_1\right]. \tag{4.6.7}$$

Combining (4.6.7) with (4.6.6) we conclude $\phi_1(r)$ is regular for small $r \geq 0$. Using (4.6.7) once more we obtain the following expression for the difference D_C between $\phi_1(r)$ and $1/r$

$$D_C(\phi_1) = \phi_1(r) - \frac{1}{r} = -\frac{4\pi}{r} \int_r^\infty (r_1 - r)\, r_1 \mathring{\psi}_1^2(r_1)\, \mathrm{d}r_1. \tag{4.6.8}$$

The relation (4.6.8) together with (4.6.2) imply

$$D_C(\phi_a) = \phi_a(r) - \frac{1}{r} = -\frac{4\pi}{r} \int_{a^{-1}r}^\infty \left(r_1 - a^{-1}r\right) r_1 \mathring{\psi}_1^2(r_1)\, \mathrm{d}r_1, \tag{4.6.9}$$

showing in particular that the difference D_C becomes small for small a. More exactly, if $\mathring{\psi}_1^2$ decays exponentially as in (4.5.9) then

$$|D_C(\phi_a)(r)| = \left|\phi_a(r) - \frac{1}{r}\right| \leq \frac{4\pi}{r} \int_{a^{-1}r}^\infty \left(r_1 - a^{-1}r\right) r_1 c_{\mathrm{e}}^2 \mathrm{e}^{-2\left(r_1^2+1\right)^{1/2}}\, \mathrm{d}r_1 \leq \tag{4.6.10}$$
$$\leq \frac{4\pi c_{\mathrm{e}}^2 \mathrm{e}^{-2a^{-1}r}}{r} \int_0^\infty \left(r_1 + a^{-1}r\right) r_1 \mathrm{e}^{-2r_1}\, \mathrm{d}r_1 = \frac{\pi c_{\mathrm{e}}^2 \left(a^{-1}r + 1\right)}{r} \mathrm{e}^{-2a^{-1}r}.$$

For instance, for $r \geq 10a$ the difference D_C between the potential $\phi_a(r)$ and the Coulomb potential $1/r$ is extremely small:

$$|D_C(\phi_a)(r)| = \pi c_{\mathrm{e}}^2 \left(a^{-1}r + 1\right) \mathrm{e}^{-2a^{-1}r} \lesssim 4.4674 \times 10^{-8} \text{ for } r \geq 10a. \tag{4.6.11}$$

Similar estimate for the power low decaying $\mathring{\psi}_1^2$ as in (4.5.1) yields

$$|D_C(\phi_a)(r)| = \left|\phi_a(r) - \frac{1}{r}\right| \leq \frac{1}{r} \int_{a^{-1}r}^\infty \frac{3\left(r_1 - a^{-1}r\right) r_1}{r_1^5}\, \mathrm{d}r_1 \tag{4.6.12}$$
$$= \frac{1}{r} \int_{a^{-1}r}^\infty \left(\frac{3}{r_1^3} - \frac{3a^{-1}r}{r_1^4}\right) \mathrm{d}r_1 = \frac{1}{r}\left(\frac{3a^2}{2r^2} - \frac{3a^3}{5r^3}\right), \text{ implying}$$
$$|D_C(\phi_a)| \leq \frac{0.009}{r}, \text{ for } r \geq 10a.$$

Notice if we would take $\mathring{\psi}_1(r) = 0$ for all $r \geq \mathsf{r}_0$, as it is the case in the Abraham-Lorentz model, the formula (4.6.7) would imply that $\mathring{\varphi}_a(r)$ would be exactly the Coulomb potential for $r \geq a\mathsf{r}_0$. But for such a $\mathring{\psi}_1(r)$ we would not be able to construct the nonlinear self-interaction component G'_φ which would satisfy (4.4.10) since it requires $\mathring{\psi}_1^2(r)$ to be strictly positive for all $r \geq 0$.

4.7 Energy related spacial scale

An attractive choice for the spacial scale can be obtained based on the requirement of the total energy $\mathcal{E}\left(\mathring{\psi}\right)$ of the resting dressed charge defined by the expression (4.2.13) to be exactly 0, which readily reduces to the requirement

$$\mathcal{E}_1\left(\mathring{\psi}\right)=\mathcal{E}_2\left(\mathring{\varphi}\right), \text{ where} \tag{4.7.1}$$

$$\mathcal{E}_1\left(\mathring{\psi}\right)=\frac{\chi^2}{2m}\int_{\mathbb{R}^3}\left|\nabla\mathring{\psi}\right|^2\,\mathrm{d}\mathbf{x},\ \mathcal{E}_2\left(\mathring{\varphi}\right)=\frac{1}{8\pi}\int_{\mathbb{R}^3}\left(\nabla\mathring{\varphi}\right)^2\,\mathrm{d}\mathbf{x}.$$

(this condition is similar to (2.1.11). Plugging $\mathring{\psi}=\mathring{\psi}_a$ and $\mathring{\varphi}=\mathring{\varphi}_a$ defined by (4.4.1) into the equalities (4.7.1) we obtain

$$\mathcal{E}_1\left(\mathring{\psi}_a\right)=a^{-2}\mathcal{E}_1\left(\mathring{\psi}_1\right),\ \mathcal{E}_2\left(\mathring{\varphi}_a\right)=a^{-1}\mathcal{E}_2\left(\mathring{\varphi}_1\right)=a^{-1}q^2\mathcal{E}_2\left(\mathring{\phi}_1\right),\ \mathring{\phi}_1=q^{-1}\mathring{\varphi}_1. \tag{4.7.2}$$

Hence, the requirement $\mathcal{E}_1\left(\mathring{\psi}\right)=\mathcal{E}_2\left(\mathring{\varphi}\right)$ in view of the relations (4.7.2) is equivalent to the following choice $a=a_\psi$ of size parameter a with

$$a_\psi=\frac{\mathcal{E}_1\left(\mathring{\psi}_1\right)}{\mathcal{E}_2\left(\mathring{\varphi}_1\right)}=\frac{4\pi\chi^2}{m}\frac{\int_{\mathbb{R}^3}\left|\nabla\mathring{\psi}_1\right|^2\,\mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^3}\left(\nabla\mathring{\varphi}_1\right)^2\,\mathrm{d}\mathbf{x}}=a_\chi\theta_\psi,\theta_\psi=\frac{4\pi\int_{\mathbb{R}^3}\left|\nabla\mathring{\psi}_1\right|^2\,\mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^3}\left(\nabla\mathring{\phi}_1\right)^2\,\mathrm{d}\mathbf{x}},\ a_\chi=\frac{\chi^2}{mq^2}. \tag{4.7.3}$$

Since functions $\mathring{\psi}_1,\mathring{\phi}_1$ in the above relations are radial the Dirichlet integrals in (4.7.3) can be recast as

$$\int_{\mathbb{R}^3}\left|\nabla\mathring{\psi}_1\right|^2\,\mathrm{d}\mathbf{x}=4\pi\int_0^\infty\left(\partial_r\left(r\mathring{\psi}_1\left(r\right)\right)\right)^2\,\mathrm{d}r\text{ with the similar formula for }\mathring{\phi}_1. \tag{4.7.4}$$

We refer the space scale a_ψ in (4.7.3) obtained based on the equality $\mathcal{E}_1\left(\mathring{\psi}\right)=\mathcal{E}_2\left(\mathring{\varphi}\right)$ as *energy-based spacial scale.*

The energy-based spatial scale a_ψ defined by (4.7.3) for power law form factor (4.5.1)

$$a_\psi=\theta_\psi a_\chi,\ \theta_\psi=\frac{4\pi\int_{\mathbb{R}^3}\left(\nabla\mathring{\psi}_1\right)^2\,\mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^3}\left(\nabla\mathring{\phi}_1\right)^2\,\mathrm{d}\mathbf{x}}=\frac{40}{7\pi}\simeq 1.8189. \tag{4.7.5}$$

For exponentially decaying form factor

$$a_\psi=\theta_\psi a_\chi,\theta_\psi\simeq 1.2473$$

Note that energy based spatial scales for power law form factor and exponential form factor are of the same order, though their variances are absolutely different (infinite variance for the power law as in (4.5.1)).

5 Accelerated motion of a single nonrelativistic charge in an external EM field

The key objective of this section is an extension of the wave-corpuscle representation defined by formulas (4.3.1), (4.3.2) to the case of a single nonrelativistic charge accelerating in an external EM field. Recall that as in (4.0.6) we neglect charge's own magnetic field and set $\mathbf{A}=0$ taking into account only external magnetic field. In the case of a general external EM field no exact closed form solution to the field equations seems to be available, but there is an approximate wave-corpuscle solution and its accuracy is a subject of our studies in this case; this solution is exact for special external fields.

The external and the total EM fields are described by their potentials

$$\bar{\varphi}=\varphi_{\mathrm{ex}}+\varphi, \bar{\mathbf{A}}=\mathbf{A}_{\mathrm{ex}} \tag{5.0.6}$$

as it is done in Section 10.7. The nonrelativistic Lagrangian $\hat{L}_0$ for the charge in external field is obtained from the one for the free charge in (4.0.6) by modifying there the covariant derivative to include the external potential, namely

$$\hat{L}_0\left(\psi,\psi^*,\varphi\right)=\frac{\chi}{2}\mathrm{i}\left[\psi^*\tilde{\partial}_t\psi-\psi\tilde{\partial}_t^*\psi^*\right]-\frac{\chi^2}{2m}\left\{\tilde{\nabla}\psi\tilde{\nabla}^*\psi^*+G\left(\psi^*\psi\right)\right\}-\frac{\left|\nabla\varphi\right|^2}{8\pi}, \tag{5.0.7}$$
$$\tilde{\partial}_t=\partial_t+\frac{\mathrm{i}q\bar{\varphi}}{\chi},\ \tilde{\nabla}=\nabla-\frac{\mathrm{i}q\mathbf{A}_{\mathrm{ex}}}{\chi c},\ \tilde{\partial}_t^*=\partial_t-\frac{\mathrm{i}q\bar{\varphi}}{\chi},\ \tilde{\nabla}^*=\nabla+\frac{\mathrm{i}q\mathbf{A}_{\mathrm{ex}}}{\chi c}.$$

This modified Lagrangian remains to be gauge invariant with respect to the transformations (4.0.9) and the general formulas (10.7.12)-(10.7.15) for the charge and current densities applied to the Lagrangian $\hat{L}_0$ yield

$$J^\mu=\left(\mathrm{c}\rho,\mathbf{J}\right),\ \rho=q\psi\psi^*,\ \mathbf{J}=\frac{\mathrm{i}\chi q}{2m}\left[\psi\tilde{\nabla}^*\psi^*-\psi^*\tilde{\nabla}\psi\right]= \tag{5.0.8}$$
$$=\frac{\chi q}{2m}\mathrm{i}\left(\nabla\psi^*\psi-\psi^*\nabla\psi\right)-\frac{q^2\mathbf{A}_{\mathrm{ex}}}{mc}\psi^*\psi=\left(\frac{\chi q}{m}\operatorname{Im}\frac{\nabla\psi}{\psi}-\frac{q^2\mathbf{A}_{\mathrm{ex}}}{mc}\right)\left|\psi\right|^2,$$

and this current satisfies the conservation/continuity equations

$$\partial_\nu J^\nu=0,\ \partial_t\rho+\nabla\cdot\mathbf{J}=0,\ J^\nu=\left(\rho\mathrm{c},\mathbf{J}\right). \tag{5.0.9}$$

The Euler-Lagrange field equations for this Lagrangian are

$$\chi\mathrm{i}\tilde{\partial}_t\psi=\frac{\chi^2}{2m}\left[-\tilde{\nabla}^2\Delta\psi+G'\left(\psi^*\psi\right)\psi\right], \tag{5.0.10}$$
$$-\Delta\varphi=4\pi q\psi\psi^*,\ \text{where } G'\left(s\right)=\partial_s G, \tag{5.0.11}$$

and as always ψ^* is complex conjugate to ψ. Then the field equations (5.0.10)-(5.0.11) can be recast as the following *field equations*

$$\mathrm{i}\chi\partial_t\psi=-\frac{\chi^2\nabla^2\psi}{2m}-\frac{\chi q\mathbf{A}_{\mathrm{ex}}\cdot\nabla\psi}{m\mathrm{c}\mathrm{i}}+q\left(\varphi+\varphi_{\mathrm{ex}}+\frac{q\mathbf{A}_{\mathrm{ex}}^2}{2m\mathrm{c}^2}\right)\psi+\frac{\chi^2G'\psi}{2m}, \tag{5.0.12}$$
$$\nabla^2\varphi=-4\pi q\left|\psi\right|^2. \tag{5.0.13}$$

As in the case of a free charge we set the total conserved charge to be exactly q and, hence, we have the following *charge normalization* condition

$$\int_{\mathbb{R}^3} \rho(t,\mathbf{x})\,\mathrm{d}\mathbf{x} = q\int_{\mathbb{R}^3} |\psi|^2\,\mathrm{d}\mathbf{x} = q \text{ or } \int_{\mathbb{R}^3} |\psi|^2\,\mathrm{d}\mathbf{x} = 1. \tag{5.0.14}$$

The presence of the external EM field turns the dressed charge into an open system with consequent subtleties in the treatment of the energy-momentum. All elements of the proper treatment of the energy and momentum densities in such a situation are provided in Section 10.7 and we apply them to the Lagrangian $\hat{L}_0$ defined by (5.0.7). *An instrumental element in the analysis of the energy-momentum tensor is its partition between the charge and the EM field. In carrying out such a partition we are guided by two principles: (i) the both energy-momenta tensors and the forces have to be gauge invariant; (ii) the forces must be of the Lorentz form. The second principal is evidently special to the EM system consisting of the charge and the EM field.*

5.1 Wave-corpuscle concept for an accelerating charge

In Section 4.3 we introduced a wave-corpuscle by the relations (4.3.1), (4.3.2) for a free moving dressed charge. In this section we extend that definition for a dressed charge in an external EM field as follows. Recall that the wave-corpuscle (4.3.1), (4.3.2) for a free moving dressed charge is an exact solution to the fields equations (2.3.3), (2.3.4), and when constructing the wave-corpuscle for a dressed charge in external EM field we also want it to be an exact solutions to the field equation (5.0.12)-(5.0.13). It turns out that if the external EM field is a homogeneous electric field that it is possible, but no closed form solution seems to be available for a general external EM field as defined by its potentials φ_{ex}, $\mathbf{A}_{\mathrm{ex}}$. Taking that into consideration we want the wave-corpuscle to be if not the exact solution to the field equations but at least an accurate approximation. Our way to accomplish that is as follows. *We construct a wave-corpuscle so that it exactly solves properly defined auxiliary field equations which differ from the original ones (5.0.12)-(5.0.13) by an explicitly defined discrepancy D, based on which we judge the accuracy of the approximation.* For certain classes of external EM fields the solutions of the auxiliary field equations are solutions of the original equations. Executing this plan we introduce the following system of *auxiliary field equations*

$$\mathrm{i}\chi\partial_t\psi = -\frac{\chi^2\nabla^2\psi}{2m} - \frac{\chi q\tilde{\mathbf{A}}_{\mathrm{ex}}\cdot\nabla\psi}{m\mathrm{c}\mathrm{i}} + q\left(\varphi+\tilde{\varphi}_{\mathrm{ex}}\right)\psi + \frac{\chi^2 G'\psi}{2m},\ \nabla^2\varphi = -4\pi q\left|\psi\right|^2. \tag{5.1.1}$$

where the auxiliary linear in $\mathbf{x}$ potentials $\tilde{\varphi}_{\mathrm{ex}}(t,\mathbf{x})$, $\tilde{\mathbf{A}}_{\mathrm{ex}}(t,\mathbf{x})$ may differ from the original potentials φ_{ex}, $\mathbf{A}_{\mathrm{ex}}$. Evidently, in addition to the alteration of potentials φ_{ex}, $\mathbf{A}_{\mathrm{ex}}$, the auxiliary field equations differ from the original ones (5.0.12), (5.0.13) only by a single term $\chi q^2\mathbf{A}_{\mathrm{ex}}^2\psi/\left(2mc^2\right)$.

We define the wave-corpuscle ψ,φ by the formula similar to (4.3.1), (4.3.2), namely

$$\psi(t,\mathbf{x}) = \mathrm{e}^{\frac{\mathrm{i}\{m\mathbf{v}(t)\cdot[\mathbf{x}-\mathbf{r}(t)]+s_{\mathrm{p}}(t)\}}{\chi}}\hat{\psi},\ \hat{\psi} = \mathring{\psi}\left(|\mathbf{x}-\mathbf{r}(t)|\right),\ \varphi = \mathring{\varphi}_0\left(|\mathbf{x}-\mathbf{r}(t)|\right), \tag{5.1.2}$$

where (i) $\mathring{\psi}$ and $\mathring{\varphi}$ are respectively the form factor and the form factor potential satisfying (2.3.5), (2.3.6); (ii) the three functions $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $s_{\mathrm{p}}(t)$ are consequently determined by the following *complimentary point charge equations*

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}^2 t}=q\mathbf{E}_{\mathrm{ex}}\left(t,\mathbf{r}\right)+\frac{q}{\mathrm{c}}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{B}_{\mathrm{ex}},\ \text{where } \mathbf{E}_{\mathrm{ex}}=-\nabla\varphi_{\mathrm{ex}}-\frac{\partial_t\mathbf{A}_{\mathrm{ex}}}{\mathrm{c}},\ \mathbf{B}_{\mathrm{ex}}=\nabla\times\mathbf{A}_{\mathrm{ex}},\tag{5.1.3}$$

$$\mathbf{r}\left(0\right)=\mathbf{r}_0,\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\left(0\right)=\dot{\mathbf{r}}_0\ \text{is initial data,}$$

$$\mathbf{v}\left(t\right)=\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\left(t\right)+\frac{q}{m\mathrm{c}}\mathbf{A}_{\mathrm{ex}}\left(\mathbf{r}\left(t\right)\right),\ \frac{\mathrm{d}s_{\mathrm{p}}}{\mathrm{d}t}=\frac{m\mathbf{v}^2\left(t\right)}{2}-q\varphi_{\mathrm{ex}}\left(\mathbf{r}\left(t\right)\right).\tag{5.1.4}$$

based on the EM potentials of the original equations (5.0.12)-(5.0.13). We readily recognize in the equation (5.1.3) the point charge motion in the external EM field equation. Notice that the *function* $\mathbf{v}\left(t\right)$ *defined by the first equation in (5.1.4) if* $\mathbf{A}_{\mathrm{ex}}\neq 0$ *is not the charge velocity* $\dot{\mathbf{r}}\left(t\right)$, but it is simply related to canonical momentum $\mathring{\mathbf{p}}$ (see (10.1.18)-(10.1.22)) by the formula

$$\mathbf{v}\left(t\right)=\frac{\mathring{\mathbf{p}}\left(t\right)}{m},\ \mathring{\mathbf{p}}=\mathbf{p}+\frac{q}{\mathrm{c}}\mathbf{A}_{\mathrm{ex}},\ \text{where } \mathbf{p}=m\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\ \text{is the kinetic momentum.}\tag{5.1.5}$$

We refer to the function $\mathbf{r}\left(t\right)$ *as wave-corpuscle center or wave-corpuscle position.*

Now we define the auxiliary linear in $\mathbf{x}$ potentials $\tilde{\varphi}_{\mathrm{ex}}\left(t,\mathbf{x}\right)$, $\tilde{\mathbf{A}}_{\mathrm{ex}}\left(t,\mathbf{x}\right)$ by the following formulas

$$\tilde{\varphi}_{\mathrm{ex}}=\varphi_{0,\mathrm{ex}}\left(t\right)+\varphi_{0,\mathrm{ex}}'\left(t\right)\cdot\left(\mathbf{x}-\mathbf{r}\left(t\right)\right),\ \text{where}\tag{5.1.6}$$

$$\varphi_{0,\mathrm{ex}}\left(t\right)=\varphi_{\mathrm{ex}}\left(t,\mathbf{r}\left(t\right)\right),\ \varphi_{0,\mathrm{ex}}'\left(t\right)=\nabla\varphi_{\mathrm{ex}}\left(t,\mathbf{r}\left(t\right)\right),$$

$$\tilde{\mathbf{A}}_{\mathrm{ex}}=\mathbf{A}_{\mathrm{ex},0}\left(t\right)+\frac{1}{2}\mathbf{B}_0\left(t\right)\times\left[\mathbf{x}-\mathbf{r}\left(t\right)\right],\ \text{where}\tag{5.1.7}$$

$$\mathbf{A}_{\mathrm{ex},0}\left(t\right)=\mathbf{A}_{\mathrm{ex}}\left(t,\mathbf{r}\left(t\right)\right),\ \mathbf{B}_0\left(t\right)=\left[\nabla\times\mathbf{A}_{\mathrm{ex}}\right]\left(t,\mathbf{r}\left(t\right)\right),$$

Verification of the fact that the wave-corpuscle defined by the relations (5.1.3)-(5.1.5) is either exact or an approximate solution to the field field equations (5.0.12)-(5.0.13) with estimated accuracy is provided in the following sections.

5.2 Energy-momentum tensor

The canonical energy-momentum $\mathring{\Theta}^{\mu\nu}$ for the EM field is obtained by applying the general formula (10.2.5) to the Lagrangian $\hat{L}_0$ yielding

$$\mathring{\Theta}^{\mu\nu}=\begin{bmatrix}\mathring{w} & \mathrm{c}\mathring{g}_1 & \mathrm{c}\mathring{g}_2 & \mathrm{c}\mathring{g}_3\\ \mathrm{c}^{-1}\mathring{s}_1 & -\mathring{\tau}_{11} & -\mathring{\tau}_{12} & -\mathring{\tau}\mathring{\sigma}_{13}\\ \mathrm{c}^{-1}\mathring{s}_2 & -\mathring{\tau}_{21} & -\mathring{\tau}_{22} & -\mathring{\tau}\mathring{\sigma}_{23}\\ \mathrm{c}^{-1}\mathring{s}_3 & -\mathring{\tau}_{31} & -\mathring{\tau}^{32} & -\mathring{\tau}\mathring{\sigma}_{33}\end{bmatrix}=\begin{bmatrix}\mathring{w} & 0 & 0 & 0\\ \mathrm{c}^{-1}\mathring{s}_1 & -\mathring{\tau}_{11} & -\mathring{\tau}_{12} & -\mathring{\tau}_{13}\\ \mathrm{c}^{-1}\mathring{s}_2 & -\mathring{\tau}_{21} & -\mathring{\tau}_{22} & -\mathring{\tau}_{23}\\ \mathrm{c}^{-1}\mathring{s}_3 & -\mathring{\tau}_{31} & -\mathring{\tau}^{32} & -\mathring{\tau}_{33}\end{bmatrix},\tag{5.2.1}$$

$$\mathring{w}=-\frac{\left|\nabla\varphi\right|^2}{8\pi},\ \mathring{g}_j=0,\ \mathring{s}_j=\mathrm{c}\frac{\partial_j\varphi\partial_0\varphi}{4\pi},\ \mathring{\tau}_{jj}=\frac{\partial_j^2\varphi}{4\pi}-\frac{\left|\nabla\varphi\right|^2}{8\pi},\ \mathring{\tau}_{ij}=\frac{\partial_i\varphi\partial_j\varphi}{4\pi}.\tag{5.2.2}$$

The gauge invariant energy-momentum of the EM field take the form

$$\Theta^{\mu\nu}=\begin{bmatrix}w & \mathrm{c}g_1 & \mathrm{c}g_2 & \mathrm{c}g_3\\ \mathrm{c}^{-1}s_1 & -\tau_{11} & -\tau_{12} & -\tau_{13}\\ \mathrm{c}^{-1}s_2 & -\tau_{21} & -\tau_{22} & -\tau_{23}\\ \mathrm{c}^{-1}s_3 & -\tau_{31} & -\tau^{32} & -\tau_{33}\end{bmatrix}=\begin{bmatrix}w & 0 & 0 & 0\\ 0 & -\tau_{11} & -\tau_{12} & -\tau_{13}\\ 0 & -\tau_{21} & -\tau_{22} & -\tau_{23}\\ 0 & -\tau_{31} & -\tau^{32} & -\tau_{33}\end{bmatrix},\tag{5.2.3}$$

$$\partial_0 w=\frac{\mathbf{J}\cdot\nabla\varphi}{\mathrm{c}}=-\frac{\mathbf{J}\cdot\mathbf{E}}{\mathrm{c}},\ g_j=0,\ s_j=0,\ \tau_{jj}=\frac{\partial_j^2\varphi}{4\pi}-\frac{\left|\nabla\varphi\right|^2}{8\pi}, \tag{5.2.4}$$

$$\tau_{ij}=\frac{\partial_i\varphi\partial_j\varphi}{4\pi},\ w\left(t,\mathbf{x}\right)=w_0\left(\mathbf{x}\right)+\int_{-\infty}^{t}\frac{\mathbf{J}\left(t',\mathbf{x}\right)\cdot\nabla\varphi\left(t',\mathbf{x}\right)}{\mathrm{c}}\,\mathrm{d}t'. \tag{5.2.5}$$

The canonical energy-momentum tensor $\mathring{T}^{\mu\nu}$ is not gauge invariant, but the following decomposition holds for it

$$\mathring{T}^{\mu\nu}=\tilde{T}^{\mu\nu}+\frac{1}{\mathrm{c}}J^{\mu}\bar{A}^{\nu},\ \bar{A}^{\nu}=\left(\varphi_{\mathrm{ex}}+\varphi,\mathbf{A}_{\mathrm{ex}}\right), \tag{5.2.6}$$

where $\tilde{T}^{\mu\nu}$ is a gauge invariant energy-momentum obtained from formula (10.7.20) applied to the Lagrangian $\hat{L}_0$ yielding

$$\tilde{T}^{\mu\nu}=\begin{bmatrix}\tilde{u} & \mathrm{c}\tilde{p}_1 & \mathrm{c}\tilde{p}_2 & \mathrm{c}\tilde{p}_3\\ \mathrm{c}^{-1}\tilde{s}_1 & -\tilde{\sigma}_{11} & -\tilde{\sigma}_{12} & -\tilde{\sigma}_{13}\\ \mathrm{c}^{-1}\tilde{s}_2 & -\tilde{\sigma}_{21} & -\tilde{\sigma}_{22} & -\tilde{\sigma}_{23}\\ \mathrm{c}^{-1}\tilde{s}_3 & -\tilde{\sigma}_{31} & -\tilde{\sigma}_{32} & -\tilde{\sigma}_{33}\end{bmatrix},\ \text{where} \tag{5.2.7}$$

$$\tilde{u}=\frac{\chi^2}{2m}\left[\tilde{\nabla}\psi\cdot\tilde{\nabla}^*\psi^*+G\left(\psi^*\psi\right)\right], \tag{5.2.8}$$

$$\tilde{p}_j=\frac{\chi\mathrm{i}}{2}\left(\psi\tilde{\partial}_j^*\psi^*-\psi^*\tilde{\partial}_j\psi\right),\ \tilde{s}_j=-\frac{\chi^2\mathrm{i}}{2m}\left(\tilde{\partial}_t\psi\tilde{\partial}_j^*\psi^*+\tilde{\partial}_t^*\psi^*\tilde{\partial}_j\psi\right),\ j=1,2,3, \tag{5.2.9}$$

and the stress tensor components σ_{ij} are represented by the formulas

$$\tilde{\sigma}_{ii}=\tilde{u}-\frac{\chi^2}{m}\tilde{\partial}_i\psi\tilde{\partial}_i^*\psi^*+\frac{\chi\mathrm{i}}{2}\left(\psi\tilde{\partial}_t\psi^*-\psi^*\tilde{\partial}_t\psi\right), \tag{5.2.10}$$

$$\tilde{\sigma}_{ij}=\tilde{\sigma}_{ji}=-\frac{\chi^2}{2m}\left(\tilde{\partial}_i\psi\tilde{\partial}_j^*\psi^*+\tilde{\partial}_j\psi\tilde{\partial}_i^*\psi^*\right)\ \text{for}\ i\neq j,\ i,j=1,2,3.$$

It follows from (5.0.8) and (5.2.9) that *the charge gauge invariant momentum* $\tilde{\mathbf{p}}$ *equals exactly the microcurrent density* $\mathbf{J}$ *multiplied by the constant* m/q, namely the following identity holds

$$\tilde{\mathbf{p}}=\frac{m}{q}\mathbf{J}=\frac{\mathrm{i}\chi}{2}\left[\psi\tilde{\nabla}^*\psi^*-\psi^*\tilde{\nabla}\psi\right]=\left(\chi\,\mathrm{Im}\,\frac{\nabla\psi}{\psi}-\frac{q\bar{\mathbf{A}}}{c}\right)\left|\psi\right|^2, \tag{5.2.11}$$

that can be viewed as the momentum density kinematic representation

$$\tilde{\mathbf{p}}=m\mathbf{v},\ \mathbf{v}=\mathbf{J}/q. \tag{5.2.12}$$

So we refer to the identities (5.2.11)-(5.2.12) as *momentum density kinematic representation.*

Using the field equations we can also verify that following conservations laws for the charge and its EM field hold

$$\partial_\mu\tilde{T}^{\mu\nu}=f^{\nu}+f_{\mathrm{ex}}^{\nu},\ \partial_\mu\Theta^{\mu\nu}=-f^{\nu},\ \partial_\mu\mathcal{T}^{\mu\nu}=\partial_\mu\left(\tilde{T}^{\mu\nu}+\Theta^{\mu\nu}\right)=f_{\mathrm{ex}}^{\nu},\ \text{where} \tag{5.2.13}$$

$$f^{\nu}=\frac{1}{\mathrm{c}}J_\mu F^{\nu\mu}=\left(\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E},\rho\mathbf{E}\right), \tag{5.2.14}$$

$$f_{\mathrm{ex}}^{\nu}=\frac{1}{\mathrm{c}}J_\mu F_{\mathrm{ex}}^{\nu\mu}=\left(\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}_{\mathrm{ex}},\rho\mathbf{E}_{\mathrm{ex}}+\frac{1}{\mathrm{c}}\mathbf{J}\times\mathbf{B}_{\mathrm{ex}}\right),$$

We readily recognize in f^{ν} and f_{ex}^{ν} in the equations (5.2.13) respectively the Lorentz force densities for the charge in its own and the external EM fields. We also see to our satisfaction from the first two equations in (5.2.13) that the Newton principle "action equals reaction" does manifestly hold for all involved densities at every point of the space-time.

5.3 Point charge mechanics via averaged quantities

Combining now the conservations laws (5.2.13) with energy-momentum tensors representations (5.2.3)-(5.2.5) and (5.2.7)-(6.2.29) we obtain the following equations for total dressed charge momentum density $\mathbf{P} = (P^1, P^2, P^3)$ and the energy density U

$$\partial_t P^i = \partial_t \left(\tilde{p}^i + g^i\right) = \sum_{j=1,2,3} \partial_j \left(\tilde{\sigma}^{ji} + \tau_{ij}\right) + \left(\rho \mathbf{E}_{\mathrm{ex}} + \frac{1}{\mathrm{c}} \mathbf{J} \times \mathbf{B}_{\mathrm{ex}}\right)^i, \; i = 1, 2, 3, \tag{5.3.1}$$

$$\partial_t U = \partial_t \left(\tilde{u} + w\right) = -\sum_{j=1,2,3} \partial_j \left(\tilde{s}_1 + s_1\right) + \mathbf{J} \cdot \mathbf{E}_{\mathrm{ex}}. \tag{5.3.2}$$

Integrating over the entire space $\mathbb{R}^3$ the above conservation laws we obtain the following equations for the total momentum P and the total energy E

$$\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \left[\rho \mathbf{E}_{\mathrm{ex}} + \frac{1}{\mathrm{c}} \mathbf{J} \times \mathbf{B}_{\mathrm{ex}}\right] (t, \mathbf{x}) \, \mathrm{d}\mathbf{x}, \; \frac{\mathrm{d}\mathsf{E}}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathbf{J} \cdot \mathbf{E}_{\mathrm{ex}} (t, \mathbf{x}) \, \mathrm{d}\mathbf{x}. \tag{5.3.3}$$

Let us introduce *charge average position* $\mathbf{r}(t)$ and *average velocity* $\mathsf{v}(t)$ by the following relations

$$\mathbf{r}(t) = \int_{\mathbb{R}^3} \mathbf{x} \left|\psi(t, \mathbf{x})\right|^2 \, \mathrm{d}\mathbf{x}, \; \mathsf{v}(t) = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J}(t, \mathbf{x}) \, \mathrm{d}\mathbf{x}. \tag{5.3.4}$$

Then using the charge conservation law (5.0.9) we find

$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathbf{x} \partial_t \left|\psi\right|^2 \, \mathrm{d}\mathbf{x} = -\frac{1}{q} \int_{\mathbb{R}^3} \mathbf{x} \nabla \cdot \mathbf{J} \, \mathrm{d}\mathbf{x} = \frac{1}{q} \int_{\mathbb{R}^3} \mathbf{J} \, \mathrm{d}\mathbf{x} = \mathsf{v}(t). \tag{5.3.5}$$

Utilizing the momentum density kinematic representation (5.2.11)-(5.2.12) and the fact the momentum density of the charge EM field is identically zero according to (5.2.4) we obtain the following kinematic representation for charge and hence the dressed charge total momentum

$$\mathsf{P}(t) = \frac{m}{q} \int_{\mathbb{R}^3} \mathbf{J}(t, \mathbf{x}) \, \mathrm{d}\mathbf{x} = m \mathsf{v}(t). \tag{5.3.6}$$

Notice now that for the spatially homogeneous EM fields $\mathbf{E}_{\mathrm{ex}}(t)$ and $\mathbf{B}_{\mathrm{ex}}(t)$ the equations (5.3.3) take a simpler form

$$\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}t} = q \mathbf{E}_{\mathrm{ex}}(t) + \frac{q \mathbf{v}(t)}{\mathrm{c}} \times \mathbf{B}_{\mathrm{ex}}(t), \; \frac{\mathrm{d}\mathsf{E}}{\mathrm{d}t} = q \mathsf{v}(t) \cdot \mathbf{E}_{\mathrm{ex}}(t). \tag{5.3.7}$$

In addition to that in this case combining the first equality in (5.3.7) with the momentum kinematic representation (5.3.6) we get

$$\frac{\mathrm{d} m \mathsf{v} \cdot \mathsf{v}}{2 \mathrm{d}t} = \mathsf{v} \cdot \frac{\mathrm{d} m \mathsf{v}}{\mathrm{d}t} = q \mathsf{v} \cdot \mathbf{E}_{\mathrm{ex}}(t), \tag{5.3.8}$$

and that combined with the second equality in (5.3.7) implies the following energy kinematic representation

$$\mathsf{E} = \frac{m \mathsf{v} \cdot \mathsf{v}}{2} + \text{constant}. \tag{5.3.9}$$

Combining the relations (5.3.5)-(5.3.7) we also obtain

$$m \frac{\mathrm{d}^2 \mathbf{r}(t)}{\mathrm{d}t^2} = q \mathbf{E}_{\mathrm{ex}}(t) + \frac{q}{\mathrm{c}} \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} \times \mathbf{B}_{\mathrm{ex}}(t), \tag{5.3.10}$$

in which we recognize the point charge in homogeneous EM field dynamic equation with the familiar expression for the Lorentz force. Notice the found above correspondence between field quantities and point mechanics quantities via the charge position and velocity defined as average values (5.3.4) is similar to the well known in quantum mechanics *Ehrenfest Theorem*, [Schiff, Sections 7, 23]. This is, of course, not accidental as one can see from the Lagrangian representation of the Schrödinger wave mechanics briefly discussed in Section 10.11. *The key argument for the Ehrenfest theorem as in our case is the momentum density kinematic representation (5.2.11)-(5.2.12).*

5.4 Accelerated motion in an external electric field

In this subsection we consider purely electric external EM field, i.e. when $\mathbf{A}_{\mathrm{ex}} = 0$, for which the field equations (5.0.12), (5.0.13) take the form

$$\mathrm{i}\chi\partial_t\psi = -\frac{\chi^2\nabla^2\psi}{2m} + q\left(\varphi + \varphi_{\mathrm{ex}}\right)\psi + \frac{\chi^2}{2m}G'\left(|\psi|^2\right)\psi,\ \nabla^2\varphi = -4\pi q\,|\psi|^2\,. \tag{5.4.1}$$

In this case the wave-corpuscle is defined by the formula (5.1.3) with the complimentary point charge equations (5.1.4), (5.1.5) taking the form

$$m\frac{\mathrm{d}^2\mathbf{r}\,(t)}{\mathrm{d}t^2} = q\mathbf{E}_{\mathrm{ex}}\,(t,\mathbf{r})\,,\ \mathbf{E}_{\mathrm{ex}}\,(t,\mathbf{r}) = -\nabla\varphi_{\mathrm{ex}}\,(t,\mathbf{r})\,,\ \mathbf{r}\,(0) = \mathbf{r}_0,\ \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,(0) = \dot{\mathbf{r}}_0, \tag{5.4.2}$$

$$\mathbf{v}\,(t) = \frac{\mathrm{d}\mathbf{r}\,(t)}{\mathrm{d}t},\ \frac{\mathrm{d}s_{\mathrm{p}}\,(t)}{\mathrm{d}t} = \frac{m\mathbf{v}^2\,(t)}{2} - q\varphi_{\mathrm{ex}}\,(t,\mathbf{r}\,(t))\,, \tag{5.4.3}$$

where $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ is initial data. In the case when the external electric field is homogeneous we show that the wave-corpuscle is an exact solution to the field equations (5.4.1), and if the external electric field is inhomogeneous we show that the wave-corpuscle is an accurate approximation following to the strategy outlined in Section 5.1. Since the electric field homogeneity plays a role in the wave-corpuscle representation, it is convenient to extract from the external electric field potential $\varphi_{\mathrm{ex}}\,(t,\mathbf{x})$ its linear in $\mathbf{x}$ part $\tilde{\varphi}_{\mathrm{ex}}\,(t,\mathbf{x})$ about trajectory $\mathbf{r}\,(t)$, namely we represent $\varphi_{\mathrm{ex}}\,(t,\mathbf{x})$ in the form

$$\begin{gathered}\varphi_{\mathrm{ex}}\,(t,\mathbf{x}) = \tilde{\varphi}_{\mathrm{ex}}\,(t,\mathbf{x}) + \varphi^{(1)}_{\mathrm{ex}}\,(t,\mathbf{x})\,,\ \text{where } \tilde{\varphi}_{\mathrm{ex}}\,(t,\mathbf{x}) = \varphi_{0,\mathrm{ex}}\,(t) + \varphi'_{0,\mathrm{ex}}\,(t)\cdot(\mathbf{x}-\mathbf{r})\,, \\ \varphi_{0,\mathrm{ex}}\,(t) = \varphi_{\mathrm{ex}}\,(t,\mathbf{r})\,,\ \varphi'_{0,\mathrm{ex}}\,(t) = \nabla_{\mathbf{x}}\varphi_{\mathrm{ex}}\,(t,\mathbf{r})\,,\ \mathbf{r} = \mathbf{r}\,(t)\,,\end{gathered} \tag{5.4.4}$$

with the remainder $\varphi^{(1)}_{\mathrm{ex}}\,(t,\mathbf{x}) = \varphi^{(1)}_{\mathrm{ex}}\,(\mathbf{x})$ satisfying the following relations

$$\begin{gathered}\varphi^{(1)}_{\mathrm{ex}}\,(t,\mathbf{x}) = \varphi_{\mathrm{ex}}\,(t,\mathbf{x}) - \varphi_{\mathrm{ex}}\,(t,\mathbf{r}) - \nabla\varphi_{\mathrm{ex}}\,(t,\mathbf{r})\,(\mathbf{x}-\mathbf{r})\,, \\ \varphi^{(1)}_{\mathrm{ex}}\,(t,\mathbf{r}) = 0,\ \nabla\varphi^{(1)}_{\mathrm{ex}}\,(t,\mathbf{r}) = 0,\ \mathbf{r} = \mathbf{r}\,(t)\,.\end{gathered} \tag{5.4.5}$$

5.4.1 Accelerated motion in an external homogeneous electric field

If the external field is purely electric and homogeneous field $\mathbf{E}_{\mathrm{ex}}\,(t)$ then its potential $\varphi_{\mathrm{ex}}\,(t,\mathbf{x})$ is linear in $\mathbf{x}$ and the representation (5.4.4) turns into

$$\begin{gathered}\varphi_{\mathrm{ex}}\,(t,\mathbf{x}) = \tilde{\varphi}_{\mathrm{ex}}\,(t,\mathbf{x}) = \varphi_{0,\mathrm{ex}}\,(t) + \varphi'_{0,\mathrm{ex}}\,(t)\cdot(\mathbf{x}-\mathbf{r}\,(t))\,,\ \text{where} \\ \varphi'_{0,\mathrm{ex}}\,(t) = \nabla_{\mathbf{x}}\varphi_{\mathrm{ex}}\,(\mathbf{r}\,(t)\,,t) = -\mathbf{E}_{\mathrm{ex}}\,(t)\,.\end{gathered} \tag{5.4.6}$$

The main result of this section is that the *wave-corpuscle as defined by formula is an exact solution to the field equation (5.4.1) that can be verified by straightforward examination.* One can alternatively establish that by considering the expression for ψ in (5.1.2) and assuming that real valued functions $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $s_{\mathrm{p}}(t)$ are unknown and to be found, if possible, from the field equations (5.4.1). Indeed, observe that the representation (5.1.2) implies

$$\partial_t \psi = \exp\left\{\frac{\mathrm{i}m}{\chi}\mathbf{v}\cdot(\mathbf{x}-\mathbf{r})+\frac{\mathrm{i}s_{\mathrm{p}}}{\chi}\right\}\times \tag{5.4.7}$$
$$\left\{\left[\frac{\mathrm{i}m}{\chi}\left(\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right)+\frac{\mathrm{i}\dot{s}_{\mathrm{p}}}{\chi}\right]\hat{\psi}-\dot{\mathbf{r}}\cdot\nabla\hat{\psi}\right\},\ \nabla\hat{\psi}=\mathring{\psi}'(\mathbf{x}-\mathbf{r})\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|},$$

and by the Leibnitz formula we have

$$\nabla^2\psi = \exp\left\{\frac{\mathrm{i}m}{\chi}\mathbf{v}\cdot(\mathbf{x}-\mathbf{r})+\frac{\mathrm{i}s_{\mathrm{p}}}{\chi}\right\}\left[\left(\frac{\mathrm{i}m\mathbf{v}}{\chi}\right)^2\hat{\psi}+2\frac{\mathrm{i}m}{\chi}\mathbf{v}\cdot\nabla\hat{\psi}+\nabla^2\hat{\psi}\right]. \tag{5.4.8}$$

To find if the expression (5.1.2) for ψ can solve the field equations (5.4.1) we substitute the expression into the field equations (5.4.1) obtaining the following equation for functions $\mathbf{v}$, $\mathbf{r}$, s_{p} :

$$\left[-m\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}-\dot{s}_{\mathrm{p}}\right]\hat{\psi}-\mathrm{i}\chi\dot{\mathbf{r}}\cdot\nabla\hat{\psi} \tag{5.4.9}$$
$$-\frac{m}{2}\mathbf{v}^2\hat{\psi}+\mathrm{i}\chi\mathbf{v}\cdot\nabla\hat{\psi}+\frac{\chi^2}{2m}\nabla^2\hat{\psi}-q\left(\tilde{\varphi}_{\mathrm{ex}}+\varphi\right)\hat{\psi}-\frac{\chi^2}{2m}G'\hat{\psi}=0.$$

Then using the charge equilibrium equation (2.3.8) we eliminate the nonlinearity G in the above equation (5.4.9) and obtain the following equivalent to it equation

$$-\left\{m\left[\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right]+\frac{m}{2}\mathbf{v}^2+\dot{s}_{\mathrm{p}}+q\tilde{\varphi}_{\mathrm{ex}}\right\}\hat{\psi}-\mathrm{i}\chi\left(\dot{\mathbf{r}}-\mathbf{v}\right)\nabla\hat{\psi}=0. \tag{5.4.10}$$

Now to determine if there is a triple of functions $\{\mathbf{r}(t),\mathbf{v}(t),s_{\mathrm{p}}(t)\}$ for which the equation (5.4.10) holds we equate to zero the coefficients before $\nabla\hat{\psi}$ and $\hat{\psi}$ in that equation resulting in two equations:

$$\mathbf{v}=\dot{\mathbf{r}},\ m\left[\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right]+\frac{m}{2}\mathbf{v}^2+\dot{s}_{\mathrm{p}}+q\tilde{\varphi}_{\mathrm{ex}}=0, \tag{5.4.11}$$

where, in view of the representation (5.4.6), the second equation in (5.4.11) can be recast as

$$m\left[\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right]+\dot{s}_{\mathrm{p}}+\frac{m\mathbf{v}^2}{2}+q\left[\varphi_{0,\mathrm{ex}}+\varphi'_{0,\mathrm{ex}}\cdot(\mathbf{x}-\mathbf{r})\right]=0. \tag{5.4.12}$$

To find out if there is a triple of functions $\{\mathbf{r}(t),\mathbf{v}(t),s_{\mathrm{p}}(t)\}$ solving the equation (5.4.12) we equate to zero the coefficient before $(\mathbf{x}-\mathbf{r})$ and the remaining coefficient and obtain the following pair of equations

$$m\dot{\mathbf{v}}=-q\varphi'_{0,\mathrm{ex}}(t),\ \dot{s}_{\mathrm{p}}-m\mathbf{v}\cdot\dot{\mathbf{r}}+\frac{m\mathbf{v}^2}{2}+q\varphi_{0,\mathrm{ex}}(t)=0. \tag{5.4.13}$$

Thus, based on the first equation (5.4.11) and the equations (5.4.13) we conclude that the wave-corpuscle defined by the formula (5.1.3) with the complimentary point charge equations (5.4.2), (5.4.3) is indeed an exact solution to the field equations (5.4.1).

In conclusion, we compare our construction of exact solutions (5.1.2) with quasi-classical approach which uses a similar ansatz (WKB ansatz). The trajectories of the charges centers as described by our model coincide with trajectories that can be found by applying well-known quasiclassical asymptotics if one neglects the nonlinearity. Note though two important effects of the nonlinearity not presented in the formal quasiclassical approach. First of all, due to the nonlinearity the charge preserves its shape in the course of evolution whereas in the linear model any wavepacket disperses over time. Second of all, the quasiclassical asymptotic expansions produce infinite asymptotic series which provide for a formal solution, whereas the properly introduced nonlinearity as in (2.3.5), (2.3.6) allows to obtain an exact solution.

5.4.2 Accelerated motion in an external inhomogeneous electric field

In this section we consider a general external electric field $\mathbf{E}_{\mathrm{ex}}(t,\mathbf{r})$ which can be inhomogeneous with the corresponding electric potential $\varphi_{\mathrm{ex}}(t,\mathbf{x})$ as described by relations (5.4.4), (5.4.5) with nonzero remainder $\varphi_{\mathrm{ex}}^{(1)}(t,\mathbf{x})$. For an inhomogeneous external electric field $\mathbf{E}_{\mathrm{ex}}(t,\mathbf{r})$ no closed form solution to the field equations (5.4.1) seems to be available but the wave-corpuscle defined by the relations (5.1.2) with complimentary point charge equations (5.4.2), (5.4.3) turns to be a good approximation with the accuracy dependent on (i) size parameter a defined by relations (4.4.1) and (ii) the degree of spatial inhomogeneity of the electric field measured by the *electric field inhomogeneity length* R_{ex} introduced below. The parameter R_{ex} is similar to the radius of curvature of the graph of $\varphi_{\mathrm{ex}}(t,\mathbf{x})$, and large or small values R_{ex} correspond respectively to almost homogeneous or highly inhomogeneous electric field. It turns out that the wave-corpuscle solves the field equations (5.4.1) with the discrepancy $D=O\left((a/R_{\mathrm{ex}})^2\right)$ for $a\ll R_{\mathrm{ex}}$ as we show below. *We fix now for the rest of this section the initial data r_0 and v_0 in (5.4.2) and consequently the function* $\mathbf{r}(t)$. We assume here that the factor $\left|\mathring{\psi}_1(|\mathbf{x}|)\right|^2$ *decays exponentially* as $|\mathbf{x}|\to\infty$, and, in particular, for some constant C_0

$$\int_{\mathbb{R}^3}\left|\mathring{\psi}_1(|\mathbf{x}|)\right|^2|\mathbf{x}|^2\,\mathrm{d}\mathbf{x}\le C_0,\quad \int_{\mathbb{R}^3}\left|\mathring{\psi}_1(|\mathbf{x}|)\right|^2|\mathbf{x}|\,\mathrm{d}\mathbf{x}\le C_0. \tag{5.4.14}$$

To assess the accuracy of the wave-corpuscle solution defined by relations (5.1.2), (5.4.2), (5.4.3) we follow to an approach discussed in Section 5.1. Namely, we introduce *auxiliary field equation*

$$\mathrm{i}\chi\partial_t\psi-\frac{\chi^2\nabla^2\psi}{2m}+q\left(\varphi+\tilde{\varphi}_{\mathrm{ex}}(t,\mathbf{x})\right)\psi-\frac{\chi^2G'\psi}{2m}=0,\ \nabla^2\varphi=-4\pi q\left|\psi\right|^2, \tag{5.4.15}$$

where $\tilde{\varphi}_{\mathrm{ex}}(t,\mathbf{x})$ is defined by (5.4.4). In view of the relations (5.4.6) the corresponding external field $\mathbf{E}_{\mathrm{ex}}(t)=-\nabla_{\mathbf{x}}\varphi_{\mathrm{ex}}(\mathbf{r}(t),t)$ is homogeneous. A straightforward examination shows that the results of Section 5.4.1 apply and the wave-corpuscle defined by (5.1.2) with complimentary point charge equations (5.4.2), (5.4.3) is an exact solution to the auxiliary field equations (5.4.15). Now notice that the auxiliary field equations differ from the original ones (5.4.1) only by replacement of $\varphi_{\mathrm{ex}}(t,\mathbf{x})$ with $\tilde{\varphi}_{\mathrm{ex}}(t,\mathbf{x})$ with the consequent *discrepancy*

$$D_0(t,\mathbf{x})=q\left[\tilde{\varphi}_{\mathrm{ex}}(t,\mathbf{x})-\varphi_{\mathrm{ex}}(t,\mathbf{x})\right]\psi(t,\mathbf{x})=-q\varphi^{(1)}(t,\mathbf{x})\,\psi(t,\mathbf{x}). \tag{5.4.16}$$

Based on the above discrepancy and taking into account dependence on size parameter a for $\mathring{\psi} = \mathring{\psi}_a$ as in (4.4.1) we introduce *integral discrepancy*

$$\bar{D}_0 = \int_{\mathbb{R}^3} D_0 (t, \mathbf{x}) \psi^* \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} -q\varphi^{(1)} (t, \mathbf{x}) \psi \psi^* \, \mathrm{d}\mathbf{x} \tag{5.4.17}$$
$$= \int_{\mathbb{R}^3} -q a^{-3} \left| \mathring{\psi}_1 \left(a^{-1} |\mathbf{x} - \mathbf{r}| \right) \right|^2 \varphi_{\mathrm{ex}}^{(1)} (t, \mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

Notice that the relations ((5.4.4) for $\tilde{\varphi}_{\mathrm{ex}} (t, \mathbf{x})$ and the charge normalization condition (5.0.14) imply that similar integral involving $\tilde{\varphi}_{\mathrm{ex}}$ equals

$$\int_{\mathbb{R}^3} q \tilde{\varphi}_{\mathrm{ex}} (t, \mathbf{x}) |\psi|^2 \, \mathrm{d}\mathbf{x} = q \varphi_{\mathrm{ex}} (t, \mathbf{r}(t)), \tag{5.4.18}$$

which coincides with the potential energy of a point charge q in electric field $\varphi_{\mathrm{ex}} (t, \mathbf{x})$, therefore it is natural to compare $\bar{D}_0$ with variation of this energy in the dynamics. To assess typical scales of inhomogeneity of the external field we introduce the potential variation quantity

$$\bar{\varphi}_{0,T} = \max_{0 \leq t \leq T} |\varphi_{\mathrm{ex}} (t, \mathbf{r}(t)) - \varphi_{\mathrm{ex}} (0, \mathbf{r}(0))| . \tag{5.4.19}$$

Note that $|q| \bar{\varphi}_{0,T}$ equals point charge potential energy variation on time interval $[0, T]$. We also introduce a parameter σ_ψ which plays a role similar to 3σ for Gaussian probability but with respect to the function $\mathring{\psi}_1 (|z|)^2$ in (4.4.1), namely

$$\mathring{\psi}_1 (r) \simeq 0 \text{ if } r \geq \sigma_\psi. \tag{5.4.20}$$

The above approximate vanishing means that the discrepancy created by replacing $\mathring{\psi}_1 (r)$ by zero for large r is smaller than the discrepancies we write below. Then we introduce the following characteristic lengths $R_\varphi (t, \mathbf{r})$ and R_φ similar to the radius of the curvature:

$$\frac{1}{R_\varphi^2 (t, \mathbf{r})} = \sup_{0 < |z| \leq a\sigma_\psi} \frac{\left| \varphi_{\mathrm{ex}}^{(1)} (t, \mathbf{r} + \mathbf{z}) \right|}{\mathbf{z}^2 |\bar{\varphi}|}, \quad \frac{1}{R_\varphi^2} = \max_{0 \leq t \leq T} \frac{1}{R_\varphi^2 (t, \mathbf{r}(t))}, \tag{5.4.21}$$
$$\text{where } |\bar{\varphi}| = \max_{0 \leq t \leq T} \max_{0 < |z| \leq a\sigma_\psi} |\varphi (t, \mathbf{r}(t) + \mathbf{z}) - \varphi (0, \mathbf{r}(0))| . \tag{5.4.22}$$

The quantity R_φ represents typical spatial scale at which spatially curvilinear component $\varphi_{\mathrm{ex}}^{(1)} (t, \mathbf{x})$ of the external field $\varphi_{\mathrm{ex}} (t, \mathbf{x})$ changes significantly around $\mathbf{r}(t)$. For small a the quantity R_φ is essentially determined by the maximal eigenvalue $|\lambda_{\max}|$ of the matrix of the second spatial derivatives of $\varphi_{\mathrm{ex}} (t, \mathbf{x})$ at $\mathbf{x} = \mathbf{r}(t)$. In addition to that, $R_\varphi (t, \mathbf{r}) \to 1/|\lambda_{\max}|$ as $a \to 0$ where $1/|\lambda_{\max}|$ is the minimal curvature radius of the graph of the normalized potential $\varphi_{\mathrm{ex}}^{(1)} (t, \mathbf{x}) / |\bar{\varphi}|$ at the point $\mathbf{r}(t)$. It follows from (5.4.5) and (5.4.21) that $1/R_\varphi^2$ is bounded as long as the curve $\mathbf{r}(t)$ is not close to singularities of the external field $\varphi_{\mathrm{ex}} (t, \mathbf{x})$ if any. Then we estimate the integral discrepancy as follows:

$$|\bar{D}_0| = |q| \left| \int_{\mathbb{R}^3} a^{-3} \left| \mathring{\psi}_1 \left(a^{-1} |\mathbf{x} - \mathbf{r}| \right) \right|^2 \varphi_{\mathrm{ex}}^{(1)} (t, \mathbf{r} + \mathbf{x} - \mathbf{r}) \, \mathrm{d}\mathbf{x} \right| \tag{5.4.23}$$
$$\lesssim \frac{|q| |\bar{\varphi}|}{R_\varphi^2} \int_{\mathbb{R}^3} a^{-3} \left| \mathring{\psi}_1 \left(a^{-1} |\mathbf{x} - \mathbf{r}| \right) \right|^2 (\mathbf{x} - \mathbf{r})^2 \, \mathrm{d}\mathbf{x} \leq C_0 \frac{a^2 |q| |\bar{\varphi}|}{R_\varphi^2}.$$

Combing inequality (5.4.23) with relations (5.4.18), (5.4.19), (5.4.22) we can judge the quality of approximation by requiring the relative dimensionless discrepancy $\left|\bar{D}_0\right| / \left(|q| \bar{\varphi}_{0,T}\right)$ to be small, namely

$$\frac{\left|\bar{D}_0\right|}{|q| \bar{\varphi}_{0,T}} \lesssim \frac{a^2 |\bar{\varphi}|}{R_\varphi^2 \bar{\varphi}_{0,T}} \ll 1 \text{ is a requirement for an accurate approximation.} \tag{5.4.24}$$

For further applications we briefly consider an effect on the discrepancy of a perturbation of the external potential $\varphi_{\mathrm{ex}}(t,\mathbf{x})$ when it is substituted with slightly different potential $\acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},\epsilon)$ with ϵ being a small perturbation parameter and the approximate solution is determined based on $\varphi_{\mathrm{ex}}(t,\mathbf{x}) = \acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},0)$. Supposing the initial data $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$ and hence the position function (trajectory) $\mathbf{r}(t)$ solving the motion equation (5.4.2) being fixed we assume that there exists fixed positive constants C, C_1, T and ϵ_1 such that for any small ϵ we have

$$\begin{gathered} \left|\varphi_{\mathrm{ex}}(t,\mathbf{x}) - \acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},\epsilon)\right| \leq C_1\epsilon, \left|\nabla\varphi_{\mathrm{ex}}(t,\mathbf{x}) - \nabla\acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},\epsilon)\right| \leq C_1\epsilon \\ \text{for any } t \text{ and } \mathbf{x} \text{ such that } |\mathbf{x}-\mathbf{r}(t)| \leq \epsilon_1,\ 0 \leq t \leq T. \end{gathered} \tag{5.4.25}$$

The above condition simply requires the external perturbed field potential to stay close to original one for a small vicinity of the trajectory $\mathbf{r}(t)$. Substitution of the original (corresponding to $\epsilon = 0$) wave-corpuscle solution $\{\psi, \varphi\}$ defined by (5.1.2) and original complimentary point charge motion equations (5.4.2) for $\mathbf{r}(t)$ into the equation (5.4.1) with the external potential $\acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},\epsilon)$ produces the total discrepancy

$$\acute{D}_0(t,\mathbf{x}) = D_0(t,\mathbf{x}) + \hat{D}_1(t,\mathbf{x}),\ \hat{D}_1(t,\mathbf{x}) = -\left[\varphi_{\mathrm{ex}}(t,\mathbf{x}) - \acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},\epsilon)\right]\psi. \tag{5.4.26}$$

Note that if a is small $a\sigma_\psi \leq \epsilon_1$ and using (5.4.25) we get

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \hat{D}_1(t,\mathbf{x})\,\psi^*\,\mathrm{d}\mathbf{x}\right| &\lesssim \left|\int_{|\mathbf{r}-\mathbf{r}(t)|\leq\epsilon_1} \left|\varphi_{\mathrm{ex}}(t,\mathbf{x}) - \acute{\varphi}_{\mathrm{ex}}(t,\mathbf{x},\epsilon)\right| |\psi|^2\,\mathrm{d}\mathbf{x}\right| \\ &\leq \sup_{|\mathbf{r}-\mathbf{r}(t)|\leq\epsilon_1} \left|\varphi_{\mathrm{ex}}(t,\mathbf{r}) - \acute{\varphi}_{\mathrm{ex}}(t,\mathbf{r},\epsilon)\right| \leq C_1\epsilon, \end{aligned} \tag{5.4.27}$$

where $C_1/|\bar{\varphi}|$ is a dimensionless constant. Combining (5.4.23), (5.4.26), (5.4.27) we get the following rough estimate

$$\frac{\left|\hat{D}_0(t,\mathbf{x})\right| + \left|\hat{D}_1(t,\mathbf{x})\right|}{|\bar{\varphi}|} \lesssim \left(C_0\frac{a^2}{R_\varphi^2} + \frac{C_1\epsilon}{|\bar{\varphi}|}\right)|q|. \tag{5.4.28}$$

It is instructive to look at the trajectory $\mathbf{r}(t)$ determined from (5.4.2) for previously constructed exact and approximate wave-corpuscle solutions from another point of view. Namely, we introduce a moving frame $\mathbf{y} = \mathbf{x} - \mathbf{r}(t)$, where $\mathbf{r} = \mathbf{r}(t)$, $s_{\mathrm{p}} = s_{\mathrm{p}}(t)$, $\mathbf{v} = \mathbf{v}(t)$ solve (5.4.2), (5.4.3). Notice that the origin of the new frame is at the center $\mathbf{r}(t)$ of the wave-corpuscle. Let change variable in equations (5.4.1):

$$\psi(t,\mathbf{x}) = \exp\left\{\frac{im\mathbf{v}\cdot\mathbf{y}}{\chi} + \frac{\mathrm{i}s_{\mathrm{p}}}{\chi}\right\}\hat{\psi}(t,\mathbf{y}),\ \varphi(\mathbf{x}) = \hat{\varphi}(\mathbf{y}), \tag{5.4.29}$$

where $\hat{\psi}(t,\mathbf{y})$ is a new unknown function. We can repeat the above calculations (without using (2.3.8)) to obtain an equivalent equation of the same form as (5.4.1), namely

$$\mathrm{i}\chi\partial_t\hat{\psi} = -\frac{\chi^2}{2m}\nabla_y^2\hat{\psi} + q\left(\hat{\varphi} + \hat{\varphi}_{\mathrm{ex}}\right)\hat{\psi} + \frac{\chi^2}{2m}G'\left(\left|\hat{\psi}\right|^2\right)\acute{\psi},\ \nabla_y^2\hat{\varphi} = -4\pi q\left|\hat{\psi}\right|^2, \tag{5.4.30}$$

with an external potential $\hat{\varphi}_{\mathrm{ex}}(t, y)$ which satisfies an additional condition $\hat{\varphi}_{\mathrm{ex}}(0) = 0$, $\nabla\hat{\varphi}_{\mathrm{ex}}(0) = 0$. If the original potential $\varphi_{\mathrm{ex}}(t, \mathbf{x})$ is linear in $\mathbf{x}$, the external potential $\hat{\varphi}_{\mathrm{ex}}(t, \mathbf{y})$ in the moving frame vanishes, i.e. $\hat{\varphi}_{\mathrm{ex}}(t, \mathbf{y}) = 0$ for all $t, \mathbf{y}$. In this case (5.4.30) coincides with the equilibrium condition (2.3.8), hence $\hat{\psi} = \mathring{\psi}$ and the wave-corpuscle rests at the origin of the moving frame.

5.5 Accelerated motion in a general external EM field

In this subsection we consider a single charge in an a general external EM field *which can have nonzero magnetic component*. The primary goal of this section is to show that wave-corpuscle defined by relations (5.1.2) and the complimentary point charge equations (5.1.3), (5.1.4) is an approximate solution to the field equations (5.0.12), (5.0.13). We accomplish that by following to the method of Section 5.1) and showing first that the wave-corpuscle is an exact solution to the auxiliary field equations (5.1.1) and then provide estimations of the discrepancy between the auxiliary and the original field equations (5.0.12), (5.0.13).

5.5.1 Wave-corpuscle as an exact solution

In this section we show that the wave-corpuscle defined by relations (5.1.2) and the complimentary point charge equations (5.1.3), (5.1.4) is an exact solution to the auxiliary field equations (5.1.1). One way to do that is to plug in ψ, φ defined by the formulas (5.1.2) into the auxiliary field equations (5.1.1) and using the complimentary point charge equations (5.1.3), (5.1.4) verify that the equality does hold. An alternative and more instructive, we believe, way to accomplish the same goal is (i) to assume for the sake of the argument that the point charge functions $\mathbf{r}(t)$, $\mathbf{v}(t)$, $s_{\mathrm{p}}(t)$ are unknown; (ii) to find out if there is way to choose those functions so that the wave-corpuscle fields ψ, φ solves exactly the auxiliary field equations (5.1.1); (iii) verify that the chosen $\mathbf{r}(t)$, $\mathbf{v}(t)$, $s_{\mathrm{p}}(t)$ satisfy the complimentary point charge equations. Following to this way we substitute (5.1.2) into (5.1.1) and obtain the following equation for $\mathbf{r}, \mathbf{v}, s_{\mathrm{p}}$:

$$\mathrm{i}\left\{\mathrm{i}m\left[\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right]+\mathrm{i}\dot{s}_{\mathrm{p}}\right\}\hat{\psi}-\mathrm{i}\chi\dot{\mathbf{r}}\cdot\hat{\psi}'\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}-\frac{m\mathbf{v}^{2}\hat{\psi}}{2}+\mathrm{i}\chi\hat{\psi}'\mathbf{v}\cdot\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}+\frac{\chi^{2}\nabla^{2}\hat{\psi}}{2m}$$
$$+\left[\frac{q\mathbf{v}\hat{\psi}}{\mathrm{c}}+\frac{\chi q\hat{\psi}'}{m\mathrm{ci}}\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}\right]\cdot\left[\mathbf{A}_{\mathrm{ex},0}+\frac{\mathbf{B}_{0}\times(\mathbf{x}-\mathbf{r})}{2}\right]-q\left(\tilde{\varphi}_{\mathrm{ex}}+\varphi\right)\hat{\psi}-\frac{\chi^{2}G'\hat{\psi}}{2m}=0. \tag{5.5.1}$$

Taking into account an obvious identity $(\mathbf{x}-\mathbf{r})\cdot[\mathbf{B}_0\times(\mathbf{x}-\mathbf{r})] = 0$ we recast the above equation as follows:

$$\begin{aligned}&\left[-m\left(\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right)-\dot{s}_{\mathrm{p}}\right]\hat{\psi}-\mathrm{i}\chi\dot{\mathbf{r}}\cdot\hat{\psi}'\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}\\&-\frac{m\mathbf{v}^{2}\hat{\psi}}{2}+\mathrm{i}\chi\hat{\psi}'\mathbf{v}\cdot\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}+\frac{\chi^{2}\nabla^{2}\hat{\psi}}{2m}+\frac{\chi q\hat{\psi}'}{m\mathrm{ci}}\frac{(\mathbf{x}-\mathbf{r})}{|\mathbf{x}-\mathbf{r}|}\cdot\mathbf{A}_{\mathrm{ex},0}\\&+\frac{q}{\mathrm{c}}\mathbf{v}\cdot\left[\mathbf{A}_{\mathrm{ex},0}+\frac{\mathbf{B}_{0}\times(\mathbf{x}-\mathbf{r})}{2}\right]\hat{\psi}-q\left(\tilde{\varphi}_{\mathrm{ex}}+\mathring{\varphi}\right)\hat{\psi}-\frac{\chi^{2}}{2m}G'\hat{\psi}=0.\end{aligned} \tag{5.5.2}$$

Using the charge equilibrium equation (2.3.8) we can eliminate G' in equation (5.5.2) obtaining the following equivalent to it equation:

$$
\begin{gathered}
-\left[m\left(\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})+\mathbf{v}\cdot\dot{\mathbf{r}}\right)+\dot{s}_{\mathrm{p}}\right]\hat{\psi}-\mathrm{i}\chi\dot{\mathbf{r}}\cdot\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}\hat{\psi}'-\frac{m\mathbf{v}^{2}\hat{\psi}}{2}+\mathrm{i}\chi\hat{\psi}'\mathbf{v}\cdot\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|} \\
+\frac{q}{\mathrm{c}}\mathbf{v}\cdot\left[\mathbf{A}_{\mathrm{ex},0}+\frac{\mathbf{B}_{0}\times(\mathbf{x}-\mathbf{r})}{2}\right]\hat{\psi}+\frac{\chi q\hat{\psi}'}{m\mathrm{ci}}\frac{\mathbf{x}-\mathbf{r}}{|\mathbf{x}-\mathbf{r}|}\cdot\mathbf{A}_{\mathrm{ex},0}-q\tilde{\varphi}_{\mathrm{ex}}\hat{\psi}=0.
\end{gathered}
\tag{5.5.3}
$$

For equation (5.5.3) to hold we may require the coefficient before $\hat{\psi}$ and $\hat{\psi}'$ in it to be zero. Executing that by collecting terms with $\hat{\psi}$ and $\hat{\psi}'$ and using $\tilde{\varphi}_{\mathrm{ex}}=\varphi_{0,\mathrm{ex}}+\varphi'_{0,\mathrm{ex}}\cdot(\mathbf{x}-\mathbf{r})$ we obtain the following equations:

$$
\dot{\mathbf{r}}=\mathbf{v}-\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}, \tag{5.5.4}
$$

$$
\begin{gathered}
-m\left[\dot{\mathbf{v}}\cdot(\mathbf{x}-\mathbf{r})-\mathbf{v}\cdot\dot{\mathbf{r}}\right]-\dot{s}_{\mathrm{p}}-\frac{m\mathbf{v}^{2}}{2}+\frac{q\mathbf{v}}{\mathrm{c}}\cdot\left[\mathbf{A}_{\mathrm{ex},0}+\frac{1}{2}\mathbf{B}_{0}\times(\mathbf{x}-\mathbf{r})\right] \\
-q\left(\varphi_{0,\mathrm{ex}}+\varphi'_{0,\mathrm{ex}}\cdot(\mathbf{x}-\mathbf{r})\right)=0.
\end{gathered}
\tag{5.5.5}
$$

To solve (5.5.5) we require coefficient $(\mathbf{x}-\mathbf{r})$ and the remaining one to be zero, and that with the help of an elementary identity $\mathbf{v}\cdot(\mathbf{B}_{0}\times(\mathbf{x}-\mathbf{r}))=(\mathbf{x}-\mathbf{r})\cdot(\mathbf{v}\times\mathbf{B}_{0})$ yields the following pair of equations:

$$
m\dot{\mathbf{v}}=-q\left[\frac{1}{2\mathrm{c}}\mathbf{B}_{0}(t)\times\mathbf{v}+\varphi'_{0,\mathrm{ex}}(t)\right], \tag{5.5.6}
$$

$$
m\mathbf{v}\cdot\dot{\mathbf{r}}-\dot{s}_{\mathrm{p}}+\frac{q}{\mathrm{c}}\mathbf{v}\cdot\mathbf{A}_{\mathrm{ex},0}-\frac{m\mathbf{v}^{2}}{2}-q\varphi_{0,\mathrm{ex}}=0. \tag{5.5.7}
$$

Now being given $v(0)$ we readily find $\mathbf{v}(t)$ from the linear equation (5.5.6). Then using $\mathbf{v}(t)$ and being given $r(0)$ we immediately find $\mathbf{r}(t)$ from equation (5.5.4). Combining equations (5.5.4) and (5.5.7) we obtain the following equations for $s_{\mathrm{p}}(t)$

$$
\dot{s}_{\mathrm{p}}=m\mathbf{v}\cdot\dot{\mathbf{r}}+\frac{q}{\mathrm{c}}\mathbf{v}\cdot\mathbf{A}_{\mathrm{ex},0}-\frac{m}{2}\mathbf{v}^{2}-q\varphi_{0,\mathrm{ex}}=\frac{m\mathbf{v}^{2}}{2}-q\varphi_{0,\mathrm{ex}}. \tag{5.5.8}
$$

It remains to verify that the triple $\{\mathbf{r}(t),\mathbf{v}(t),s_{\mathrm{p}}(t)\}$ satisfies the complimentary point charge equations (5.1.3), (5.1.4). Indeed, combining the relations (5.5.6) and (5.5.4) we obtain

$$
m\ddot{\mathbf{r}}=-q\left[\frac{1}{2\mathrm{c}}\mathbf{B}_{0}\times\left(-\dot{\mathbf{r}}-\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}\right)+\varphi'_{0,\mathrm{ex}}+\frac{1}{\mathrm{c}}\partial_{t}\mathbf{A}_{\mathrm{ex},0}\right]. \tag{5.5.9}
$$

A straightforward comparison taking into account (5.1.6) shows that the above equation (5.5.9) coincides with the point charge motion equation (5.1.3), and the equations (5.5.4) and (5.5.8) provide for the point charge equations (5.1.4). *With that we completed the desired verification of the fact the wave-corpuscle does solve exactly the auxiliary field equations (5.1.1).*

As to the *exact solvability* issue let us compare the coefficients of auxiliary equations (5.1.1) and the original field equations (5.0.12), (5.0.13). Taking into account the relation (5.1.6)-(5.1.7) we find that the EM potentials $\varphi_{\mathrm{ex}}(t,\mathbf{x})$, $\mathbf{A}_{\mathrm{ex}}(t,\mathbf{x})$ in the field equations (5.0.12), (5.0.13) are compatible with the auxiliary system (5.1.1) for the wave-corpuscle defined by

(5.1.2) and the complimentary point charge equations (5.1.3), (5.1.4) if the following *exact solvability condition* holds

$$\varphi_{\mathrm{ex}}(t,\mathbf{x})=\varphi_{0,\mathrm{ex}}+\varphi_{0,\mathrm{ex}}^{\prime}\cdot(\mathbf{x}-\mathbf{r})-\frac{q}{2mc^{2}}\left(\mathbf{A}_{\mathrm{ex},0}+\frac{1}{2}\mathbf{B}_{0}(t)\times(\mathbf{x}-\mathbf{r}(t))\right)^{2}, \qquad (5.5.10)$$
$$\mathbf{A}_{\mathrm{ex}}(t,\mathbf{x})=\tilde{\mathbf{A}}_{\mathrm{ex}}(t,\mathbf{x})=\mathbf{A}_{\mathrm{ex},0}+\frac{1}{2}\mathbf{B}_{0}(t)\times(\mathbf{x}-\mathbf{r}(t)).$$

Note that if $\mathbf{B}_0 \neq 0$ the electrical field potential φ_{ex} involves a quadratic term which vanishes at the center of the wave-corpuscle. One can naturally ask how broad is the class of external EM fields as in (5.5.10) for which there are exact solutions as the wave-corpuscles? The class of such EM field is sufficiently broad in the sense that for any accelerated motion of a point charge in an arbitrary EM field there is wave-corpuscle as an exact solution to the field equations with an external field from the class. To see that let us consider a point charge in *arbitrary* external EM field and find its trajectory $\mathbf{r}(t)$. Then we introduce a special EM field defined by (5.1.6), (5.1.7) with $\mathbf{B}_0=\mathbf{B}(t)$ and $\varphi_{0,\mathrm{ex}}^{\prime}(t)$ defined by (5.5.11) and for this external field the wave-corpuscle (5.1.2) is an exact solution to the field equations(5.0.12), (5.0.13) its center moves exactly according to the trajectory $\mathbf{r}(t)$. More than that, an arbitrary vector-function $\mathbf{r}(t)$ can be obtained as a solution of (5.1.3) with appropriate choice of $\acute{\mathbf{E}}(t,\mathbf{r})$, $\acute{\mathbf{B}}(t,\mathbf{r})$. Indeed, let $\acute{\mathbf{E}}(t,\mathbf{r})=m\ddot{\mathbf{r}}(t)/q$ and such $\mathbf{r}(t)$ is a solution of (5.1.3). Note that for the given $\acute{\mathbf{E}}(t)$ along the trajectory we can take $\mathbf{A}_{\mathrm{ex},0}(t)$ to be arbitrary and determine $\varphi_{0,\mathrm{ex}}^{\prime}(t)$ by the following formula

$$\varphi_{0,\mathrm{ex}}^{\prime}(t)=-\left\{\frac{\partial_t\mathbf{A}_{\mathrm{ex},0}(t)}{\mathrm{c}}-\frac{\mathbf{B}_{0}(t)\times\dot{\mathbf{r}}(t)}{2\mathrm{c}}+\acute{\mathbf{E}}(t)+\frac{q}{mc^{2}}\left[\frac{1}{2}\mathbf{B}_{0}(t)\times\mathbf{A}_{\mathrm{ex},0}(t)\right]\right\}. \qquad (5.5.11)$$

Thus, we can conclude that the wave-corpuscle (5.1.2) as an exact solution to the field equations(5.0.12), (5.0.13) with an appropriate choice of the external EM field can model any motion of a point charge.

5.5.2 de Broglie factor for accelerating charge

In this subsection we would like to take a look at the de Broglie exponential factor in the wave-corpuscle defined by (5.1.2) the complimentary point charge equations (5.1.3), (5.1.4). Let $\check{\psi}(\mathbf{k})=[\mathcal{F}\psi](\mathbf{k})$ be the Fourier transform of the wave function $\psi(\mathbf{x})$

$$\check{\psi}(\mathbf{k})=[\mathcal{F}\psi](\mathbf{k})=\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{x})\,\mathrm{d}\mathbf{x}. \qquad (5.5.12)$$

Then in view of the charge normalization condition (5.0.14) and by the Parseval theorem $\check{\psi}(\mathbf{k})$ satisfies similar condition, namely

$$(2\pi)^{-3}\int_{\mathbb{R}^3}\left|\check{\psi}(\mathbf{k})\right|^{2}\,\mathrm{d}\mathbf{k}=1, \qquad (5.5.13)$$

and we can introduce the center $\mathbf{k}_*(\psi)$ for $\check{\psi}(\mathbf{k})$ as follows:

$$\mathbf{k}_*(\psi)=\mathbf{k}_*=(2\pi)^{-3}\int_{\mathbb{R}^3}\mathbf{k}\left|\check{\psi}(\mathbf{k})\right|^{2}\,\mathrm{d}\mathbf{k}. \qquad (5.5.14)$$

Note that the following identity holds

$$\mathbf{k}\left|\check{\psi}\left(\mathbf{k}\right)\right|^{2}=\mathbf{k}\check{\psi}\left(\mathbf{k}\right)\check{\psi}^{*}\left(\mathbf{k}\right)=\frac{1}{2\mathrm{i}}\left\{\mathrm{i}\mathbf{k}\check{\psi}\left(\mathbf{k}\right)\check{\psi}^{*}\left(\mathbf{k}\right)-\check{\psi}\left(\mathbf{k}\right)\left[\mathrm{i}\mathbf{k}\check{\psi}\left(\mathbf{k}\right)\right]^{*}\right\}$$

implying together with (5.5.14) the following representation

$$\mathbf{k}_{*}=\frac{1}{2\mathrm{i}}\int_{\mathbb{R}^{3}}\left[\nabla\psi\left(\mathbf{x}\right)\psi^{*}\left(\mathbf{x}\right)-\nabla\psi^{*}\left(\mathbf{x}\right)\psi\left(\mathbf{x}\right)\right]\,\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^{3}}\operatorname{Im}\frac{\nabla\psi\left(\mathbf{x}\right)}{\psi\left(\mathbf{x}\right)}\left|\psi\left(\mathbf{x}\right)\right|^{2}\,\mathrm{d}\mathbf{x}. \tag{5.5.15}$$

Observe that the Fourier transform (5.5.12) of the wave-corpuscle defined by (5.1.2) is

$$\check{\psi}\left(t,\mathbf{k}\right)=\exp\left\{\mathrm{i}\mathbf{r}\left(t\right)\mathbf{k}-\frac{\mathrm{i}s_{\mathrm{p}}\left(t\right)}{\chi}\right\}\left(\mathcal{F}\left[\mathring{\psi}\right]\right)\left(\mathbf{k}-\frac{m\mathbf{v}\left(t\right)}{\chi}\right), \tag{5.5.16}$$

and since $\mathring{\psi}=\mathring{\psi}\left(\left|\mathbf{x}\right|\right)$ is a radial function its Fourier transform $\mathcal{F}\left[\mathring{\psi}\right]\left(\mathbf{k}\right)$ is a radial function as well. Let us consider $\mathbf{k}_{*}$ defined by the formula (5.5.14) that corresponds to the wave-corpuscle (5.5.16). Using the fact $\mathcal{F}\left[\mathring{\psi}\right]\left(\mathbf{k}\right)$ is a radial function and the relations (5.1.4), (5.1.5) we readily find that

$$\mathbf{k}_{*}=\frac{m\mathbf{v}\left(t\right)}{\chi},\ \ \mathbf{v}\left(t\right)=\dot{\mathbf{r}}\left(t\right)+\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}=\frac{\mathring{\mathbf{p}}}{m}. \tag{5.5.17}$$

The dispersion relation $\omega\left(\mathbf{k}\right)$ for the linear part of equation (5.1.1) and the corresponding group velocity $\nabla_{\mathbf{k}}\omega\left(\mathbf{k}\right)$ are respectively

$$\omega\left(\mathbf{k}\right)=\frac{\chi}{2m}\mathbf{k}^{2}-\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}\cdot\mathbf{k},\ \ \nabla_{\mathbf{k}}\omega\left(\mathbf{k}\right)=\frac{\chi}{m}\mathbf{k}-\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}. \tag{5.5.18}$$

Combining relations (5.5.18) and (5.5.17)

$$\nabla_{\mathbf{k}}\omega\left(\mathbf{k}_{*}\right)=\frac{\chi}{m}\mathbf{k}_{*}-\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}=\mathbf{v}\left(t\right)-\frac{q}{mc}\mathbf{A}_{\mathrm{ex},0}=\dot{\mathbf{r}}\left(t\right), \tag{5.5.19}$$

we find that the charge velocity $\mathbf{v}\left(t\right)$ is identical with the group velocity $\nabla_{\mathbf{k}}\omega\left(\mathbf{k}_{*}\right)$ indicating the wave origin of the charge motion.

5.5.3 General external EM field

The subject of this section and the treatment are similar to ones in Section 5.4.2, but estimates in the presence of external magnetic field are more involved. Below we provide the most essential estimates related to this case omitting tedious details. The main result of this section is that the wave-corpuscle defined by (5.1.2) and the complimentary point charge equations (5.1.3), (5.1.4) is an approximate solution to the field equations (5.0.12), (5.0.13) with a discrepancy of the order $O\left(\left(a/R_{\mathrm{ex}}\right)^{2}\right)$ for $a\ll R_{\mathrm{ex}}$ where a is the size parameter defined by relations (4.4.1) and R_{ex} is a typical length for inhomogeneity of the external field. We assume here that the function $\left|\mathring{\psi}_{1}\left(s\right)\right|^{2}$ decays exponentially as $s\to\infty$, and (5.4.14) holds.

We define the coefficients (5.1.6)- (5.1.7) of auxiliary field equations in (5.1.1) as follows

$$\varphi_{0,\mathrm{ex}}\left(t\right)=\varphi_{\mathrm{ex}}\left(\mathbf{r}\left(t\right),t\right)+\frac{q}{2mc^{2}}\mathbf{A}_{\mathrm{ex}}^{2}\left(\mathbf{r}\left(t\right),t\right), \tag{5.5.20}$$

$$\varphi_{0,\mathrm{ex}}^{\prime}\left(t\right)=\nabla_{x}\varphi_{\mathrm{ex}}\left(\mathbf{r}\left(t\right),t\right)+\frac{q}{2mc^{2}}\left[\mathbf{A}_{\mathrm{ex},0}\times\mathbf{B}_{0}\left(t\right)\right],\ \text{implying}$$

$$\tilde{\varphi}_{\mathrm{ex}}(t,\mathbf{x})=\varphi_{0,\mathrm{ex}}(t)+\varphi'_{0,\mathrm{ex}}\cdot(\mathbf{x}-\mathbf{r}(t)),\ \varphi'_{0,\mathrm{ex}}=\varphi'_{0,\mathrm{ex}}(t), \tag{5.5.21}$$
$$\mathbf{B}_0(t)=\mathbf{B}(\mathbf{r}(t),t),\ \mathbf{A}_{\mathrm{ex},0}(t)=\mathbf{A}_{\mathrm{ex}}(\mathbf{r}(t),t).$$

Note that the wave-corpuscle defined by (5.1.2) and the complimentary point charge equations (5.1.3), (5.1.4) generally speaking is not an exact solution of (5.0.12) since the potentials of (5.0.12) satisfy more general relations than (5.5.10):

$$\varphi_{\mathrm{ex}}(t,\mathbf{x})=\varphi_{0,\mathrm{ex}}(t)+\varphi'_{0,\mathrm{ex}}\cdot(\mathbf{x}-\mathbf{r}(t))-\frac{q}{2m\mathrm{c}^2}\left(\mathbf{A}_{\mathrm{ex},0}+\frac{1}{2}\mathbf{B}_0(t)\times(\mathbf{x}-\mathbf{r}(t))\right)^2+\varphi^{(1)}_{\mathrm{ex}}(t,\mathbf{x}), \tag{5.5.22}$$
$$\mathbf{B}(t,\mathbf{x})=\mathbf{B}_0(t)+\mathbf{B}_1(t,\mathbf{x}),\ \mathbf{A}_{\mathrm{ex}}(t,\mathbf{x})=\mathbf{A}_{\mathrm{ex},0}(t)+\frac{1}{2}\mathbf{B}_0(t)\times(\mathbf{x}-\mathbf{r}(t))+\mathbf{A}_{\mathrm{ex},1}(t,\mathbf{x}),$$

with non-zero terms $\varphi^{(1)}_{\mathrm{ex}}(t,\mathbf{x})$, $\mathbf{B}_1(t,\mathbf{x})$, $\mathbf{A}_{\mathrm{ex},1}(t,\mathbf{x})$. These extra terms are small in the vicinity of $\mathbf{r}(t)$ where ψ is localized since

$$\varphi^{(1)}_{\mathrm{ex}}(x,\mathbf{r}(t))=0,\ \nabla\varphi^{(1)}_{\mathrm{ex}}(x,\mathbf{r}(t))=0,\ \mathbf{B}_1(\mathbf{r}(t),t)=0,\ \mathbf{A}_{\mathrm{ex},1}(\mathbf{r}(t),t)=0. \tag{5.5.23}$$

To estimate discrepancy resulting from magnetic field we introduce quantities

$$\left|\bar{\mathbf{B}}\right|=\max_{0\le t\le T}\mathbf{B}(\mathbf{r}(t),t),\ \left|\bar{\mathbf{A}}\right|=\max_{0\le t\le T}\max_{0<|z|\le a\sigma_\psi}\left|\mathbf{A}(\mathbf{r}(t)+\mathbf{z},t)-\mathbf{A}(\mathbf{r}(0),0)\right|, \tag{5.5.24}$$
$$\frac{1}{R_B(t,\mathbf{r})}=\sup_{0<|z|\le a\sigma_\psi}\frac{|\mathbf{B}_1(\mathbf{r}(t)+\mathbf{z},t)|}{|\mathbf{z}|\left|\bar{\mathbf{B}}\right|},$$
$$\frac{1}{R_A(t,\mathbf{r})}=\sup_{0<|z|\le a\sigma_\psi}\frac{|\mathbf{A}_{\mathrm{ex},1}(\mathbf{r}(t)+\mathbf{z},t)|}{|\mathbf{z}|\left|\bar{\mathbf{A}}\right|},\ \frac{1}{R_M}=\max_{0\le t\le T}\left(\frac{1}{R_B(t,\mathbf{r})},\frac{1}{R_A(t,\mathbf{r})}\right).$$

The quantity $R_A(t,\mathbf{r})$ is a spatial distance at which the local variation of $\mathbf{A}$ is of the same order as the global variation $\left|\bar{\mathbf{A}}\right|$ of $\mathbf{A}$ itself, and consequently it represents a spatial scale at which the local variation of $\mathbf{A}$ is not negligible. By (5.5.23) the quantity $1/R_M$ is bounded. We substitute (5.1.2) in (5.0.12) and observe that according Subsection 5.5, ψ,φ exactly satisfy (5.0.12) if $\varphi^{(1)}_{\mathrm{ex}}(t,\mathbf{x})$, $\mathbf{B}_1(t,\mathbf{x})$, $\mathbf{A}_{\mathrm{ex},1}(t,\mathbf{x})$ are identically zero. If they are not zero, we obtain discrepancy $D=D_0+D_1$ with D_0 as in Section 5.4.2 and

$$D_1=\frac{\chi q}{m\mathrm{ci}}\left[\mathbf{A}_{\mathrm{ex},1}+\mathbf{B}_1(t,\mathbf{x})\times(\mathbf{x}-\mathbf{r}(t))\right]\cdot\nabla\psi. \tag{5.5.25}$$

Similarly to (5.4.17) we introduce *integral discrepancy* $\bar{D}_1$*:*

$$\bar{D}_1=\mathrm{Re}\int_{\mathbb{R}^3}\frac{\chi q}{m\mathrm{ci}}\left(\mathbf{A}_{\mathrm{ex},1}+\mathbf{B}_1(t,\mathbf{x})\times(\mathbf{x}-\mathbf{r}(t))\right)\cdot\nabla\psi\psi^*\,\mathrm{d}\mathbf{x} \tag{5.5.26}$$
$$=\int_{\mathbb{R}^3}\frac{\chi q}{mc}\left[\mathbf{A}_{\mathrm{ex},1}+\mathbf{B}_1(t,\mathbf{x})\times(x-\mathbf{r}(t))\right]\cdot\mathrm{Im}\left(\frac{\nabla\psi}{\psi}\right)|\psi|^2\,\mathrm{d}\mathbf{x}$$

Note that for solutions of the form (5.1.2) we have

$$\mathrm{Im}\left(\frac{\nabla\psi}{\psi}\right)=\mathrm{Im}\,\nabla\left(\frac{\mathrm{i}m}{\chi}\mathbf{v}\cdot(\mathbf{x}-\mathbf{r})+\frac{\mathrm{i}}{\chi}s_p\right)=\frac{m}{\chi}\mathbf{v},$$

implying when combined with (5.5.26)

$$\bar{D}_1 = \frac{q}{\mathrm{c}} \int_{\mathbb{R}^3} [\mathbf{A}_{\mathrm{ex},1} + \mathbf{B}_1 (t, \mathbf{x}) \times (\mathbf{x} - \mathbf{r}(t))] \cdot \mathbf{v} |\psi|^2 \, \mathrm{d}\mathbf{x},$$

which, in turn, yields the following estimate

$$\begin{aligned} \left|\bar{D}_1\right| &\lesssim \frac{|\mathbf{v}|}{\mathrm{c}} q \int_{\mathbb{R}^3} \frac{\left|\bar{\mathbf{A}}\right| |x - \mathbf{r}(t)| |\psi_a|^2}{R_A (t, \mathbf{r})} \mathrm{d}\mathbf{x} + \frac{|\mathbf{v}|}{\mathrm{c}} q \left|\bar{\mathbf{B}}\right| \int_{\mathbb{R}^3} \frac{1}{R_B (t, \mathbf{r})} |\mathbf{x} - \mathbf{r}(t)|^2 |\psi_a|^2 \, \mathrm{d}\mathbf{x} \quad (5.5.27) \\ &\leq C_0 \frac{|\mathbf{v}|}{\mathrm{c}} \frac{a}{R_M} \left(|q| \left|\bar{\mathbf{A}}\right| + qa \left|\bar{\mathbf{B}}\right| \right), \ 0 \leq t \leq T. \end{aligned}$$

Combining relation (5.4.23) and (5.5.27) we get

$$\left|\bar{D}_0 + \bar{D}_1\right| \lesssim C_0 \frac{a^2 |q| |\bar{\varphi}|}{R_\varphi^2} + \frac{|\mathbf{v}|}{\mathrm{c}} \frac{a}{R_M} C_0 \left(|q| \left|\bar{\mathbf{A}}\right| + qa \left|\bar{\mathbf{B}}\right| \right). \quad (5.5.28)$$

yielding the following conditions for the discrepancy to be relatively small

$$\frac{|\mathbf{v}|}{\mathrm{c}} \frac{a}{R_M} \ll 1, \ \frac{a^2}{R_\varphi^2} \ll 1. \quad (5.5.29)$$

5.6 Stability issues

A comprehensive analysis of the stability is complex, involved and beyond of the scope of this paper. Nevertheless, we would like to give a concise consideration to some three aspects of stability for well separated charges in the nonrelativistic regime : (i) no "blow-up" or "collapse"; (ii) preservation with high accuracy of the form of a wave-corpuscle solution for a limited time; (iii) preservation of spatial localization for certain solutions on long time intervals.

Here is an argument why there can not a "blow-up" in finite time. A "blow-up" is an issue since the nonlinearity $G'(s)$ provides for focusing properties with consequent soliton-like solutions $\mathring{\psi}^\ell$, $\mathring{\varphi}^\ell$. In our model the possibility of "blow up" is excluded when we define $\left[G_a^\ell\right]'$ to be a constant for large amplitudes of the fields, namely for $s \geq \left(\mathring{\psi}^\ell\right)^2(0)$ as in (4.4.8). This factor combined with the charge normalization condition (5.0.14) imply that the energy is bounded from below. Indeed, according to (6.1.8),(6.1.10) and (4.1.18) the energy of a free charge can be written in the form

$$\mathcal{E}(\psi, \varphi) = \int_{\mathbb{R}^3} (w + \tilde{u}) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} \frac{|\nabla \varphi|^2}{8\pi} + \frac{\chi^2}{2m} \left[|\nabla \psi|^2 + G\left(|\psi|^2\right) \right] \mathrm{d}\mathbf{x}, \quad (5.6.1)$$

where $\varphi = \varphi_\psi$ is determined from (2.4.6). In view of relations (4.4.8) the nonlinearity derivative $G'(s)$ is bounded implying $G\left(|\psi|^2\right) \geq -C |\psi|^2$ for a constant C. That combined with the charge normalization condition (5.0.14) implies boundedness of the energy from below, namely

$$\mathcal{E}(\psi, \varphi) \geq -C \text{ for all } \psi, \varphi, \ \|\psi\|^2 = \int_{\mathbb{R}^3} |\psi|^2 \, \mathrm{d}\mathbf{x} = 1. \quad (5.6.2)$$

A similar argument in the case of many interacting charges also shows that the energy is bounded from below. Since energy is a conserved quantity, using the boundedness of the energy from below one can prove along lines of [Kato89] the global existence of a unique solution $\psi^{\ell}(t,\mathbf{x})$, $\varphi^{\ell}(t,\mathbf{x})$ to (2.4.2), (2.4.7) for all times $0 \leq t < \infty$ for given initial data $\psi^{\ell}(0,\mathbf{x})$.

The second aspect of stability is about a preservation of the wave-corpuscle with high accuracy for limited times. A basis for it is provided in Section 5.4.2. Since discrepancies in the equations (5.4.15) are of the order $|q|\,|\bar{\varphi}|\,a^2/R^2$ the charge in an external EM field the fields ψ, φ have to be close to the wave-corpuscle of the form (5.1.2) on time intervals of order $|q|\,|\bar{\varphi}|\,a^2/(\chi R^2)$ where R is a spatial scale of inhomogeneity of the external field, and $|q|\,|\bar{\varphi}|$ is a global variation of external field potential energy near the trajectory of wave-corpuscle.

The third aspect is a stability on very long time intervals which is understood in a broader sense, namely when a charge maintains its spatial localization without necessarily preserving the exact form of a wave-corpuscle It is shown in Section 6.1.1 such a broad localization assumption is sufficient to identify the corresponding point charge trajectory. Now let us consider the following argument for the charge stability based on properties of the energy. For simplicity let us consider a single free charge with energy (5.6.1). The energy conservation law implies

$$\mathcal{E}(\psi(t),\varphi(t)) = \mathcal{E}(\psi(0),\varphi(0)), \text{ for all } 0 \leq t < \infty. \tag{5.6.3}$$

Note that the rest solution $\mathring{\psi}$ as in (2.3.5), (2.3.6) is a critical point of $\mathcal{E}$ defined by (5.6.1). Let us assume that the rest solution $\mathring{\psi}$ is the global minimum under the charge normalization constraint, namely

$$\mathcal{E}\left(\mathring{\psi}, \varphi_{\mathring{\psi}}\right) = \min_{\|\psi\|=1} \mathcal{E}\left(\psi, \varphi_{\psi}\right) = \mathsf{E}_0. \tag{5.6.4}$$

Consider then the initial data ψ_0 for (4.0.11)-(4.0.12) at $t=0$ that (i) satisfies the charge normalization condition (5.0.14); (ii) is close to $\mathring{\psi}$ and has almost the same energy, i.e. $|\mathcal{E}(\psi(0),\varphi(0)) - \mathsf{E}_0| \ll 1$. Note that since every spatial shift $\mathring{\psi}(\mathbf{x}-\mathbf{r})$, $\varphi_{\mathring{\psi}}(\mathbf{x}-\mathbf{r})$ of $\mathring{\psi}(\mathbf{x}), \varphi_{\mathring{\psi}}(\mathbf{x})$ produces fields satisfying the charge normalization condition (5.0.14) and of the the same energy, the minimum in (5.6.4) has to be degenerate. But if we assume that all the degeneracy is due to to spatial translations, rotations and the multiplication by e^{is} then the condition $|\mathcal{E}(\psi(t),\varphi(t)) - \mathsf{E}_0| \ll 1$ though allows for spatial translation of $\mathring{\psi}(\mathbf{x}), \varphi_{\mathring{\psi}}(\mathbf{x})$ to large distances and times still implies that form of $|\psi(t,\mathbf{x}-\mathbf{r}(t))|$, $\varphi(t,\mathbf{x}-\mathbf{r}(t))$ has to be almost the same as the form of $\mathring{\psi}(\mathbf{x}), \mathring{\varphi}(\mathbf{x})$.

6 Many interacting charges

A qualitatively new physical component in the theory of two or more charges compare to the theory of a single charge is obviously interaction between them. In our approach *any individual "bare" interacts directly only with the EM* field and consequently different charges interact with each other only indirectly trough the EM field. In this section we develop the Lagrangian theory for many interacting charges for the both relativistic and nonrelativistic cases based on Lagrangians for single charges studies in Sections 3, 4, 5. The primary focus of our studies on many charges is ways of integration into our wave theory the point charge mechanics in the regime of remote interaction when the charges are separated by large distances compare to their sizes. A system of many charges can have charges of the same type, for instance electrons and protons. In that case we naturally assume that individual Lagrangians for charges of the same type to have identical Lagrangians with the same mass m, charge q, form factor $\mathring{\psi}$ and consequently the same nonlinear self-interaction G. We use here general results of the Lagrangian field theory for many interacting charges including symmetries, conservation laws and the construction of gauge-invariant and symmetric energy-momentum tensors described in Section 10.5.

Let us introduce a system of N charges interacting directly only with the EM field described by its 4-vector potential $A^{\mu} = (\varphi, \mathbf{A})$. The charges are described by their wave functions ψ^{ℓ} with the superscript index $\ell = 1, \ldots, N$ labeling them. In this section we study the dynamics of the system of charges in the *regime of remote interaction*, that is when any two different charges of the system are well separated so that the distance between them is much larger compare to their typical sizes. We show here that under the assumption of remote interaction the charges interact which is other by Lorentz forces and that in non-relativistic case their dynamics is perfect correspondence with the dynamics of the corresponding system of point charges. In the relativistic case the correspondence with the point mechanics is more subtle because of fundamental limitations. In turns out that in the regime of remote interaction the nonlinear self-interaction terms associated with charges do not manifest themselves in any way but making charges to behave as wave-corpuscles similar to ones studied in Sections 3, 4, 5.

In non-relativistic case provide a big picture of interaction and dynamics via a single charge in an external field and charges interacting instantaneously via their individual electric fields.

6.1 Non-relativistic theory of interacting charges

The purpose of this section is to develop a non-relativistic theory of many interacting charges that would be sufficient for establishing its intimate relation to the point charges mechanics. Developed here non-relativistic theory for many interacting charges naturally integrates the theory of single non-relativistic charge developed in Sections 4 and 5, including the set up for the interaction between the bare charge and the EM field as described by its electric potential φ. Our nonrelativistic Lagrangian $\hat{\mathcal{L}}$ for many charges is constructed based on (i) individual charges nonrelativistic Lagrangians $\hat{L}^{\ell}$ of the form (4.0.6) and (ii) the assumption that every charge interacts directly only with the EM field as defined by its electric potential φ , namely

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}\left(\psi^{\ell}, \psi^{\ell}_{,\mu}, \psi^{\ell *}, \psi^{\ell *}_{,\mu}, \nabla\varphi, \varphi, x^{\mu}\right) = \frac{\left|\nabla\varphi\right|^{2}}{8\pi} + \sum_{\ell} \hat{L}^{\ell}\left(\psi^{\ell}, \psi^{\ell *}, \varphi\right), \tag{6.1.1}$$

$$\text{where } \hat{L}^{\ell} = \frac{\chi}{2}\mathrm{i}\left[\psi^{\ell *}\tilde{\partial}^{\ell}_{t}\psi^{\ell} - \psi^{\ell}\tilde{\partial}^{\ell *}_{t}\psi^{\ell *}\right] - \frac{\chi^{2}}{2m^{\ell}}\left\{\tilde{\nabla}\psi^{\ell}\tilde{\nabla}^{*}\psi^{\ell *} + G^{\ell}\left(\psi^{\ell *}\psi^{\ell}\right)\right\},$$

$$\begin{aligned}
\tilde{\partial}^{\ell}_{t} &= \partial_{t} + \frac{\mathrm{i}q^{\ell}\left(\varphi + \varphi_{\mathrm{ex}}\right)}{\chi}, \ \tilde{\nabla}^{\ell} = \nabla - \frac{\mathrm{i}q^{\ell}\mathbf{A}_{\mathrm{ex}}}{\chi c}, \\
\tilde{\partial}^{\ell *}_{t} &= \partial_{t} - \frac{\mathrm{i}q^{\ell}\left(\varphi + \varphi_{\mathrm{ex}}\right)}{\chi}, \ \tilde{\nabla}^{\ell *} = \nabla + \frac{\mathrm{i}q^{\ell}\mathbf{A}_{\mathrm{ex}}}{\chi c},
\end{aligned}$$

where $(\varphi_{\mathrm{ex}}, \mathbf{A}_{\mathrm{ex}})$ is the potential of the external EM field. Evidently according to this Lagrangian every charge is coupled to the EM field exactly as it would a single charge but since there is just one EM field all charge are coupled. The Euler-Lagrange field equations for this

Lagrangian are

$$\chi \mathrm{i}\tilde{\partial}_t \psi^\ell = \frac{\chi^2}{2m^\ell}\left[-\tilde{\nabla}^{\ell 2}\psi^\ell + \left[G^\ell\right]'\left(\left|\psi^\ell\right|^2\right)\psi\right], \tag{6.1.2}$$

$$-\Delta\varphi = 4\pi\sum_\ell q^\ell\left|\psi^\ell\right|^2, \text{ where } \left[G^\ell\right]'(s) = \partial_s G^\ell(s), \tag{6.1.3}$$

and as in the case of a single charge $\psi^{\ell *}$ is complex conjugate to ψ^ℓ for all ℓ. The nonlinear self-interaction terms G^ℓ in (6.1.1) are determined based on the corresponding form factors $\mathring{\psi}^\ell$ from ℓ-th charge equilibrium equation (2.3.8).

The Lagrangian $\hat{\mathcal{L}}$ defined by (6.1.1) is gauge invariant with respect to the first and the second gauge transformations (10.7.6), (10.7.7) and consequently every ℓ-th charge has a conserved current $J^{\ell\mu} = \left(c\rho^\ell, \mathbf{J}^\ell\right)$ which can be found from relations (10.7.12), (10.7.13) yielding the following formulas similar to (5.0.8)

$$J^{\ell\mu} = \left(\mathrm{c}\rho^\ell, \mathbf{J}^\ell\right), \ \rho^\ell = q\left|\psi^\ell\right|^2, \ \mathbf{J}^\ell = \frac{\mathrm{i}\chi q^\ell}{2m^\ell}\left[\psi\tilde{\nabla}^{\ell *}\psi^{\ell *} - \psi^{\ell *}\tilde{\nabla}^\ell\psi^\ell\right] = \tag{6.1.4}$$

$$= \frac{\chi q^\ell}{2m^\ell}\mathrm{i}\left(\nabla\psi^{\ell *}\psi^\ell - \psi^{\ell *}\nabla\psi^\ell\right) - \frac{q^{\ell 2}\mathbf{A}_{\mathrm{ex}}}{m^\ell \mathrm{c}}\left|\psi^\ell\right|^2 = \left(\frac{\chi q^\ell}{m^\ell}\operatorname{Im}\frac{\nabla\psi^\ell}{\psi^\ell} - \frac{q^{\ell 2}\mathbf{A}_{\mathrm{ex}}}{m^\ell \mathrm{c}}\right)\left|\psi^\ell\right|^2,$$

Every current $J^{\ell\mu}$ satisfies the conservation/continuity equations

$$\partial_\nu J^{\ell\nu} = 0, \ \partial_t\rho^\ell + \nabla\cdot\mathbf{J}^\ell = 0 \text{ or} \tag{6.1.5}$$

$$m^\ell\partial_t\left|\psi^\ell\right|^2 + \nabla\cdot\left(\chi\operatorname{Im}\frac{\nabla\psi^\ell}{\psi^\ell}\left|\psi^\ell\right|^2 - \frac{q^\ell}{\mathrm{c}}\mathbf{A}_{\mathrm{ex}}\left|\psi^\ell\right|^2\right) = 0.$$

The equations (6.1.5) imply the conservation of the total ℓ-th conserved charge. Similarly to the case of a single charge we require every total ℓ-th conserved charge to be exactly q^ℓ and, hence, to satisfy the following *charge normalization* condition of the form (2.4.5)

$$\int_{\mathbb{R}^3}\left|\psi^\ell\right|^2 \mathrm{d}\mathbf{x} = 1 \text{ for all } t. \tag{6.1.6}$$

Next based on the equation (6.1.3) it is natural to introduce for every ℓ-th charge the corresponding potential φ^ℓ using the Green function (11.0.11), namely

$$\varphi^\ell(t,\mathbf{x}) = q^\ell\int_{\mathbb{R}^3}\frac{\left|\psi^\ell\right|^2(t,\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}\,\mathrm{d}\mathbf{y}\ , \varphi = \sum_\ell \varphi^\ell. \tag{6.1.7}$$

Taking into account the expression for the covariant derivatives from (6.1.1) we can recast the field equations (6.1.2), (6.1.3) as (2.4.2). The charge conservation equations (6.1.5) can be alternatively derived directly from the field equations (2.4.2) by multiplying them and their complex conjugate respectively by $\psi^{\ell *}$, ψ^ℓ and subtracting from one another.

One can see in the integral expression (6.1.7) instantaneous action-at-a-distance, a feature which occurs in the nonrelativistic point Lagrangian mechanics.

Many technical aspects needed for the treatment of many charges in the regime of remote interaction are already developed in our studies of a single non-relativistic charge in an external EM field in Section 5, and to avoid repetition whenever the case we use relevant results from there. *In fact, an accurate guiding principle for the treatment of distant interaction of non-relativistic charges is to view every charge as essentially a single one in the slowly varying in the space and time external EM field created by other charges.*

To study the motion of the energy and momentum for involved charges and the EM field we introduce for every ℓ-th charge its gauge invariant energy-momentum tensor $\tilde{T}^{\ell\mu\nu}$ based on the formulas (5.2.7)-(5.2.10) substituting there ψ^{ℓ}, $\psi^{\ell *}$ in place of ψ, ψ^* and the covariant derivatives with the index ℓ from (6.1.1) in place of the covariant derivatives for ψ, ψ^*. That yields the following formulas for energy and momentum densities for individual charges

$$\tilde{u}^{\ell} = \frac{\chi^2}{2m}\left[\tilde{\nabla}^{\ell}\psi^{\ell}\cdot\tilde{\nabla}^{\ell *}\psi^{\ell *} + G^{\ell}\left(\psi^{\ell *}\psi^{\ell}\right)\right], \tag{6.1.8}$$

$$\tilde{p}_j^{\ell} = \frac{\chi \mathrm{i}}{2}\left(\psi^{\ell}\tilde{\partial}_j^{\ell *}\psi^{\ell *} - \psi^{\ell *}\tilde{\partial}_j^{\ell}\psi^{\ell}\right),\ j = 1, 2, 3. \tag{6.1.9}$$

The gauge invariant energy-momentum tensor $\Theta^{\mu\nu}$ for the EM field is defined by (5.2.3)-(5.2.5) and, in particular, its energy, momentum and energy flux densities are

$$\partial_0 w = \frac{\mathbf{J}\cdot\nabla\varphi}{\mathrm{c}} = -\frac{\mathbf{J}\cdot\mathbf{E}}{\mathrm{c}},\ g_j = 0,\ s_j = 0. \tag{6.1.10}$$

Using the field equations (6.1.2), (6.1.3) and the representation (4.0.8) for $F^{\nu\mu}$ we can also verify that following conservations laws for the individual charges and the EM field

$$\partial_\mu\tilde{T}^{\ell\mu\nu} = f^{\ell\nu} + f_{\mathrm{ex}}^{\ell\nu},\ \partial_\mu\Theta^{\mu\nu} = -\sum_{\ell} f^{\ell\nu},\ \text{where } f^{\ell\nu} = \frac{1}{\mathrm{c}}J_\mu^{\ell}F^{\nu\mu} = \left(\frac{1}{\mathrm{c}}\mathbf{J}^{\ell}\cdot\mathbf{E}, \rho^{\ell}\mathbf{E}\right), \tag{6.1.11}$$
$$f_{\mathrm{ex}}^{\ell\nu} = \frac{1}{\mathrm{c}}J_\mu^{\ell}F_{\mathrm{ex}}^{\nu\mu} = \left(\frac{1}{\mathrm{c}}\mathbf{J}^{\ell}\cdot\mathbf{E}_{\mathrm{ex}}, \rho^{\ell}\mathbf{E}_{\mathrm{ex}} + \frac{1}{\mathrm{c}}\mathbf{J}^{\ell}\times\mathbf{B}_{\mathrm{ex}}\right).$$

The energy-momentum conservation equations (6.1.11) can be viewed as equations of motion in elastic continuum, [Moller, Section 6.4, (6.56), (6.57)], similar to the case of kinetic energy-momentum tensor for a single relativistic particle, [Pauli RT, Section 37, (3.24)]. *It is important to remember though that in contrast to the point mechanics the motion equations (6.1.11) must always be complemented with the field equations (6.1.2), (6.1.3) or (2.4.2) without which they do not have to hold and are not alone sufficient to determine the motion.* We also recognize in $f^{\ell\nu}$ and $f_{\mathrm{ex}}^{\ell\nu}$ in the motion equations (6.1.11) respectively the Lorentz force densities for the charge in the EM field (of charges) and the same for the external EM field. Observe that equations (6.1.11) satisfy manifestly the Newton principle "action equals reaction" for all involved densities at every point of the space-time.

In the regime of remote interactions it makes sense to introduce dressed charges and attribute to every charge its EM field via the potential φ^{ℓ} as defined by relations (2.4.2) and (6.1.7). Based on the potentials φ^{ℓ} we define the corresponding energy-momentum tensor $\Theta^{\ell\mu\nu}$ by formulas (5.2.3)-(5.2.5) where we substitute φ^{ℓ} and $\mathbf{J}^{\ell}$ defined by equalities (6.1.7), (6.1.4) in place of φ and $\mathbf{J}$. One can verify then that the conservation law (5.2.13) for $\Theta^{\ell\mu\nu}$ takes here the form

$$\partial_\mu\Theta^{\ell\mu\nu} = -\frac{1}{\mathrm{c}}J_\mu^{\ell}F^{\ell\nu\mu}. \tag{6.1.12}$$

Now for every ℓ-th dressed charge we define its energy-momentum tensor $\mathrm{T}^{\ell\mu\nu}$ by the formula

$$\mathrm{T}^{\ell\mu\nu} = \tilde{T}^{\ell\mu\nu} + \Theta^{\ell\mu\nu}. \tag{6.1.13}$$

It is also natural and useful to introduce for every ℓ-th charge the EM field $\mathbf{E}_{\mathrm{ex}}^{\ell}$ and $F_{\mathrm{ex}}^{\ell\nu\mu}$ of all other charges $\ell' \neq \ell$ by

$$\mathbf{E}_{\mathrm{ex}}^{\ell} = \sum_{\ell' \neq \ell} \mathbf{E}^{\ell'},\ F_{\mathrm{ex}}^{\ell\nu\mu} = \sum_{\ell' \neq \ell} F^{\ell'\nu\mu},\ \mathbf{E}^{\ell} = -\nabla\varphi^{\ell}. \tag{6.1.14}$$

Then combining relations (6.1.11), (6.1.12) with (4.0.8) we obtain the following *motion equations for dressed charges*

$$\partial_{\mu}\mathrm{T}^{\ell\mu\nu} = \mathrm{f}^{\ell\nu} + f_{\mathrm{ex}}^{\ell\nu},\ \mathrm{f}^{\ell\nu} = \frac{1}{\mathrm{c}} J_{\mu}^{\ell} \sum_{\ell' \neq \ell} F^{\ell'\nu\mu} = \left(\frac{1}{\mathrm{c}}\mathbf{J}^{\ell} \cdot \mathbf{E}_{\mathrm{ex}}^{\ell}, \rho^{\ell}\mathbf{E}_{\mathrm{ex}}^{\ell}\right), \tag{6.1.15}$$

describing the motion of energies and momenta of the dressed charges in the space-time continuum. Importantly, *the Lorentz force* $\mathrm{f}^{\ell\nu}$ *in the right-hand of (6.1.15) excludes manifestly the self-interaction* in contrast to the Lorentz force acting upon bare charge as in (6.1.11) which explicitly includes the self-interaction term $\frac{1}{\mathrm{c}} J_{\mu}^{\ell} F^{\ell\nu\mu}$. Thus, we can conclude that when the charge and its EM field are treated as a single entity, namely dressed charge, there is no self-interaction as signified by the exact equations (6.2.26).

It follows from (6.1.4) and (6.1.9) that *the charge gauge invariant momentum density* $\tilde{\mathbf{p}}^{\ell}$ *equals exactly the microcurrent density* $\mathbf{J}^{\ell}$ *multiplied by the constant* m^{ℓ}/q^{ℓ}, namely the following identity holds

$$\tilde{\mathbf{p}}^{\ell} = \frac{m^{\ell}}{q^{\ell}}\mathbf{J}^{\ell} = \frac{\mathrm{i}\chi}{2}\left[\psi^{\ell}\tilde{\nabla}^{\ell*}\psi^{\ell*} - \psi^{\ell*}\tilde{\nabla}^{\ell}\psi^{\ell}\right] = \left(\chi\,\mathrm{Im}\,\frac{\nabla\psi^{\ell}}{\psi^{\ell}} - \frac{q^{\ell}\bar{\mathbf{A}}}{\mathrm{c}}\right)\left|\psi^{\ell}\right|^{2}, \tag{6.1.16}$$

that can be viewed as the momentum density kinematic representation. We can also recast the above equality as

$$\tilde{\mathbf{p}}^{\ell}(t,\mathbf{x}) = m\mathbf{v}^{\ell}(t,\mathbf{x}),\ \text{where } \mathbf{v}^{\ell}(t,\mathbf{x}) = \mathbf{J}^{\ell}(t,\mathbf{x})/q^{\ell} \text{ is the velocity density.} \tag{6.1.17}$$

Up to this point we introduced the basic elements of theory of interacting charges described as fields via the Lagrangian (6.1.1). A natural question then is in what ways the point charge mechanics is integrated into this Lagrangian classical field theory? There are two distinct ways to correspond our field theory to the point charge mechanics: (i) via averaged quantities in spirit of the well known in quantum mechanics *Ehrenfest Theorem,* [Schiff, Sections 7, 23]; (ii) via a construction of approximate solutions to the field equations (6.1.2), (6.1.3), (2.4.2) in terms of radial wave-corpuscles similar to (5.1.2). We consider these two ways in the next subsections.

6.1.1 Point mechanics via averaged quantities

Introducing the total individual momenta P^{ℓ} and energies E^{ℓ} for ℓ-th dressed charge

$$\mathsf{P}^{\ell} = \int_{\mathbb{R}^3} \tilde{\mathbf{p}}^{\ell}\,\mathrm{d}\mathbf{x},\ \mathsf{E}^{\ell} = \int_{\mathbb{R}^3} \tilde{u}^{\ell}\,\mathrm{d}\mathbf{x}, \tag{6.1.18}$$

and using arguments similar to (5.3.1)-(5.3.3) combined with relations (6.1.14), (6.1.15) we obtain the following equations

$$\frac{\mathrm{d}\mathsf{P}^{\ell}}{\mathrm{d}t} = q^{\ell}\int_{\mathbb{R}^3}\left[\left(\sum\nolimits_{\ell' \neq \ell}\mathbf{E}^{\ell'} + \mathbf{E}_{\mathrm{ex}}\right)\left|\psi^{\ell}\right|^{2} + \frac{1}{\mathrm{c}}\mathbf{v}^{\ell} \times \mathbf{B}_{\mathrm{ex}}\right]\mathrm{d}\mathbf{x},\ \text{where} \tag{6.1.19}$$

$$\int_{\mathbb{R}^3}\sum\nolimits_{\ell' \neq \ell}\mathbf{E}^{\ell'}\left|\psi^{\ell}\right|^{2}\mathrm{d}\mathbf{x} = -\sum\nolimits_{\ell' \neq \ell} q^{\ell'}\int_{\mathbb{R}^3 \times \mathbb{R}^3}\frac{(\mathbf{y}-\mathbf{x})\left|\psi^{\ell'}\right|^{2}(t,\mathbf{y})\left|\psi^{\ell}(t,\mathbf{x})\right|^{2}}{|\mathbf{y}-\mathbf{x}|^{3}}\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}, \tag{6.1.20}$$

$$\frac{\mathsf{dE}^{\ell}}{\mathrm{d}t}=\int_{\mathbb{R}^{3}}\mathbf{J}^{\ell}\cdot\mathbf{E}_{\mathrm{ex}}^{\ell}\left(t,\mathbf{x}\right)\,\mathrm{d}\mathbf{x}=q^{\ell}\int_{\mathbb{R}^{3}}\mathbf{v}^{\ell}\cdot\left(\sum\nolimits_{\ell'\neq\ell}\mathbf{E}^{\ell'}+\mathbf{E}_{\mathrm{ex}}\right)\,\mathrm{d}\mathbf{x}. \tag{6.1.21}$$

Let us introduce the ℓ-th charge position $\mathbf{r}^{\ell}(t)$ and velocity $\mathsf{v}^{\ell}(t)$ as the following averages

$$\mathbf{r}^{\ell}\left(t\right)=\int_{\mathbb{R}^{3}}\mathbf{x}\left|\psi^{\ell}\left(t,\mathbf{x}\right)\right|^{2}\,\mathrm{d}\mathbf{x},\;\;\mathsf{v}^{\ell}\left(t\right)=\frac{1}{q}\int_{\mathbb{R}^{3}}\mathbf{J}^{\ell}\left(t,\mathbf{x}\right)\,\mathrm{d}\mathbf{x}. \tag{6.1.22}$$

Then using the charge conservation law (6.1.5) we the following identities

$$\frac{\mathrm{d}\mathbf{r}^{\ell}\left(t\right)}{\mathrm{d}t}=\int_{\mathbb{R}^{3}}\mathbf{x}\partial_{t}\left|\psi^{\ell}\right|^{2}\,\mathrm{d}\mathbf{x}=-\frac{1}{q^{\ell}}\int_{\mathbb{R}^{3}}\mathbf{x}\nabla\cdot\mathbf{J}^{\ell}\,\mathrm{d}\mathbf{x}=\frac{1}{q^{\ell}}\int_{\mathbb{R}^{3}}\mathbf{J}^{\ell}\mathrm{d}\mathbf{x}=\mathsf{v}^{\ell}\left(t\right), \tag{6.1.23}$$

showing the positions and velocities defined by formulas (6.1.22) are related exactly as in the point charge mechanics. Then utilizing the momentum density kinematic representation (6.1.16)-(6.1.17) and the fact the momentum density of the ℓ-th charge EM field is identically zero according to (5.2.4) we obtain the following kinematic representation for charge and hence the dressed charge total momentum

$$\mathsf{P}^{\ell}\left(t\right)=\frac{m^{\ell}}{q^{\ell}}\int_{\mathbb{R}^{3}}\mathbf{J}^{\ell}\left(t,\mathbf{x}\right)\,\mathrm{d}\mathbf{x}=m^{\ell}\mathsf{v}^{\ell}\left(t\right), \tag{6.1.24}$$

which also exactly the same as in point charges mechanics. Combining relations (6.1.19), (6.1.23) and (6.1.24) we obtain the following system of motion equations for N charges:

$$m^{\ell}\frac{\mathrm{d}^{2}\mathbf{r}^{\ell}\left(t\right)}{\mathrm{d}^{2}t}=\frac{\mathsf{dP}^{\ell}}{\mathrm{d}t}=q^{\ell}\int_{\mathbb{R}^{3}}\left[\left(\sum\nolimits_{\ell'\neq\ell}\mathbf{E}^{\ell'}+\mathbf{E}_{\mathrm{ex}}\right)\left|\psi^{\ell}\right|^{2}+\frac{1}{\mathrm{c}}\mathbf{v}^{\ell}\times\mathbf{B}_{\mathrm{ex}}\right]\,\mathrm{d}\mathbf{x}, \tag{6.1.25}$$
$$\text{where }\mathbf{E}^{\ell}\left(t,\mathbf{x}\right)=-\nabla\varphi^{\ell}\left(t,\mathbf{x}\right),\;\mathbf{E}_{\mathrm{ex}}=-\nabla\varphi_{\mathrm{ex}}\left(t,\mathbf{x}\right).$$

The above system is analogous to the well known in quantum mechanics *Ehrenfest Theorem*, [Schiff, Sections 7, 23]. Notice that the system of the motion equations (6.1.25) departs from the corresponding system for point charges by having the averaged Lorentz density force instead of the Lorentz force at exact position $\mathbf{r}^{\ell}(t)$. Observe, also that the system of motion equations (6.1.25) is consistent with the Newton third law of motion "action equals reaction" as it follows from the relations (6.1.20).

Let us suppose now that charges and current densities $\left|\psi^{\ell}\right|^{2}$ and $q^{\ell}\mathbf{v}^{\ell}$ are localized near a point $\mathbf{r}^{\ell}(t)$, and have localization scales a^{ℓ} which are small compared with the typical variation scale R_{EM} of the EM field. Then for a bounded R_{EM} and a^{ℓ} converging to 0 we have

$$\left|\psi^{\ell}\right|^{2}\left(t,\mathbf{x}\right)\rightarrow\delta\left(\mathbf{x}-\mathbf{r}^{\ell}\left(t\right)\right),\;\mathbf{v}^{\ell}\left(t,x\right)\rightarrow\mathsf{v}^{\ell}\left(t\right)\delta\left(\mathbf{x}-\mathbf{r}^{\ell}\left(t\right)\right)\;\text{ as }a^{\ell}\rightarrow0, \tag{6.1.26}$$

where coefficients at the delta-functions are determined from the charge normalization conditions (6.1.6) and relations (6.1.22). Using potential representations (6.1.7) we infer from (6.1.26) the convergence of the potentials φ^{ℓ} to the corresponding Coulomb potentials, namely

$$\varphi^{\ell}\left(t,\mathbf{x}\right)\rightarrow\varphi_{0}^{\ell}\left(t,\mathbf{x}\right)=\frac{q^{\ell}}{\left|\mathbf{x}-\mathbf{r}^{\ell}\left(t\right)\right|}\text{ as }a^{\ell}\rightarrow0. \tag{6.1.27}$$

Hence, we can recast the motion equations (6.1.25) as the system

$$m^{\ell}\frac{\mathrm{d}^{2}\mathbf{r}^{\ell}}{\mathrm{d}t^{2}}=\frac{\mathrm{d}\mathsf{P}^{\ell}}{\mathrm{d}t}=\mathsf{f}^{\ell}+\boldsymbol{\epsilon}_{\mathsf{P}^{\ell}},\ \text{where} \tag{6.1.28}$$

$$\mathsf{f}^{\ell}=\sum_{\ell'\neq\ell}q^{\ell}\mathbf{E}_{0}^{\ell'}+q^{\ell}\mathbf{E}_{\mathrm{ex}}\left(\mathbf{r}^{\ell}\right)+\frac{1}{\mathrm{c}}\mathsf{v}^{\ell}\times\mathbf{B}_{\mathrm{ex}}\left(\mathbf{r}^{\ell}\right),\ \ell=1,...,N,$$

with small discrepancies $\epsilon_{\mathsf{P}^{\ell}}\to 0$ as $a^{\ell}/R_{\mathrm{EM}}\to 0$. Notice that terms $q^{\ell}\mathbf{E}_{0}^{\ell'}$ in equations (6.1.28) coincide with the Lorentz forces for the Coulomb potentials

$$q^{\ell}\mathbf{E}_{0}^{\ell'}=-q^{\ell}\nabla\varphi_{0}^{\ell'}\left(t,\mathbf{x}\right)=-\frac{q^{\ell}q^{\ell'}\left(\mathbf{r}^{\ell'}-\mathbf{r}^{\ell}\right)}{\left|\mathbf{r}^{\ell'}-\mathbf{r}^{\ell}\right|^{3}}. \tag{6.1.29}$$

In the case when there is no external EM field the point charges motion equations (6.1.28) in the limit $a^{\ell}\to 0$ are associated with the following Lagrangian ("static limit, zeroth order in (v/c)", [Jackson, Section 12.6])

$$\mathcal{L}_{\mathrm{p}}=\sum_{\ell}\frac{m^{\ell}\left(\partial_{t}\mathbf{r}^{\ell}\right)^{2}}{2}-\sum_{\ell'\neq\ell}\frac{q^{\ell}q^{\ell'}}{\left|\mathbf{r}^{\ell'}-\mathbf{r}^{\ell}\right|}, \tag{6.1.30}$$

and the motion equations (6.1.28) take the form

$$m^{\ell}\frac{\mathrm{d}^{2}\mathbf{r}^{\ell}}{\mathrm{d}t^{2}}=-\sum_{\ell'\neq\ell}\frac{q^{\ell}q^{\ell'}\left(\mathbf{r}^{\ell'}-\mathbf{r}^{\ell}\right)}{\left|\mathbf{r}^{\ell'}-\mathbf{r}^{\ell}\right|^{3}},\ \ell=1,\ldots,N. \tag{6.1.31}$$

Using similar arguments we obtain from (6.1.21)

$$\frac{\mathrm{d}\mathsf{E}^{\ell}}{\mathrm{d}t}=\mathsf{v}^{\ell}\cdot\mathsf{f}^{\ell}+\epsilon_{\mathsf{E}^{\ell}},\ \text{with small discrepancies}\ \epsilon_{\mathsf{E}^{\ell}}\to 0\ \text{as}\ a^{\ell}/R_{\mathrm{EM}}\to 0. \tag{6.1.32}$$

Combining equalities (6.1.24) (6.1.28) (6.1.32) we get

$$\frac{\mathrm{d}m^{\ell}\mathbf{v}^{\ell}\cdot\mathbf{v}^{\ell}}{2\mathrm{d}t}=\mathbf{v}^{\ell}\cdot\frac{\mathrm{d}m\mathbf{v}^{\ell}}{\mathrm{d}t}=\mathbf{v}^{\ell}\cdot\mathsf{f}^{\ell}+\mathbf{v}^{\ell}\cdot\boldsymbol{\epsilon}_{\mathsf{P}^{\ell}}= \tag{6.1.33}$$

$$=\frac{\mathrm{d}\mathsf{E}^{\ell}}{\mathrm{d}t}+\mathbf{v}^{\ell}\cdot\boldsymbol{\epsilon}_{\mathsf{P}^{\ell}}-\epsilon_{\mathsf{E}^{\ell}},\ \text{implying}\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathsf{E}^{\ell}-\frac{m^{\ell}\mathbf{v}^{\ell}\cdot\mathbf{v}^{\ell}}{2}\right)=\epsilon_{\mathsf{E}^{\ell}}-\mathbf{v}^{\ell}\cdot\boldsymbol{\epsilon}_{\mathsf{P}^{\ell}},$$

which, up to small errors, are well known kinematic representation for the energies of individual charges from the point charge mechanics. Note that to obtain point charges motion equations (6.1.28) it is sufficient to assume localization only for ψ^{ℓ}.

6.1.2 Point mechanics via wave-corpuscles

In this section we construct approximate solution of the field equations (6.1.2), (6.1.3) (or, equivalently, (2.4.2)) for N interacting charges in which every charge is a wave-corpuscle defined by (5.1.2) with properly chosen complimentary point charges motion equations. The proposed here construction is valid for any external EM field but to avoid involved expressions we consider the case when the external EM field is purely electric with $\mathbf{A}_{\mathrm{ex}}=0$. The general case when $\mathbf{A}_{\mathrm{ex}}\neq 0$ is treated similarly based on the results of using results of Section

5.5. We assume here that the shape factor $\left|\mathring{\psi}_1^{\ell}(|\mathbf{x}|)\right|^2$ *decays exponentially* as $|\mathbf{x}| \to \infty$ for every ℓ-th charge and (5.4.14) holds. The wave-corpuscle approximation (2.4.18) is based on trajectories $\mathbf{r}_0^{\ell}$ for the wave-corpuscle centers determined from equations (2.4.15) which involve the exact Coulomb potentials $\mathring{\varphi}_0^{\ell}$ corresponding to the size parameter $a = 0$. To show that the approximation is accurate for small $a > 0$ we use the results obtained for a single charge motion in an external field. As the first step for an estimate we introduce an auxiliary system of equations to determine all center trajectories. This system has the following property. If ℓ-th charge is singled out and the potentials $\mathring{\varphi}_a^{\ell'}\left(\mathbf{x} - \mathbf{r}_0^{\ell'}(t)\right)$ of remaining charges are replaced by the linear approximation of $\mathring{\varphi}_0^{\ell'}\left(\mathbf{x} - \mathbf{r}_0^{\ell'}(t)\right)$ based on the position of the ℓ-th charge, then the exact wave-corpuscle solution for ℓ-th charge is available in so modified field. In addition to that, the motion of the center $\mathbf{r}^{\ell}$ of the exact solution to the auxiliary equation has the same trajectories $\mathbf{r}_0^{\ell}$. Replacing $\mathring{\varphi}_a^{\ell'}$ by the exact Coulomb potential $\mathring{\varphi}_0^{\ell'}$ produces a contribution to the discrepancy, and the second source of the discrepancy is the field linearization at $\mathbf{r}_0^{\ell}$. To estimate these discrepancies we use results of Section 5.4.2. First, we find trajectories $\mathbf{r}_0^{\ell}(t)$ from the auxiliary equations

$$m^{\ell}\frac{\mathrm{d}^2\mathbf{r}_0^{\ell}}{\mathrm{d}t^2} = -q^{\ell}\nabla\varphi_{\mathrm{ex},0}^{\ell}\left(\mathbf{r}_0^{\ell}\right), \ \mathbf{r}_0^{\ell}(0) = \acute{\mathbf{r}}_0^{\ell}, \ \frac{\mathrm{d}\mathbf{r}_0^{\ell}}{\mathrm{d}t}(0) = \acute{\mathbf{v}}_0^{\ell}, \ \ell = 1, ..., N. \tag{6.1.34}$$

The electrostatic potential $\varphi_{\mathrm{ex},0}^{\ell}$ in (6.1.34) is the Coulomb potential as in (2.4.14) and for $a > 0$ we introduce an intermediate external potential for ℓ-th charge as follows:

$$\mathring{\varphi}_{\mathrm{ex},a}^{\ell}(t, \mathbf{x}) = \varphi_{\mathrm{ex}}(t, \mathbf{x}) + \sum\nolimits_{\ell' \neq \ell} \mathring{\varphi}_a^{\ell'}\left(\mathbf{x} - \mathbf{r}_0^{\ell'}\right). \tag{6.1.35}$$

We define then an approximate solution $\psi_{\mathrm{ap}}^{\ell}$ to be of the form of wave-corpuscles (2.4.18), (2.4.19), namely:

$$\psi_{\mathrm{ap}}^{\ell}(t, \mathbf{x}) = \mathrm{e}^{\frac{\mathrm{i}S^{\ell}}{\chi}}\mathring{\psi}\left(\left|\mathbf{x} - \mathbf{r}_0^{\ell}\right|\right), \ \varphi_{\mathrm{ap}}^{\ell}(t, \mathbf{x}) = q^{\ell}\mathring{\varphi}_a^{\ell}\left(\mathbf{x} - \mathbf{r}_0^{\ell}\right), \text{ where} \tag{6.1.36}$$
$$\mathbf{p}^{\ell} = m^{\ell}\frac{\mathrm{d}\mathbf{r}_0^{\ell}}{\mathrm{d}t}, \ S^{\ell}(t, \mathbf{x}) = \mathbf{p}^{\ell} \cdot \mathbf{x} - \int_0^t \frac{\mathbf{p}^{\ell 2}}{2m}\,\mathrm{d}t' - q\int_0^t \varphi_{\mathrm{ex},0}\left(t', \mathbf{r}_0^{\ell}\right)\,\mathrm{d}t',$$

and $\mathbf{r}_0^{\ell}$ is solution of (6.1.34). Recall that the dependence on the size parameter a of the form factor $\mathring{\psi}^{\ell} = \mathring{\psi}_a^{\ell}$ and corresponding potential $\mathring{\varphi}^{\ell} = \mathring{\varphi}_a^{\ell}$ is given by (4.4.1). Notice that according to relations (4.6.7) the interaction force term in (6.1.34) approaches the Lorentz forces based on the Coulomb potential, namely

$$\mathring{\varphi}_a^{\ell}(\mathbf{x}) = q^{\ell}\int_{\mathbb{R}^3} \frac{\left|\mathring{\psi}_a^{\ell}\right|^2(t, \mathbf{y})}{|\mathbf{y} - \mathbf{x}|}\,\mathrm{d}\mathbf{y} \to \mathring{\varphi}_0^{\ell}(\mathbf{x}) = \frac{q^{\ell}}{|\mathbf{x}|} \text{ as } a \to 0. \tag{6.1.37}$$

Let us introduce an auxiliary spatially linear potential $\tilde{\varphi}_{\mathrm{ex},0}^{\ell}(t, \mathbf{x})$ for ℓ-th charge by formula (5.4.4) with φ_{ex} replaced by $\varphi_{\mathrm{ex},0}^{\ell}$. Observe that for every ℓ the pair $\left\{\psi_{\mathrm{ap}}^{\ell}, \varphi_{\mathrm{ap}}^{\ell}\right\}$ is an exact solution to the auxiliary equation (5.4.15) with the external potential $\tilde{\varphi}_{\mathrm{ex},0}^{\ell}(t, \mathbf{x})$ obtained by the linearization of $\varphi_{\mathrm{ex},0}^{\ell}$ at $\mathbf{r}_0^{\ell}(t)$ according to (5.4.4). It remains to examine if so defined $\left\{\psi_{\mathrm{ap}}^{\ell}, \varphi_{\mathrm{ap}}^{\ell}\right\}$ yield an approximation to the field equations (2.4.2). Indeed, the ℓ-th wave-corpuscles $\left\{\psi_{\mathrm{ap}}^{\ell}, \varphi_{\mathrm{ap}}^{\ell}\right\}$ is an exact solution to equations (5.4.15). To obtain from

(5.4.15) the ℓ-th equation (2.4.2) we have to replace $\tilde{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})$ by $\mathring{\varphi}_{\mathrm{ex},a}^{\ell}(t,\mathbf{x})$ resulting in a discrepancy

$$\begin{gathered}\left(\tilde{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})-\mathring{\varphi}_{\mathrm{ex},a}^{\ell}(t,\mathbf{x})\right)\mathring{\varphi}_{a}^{\ell}(\mathbf{x})\\ =\left(\tilde{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})-\mathring{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})\right)\mathring{\varphi}_{a}^{\ell}(\mathbf{x})+\left(\mathring{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})-\mathring{\varphi}_{\mathrm{ex},a}^{\ell}(t,\mathbf{x})\right)\mathring{\varphi}_{a}^{\ell}(\mathbf{x}).\end{gathered}$$

The first term of the above discrepancy is the same as in (5.4.16) and corresponding integral discrepancy is estimated as in (5.4.23). The second term has the form

$$\begin{gathered}\left(\mathring{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})-\mathring{\varphi}_{\mathrm{ex},a}^{\ell}(t,\mathbf{x})\right)\mathring{\varphi}_{a}^{\ell}(\mathbf{x})\\ =\sum_{\ell'\neq\ell}q^{\ell}\left(\mathring{\varphi}_{a}^{\ell'}\left(\mathbf{x}-\mathbf{r}_{0}^{\ell'}\right)-\mathring{\varphi}_{0}^{\ell'}\left(\mathbf{x}-\mathbf{r}_{0}^{\ell'}\right)\right)\mathrm{e}^{\frac{\mathrm{i}S^{\ell}}{\chi}}\mathring{\psi}_{a}^{\ell}\left(\left|\mathbf{x}-\mathbf{r}_{0}^{\ell}\right|\right).\end{gathered}\tag{6.1.38}$$

Taking into account that $\mathring{\psi}_{a}^{\ell}(|\mathbf{x}|)$ decays exponentially as $|\mathbf{x}|\to\infty$ we find that every term in (6.1.38) is small if $\mathbf{r}_{0}^{\ell}$ is separated from $\mathbf{r}_{0}^{\ell'}$ for $\ell'\neq\ell$. To take into account point charges separation we introduce a quantity

$$R_{\min}=\min_{\ell\neq\ell',\ 0\leq t\leq T}\left|\mathbf{r}_{0}^{\ell}(t)-\mathbf{r}_{0}^{\ell'}(t)\right|\tag{6.1.39}$$

and assume it to be positive, i.e. $R_{\min}>0$. Under this condition using (4.4.5) we obtain

$$\begin{gathered}\left|\mathring{\varphi}_{a}^{\ell'}\left(\mathbf{x}-\mathbf{r}_{0}^{\ell'}\right)-\mathring{\varphi}_{0}^{\ell'}\left(\mathbf{x}-\mathbf{r}_{0}^{\ell'}\right)\right|\leq C\varepsilon\left(a/R_{\min}\right),\\ \text{if }\left|\mathbf{x}-\mathbf{r}_{0}^{\ell}\right|\leq a\sigma_{\psi}\text{ where }\varepsilon\left(a/R_{\min}\right)\to 0\text{ as }a/R_{\min}\to 0,\end{gathered}\tag{6.1.40}$$

where constant C, $\sigma_{\psi}=\sigma_{\psi^{\ell}}$ are the same as in (5.4.20) (note that $\varepsilon(a/R_{\min})\to 0$ exponentially fast if $\mathring{\psi}^{\ell}(r)$ decays exponentially). Hence, we can take $\mathring{\varphi}_{\mathrm{ex},a}^{\ell}(t,\mathbf{x})$ as $\acute{\varphi}_{\mathrm{ex}}(t,\mathbf{r},\epsilon)$ and $\mathring{\varphi}_{\mathrm{ex},0}^{\ell}(t,\mathbf{x})$ as $\varphi_{\mathrm{ex}}(t,\mathbf{x})$ in (5.4.25) with $\epsilon=\varepsilon(a/R_{\min})$. Then we use the integral discrepancy estimate (5.4.28) which implies

$$\left|\hat{D}_{0}(x,t)+\hat{D}_{1}(x,t)\right|\lesssim\left(C_{0}\frac{a^{2}}{R_{\varphi}^{2}}+\frac{C_{1}}{|\bar{\varphi}|}\varepsilon\left(\frac{a}{R_{\min}}\right)\right)\max_{\ell}\left|q^{\ell}\right|\left|\bar{\varphi}^{\ell}\right|,\tag{6.1.41}$$

where the potential curvature radius R_{φ}^{2} is based on $\varphi_{\mathrm{ex},0}(\mathbf{x},t)$ as in (6.1.35), $\bar{\varphi}^{\ell}$ is based on $\varphi_{\mathrm{ex},0}^{\ell}$ and $\mathbf{r}_{0}^{\ell}$. The factors $\left|q^{\ell}\right|\left|\bar{\varphi}^{\ell}\right|$ are bounded uniformly in a. Consequently, we conclude that the integral discrepancy resulting from the substitution of $\psi_{\mathrm{ap}}^{\ell}(t,\mathbf{x})$ into (2.4.2) and given by (6.1.41) tends to zero as $a/R\to 0$ where $R=\min(R_{\min},R_{\varphi})$. Note that for exponentially decaying $\mathring{\psi}^{\ell}$ the function $\varepsilon(a/R_{\min})$ decays exponentially as $a/R_{\min}\to 0$ and hence

$$\left|\hat{D}_{0}(x,t)+\hat{D}_{1}(x,t)\right|\simeq O\left(\frac{a^{2}}{R_{\varphi}^{2}}\right)U,\ U=\max_{\ell}\left|q^{\ell}\right|\left|\bar{\varphi}^{\ell}\right|.$$

Interestingly, an additional analysis of the exact motion equations (2.4.12) shows that though the integral discrepancy decays as a^{2}/R_{φ}^{2}, the positions $\mathbf{r}^{\ell}(t)$ given by (6.1.22) are approximated by $\mathbf{r}_{0}^{\ell}(t)$ with accuracy of the order a^{3}/R_{φ}^{3}.

6.2 Relativistic theory of interacting charges

Relativistic theory of many interacting point particles is known to have fundamental difficulties. "The invariant formulation of the motion of two or more interacting particles is complicated by the fact that each particle will have a different proper time. ... No exact general theory seems to be available", [Barut, Section II.1, System of colliding particles]. Some of these difficulties are analyzed by H. Goldstein in his classical monograph, [Goldstein, Section 7.10]: "The great stumbling block however is the treatment of the type of interaction that is so natural and common in nonrelativistic mechanics - direct interaction between particles. ... To say that the force on a particle depends upon the positions or velocities of other particles at the same time implies propagation of effects with infinite velocity from one particle to another - "action at a distance." In special relativity, where signals cannot travel faster than the speed of light, action-at-a-distance seems outlawed. And in a certain sense this seems to be the correct picture. It has been proven that if we require certain properties of the system to behave in the normal way (such as conservation of total linear momentum), then there can be no covariant direct interaction between particles except through contact forces." Another argument, due von Laue, [von Laue], on the incompatibility of the relativity with any finite dimensional mechanical system was articulated by W. Pauli, [Pauli RT, Section 45]: "...This in itself raised strong doubts as to the possibility of introducing the concept of a rigid body into relativistic mechanics[247]. The final clarification was brought about in a paper by Laue[248], who showed by quite elementary arguments that the number of kinematic degrees of freedom of a body cannot be limited, according to the theory of relativity. For, since no action can be propagated with a velocity greater than that of light, an impulse which is given to the body simultaneously at n different places, will, *to start off with*, produce a motion to which at least n degrees of freedom must be ascribed."

Now we ask ourselves what features of point charges mechanics can be integrated into a relativistic mechanics of fields? It seems that the above *arguments by Goldstein, von Laue and Pauli completely rule out any Lagrangian mechanics with finitely many degrees of freedom even as an approximation* because of its incompatibility with a basic relativity requirement for the signal speed not to exceed the speed of light. On the constructive side, these arguments suggest that (i) the EM field has to be an integral part of charges mechanics, (ii) every charge of the system has to be some kind of elastic continuum coupled to the EM field. We anticipate though that point mechanics features that can be integrated into a relativistic field mechanics to be limited and to have subtler manifestation compared to the nonrelativistic theory. We expect point mechanics features to manifest themselves in (i) identification of the energy-momentum tensor for every individual bare charge; (ii) certain partition of the EM field into a sum of EM fields attributed to individual charges with consequent formation of dressed charges, that is bare charges with attached to them EM fields. *That energy-momentum partition between individual charges must comply with the Newton "action equals to reaction" law, the interaction forces densities have to be of the Lorentzian form and every dressed charge has not to interact with itself.*

In proposed here theory we address the above challenges by (i) principle departure from the concept of point charge, which is substituted by a concept of wave-corpuscle described by a complex valued function in the space-time; (ii) requirement for every charge to interact directly to only the EM field implying that different charges interact only via the EM field. With all that in mind we introduce the system Lagrangian $\mathcal{L}$ to be of the general form as in

Section 10.5

$$\mathcal{L}\left(\left\{\psi^{\ell},\psi^{\ell}_{;\mu}\right\},\left\{\psi^{\ell*},\psi^{\ell*}_{;\mu}\right\},A^{\mu}\right)=\sum_{\ell}L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right)-\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi},\tag{6.2.1}$$
$$F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu},$$

with every ℓ-th charge Lagrangian L^{ℓ} to be of the form of single relativistic charge (3.0.1)-(3.0.2)

$$L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right)=\frac{\chi^{2}}{2m^{\ell}}\left\{\psi^{\ell*}_{;\mu}\psi^{\ell;\mu}-\kappa^{\ell 2}\psi^{\ell*}\psi^{\ell}-G^{\ell}\left(\psi^{\ell*}\psi^{\ell}\right)\right\},\tag{6.2.2}$$
$$\kappa^{\ell}=\frac{\omega^{\ell}}{\mathrm{c}}=\frac{m^{\ell}\mathrm{c}}{\chi},\ \omega^{\ell}=\frac{m^{\ell}\mathrm{c}^{2}}{\chi},$$

where $\psi^{\ell}_{;\mu}$ and $\psi^{\ell*}_{;\mu}$ are the *covariant derivatives* defined by the following formulas

$$\psi^{\ell}_{;\mu}=\tilde{\partial}^{\ell\mu}\psi^{\ell},\ \psi^{\ell*}_{;\mu}=\tilde{\partial}^{\ell\mu*}\psi^{\ell},\ \tilde{\partial}^{\ell\mu}=\partial^{\mu}+\frac{\mathrm{i}q^{\ell}A^{\mu}}{\chi\mathrm{c}},\ \tilde{\partial}^{\ell\mu*}=\partial^{\mu}-\frac{\mathrm{i}q^{\ell}A^{\mu}}{\chi\mathrm{c}},\tag{6.2.3}$$

and $\tilde{\partial}^{\ell\mu}$ and $\tilde{\partial}^{\ell\mu*}$ are called the *covariant differentiation operators*. We also assume that for every ℓ: (i) $m^{\ell}>0$ is the charge mass; (ii) q^{ℓ} is a real valued (positive or negative) charge; (iii) $\kappa^{\ell}>0$; (iv) G^{ℓ} is a nonlinear self-interaction function. Notice that charges interaction with the EM field enters the Lagrangian $\mathcal{L}$ via the covariant derivatives (6.2.3). The Lagrangian $\mathcal{L}$ defined by (6.2.1)-(6.2.4) is manifestly local, Lorentz invariant, and gauge invariant with respect to the second-kind (local) gauge transformation

$$\psi^{\ell}\rightarrow\mathrm{e}^{-\frac{\mathrm{i}q^{\ell}\lambda(x)}{\chi\mathrm{c}}}\psi^{\ell},\ \psi^{\ell*}\rightarrow\mathrm{e}^{\frac{\mathrm{i}q^{\ell}\lambda(x)}{\chi\mathrm{c}}}\psi^{\ell*},\ A^{\mu}\rightarrow A^{\mu}+\partial^{\mu}\lambda\left(x\right),\tag{6.2.4}$$

as well as with respect to the group of global (the first-kind) gauge transformations

$$\psi^{\ell}\rightarrow\mathrm{e}^{-\mathrm{i}q^{\ell}\lambda^{\ell}}\psi^{\ell},\ \psi^{\ell*}\rightarrow\mathrm{e}^{-\mathrm{i}q^{\ell}\lambda^{\ell}}\psi^{\ell*},\ A^{\mu}\rightarrow A^{\mu}\text{ for any real numbers }\lambda^{\ell}.\tag{6.2.5}$$

The Euler-Lagrange field equations (10.5.12)-(10.5.13) for the Lagrangian $\mathcal{L}$ defined by (6.2.1)-(6.2.4) take the form

$$\left[\tilde{\partial}^{\ell}_{\mu}\tilde{\partial}^{\ell\mu}+\kappa^{\ell 2}+G^{\ell\prime}\left(\left|\psi^{\ell}\right|^{2}\right)\right]\psi^{\ell}=0,\ \tilde{\partial}^{\ell\mu}=\partial^{\mu}+\frac{\mathrm{i}q^{\ell}A^{\mu}}{\chi\mathrm{c}},\tag{6.2.6}$$

$$\left[\tilde{\partial}^{\ell*}_{\mu}\tilde{\partial}^{\ell*\mu}+\kappa^{\ell 2}+G^{\ell\prime}\left(\left|\psi^{\ell}\right|^{2}\right)\right]\psi^{*\ell}=0,\ \tilde{\partial}^{\ell*\mu}=\partial^{\mu}-\frac{\mathrm{i}q^{\ell}A^{\mu}}{\chi\mathrm{c}},\tag{6.2.7}$$

$$\partial_{\mu}F^{\mu\nu}=\frac{4\pi}{\mathrm{c}}J^{\nu},\ J^{\nu}=\sum_{\ell}J^{\ell\nu},\ F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu},\tag{6.2.8}$$

where the ℓ-th charge *4-vector EM micro-current* $J^{\ell\nu}$ defined by (10.5.14) takes here the form

$$J^{\ell\mu\nu}=-\mathrm{i}\frac{q^{\ell}\chi\left[\left(\tilde{\partial}^{\ell\nu*}\psi^{\ell*}\right)\psi^{\ell}-\psi^{\ell*}\tilde{\partial}^{\ell\nu}\psi^{\ell}\right]}{2m^{\ell}}=-\frac{q^{\ell}\chi\left|\psi^{\ell}\right|^{2}}{m^{\ell}}\operatorname{Im}\frac{\tilde{\partial}^{\ell\nu}\psi^{\ell}}{\psi^{\ell}}=\tag{6.2.9}$$
$$=-\mathrm{i}\frac{q^{\ell}\chi\left(\partial^{\nu}\psi^{\ell*}\psi^{\ell}-\psi^{\ell*}\partial^{\nu}\psi^{\ell}\right)}{2m^{\ell}}-\frac{q^{\ell 2}A^{\nu}\psi^{\ell*}\psi^{\ell}}{m^{\ell}\mathrm{c}}=-\frac{q^{\ell}\chi\left|\psi^{\ell}\right|^{2}}{m^{\ell}}\left(\operatorname{Im}\frac{\partial^{\nu}\psi^{\ell}}{\psi^{\ell}}+\frac{q^{\ell}A^{\mu}}{\chi\mathrm{c}}\right).$$

Observe that the equations (6.2.8) are the Maxwell equations (10.4.7) with currents $J^{\ell\mu}$. As in the case of the more general Lagrangian (10.5.5) the gauge invariance (6.2.5) implies that every individual ℓ-th charge 4-vector micro-current $J^{\ell\mu}$ satisfies the continuity equation

$$\partial_\nu J^{\ell\nu} = 0,\ \partial_t \rho^\ell + \nabla\cdot\mathbf{J}^\ell = 0,\ J^{\ell\nu} = \left(\rho^\ell \mathrm{c}, \mathbf{J}^\ell\right), \tag{6.2.10}$$

under the assumption that the fields $\left\{\psi^\ell, F^{\mu\nu}\right\}$ satisfy the Euler-Lagrange field equations (6.2.6)-(6.2.8). Notice that in view of (6.2.9)

$$\rho^\ell = -\mathrm{i}\frac{\chi q^\ell\left(\partial_t\psi^{\ell*}\psi^\ell - \psi^{\ell*}\partial_t\psi^\ell\right)}{2m^\ell \mathrm{c}^2} - \frac{q^2\varphi\psi^{\ell*}\psi^\ell}{m\mathrm{c}^2} = -\frac{q^\ell\left|\psi^\ell\right|^2}{m^\ell\mathrm{c}^2}\left(\chi\operatorname{Im}\frac{\partial_t\psi^\ell}{\psi^\ell} + q^\ell\varphi\right), \tag{6.2.11}$$

$$\mathbf{J}^\ell = \mathrm{i}\frac{\chi q^\ell\left(\nabla\psi^{\ell*}\psi^\ell - \psi^{\ell*}\nabla\psi^\ell\right)}{2m^\ell} - \frac{q^2\mathbf{A}\psi^{\ell*}\psi^\ell}{m^\ell\mathrm{c}} = \frac{q^\ell\left|\psi^\ell\right|^2}{m^\ell}\left(\chi\operatorname{Im}\frac{\nabla\psi^\ell}{\psi^\ell} - \frac{q^\ell\mathbf{A}}{\mathrm{c}}\right).$$

As a consequence of the continuity equations (6.2.10) *the space integral of every $\rho^\ell(x)$ is a conserved quantity which we assign to be exactly q^ℓ*, i.e. we assume the following *charge normalization*

$$\int_{\mathbb{R}^3}\frac{\rho^\ell(x)}{q^\ell}\,\mathrm{d}\mathbf{x} = -\frac{1}{m^\ell\mathrm{c}^2}\int_{\mathbb{R}^3}\left(\chi\operatorname{Im}\frac{\partial_t\psi^\ell}{\psi^\ell} + q^\ell\varphi\right)\left|\psi^\ell\right|^2\,\mathrm{d}\mathbf{x} = 1,\ \ \ell = 1,\ldots N. \tag{6.2.12}$$

To summarize, the equations (6.2.6)-(6.2.8) together with the normalization (6.2.12) constitute a complete set of equations describing the state of the all fields $\left\{\psi^\ell, F^{\mu\nu}\right\}$ in the space-time. Notice that (6.2.6) and the Maxwell equations can be recast as

$$\left[\mathrm{c}^{-2}\tilde{\partial}_t^{\ell 2} - \tilde{\nabla}^{\ell 2} + \kappa^{\ell 2} + G^{\ell\prime}\left(\left|\psi^\ell\right|^2\right)\right]\psi^\ell = 0,\ \text{where} \tag{6.2.13}$$

$$\tilde{\partial}_t^\ell = \partial_t + \frac{\mathrm{i}q^\ell\varphi}{\chi},\ \tilde{\nabla}^\ell = \nabla - \frac{\mathrm{i}q^\ell\mathbf{A}}{\chi\mathrm{c}},$$

$$\nabla\cdot\left(\partial_t\mathbf{A} + \nabla\varphi\right) = 4\pi\sum_\ell\frac{q^\ell\left|\psi^\ell\right|^2}{m^\ell\mathrm{c}^2}\left(\chi\operatorname{Im}\frac{\partial_t\psi^\ell}{\psi^\ell} + q^\ell\varphi\right), \tag{6.2.14}$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{\mathrm{c}}\partial_t\left(\partial_t\mathbf{A} + \nabla\varphi\right) = \frac{4\pi}{\mathrm{c}}\sum_\ell\left(\frac{\chi q^\ell}{m^\ell}\operatorname{Im}\frac{\nabla\psi^\ell}{\psi^\ell} - \frac{q^{\ell 2}\mathbf{A}}{m^\ell\mathrm{c}}\right)\left|\psi^\ell\right|^2. \tag{6.2.15}$$

In order to see point charge features in the charges described as fields over the continuum of the space-time we have to identify the energy-momentum tensor $T^{\ell\mu\nu}$ for every ℓ-th charge and the EM field energy-momentum $\Theta^{\mu\nu}$. Notice that the system Lagrangian $\mathcal{L}$ defined by (6.2.1)-(6.2.4) satisfies the symmetry condition (10.5.9) and consequently the general construction of the symmetric energy-momenta from Section 10.5 applies here. Namely, using the formulas (10.5.24)-(10.5.25) we get the following representation for the energy-momenta

$$T^{\ell\mu\nu} = \frac{\chi^2}{2m^\ell}\left\{\left(\psi^{\ell;\mu*}\psi^{\ell;\nu} + \psi^{\ell;\mu}\psi^{\ell;\nu*}\right) - \left[\psi^{\ell*}_{;\mu}\psi^{\ell;\mu} - \kappa^{\ell 2}\psi^{\ell*}\psi^\ell - G^\ell\left(\psi^{\ell*}\psi^\ell\right)\right]\delta^{\mu\nu}\right\} \tag{6.2.16}$$

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(g^{\mu\gamma}F_{\gamma\xi}F^{\xi\nu} + \frac{1}{4}g^{\mu\nu}F_{\gamma\xi}F^{\gamma\xi}\right),\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{6.2.17}$$

The defined above energy-momentum tensors satisfy the equations (10.5.29)-(10.5.30), namely

$$\partial_\mu T^{\ell\mu\nu} = f^{\ell\nu}, \text{ where } f^{\ell\nu} = \frac{1}{\mathrm{c}} J^\ell_\mu F^{\nu\mu} \tag{6.2.18}$$

$$\partial_\mu \Theta^{\mu\nu} = -f^\nu, \text{where } f^\nu = \sum_\ell f^{\ell\nu} = \frac{1}{\mathrm{c}} J_\mu F^{\nu\mu},\ J_\mu = \sum_\ell J^\ell_\mu. \tag{6.2.19}$$

The energy-momentum conservation equations (6.2.18)-(6.2.19) can be viewed as equations of motion in elastic continuum, [Moller, Section 6.4, (6.56), (6.57)], similar to the case of kinetic energy-momentum tensor for a single relativistic particle, [Pauli RT, Section 37, (3.24)]. We recognize in the 4-vectors $f^{\ell\nu}$ in right-hand side of conservation equations (6.2.18) the density of the Lorentz force acting upon ℓ-th charge, and we also see the density of the Lorentz force with the minus sign in the right-hand side of EM energy-momentum conservation equation (6.2.18). We remind that equations (6.2.18)-(6.2.19) hold only under the assumption that the involved fields satisfy the field equations (6.2.6)-(6.2.9). Consequently, in contrast to the case of the point mechanics the conservation/motion equations (6.2.18)-(6.2.19) in the elastic continuum alone can not substitute for the field equations and determine the motion.

Now we would like to identify the EM field attributed to every individual bare charge. That can be naturally accomplished by partitioning the total EM $F^{\mu\nu}$ defined as a causal solution to the linear Maxwell equation (6.2.8) (see Section 10.4.1) with a source $\frac{4\pi}{\mathrm{c}} J^\mu$ according to the partition of the current $J^\mu = \sum_\ell J^{\ell\mu}$. Namely we introduce the EM potentials $A^{\ell\mu}$ and the corresponding EM field $F^{\ell\mu\nu}$ for every individual ℓ-th charge as the causal solution of the form (10.4.39) to the following Maxwell equation

$$\partial_\mu F^{\ell\mu\nu} = \frac{4\pi}{\mathrm{c}} J^{\ell\nu}. \tag{6.2.20}$$

In view of the linearity of the Maxwell equation (6.2.8) we evidently always have

$$F^{\mu\nu} = \sum_\ell F^{\ell\mu\nu}. \tag{6.2.21}$$

Being given individual EM fields $F^{\ell\mu\nu}$ we introduce the corresponding EM energy-momentum tensor $\Theta^{\ell\mu\nu}$ via the general formula (10.4.20), namely

$$\Theta^{\ell\mu\nu} = \frac{1}{4\pi}\left(g^{\mu\gamma} F^\ell_{\gamma\xi} F^{\ell\xi\nu} + \frac{1}{4} g^{\mu\nu} F^\ell_{\gamma\xi} F^{\ell\gamma\xi} \right). \tag{6.2.22}$$

Then combining (6.2.20), (6.2.22) with (10.4.27) we obtain

$$\partial_\mu \Theta^{\ell\mu\nu} = -\frac{1}{\mathrm{c}} J^\ell_\mu F^{\ell\nu\mu}. \tag{6.2.23}$$

Notice also that from (6.2.21) and (6.2.19) we have

$$\partial_\mu T^{\ell\mu\nu} = \frac{1}{\mathrm{c}} J^\ell_\mu F^{\ell\nu\mu} + \frac{1}{\mathrm{c}} J^\ell_\mu \sum_{\ell'\neq\ell} F^{\ell'\nu\mu}. \tag{6.2.24}$$

If we introduce now the energy-momentum $\mathrm{T}^{\ell\mu\nu}$ of the dressed charge, i.e. the charge with its EM field, by the formula

$$\mathrm{T}^{\ell\mu\nu} = T^{\ell\mu\nu} + \Theta^{\ell\mu\nu}, \tag{6.2.25}$$

then the sum of two equalities (6.2.23)-(6.2.24) readily yields the following *motion equations for dressed charges*

$$\partial_\mu \mathrm{T}^{\ell\mu\nu} = \frac{1}{\mathrm{c}} J^\ell_\mu \sum_{\ell' \neq \ell} F^{\ell'\nu\mu},\ \ell = 1, \ldots, N. \tag{6.2.26}$$

describing the motion of energies and momenta of the dressed charges in the space-time continuum. Importantly, *the Lorentz force in the right-hand of (6.2.26) excludes manifestly the self-interaction* in contrast to the Lorentz force acting upon bare charge as in (6.2.25) which explicitly includes the self-interaction term $\frac{1}{\mathrm{c}} J^\ell_\mu F^{\ell\nu\mu}$. Thus, we can conclude that when the charge and its EM field are treated as a single entity, namely dressed charge, there is no self-interaction as signified by the exact equations (6.2.26).

We would like to point out though to certain subtleties related to the individual EM fields $F^{\ell\nu\mu}$. Namely, the individual currents J^ℓ_μ constructed via solutions to the Euler-Lagrange field equations (6.2.6)-(6.2.8) are implicitly related to each other. Those implicit relations manifest themselves in particular in the fact the sum of the energy-momentum tensors of the individual EM fields does not exactly equal to the energy-momentum tensor of the total EM field, i.e.

$$\Theta^{\mu\nu} \neq \sum_\ell \Theta^{\ell\mu\nu}, \tag{6.2.27}$$

since $\Theta^{\mu\nu}$ defined by (6.2.18) is a quadratic, and hence nonlinear, function of the EM $F^{\mu\nu}$. Nevertheless, the approximate equality holds

$$\Theta^{\mu\nu} \approx \sum_\ell \Theta^{\ell\mu\nu}, \tag{6.2.28}$$

when the supports of charges wave functions ψ^ℓ are well separated in the space for the time period of interest and, hence,

$$F^\ell_{\gamma\xi} F^{\ell'\xi\nu} \approx 0 \text{ for } \ell' \neq \ell \tag{6.2.29}$$

implying (6.2.28). In other words, for well separated charges the above mentioned subtle correlations between individual currents become insignificant for their EM energy-momenta.

We can ask now how far one can go in extracting from the equations of motion (6.2.26) equations of point charges similarly to the non-relativistic theory in previous subsection. We can argue that in the relativistic theory in the regime of remote interaction we study every charge can behave similarly to a wave-corpuscle but their motion can not be reduced to a system of differential equations obtained from a conventional finite-dimensional Lagrangian since it is prohibited by presented above arguments by Goldstein, von Laue and Pauli. The next in simplicity option can be the motion governed by a system of integro-differential equations which can account for retardation effects similar to the Sommerfeld integro-differential-difference motion equation for the nonrelativistically rigid electron and its relativistic generalizations, [Pearle1, Sections 8-10], but we are not going to pursue this problem any further in this paper.

7 Equations in dimensionless form

We introduce here changes of variables allowing to recast the original field equations into a dimensionless form. These equations in dimensionless form allow to clarify three aspects of the theory for a single charge: (i) out of all the constants involved there is only one parameter of significance denoted by α, and it coincides with the Sommerfeld fine structure constant $\alpha_S \simeq 1/137$ if $\chi = \hbar$ and q, m are the electron charge and mass respectively; (ii) the non-relativistic Lagrangian (4.0.6) can be obtained from the relativistic one via the frequency-shifted Lagrangian (3.3.3) by setting there $\alpha = 0$; (iii) the fulfillment of charge and energy normalization conditions in relativistic case follows from smallness of α.

Recall that the single charge nonrelativistic Lagrangian $\mathring{L}_0$ defined by (4.0.6) is constructed in Section 4 based on the relativistic one via the the frequency shifted Lagrangian L_{ω_0} defined by (3.3.1)-(3.3.3) (see also Sections 3.3, 10.8.1). The corresponding to L_{ω_0} Euler-Lagrange field equations are

$$\frac{\tilde{\partial}_t^2 \psi}{\mathrm{c}^2} - \frac{\mathrm{i}m}{\chi}\left(2\partial_t \psi + 2\frac{\mathrm{i}q\bar{\varphi}}{\chi}\hat{\psi}\right) - \tilde{\nabla}^2 \psi + G'\left(|\psi|^2\right)\psi = 0, \tag{7.0.30}$$

$$\frac{1}{4\pi}\nabla\cdot\left(\frac{\partial_t \mathbf{A}}{\mathrm{c}} + \nabla\varphi\right) = \left(\frac{\chi q}{m\mathrm{c}^2}\operatorname{Im}\frac{\partial_t\hat{\psi}}{\hat{\psi}} + \frac{q^2\bar{\varphi}}{m\mathrm{c}^2}\right)|\psi|^2 - q\,|\psi|^2,$$

$$-\left(\nabla\times(\nabla\times\mathbf{A}) + \frac{\partial_t}{\mathrm{c}}\left(\frac{\partial_t\mathbf{A}}{\mathrm{c}} + \nabla\varphi\right)\right) = \frac{4\pi}{\mathrm{c}}\left(-\frac{\chi q}{m}\operatorname{Im}\frac{\nabla\psi}{\psi} - \frac{q^2\bar{\mathbf{A}}}{m\mathrm{c}}\right)|\psi|^2,$$

where

$$\tilde{\partial}_t = \partial_t + \frac{\mathrm{i}q\bar{\varphi}}{\chi},\ \bar{\varphi} = \varphi + \varphi_{\mathrm{ex}},\ \bar{\mathbf{A}} = \mathbf{A} + \mathbf{A}_{\mathrm{ex}}. \tag{7.0.31}$$

Let us introduce the following constants and new variables:

$$a_\chi = \frac{\chi^2}{mq^2},\ \alpha = \frac{q^2}{\chi\mathrm{c}},\ \omega_0 = \frac{m\mathrm{c}^2}{\chi} = \frac{\mathrm{c}}{\alpha a_\chi}, \tag{7.0.32}$$

$$\alpha^2\omega_0 t = \tau,\ \mathbf{x} = a_\chi \mathbf{y}, \tag{7.0.33}$$

$$\psi(\mathbf{x}) = \frac{1}{a_\chi^{3/2}}\Psi\left(\frac{\mathbf{x}}{a_\chi}\right),\ \varphi(\mathbf{x}) = \frac{q}{a_\chi}\Phi\left(\frac{\mathbf{x}}{a_\chi}\right),\ \mathbf{A}(\mathbf{x}) = \frac{q}{a_\chi}\mathsf{A}\left(\frac{\mathbf{x}}{a_\chi}\right).$$

In those new variables the field equations (7.0.30) turn into the following dimensionless form:

$$\alpha^2\left(\partial_\tau + \mathrm{i}\bar{\Phi}\right)^2\Psi - 2\mathrm{i}\left(\partial_\tau + \mathrm{i}\bar{\Phi}\right)\Psi - \left(\nabla_y - \mathrm{i}\alpha\bar{\mathsf{A}}\right)^2\Psi + \mathsf{G}'\left(|\Psi|^2\right)\Psi = 0, \tag{7.0.34}$$

$$\frac{1}{4\pi}\nabla_y\cdot\left(\alpha\partial_\tau\mathsf{A} + \nabla_y\Phi\right) = \left(\alpha^2\operatorname{Im}\frac{\partial_\tau\Psi}{\Psi} + \alpha^2\Phi\right)|\Psi|^2 - |\Psi|^2,$$

$$-\left(\nabla_y\times(\nabla_y\times\mathsf{A}) + \alpha\partial_\tau\left(\alpha\partial_\tau\mathsf{A} + \nabla_y\Phi\right)\right) = -4\pi\alpha\left(\operatorname{Im}\frac{\nabla_y\Psi}{\Psi} + \alpha\bar{\mathsf{A}}\right)|\Psi|^2,$$

We would like to show that the dimensionless form of the non-relativistic equations field equations (5.0.12), (5.0.13) can be obtained from the field equations (7.0.34) in the limit $\alpha \to 0$. To have a nonvanishing external magnetic field in the limit $\alpha \to 0$ we set

$$\mathsf{A}_{\mathrm{ex}} = \alpha^{-1}\mathsf{A}_{\mathrm{ex}}^0 \tag{7.0.35}$$

Plugging in the expression (7.0.35) into the equations (7.0.34) we obtain in the limit $\alpha \to 0$ the following dimensionless version of the field equations (5.0.12), (5.0.13):

$$\mathrm{i}\partial_\tau\Psi = -\frac{1}{2}\left(\nabla_y - \mathrm{i}\mathsf{A}_{\mathrm{ex}}^0\right)^2\Psi + \left(\Phi + \Phi_{\mathrm{ex}}\right)\Psi + \frac{1}{2}\mathsf{G}'\left(|\Psi|^2\right)\Psi, \tag{7.0.36}$$

$$-\nabla_y^2\cdot\Phi = 4\pi|\Psi|^2,\ \left(\nabla_y\times(\nabla_y\times\mathsf{A})\right) = 0.$$

To get an insight in the nonrelativistic case as an approximation to the relativistic one we would like to make a few comments on the relative magnitude of terms that have to be omitted in equation (7.0.34) in order to obtain equation (7.0.36). The nonrelativistic case is defined as one when the charge velocity v is much smaller than the speed of light c, and a careful look at those omitted terms in (7.0.34) that have factors α and α^2 shows that they can be small not only because of α, but also because of the smallness of typical values of velocities compared to the speed of light. Indeed, every term that has factor α involves time derivatives with only one exception: the term $\alpha^2 \left(\mathrm{i}\bar{\Phi}\right)^2 \Psi$. An estimation of the magnitude of the omitted terms for solutions of the form of wave-corpuscles (5.1.2) indicated that every such a term is of order $\alpha \left|\mathbf{v}\right| /\mathrm{c}$ where $\mathbf{v}$ is the wave-corpuscle velocity. The only omitted term in (7.0.34) which does not involve time derivatives is $\alpha^2\Phi^2\Psi$ and, in fact, it can be preserved in the nonrelativistic system which would be similar to (7.0.36) with properties analogous to (5.0.12). Analysis of that system is more involved and the treatment is similar to the one for the rest solution of the relativistic equation involving that term considered in next Subsection 7.1.

7.1 Single relativistic charge at rest

In this section we a consider a single relativistic charge. Using the new constants and variables (7.0.32), (7.0.33) we get the following dimensionless version of the resting charge equations (2.0.12), (2.0.13) and the charge normalization condition (2.0.20)

$$-\frac{1}{2}\nabla_y^2\Psi + \left(\Phi - \frac{\alpha^2\Phi^2}{2}\right)\Psi + \frac{\hat{G}'\left(\Psi^2\right)}{2}\Psi = 0, \tag{7.1.1}$$

$$-\nabla^2\Phi = 4\pi\left(1-\alpha^2\Phi\right)\left|\Psi\right|^2 = 0, \tag{7.1.2}$$

$$\int_{\mathbb{R}^3}\left[\left(1-\alpha^2\Phi\right)\left|\Psi\right|^2\right]\,\mathrm{d}\mathbf{x} = 1. \tag{7.1.3}$$

Setting in the above equations $\alpha = 0$ we obtain the dimensionless form nonrelativistic equilibrium equations (2.3.5), (2.3.6) and the charge normalization condition (2.3.12), namely

$$-\frac{1}{2}\nabla_y^2\Psi + \Phi\Psi + \hat{G}_0'\left(\left|\Psi\right|^2\right)\check{\Psi} = 0,\ -\nabla^2\Phi = 4\pi\left|\Psi\right|^2, \tag{7.1.4}$$

$$\int_{\mathbb{R}^3}\left|\Psi\right|^2\,\mathrm{d}\mathbf{x} = 1. \tag{7.1.5}$$

Now using perturbations analysis we argue that for small α the solution Ψ_α, Φ_α to the equations (7.1.2) is close to the solution Ψ_0, Φ_0 of the equations (7.1.4). Indeed, for zero approximation $\Phi_0\left(\mathbf{x}\right)$

$$\Phi_0\left(\mathbf{x}\right) = \int_{\mathbb{R}^3}\frac{\left|\Psi_0\left(\mathbf{y}\right)\right|^2}{\left|\mathbf{x}-\mathbf{y}\right|}\,\mathrm{d}\mathbf{y},$$

and the first order approximation Φ_1 is a solution to

$$-\nabla^2\Phi_1 = 4\pi\left(1-\alpha^2\Phi_0\right)\left|\Psi_0\right|^2.$$

Using the Maximum principle we get

$$0 < \Phi_1\left(\mathbf{x}\right) < \Phi\left(\mathbf{x}\right) < \Phi_0\left(\mathbf{x}\right) = \check{\Phi}\left(\mathbf{x}\right)\ \text{ for all } \mathbf{x}. \tag{7.1.6}$$

Obviously,

$$\Phi_1(\mathbf{x}) = \Phi_0(\mathbf{x}) + \alpha^2\Phi_{01}(\mathbf{x}), \text{ where } \nabla^2\Phi_{01} = 4\pi\Phi_0 |\Psi_0|^2, \tag{7.1.7}$$

and hence

$$\Phi_{01}(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{\Phi_0(\mathbf{y})|\Psi_0(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|} \,\mathrm{d}\mathbf{y}.$$

Consequently, inequalities (7.1.6) imply explicit estimate

$$\alpha^2\Phi_{01}(\mathbf{x}) < \Phi(\mathbf{x}) - \Phi_0(\mathbf{x}) < 0 \text{ for all } \mathbf{x} \in \mathbb{R}^d. \tag{7.1.8}$$

7.2 Charge and energy simultaneous normalization

We consider here some technical details related to the size representation (2.1.8), (2.1.9) and the problem of simultaneous normalization of the charge and the energy by equations (2.1.10) and (2.1.11). Let function $\Psi_\theta(\mathbf{y})$ be the dimensionless version of the function ψ_a in (2.1.8), (2.1.9), namely

$$\psi_a(\mathbf{x}) = \frac{\theta^{3/2}}{a_\chi^{3/2}}\psi_1\left(\frac{\theta^{3/2}\mathbf{x}}{a_\chi^{3/2}}\right) = \Psi_\theta(\mathbf{y}) = C_\Psi\theta^{3/2}\Psi_1(\theta\mathbf{y}), \tag{7.2.1}$$

$$\text{where } \theta = \frac{a_\chi}{a}, \text{ and } \int_{\mathbb{R}^3} |\Psi_1|^2 \,\mathrm{d}\mathbf{y} = 1.$$

Then the charge and the energy normalization conditions (2.1.10) and (2.1.11) take the form

$$\mathcal{N} = \int_{\mathbb{R}^3} \left(1 - \alpha^2\Phi_\theta\right)|\Psi_\theta|^2 \,\mathrm{d}\mathbf{y} = 1. \tag{7.2.2}$$

Using the relation (10.8.36) and (10.8.40) we obtain the following representation for the energy $\mathcal{E}(\psi_a, \varphi_a)$, which is a version of the Pohozhaev-Derrik formula (see Section 10.8.1),

$$\mathcal{E}_0(\psi_a, \varphi_a) = mc^2 + \mathcal{E}_0'(\psi_a, \varphi_a), \ \mathcal{E}_0'(\psi_a, \varphi_a) = \frac{2}{3}\int_{\mathbb{R}^3} \left(\frac{\chi^2|\nabla\psi_a|^2}{2m} - \frac{|\nabla\varphi_a|^2}{8\pi}\right) \mathrm{d}\mathbf{x}. \tag{7.2.3}$$

Then the energy normalization condition (2.1.11), namely $\mathcal{E}_0' = 0$, turns into the dimensionless variables into the condition

$$\mathcal{E}_0'(\Psi_\theta, \Phi_\theta) = \frac{q^2}{3a_\chi}\int_{\mathbb{R}^3} \left(|\nabla\Psi_\theta|^2 - \frac{|\nabla\Phi_\theta|^2}{4\pi}\right) \mathrm{d}\mathbf{y} = 0. \tag{7.2.4}$$

First, let us consider a simple case $\alpha = 0$ using for it the notation $\check{\Phi}_\theta = \Phi_\theta|_{\alpha=0}$, $\check{\Psi}_\theta = \Psi_\theta|_{\alpha=0}$. In this case the charge normalization condition (7.2.2), in view of the normalization condition in (7.2.1), is satisfied for $C_\Psi = 1$ and

$$\check{\Psi}_\theta(\mathbf{y}) = \theta^{3/2}\check{\Psi}_1(\theta\mathbf{y}), \ \check{\Phi}_\theta(\mathbf{y}) = \theta\check{\Phi}_1(\theta\mathbf{y}) \text{ for } \alpha = 0. \tag{7.2.5}$$

It follows then from (7.2.4) that

$$\mathcal{E}_0'\left(\check{\Psi}_\theta, \check{\Phi}_\theta\right) = \frac{q^2}{3a_\chi}\int_{\mathbb{R}^3} \left(\theta^2\left|\nabla\check{\Psi}_1\right|^2 - \frac{\left|\nabla\check{\Phi}_1\right|^2}{4\pi}\theta\right) \mathrm{d}\mathbf{y}. \tag{7.2.6}$$

implying that $\mathcal{E}_0'\left(\check{\Psi}_\theta, \check{\Phi}_\theta\right)$ is a quadratic function of the parameter θ. Since Ψ is fixed we set in (7.2.1) $C_\Psi = 1$ and $\theta = \theta_0$, where θ_0 is defined by

$$\theta_0 = \frac{b_\Phi}{b_\Psi},\ b_\Phi = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left|\nabla\check{\Phi}_1\right|^2 \mathrm{d}\mathbf{y},\ b_\Psi = \int_{\mathbb{R}^3}\left|\nabla\check{\Psi}_1\right|^2 \mathrm{d}\mathbf{y}, \tag{7.2.7}$$

and obtain the desired energy normalization condition (7.2.4), namely

$$\mathcal{E}_0'\left(\check{\Psi}_\theta, \check{\Phi}_\theta\right) = 0,\ \int_{\mathbb{R}^3}\left|\check{\Psi}_\theta\right|^2 \mathrm{d}\mathbf{x} = 1. \tag{7.2.8}$$

Note that θ_0 as in (7.2.7) coincides with θ_ψ as in (4.7.3).

Let us consider now the case $\alpha > 0$. We would like to show that for a given form factor Ψ and small $\alpha > 0$ there exist constants C_Ψ and θ such that the following two equation hold

$$\mathcal{E}_0'\left(\check{\Psi}_\theta, \check{\Phi}_\theta, \alpha\right) = 0,\ \mathcal{N}\left(C_\Psi, \theta, \alpha\right) - 1 = 0, \tag{7.2.9}$$

where $\mathcal{E}_0'$ and $\mathcal{N}_1$ are defined respectively by relations (7.2.4) and (7.2.2). In other words he need to find two parameters C_Ψ and θ from a system of two nonlinear equations (7.2.9). We have already established that for $\alpha = 0$: the solution is $C_\Psi = 1$, $\theta = \theta_0$ as in (7.2.7). A geometrical argument shows that *for sufficiently small* α equations (7.2.9) have a solution $\{C, \theta\}$ which is close for small α to the solution $C_\Psi = 1$, $\theta = \theta_0$. The complete argument is based on the inequality (7.1.8) but its details are rather technical and we omit them here.

8 Hydrogen atom model

The purpose of this section is to introduce a Hydrogen atom model within the framework of our theory for two interacting charges: an electron and a proton. We have no intend though to study this model in detail here. Our modest effort on the subject in this paper is, first, to indicate a similarity between our and Schrodinger's Hydrogen atom models and to contrast it to any kind of Kepler model. Another point we can make based on our Hydrogen atom model is that our theory does provide a basis for a regime of close interaction between two charges which differs significantly from the regime of remote interaction which is the primary focus of this paper. We cannot apply the results on interaction of many charges as in Section 6.1.2 since now the charges are not separated in space and the potentials can vary significantly, and other methods have be developed.

To model interaction of two charges *at a short distance* we must consider the original system (2.4.2) for two charges with $-q_1 = q_2 = q > 0$, that is

$$\begin{aligned}\mathrm{i}\chi\partial_t\psi_1 &= -\frac{\chi^2\nabla^2\psi_1}{2m_1} + q\left(\varphi_1+\varphi_2\right)\psi_1 + \frac{\chi^2 G_1'\left(\left|\psi_1\right|^2\right)\psi_1}{2m_1},\ -\nabla^2\varphi_1 = -4\pi q\left|\psi_1\right|^2, \\ \mathrm{i}\chi\partial_t\psi_2 &= -\frac{\chi^2\nabla^2\psi_2}{2m_2} - q\left(\varphi_1+\varphi_2\right)\psi_2 + \frac{\chi^2 G_2'\left(\left|\psi_2\right|^2\right)\psi_2}{2m_2},\ \nabla^2\varphi_2 = -4\pi q\left|\psi_2\right|^2.\end{aligned} \tag{8.0.10}$$

Note that the model describes proton-electron interaction if $q = e$ is electron charge, $\chi = \hbar$ is Planck's constant, m_1 and m_2 are electron and proton masses respectively. Let us look now at time-harmonic solutions to the system (8.0.10) in the form

$$\psi_1\left(t, \mathbf{x}\right) = \mathrm{e}^{-\mathrm{i}\omega_1 t}u_1\left(\mathbf{x}\right),\ \psi_2 = \mathrm{e}^{-\mathrm{i}\omega_2 t}u_2\left(\mathbf{x}\right), \Phi_1 = \varphi_1/q,\ \Phi_2 = -\varphi_2/q \tag{8.0.11}$$

The system (8.0.10) for such solutions turns into the following nonlinear eigenvalue problem:

$$
\begin{aligned}
-\frac{a_1}{2}\nabla^2 u_1 + (\Phi_1 - \Phi_2)\, u_1 + \frac{a_1}{2} G_1'\left(|u_1|^2\right) u_1 = \frac{\chi}{q^2}\omega_1 u_1,\ a_1 = \frac{\chi^2}{q^2 m_1}, \\
-\frac{a_2}{2}\nabla^2 u_2 + (\Phi_2 - \Phi_1)\, u_2 + \frac{a_2}{2} G_2'\left(|u_2|^2\right) u_2 = \frac{\chi}{q^2}\omega_2 u_2,\ a_2 = \frac{\chi^2}{q^2 m_2}.
\end{aligned}
\tag{8.0.12}
$$

Here a_1 coincides with Bohr radius. We seek solutions of (8.0.12)) satisfying the charge normalization conditions

$$
\int_{\mathbb{R}^3} |u_1|^2 \, \mathrm{d}\mathbf{x} = 1,\ \int_{\mathbb{R}^3} |u_2|^2 \, \mathrm{d}\mathbf{x} = 1. \tag{8.0.13}
$$

The potentials Φ_i are presented using (8.0.10) as follows:

$$
\Phi_i = 4\pi \left(-\nabla^2\right)^{-1} |u_i|^2 = 4\pi \left(-1\right)^i \int_{\mathbb{R}^3} \frac{|u_i|^2(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} \, \mathrm{d}\mathbf{y},\ i = 1, 2. \tag{8.0.14}
$$

Let us introduce now the following energy functional

$$
\begin{aligned}
\mathcal{E}(u_1, u_2) \ &= \ q^2 \int_{\mathbb{R}^3} \frac{a_1\left[|\nabla u_1|^2 + G_1\left(|u_1|^2\right)\right]}{2} + \frac{a_2\left[|\nabla u_2|^2 + a_2 G_2\left(|u_2|^2\right)\right]}{2} + \\
&\quad + (\Phi_2 + \Phi_1)\left(|u_1|^2 + |u_2|^2\right) - \frac{|\nabla \Phi_1|^2}{8\pi} - \frac{|\nabla \Phi_2|^2}{8\pi} \, \mathrm{d}\mathbf{x},
\end{aligned}
\tag{8.0.15}
$$

where Φ_1, Φ_2 are determined in terms of u_1, u_2 by (8.0.14). Notice that the equations (8.0.12) describe stationary points of the functional $\mathcal{E}$ and can be obtained by its variation under constraints (8.0.13) with the frequencies ω_1, ω_2 being the Lagrange multipliers. Observe that the energy functional and the constraints are invariant with respect to multiplication by -1, and that allows to apply the Lusternik-Schnirelman theory which guarantees the existence of an infinite set of critical points under appropriate conditions. The critical points are the eigenfunctions of the corresponding Schrödinger operators (see, for example, [Heid]). Our preliminary analysis shows that properties of solutions are reminiscent of those in the spectral theory of the Hydrogen atom described by the linear Schrödinger equation. The smallness of the ratio $m_1/m_2 \cong 1/1836$ of electron to proton masses plays an important role in the analysis. For the critical points with low energies the potential Φ_2 of the proton is close to the Coulomb potential $1/|\mathbf{x}|$ at spatial scales of order a_1. For a properly chosen nonlinearity the quadratic part of the energy functional and the corresponding linear Schrödinger equation can be used to find discrete low levels of energy. Rough estimates of the energy levels of approximate solutions of the nonlinear problem based on eigenfunctions of the linear Schrödinger operator show qualitative agreement with well-known lower energy levels for the Hydrogen atom with several percent accuracy.

9 Comparison with the Schrödinger wave theory

We already made some points on common features and differences between our theory and the Schrödinger wave mechanics in Section 2.5, and here we discuss in more detail a few significant differences between the two theories. Recall that the Schrödinger wave mechanics is constructed based on the point particle Hamiltonian

$$\mathcal{E} = H(\mathbf{p}, \mathbf{x}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}) \tag{9.0.16}$$

by substituting there, [Pauli WM, Section 2, 11], [Pauli PWM, Sections 3, 4],

$$\mathbf{p} \to -\mathrm{i}\hbar\nabla,\ \mathcal{E} \to \mathrm{i}\hbar\frac{\partial}{\partial t}, \tag{9.0.17}$$

that yields the celebrated Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2\nabla^2\psi}{2m} + V(\mathbf{x})\,\psi. \tag{9.0.18}$$

The substitution (9.0.17) is essentially the quantization procedure allowing to correspond the classical point Hamiltonian (9.0.16) to the quantum mechanical wave equation (9.0.18). If we introduce the polar representation

$$\psi = \mathrm{e}^{\mathrm{i}\frac{S}{\hbar}}R,\ \ R \geq 0 \tag{9.0.19}$$

then the Schrödinger equation (9.0.18) can be recast as following system of two coupled equations for the real valued phase function S and the amplitude R, [Holland, Section 3.2.1],

$$\frac{\partial S}{\partial t} + \frac{\hbar^2(\nabla S)^2}{2m} + V - \frac{\hbar^2\nabla^2 R}{2mR} = 0, \tag{9.0.20}$$

$$\frac{\partial R^2}{\partial t} + \nabla\cdot\frac{R^2\nabla S}{m} = 0. \tag{9.0.21}$$

If we expand the phase S into power series with respect to $\hbar$ i.e.

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots, \tag{9.0.22}$$

we obtain from the equation (9.0.20) so called WKB approximation, [Pauli PWM, Section 12]. In particular, the function S_0 satisfy the Hamilton-Jacobi equation

$$\frac{\partial S_0}{\partial t} + \frac{\hbar^2(\nabla S_0)^2}{2m} + V = 0, \tag{9.0.23}$$

that shows, in particular, that the Schrödinger wave equation (9.0.18) does "remember" how it was constructed by "returning back" the original Hamiltonian H via the Hamilton-Jacobi equation (9.0.23) for S_0.

Our approach works other way around. We introduce the Lagrangian (3.0.1)-(3.0.3) and the corresponding field equations as a fundamental basis and deduce from them the classical Newtonian mechanics as a certain approximation (see Sections 5, 6.1). To appreciate the difference let us look at a system of N charges. The introduced here wave-corpuscle mechanics would have N wave functions and the EM fields defined over the same 3 dimensional space, whereas the same system of N charges in the Schrödinger wave mechanics has a single wave function defined over a $3N$-dimensional "configuration space".

It is quite instructive to compare the polar representation (2.3.16) for wave-corpuscle ψ with the same for the Schrödinger wave function ψ for the potential $V(\mathbf{x}) = -q\mathbf{E}_{\mathrm{ex}} \cdot \mathbf{x}$ corresponding to a homogeneous external electric field (the eigenfunctions of the corresponding Schrödinger equations can be expressed in terms of the Airy functions, [Pauli WM, Section 26]). The amplitude $\mathring{\psi}(|\mathbf{x} - \mathbf{r}(t)|)$ of the wave-corpuscle in the expression (2.3.16) for ψ is a soliton-like field moving exactly as the point charge described by its position $\mathbf{r}(t)$ in contrast to the amplitude R of the Schrödinger wave function which describes the location of the charge rather implicitly via the coupled equations (9.0.21). The difference between the phases is equally significant. Indeed, the exponential factor $e^{\frac{\mathrm{i}S}{\chi}}$ for the accelerating wave-corpuscle is just a plane wave with the phase S that depends only on the point charge position r and momentum p whereas the same for the Schrödinger equations captures the features of the point charge only via WKB approximation and the Hamilton-Jacobi equation (9.0.23) which holds for the phase S only in the limit $\hbar \to 0$.

Let us now take a look now at uncertainty relations which constitutes a very important consequence of the Schrodinger wave mechanics as a linear wave theory. Detailed studies of this subject is not in the scope of this paper but we can already see significant alterations of the uncertainty relations brought by the nonlinearity. W. Pauli writes in section "The uncertainty principle", [Pauli WM, Section 3]): "The kinematics of waves does not allow the simultaneous specification of the location and the exact wavelength of a wave. Indeed, one can only speak of the location of a wave in the case of a spatially localized wave packet. The number of different wavelengths contained in the Fourier spectrum increases as the wave packet becomes more localized. A relation of the form $\Delta k_i \Delta x_i \geq$ constant seems reasonable, and we now want to derive this relation quantitatively". Then in the same section he derives the well known Heisenberg uncertainty principle for a wavepacket in the form

$$\Delta k \Delta x \geq \frac{1}{2}, \; \Delta p \Delta x \geq \frac{\hbar}{2}, \tag{9.0.24}$$

where Δx, Δk and $\Delta p = \hbar \Delta k$ are respectively the spacial range, the wave number range and the momentum range for the wavepacket $\psi(\mathbf{x})$ defined by

$$\Delta x^2 = \int_{\mathbb{R}^3} (\mathbf{x} - \bar{\mathbf{x}})^2 |\psi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}, \; \bar{\mathbf{x}} = \int_{\mathbb{R}^3} \mathbf{x} |\psi(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}, \tag{9.0.25}$$

$$\Delta k^2 = \int_{\mathbb{R}^3} \left(\mathbf{k} - \bar{\mathbf{k}}\right)^2 \left|\hat{\psi}(\mathbf{k})\right|^2 \, \mathrm{d}\mathbf{k}, \; \bar{\mathbf{k}} = \int_{\mathbb{R}^3} \psi^*(\mathbf{x}) \left(-\mathrm{i}\frac{\partial}{\partial x}\right) \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{k} \left|\hat{\psi}(\mathbf{k})\right|^2 \, \mathrm{d}\mathbf{k},$$

$$\hat{\psi}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ikx} \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

Importantly, in the classical quantum wave mechanics, which is a linear theory, if $\psi(\mathbf{x})$ is the wave function $|\psi(\mathbf{x})|^2$ is interpreted as the probability density for a point particle to be at a location $\mathbf{x}$. Hence, in this theory the uncertainty is already in the very interpretation of the wave function, and Δx as in (9.0.25) is an uncertainty in the location of the point particle with a similar interpretation holding for Δp. *Thus the exact form of the Heisenberg uncertainty relation (9.0.24) is a direct consequence of the fundamental definition (9.0.17) of the momentum and the definition of the uncertainty as in (9.0.25) based on the probabilistic interpretation of the wave function.*

An important feature of the uncertainty relations in the linear theory is that any freely propagating wavepacket spreads out as a quadratic function of the time t and such a spread out takes particularly simple form for a Gaussian wavepacket, [Pauli WM, Section 3],

$$(\Delta x)^2 = \frac{1}{4(\Delta k)^2} + \frac{\hbar^2 (\Delta k)^2}{m^2} t^2. \tag{9.0.26}$$

We would like to point out that the very concept of wavepacket is based on the medium linearity and the same is true for of the uncertainty relations (9.0.24), (9.0.26) as general wave phenomena.

Let us consider now the proposed here wave-corpuscle mechanics from the uncertainty relations point of view. In wave-corpuscle mechanics we denote uncertainties in the position x and the momentum p by respectively by $\tilde{\Delta} x$ and $\tilde{\Delta} p$ using a different symbol $\tilde{\Delta}$ to emphasize the difference in their definition since the wave-function does not carry a probabilistic interpretation. *We also limit our considerations of the uncertainties $\tilde{\Delta} x$ and $\tilde{\Delta} p$ to special cases when a wave-corpuscle is an exact solution to either relativistic or nonrelativistic field equations since in these cases we can argue more convincingly what constitutes uncertainty without giving its definition in a general case.*

Notice that a common feature of the dressed charge (wave-corpuscle) and a wavepacket is wave origin of their kinematics as manifested by the equality of the velocity to the group velocity of the underlying linear medium. But, in contrast to a wavepacket in a linear medium, the free relativistic dressed charge described by the relations (3.4.1)-(3.4.4) does not disperse as it moves and preserves its shape up the Lorentz contraction. The de Broglie vector $\mathbf{k}$ and the frequency ω can be determined from the factor $\mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{k}\cdot\mathbf{x})}$ in (3.4.1), and consequently the same applies to the total momentum $\mathsf{P} = \hbar\mathbf{k}$. In the case of a nonrelativistic wave-corpuscle as defined by relations (4.3.1), (4.3.2) similarly its total momentum is $\mathsf{P} = m\mathbf{v} = \hbar\mathbf{k}$. More than that as we show in Section 5 the wave-corpuscle as defined by relations (5.1.2) is an exact solution to the field equations and it propagates in the space without dispersion even when accelerates.

To further clarify differences with the uncertainties in Schrodinger wave mechanics and the wave-corpuscle mechanics introduced here notice the following. The dressed charge described by the relations (3.4.1)-(3.4.4) is a material wave for which we can reasonably assign size D in relativistic and nonrelativistic cases respectively by the formula

$$D^2(t) = \int_{\mathbb{R}^3} (\mathbf{x} - \bar{\mathbf{x}})^2 \frac{\rho(t,\mathbf{x})}{q}\,\mathrm{d}\mathbf{x}, \text{ where } \bar{\mathbf{x}} = \int_{\mathbb{R}^3} \mathbf{x}\frac{\rho(t,\mathbf{x})}{q}\,\mathrm{d}\mathbf{x} \text{ and} \tag{9.0.27}$$
$$\frac{\rho(t,\mathbf{x})}{q} = q\left(1 - \frac{q\mathring{\varphi}(\mathbf{x} - \mathbf{v}t)}{m\mathrm{c}^2}\right)\mathring{\psi}^2(\mathbf{x} - \mathbf{v}t) \text{ or } \frac{\rho(t,\mathbf{x})}{q} = \mathring{\psi}^2(\mathbf{x} - \mathbf{v}t).$$

In the case of a relativistic or nonrelativistic wave-corpuscle we define "safely" the uncertainty $\tilde{\Delta} x = D(t)$ and notice that it follows from the definition (9.0.24) and charge normalization conditions (3.0.11), (4.0.14) that in fact $\tilde{\Delta} x = D(t) = D$ does not depend on t. Hence, the uncertainty $\tilde{\Delta} x = D$ unlike the uncertainty Δx from (9.0.26) does not grow without bound for large times. As to the charges momentum $\mathsf{P} = m\mathbf{v} = \hbar\mathbf{k}$ we can argue that $\tilde{\Delta} p = 0$ for the exact wave-corpuscle solutions. Indeed, for a wave-corpuscle as defined by relations (3.4.1)-(3.4.4) or (4.3.1), (4.3.2) respectively in the relativistic and nonrelativistic cases the motion of dressed charge is obtained by application of space translations (or Lorentz transformations) to a fixed rest charge, therefore "every point" of the dressed charge moves with exactly the same velocity $\mathbf{v}$ similarly to a rigid body, which allows naturally define its velocity and momentum without uncertainty. Summarizing, we can conclude that in the wave-corpuslce mechanics the Heisenberg uncertainty principle can not be a universal principle.

10 Classical field theory

In this section we discuss important elements of the classical field theory including the variational principles and the Lagrangian formalism, gauge invariance and conservation laws. When writing this section we started first with just a few formulas, but then, under the pressure of having complete and self-contained arguments we extended it to its current form.

There are many classical references on the classical field theory: [Barut], [Goldstein], [LandauLif F], [Morse Feshbach 1, Section 3.4], [Pauli RFTh] and more. So we picked and chose different parts of theory from different sources to emphasize concepts and constructions important for our own arguments. Often we gave multiple references to provide different and complementary points of view of the same subject. We also extended some parts of the theory as needed. In particularly, we did that for an important for us subject of many charges interacting with the EM.

10.1 Relativistic Kinematics

Here where provide very basic facts and notations related to the relativistic kinematics following to [Barut, Section 1], [LandauLif F, Sections 1.1-1.4, 2], [Jackson, Section 11.3], [Goldstein, Section 7].

The time-space four vector in its contravariant x^{μ} and covariant x_{μ} forms are represented as follows

$$x = x^{\mu} = \left(x^0, x^1, x^2, x^3\right) = (\mathrm{c}t, \mathbf{x}), \ \mu = 0, 1, 2, 3; \tag{10.1.1}$$

$$x_{\mu} = g_{\mu\nu} x^{\nu} = \left(x^0, -x^1, -x^2, -x^3\right), g_{\mu\nu}, \tag{10.1.2}$$

with the common convention on the summation of the same indices, and where $g_{\mu\nu}$ or $g^{\mu\nu}$, called metric tensor, 4×4 matrix, are defined by

$$\{g_{\mu\nu}\} = \{g^{\mu\nu}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{10.1.3}$$

Notice also that, [Jackson, Section 11.6], [Schwabl, Section 6.1],

$$g^{\mu}{}_{\nu} = g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}{}_{\nu} \text{ where } \delta^{\mu}{}_{\nu} \text{ is the Dirac symbol, } \{\delta^{\mu}{}_{\nu}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{10.1.4}$$

and

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{\mathrm{c}}\partial_t, \nabla\right), \ \partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{1}{\mathrm{c}}\partial_t, -\nabla\right). \tag{10.1.5}$$

The elementary Lorentz transformation to a moving with a velocity $\mathbf{v}$ frame is

$$x^{0\prime} = \gamma\left(x^0 - \boldsymbol{\beta}\cdot\mathbf{x}\right), \ \mathbf{x}' = \mathbf{x} + \frac{\gamma - 1}{\beta^2}\left(\boldsymbol{\beta}\cdot\mathbf{x}\right)\boldsymbol{\beta} - \gamma\boldsymbol{\beta} x^0, \tag{10.1.6}$$

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{\mathrm{c}}, \ \beta = |\boldsymbol{\beta}|, \ \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{\mathrm{c}}\right)^2}}.$$

If for a space vector $\mathbf{x}$ we introduce $\mathbf{x}_{\|}$ and $\mathbf{x}_{\perp}$ so that they are respectively its components parallel and perpendicular to the velocity $\mathbf{v}$, i.e. $\mathbf{x} = \mathbf{x}_{\|} + \mathbf{x}_{\perp}$, then (10.1.6) can be recast as

$$x^{0\prime} = \gamma \left(x^0 - \boldsymbol{\beta} \cdot \mathbf{x}\right), \ \mathbf{x}'_{\|} = \gamma \left(\mathbf{x}_{\|} - \boldsymbol{\beta} x^0\right), \ \mathbf{x}'_{\perp} = \mathbf{x}_{\perp}, \tag{10.1.7}$$

which in the case when $\mathbf{v}$ is parallel to the axis x^1 turns into

$$x^{0\prime} = \gamma \left(x^0 - \beta x^1\right), \ x^{1\prime} = \gamma \left(x^1 - \beta x^0\right), \ x^{2\prime} = x^2, \ x^{3\prime} = x^3. \tag{10.1.8}$$

The Lorentz invariance then of a 4-vector x under the above transformation reduces to

$$\left(x^{0\prime}\right)^2 - \left|\mathbf{x}'\right|^2 = \left(x^0\right)^2 - \left|\mathbf{x}\right|^2. \tag{10.1.9}$$

The general infinitesimal form of the inhomogeneous Lorentz transformation is, [Moller, Section 6.1],

$$x'^{\mu} = x^{\mu} + \xi^{\mu\nu} x_{\nu} + a^{\mu}, \ \xi^{\mu\nu} = -\xi^{\nu\mu}, \tag{10.1.10}$$

where $\omega^{\mu\nu}$ and a^{μ} are its ten parameters.

The Lagrangian L_{p} of the moving point mass is

$$L_{\mathrm{p}} = -m\mathrm{c}^2 \sqrt{1 - \left(\frac{\mathbf{v}}{\mathrm{c}}\right)^2}, \tag{10.1.11}$$

implying the following nonrelativistic approximation

$$L_{\mathrm{p}} \cong -m\mathrm{c}^2 + \frac{m\mathbf{v}^2}{2}, \ \text{for} \ \left|\frac{\mathbf{v}}{\mathrm{c}}\right| \ll 1. \tag{10.1.12}$$

The momentum (the ordinary kinetic momentum) and the energy of the point mass for the relativistic Lagrangian L_{p} defined by (10.1.11) are, [LandauLif F, Section 2.9],

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \left(\frac{\mathbf{v}}{\mathrm{c}}\right)^2}}, \ \mathcal{E} = p^0 \mathrm{c} = \mathbf{p} \cdot \mathbf{v} - L = \frac{m\mathrm{c}^2}{\sqrt{1 - \left(\frac{\mathbf{v}}{\mathrm{c}}\right)^2}} = \mathrm{c}\sqrt{\mathbf{p}^2 + m^2\mathrm{c}^2}. \tag{10.1.13}$$

The relativistic Lagrangian L_{p} of a point charge q with a mass m in an external EM field as described by electric potential φ and vector potential $\mathbf{A}$ is defined by, [Jackson, Section 12.1]

$$L_{\mathrm{p}} = -m\mathrm{c}^2 \sqrt{1 - \left(\frac{\mathbf{v}}{\mathrm{c}}\right)^2} - q\varphi + \frac{q}{\mathrm{c}} \mathbf{v} \cdot \mathbf{A}. \tag{10.1.14}$$

For this Lagrangian the ordinary kinetic momentum $\mathbf{p}$, the canonical (conjugate) momentum $\mathring{\mathbf{p}}$, and the Hamiltonian H_{p} are defined by the following relations

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \left(\frac{\mathbf{v}}{\mathrm{c}}\right)^2}}, \ \mathring{\mathbf{p}} = \mathbf{p} + \frac{q}{\mathrm{c}} \mathbf{A}, \tag{10.1.15}$$

$$H_{\mathrm{p}} = \mathring{\mathbf{p}} \cdot \mathbf{v} - L = \sqrt{\left(\mathrm{c}\mathbf{p} - q\mathbf{A}\right)^2 + m^2\mathrm{c}^4} + q\varphi, \tag{10.1.16}$$

and the Euler-Lagrange equations are

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} = q\mathbf{E} + \frac{q}{\mathrm{c}} \mathbf{v} \times \mathbf{B}, \ \text{and} \ \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\mathbf{v} \cdot \mathbf{E}, \ \text{where} \ \mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \ \mathbf{B} = \nabla \times \mathbf{A}, \tag{10.1.17}$$

where $\mathbf{F}$ is the *Lorentz force*, and $\mathbf{E}$ and $\mathbf{B}$ are respectively the electric field and the magnetic induction.

The nonrelativistic version of the above Lagrangian in view of (10.1.12) is, [Goldstein, Section 1.5]

$$L_{\mathrm{p}} = \frac{m\dot{\mathbf{r}}^2}{2} - q\varphi\left(\mathbf{r}\right) + \frac{q}{\mathrm{c}}\mathbf{A}\left(\mathbf{r}\right)\cdot\dot{\mathbf{r}},\ \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}. \tag{10.1.18}$$

The corresponding ordinary kinetic momentum $\mathbf{p}$, the canonical (conjugate) momentum $\mathring{\mathbf{p}}$, and the Hamiltonian H_{p} are defined by the following relations

$$\mathbf{p} = m\dot{\mathbf{r}},\ \mathring{\mathbf{p}} = \mathbf{p} + \frac{q}{\mathrm{c}}\mathbf{A},\ H_{\mathrm{p}} = \mathring{\mathbf{p}}\cdot\dot{\mathbf{r}} - L = \frac{m\dot{\mathbf{r}}^2}{2} + q\varphi. \tag{10.1.19}$$

The canonical Euler-Lagrange equations for the nonrelativistic Lagrangian (10.1.18) take the form

$$\frac{\mathrm{d}\mathring{\mathbf{p}}}{\mathrm{d}t} = -q\nabla\varphi + \frac{q}{\mathrm{c}}\left[D\mathbf{A}\right]\dot{\mathbf{r}},\ \left[D\mathbf{A}\right]_j = \frac{\partial}{\partial x^j}\mathbf{A},\ j = 1,2,3, \tag{10.1.20}$$

or, if use $\mathring{\mathbf{p}} = \mathbf{p} + \frac{q}{\mathrm{c}}\mathbf{A}$ and the identity

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\partial\mathbf{A}}{\partial t} + \left[D\mathbf{A}\right]\dot{\mathbf{r}}, \tag{10.1.21}$$

we can recast (10.1.20) as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}^2t} = \mathbf{F} = q\mathbf{E} + \frac{q}{\mathrm{c}}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\times\mathbf{B}, \tag{10.1.22}$$

where the right-hand side of the equation (10.1.22) is the Lorentz force. Importantly for what we study in this paper the canonical Euler-Lagrange equation (10.1.20) involves the canonical momentum $\mathring{\mathbf{p}}$ and the canonical force $-q\nabla\varphi + \frac{q}{\mathrm{c}}\left[D\mathbf{A}\right]\dot{\mathbf{r}}$ which manifestly depend on the EM potentials φ and $\mathbf{A}$ rather than EM fields $\mathbf{E}$ and $\mathbf{B}$. Consequently, the equation (10.1.20) involves quantities which are not directly measurable in contrast to the equivalent to it equation (10.1.22) which is gauge invariant and involves measurable quantities, namely the kinematic momentum $\mathbf{p}$ and the Lorentz force $q\mathbf{E} + \frac{q}{\mathrm{c}}\dot{\mathbf{r}}\times\mathbf{B}$.

10.2 Lagrangians, field equations and conserved quantities

In this section we collect basic well known facts on the Lagrangian formalism for classical fields following to [Barut, Section III.3], [Morse Feshbach 1, Section 3.4], [Pauli RFTh] and other classical sources. Let us assume that physical systems of interest are described by fields real-valued $q^{\ell}\left(x\right)$, $\ell = 1,\ldots,N$, with the Lagrangian density

$$\mathcal{L}(\left\{q^{\ell}\left(x\right)\right\},\left\{q^{\ell}_{,\mu}\left(x\right)\right\},x),\ q^{\ell}_{,\mu}\left(x\right) = \partial_{\mu}q^{\ell}\left(x\right),\ \mu = 0,1,2,3, \tag{10.2.1}$$

According to the Lagrangian formalism the field equation are derived from the variational principle

$$\delta\int_R \mathcal{L}\,\mathrm{d}x = 0 \tag{10.2.2}$$

where R is four-dimensional *space-like region* with a three-dimensional boundary ∂R. Importantly, the variation δ is such that δq^{ℓ} vanish on the boundary ∂R. Then the corresponding Euler-Lagrange field equations take the form

$$\Lambda_{\ell} = \frac{\partial\mathcal{L}}{\partial q^{\ell}} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial q^{\ell}_{,\mu}} = 0. \tag{10.2.3}$$

The equation (10.2.3) can be recast in a Hamiltonian form as

$$\frac{\partial \pi_\ell}{\partial x_0} = F_\ell,\ \pi_\ell = \frac{\partial \mathcal{L}}{\partial q^\ell_{,0}},\ F_\ell = \frac{\partial \mathcal{L}}{\partial q^\ell} - \sum_{\mu=1}^{3} \partial_\mu \frac{\partial \mathcal{L}}{\partial q^\ell_{,\mu}},\ x_0 = \mathrm{ct}, \tag{10.2.4}$$

where π_ℓ is interpreted as the *canonical momentum density* of the field q^ℓ and F_ℓ is the canonical force density acting upon the field. The term $\frac{\partial L}{\partial q^\ell}$ of the canonical force density F_ℓ has to do with external forces acting on the field q^ℓ. *In the view of the last remark, if the Lagrangian L involves a nonlinear terms $G_\ell\left(q^\ell\right)$ its derivative $\frac{\partial G}{\partial q^\ell}$ can be interpreted as a self-force.*

In our considerations the fields $q^\ell(x)$ describe elementary charges (and in fact they are complex-valued) and the potential 4-vector field A_μ describes the classical EM field. The extension of the Lagrangian formalism to complex-valued in considered in a following section.

The *canonical energy-momentum tensor* (also called *stress-energy tensor* or *stress-tensor*) is defined by the following formula, [Barut, (3.63)], [LandauLif F, Section 32, (32.5)], [Goldstein, Section 13.3, (13.30)], [Schwabl, Section 12.4, (12.4.1)]

$$\mathring{T}^{\mu\nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{,\mu}} q^{\ell,\nu} - g^{\mu\nu}\mathcal{L}, \tag{10.2.5}$$

with the energy conservation laws in the form, [Barut, (3.94)], [Goldstein, (13.28)]

$$\partial_\mu \mathring{T}^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_\nu}. \tag{10.2.6}$$

Notice that in [Morse Feshbach 1, (3.4.2)], [Goldstein, Section 13.3] the canonical energy-momentum tensor is defined as matrix-transposed to $\mathring{T}^{\mu\nu}$ defined by (10.2.5), namely $\mathring{T}^{\mu\nu} \to \mathring{T}^{\nu\mu}$. The conservation laws for the energy-momentum are examples of conservation laws obtained from the Noether's theorem considered below in Section 10.3.

In the case of the Lagrangian $\mathcal{L}$ does not depend explicitly on x_ν, in other words invariant under translations $x_\nu \to x_\nu + a_\nu$, $\frac{\partial L}{\partial x_\nu} = 0$ and the conservation laws (10.2.6) turn into the following *continuity equations*,

$$\partial_\mu \mathring{T}^{\mu\nu} = 0. \tag{10.2.7}$$

A typical situation when the general conservation laws (10.2.6) apply rather than (10.2.7) is the presence of external forces which can "drive" our field. For instance, an external EM field driving a charge or an external "imposed" current which becomes a source for the EM field.

We would like to remind the reader that the canonical tensor of energy-momentum defined by (10.2.5) is not the only one that satisfies the conservation laws (10.2.6) or (10.2.7). For instance, any tensor of the form, [Barut, (3,73)], [LandauLif F, (32.7)], [Pauli RFTh, (14)]

$$T^{\mu\nu} = \mathring{T}^{\mu\nu} - \partial_\gamma f^{\mu\gamma\nu},\ \text{where } f^{\mu\gamma\nu} = -f^{\gamma\mu\nu}, \tag{10.2.8}$$

would satisfy (10.2.7) as long as $\mathring{T}^{\mu\nu}$ does. In view of the (10.2.8) the energy-momentum tensor $T^{\mu\nu}$ satisfy the same conservation laws (10.2.6) or (10.2.7) as $\mathring{T}^{\mu\nu}$, namely

$$\partial_\mu T^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial x_\nu}, \tag{10.2.9}$$

or, if the Lagrangian L does not depend explicitly on x_ν and, hence, invariant under time and space translations, the above conservation laws turn into

$$\partial_\mu T^{\mu\nu} = 0. \tag{10.2.10}$$

In fact, this flexibility in choosing the energy-momentum can be used to define $f^{\gamma\mu\nu}$ and construct a *symmetric energy-momentum tensor* $T^{\mu\nu}$, i.e. $T^{\mu\nu} = T^{\nu\mu}$, which is a necessary and sufficient condition for the field angular momentum density to be represented by the usual formula in terms of the field momentum density, [LandauLif F, Section 32], [Barut, Section III.4]. The symmetry of the energy-momentum tensor for matter fields is also fundamentally important since it is a source for the gravitational field, [Nair, Section 3.8], [Misner, Section 5.7]. As to the uniqueness of the energy-momentum we can quote [Misner, Section 21.3]: "... the theory of gravity in the variational formulation gives a unique prescription for fixing the stress-energy tensor, a prescription that, besides being symmetric, also automatically satisfies the laws of conservation of momentum and energy". This unique form of the symmetric energy-momentum can be derived based on a variational principle involving charge of boundary, with varied boundary, [Barut, Section III.3(B)], and under the following assumptions: (i) the Lagrangian does not depend explicitly on x; (ii) the fields $q^\ell(x)$ satisfy the fields equations (10.2.3); (iii) the fields vanish at the spacial infinity sufficiently fast. The result is a symmetric *Belinfante-Rosenfeld energy-momentum tensor* $T^{\mu\nu}$, [Barut, (3.73)-(3.75)], [Pauli RFTh, (13a), (13b), (13c), (14)], [Belinfante1], [Belinfante2], [Rosenfeld], namely

$$T^{\mu\nu} = \mathring{T}^{\mu\nu} - \partial_\gamma f^{\mu\gamma\nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{,\mu}} q^{\ell,\nu} - g^{\mu\nu}\mathcal{L} - \partial_\gamma f^{\mu\nu\gamma}, \text{ where} \tag{10.2.11}$$

$$f^{\mu\gamma\nu} = -f^{\gamma\mu\nu} = \frac{1}{2}\left[\sum_{\ell\ell'}\left(L^\mu_\ell S^{\ell\gamma\nu}_{\ell'} + L^\gamma_\ell S^{\ell\nu\mu}_{\ell'} - L^\nu_\ell S^{\ell\mu\gamma}_{\ell'}\right) q^{\ell'}\right], \; L^\mu_\ell = \frac{\partial \mathcal{L}}{\partial q^\ell_{,\mu}},$$

where the tensor $S^{\ell\mu\nu}_{\ell'}$ describes the infinitesimal transformation of the involved fields $q^\ell(x) \to q'^\ell(x')$ along with the infinitesimal inhomogeneous Lorentz transformation of the space time vector $x \to x'$ as described by (10.1.10), namely,

$$x'^\mu = x^\mu + \xi^{\mu\nu} x_\nu + a^\mu, \; \xi^{\mu\nu} = -\xi^{\nu\mu}, \tag{10.2.12}$$

where $\xi^{\mu\nu}$ and a^μ are the ten parameters, and

$$q'^\ell(x') = q^\ell(x) + \xi_{\mu\nu} \sum_{\ell'} S^{\ell\mu\nu}_{\ell'} q^{\ell'}(x), \; S^{\ell\mu\nu}_{\ell'} = -S^{\ell\nu\mu}_{\ell'}. \tag{10.2.13}$$

In particular, [Barut, III.4(A)],

$$S^{\ell\mu\nu}_{\ell'} = 0 \text{ if } q \text{ is a scalar field,} \tag{10.2.14}$$

$$S^{\alpha\mu\nu}_{\beta} = g^{\alpha\mu} g^\nu_\beta - g^{\alpha\nu} g^\mu_\beta \text{ if } q \text{ is a vector field.} \tag{10.2.15}$$

The conserved quantities are, [Barut, (3.76)-(3.77)], [Pauli RFTh, (6)]

$$P^\nu = \int_\sigma T^{\mu\nu}\, \mathrm{d}\sigma_\mu, \; J^{\nu\gamma} = \int_\sigma M^{\mu\nu\gamma}\, \mathrm{d}\sigma_\mu, \; M^{\mu\nu\gamma} = T^{\mu\nu} x^\gamma - T^{\mu\gamma} x^\nu, \tag{10.2.16}$$

where σ is any space-like surface, for instance $x_0 = \text{const}$. P^ν is *four-vector of the total energy-momentum* and $J^{\nu\gamma} = -J^{\gamma\nu}$ is *the total angular momentum tensor*. The differential form of the conservation laws is

$$\partial_\mu T^{\mu\nu} = 0,\ \partial_\mu M^{\mu\nu\gamma} = T^{\gamma\nu} - T^{\nu\gamma} = 0. \tag{10.2.17}$$

Observe that the conservation of the angular momentum $M^{\mu\nu\gamma}$ in (10.2.17) implies the symmetry of the energy-momentum tensor and, in view of (10.2.11), the following identities, [Barut, (3.81′)],

$$T^{\mu\nu} = T^{\nu\mu},\ \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{,\mu}} q^{\ell,\nu} - \partial_\gamma f^{\mu\gamma\nu} = \sum_\ell \frac{\partial \mathcal{L}}{\partial q^\ell_{,\nu}} q^{\ell,\mu} - \partial_\gamma f^{\nu\gamma\mu}. \tag{10.2.18}$$

For an alternative insightful derivation of the symmetric energy-momentum tensor based on kinosthenic (ignorable) variables and the Noether's method as a way to generate such variables we refer to [Lanczos VP, Section 3.5, 3.6, 3.10]. Interestingly under this approach the conservations laws take the form of the Euler-Lagrange equations for those kinosthenic variables. *We would like to point out that the symmetry of the energy-momentum tensor and the corresponding identities (10.2.18) are nontrivial relations which hold provided that the involved fields satisfy the field equations (10.2.3).*

Since the symmetric energy-momentum tensor $T^{\mu\nu}$ *is the one used in most of the cases we often refer to it just as the energy-momentum tensor,* while the tensor $\mathring{T}^{\mu\nu}$ defined by (10.2.5) is referred to as the *canonical energy-momentum tensor*.

The interpretation of the symmetric energy-momentum tensor $T^{\mu\nu}$ entries is as follows, [LandauLif F, Section 32], [Goldstein, Sections 13.2, 13.3], [Morse Feshbach 1, Chapter 3.4]

$$T^{\mu\nu} = \begin{bmatrix} u & \mathrm{c}p_1 & \mathrm{c}p_2 & \mathrm{c}p_3 \\ \mathrm{c}^{-1}s_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ \mathrm{c}^{-1}s_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ \mathrm{c}^{-1}s_3 & -\sigma_{31} & -\sigma^{32} & -\sigma_{33} \end{bmatrix}, \tag{10.2.19}$$

where

u	field energy density,
$p_j,\ j = 1, 2, 3,$	field momentum density,
$s_j = \mathrm{c}^2 p_j,\ j = 1, 2, 3,$	field energy flux density,
$\sigma_{ij} = \sigma_{ji},\ j = 1, 2, 3,$	field symmetric stress tensor.

(10.2.20)

If we introduce the 4-momentum and 4-flux densities by the formulas

$$p^\nu = \frac{1}{\mathrm{c}} T^{0\nu} = \left(\frac{1}{\mathrm{c}} u, \mathbf{p}\right),\ s^\nu = \mathrm{c} T^{\nu 0} = (\mathrm{c}u, \mathbf{s}).$$

then the energy-momentum conservation law (10.2.9) can be recast as follows

$$\partial_t p^\nu + \sum_{j=1,2,3} \partial_j T^{j\nu} = -\frac{\partial \mathcal{L}}{\partial x_\nu}, \tag{10.2.21}$$

or, in other words,

$$\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji} - \frac{\partial \mathcal{L}}{\partial x^i},\ \text{where } p_i = \frac{1}{\mathrm{c}} T^{0i},\ \sigma_{ji} = -T^{ji},\ i, j = 1, 2, 3, \tag{10.2.22}$$

$$\partial_t u + \sum_{j=1,2,3} \partial_j s_j = -\frac{\partial \mathcal{L}}{\partial t}, \text{ where } u = T^{00},\ s_i = \mathrm{c} T^{i0} = \mathrm{c}^2 p_i,\ i = 1,2,3. \tag{10.2.23}$$

If the Lagrangian $\mathcal{L}$ does not depend explicitly on x_ν the above energy-momentum conservation laws turn into

$$\partial_t p_i = \sum_{j=1,2,3} \partial_j \sigma_{ji}, \text{ where } p_i = \frac{1}{\mathrm{c}} T^{0i},\ \sigma_{ji} = -T^{ji},\ i,j = 1,2,3, \tag{10.2.24}$$

$$\partial_t u + \sum_{j=1,2,3} \partial_j s_j = 0, \text{ where } u = T^{00},\ s_i = \mathrm{c} T^{i0} = \mathrm{c}^2 p_i,\ i = 1,2,3. \tag{10.2.25}$$

Consequently, the *total conserved energy momentum* 4-vector takes the form

$$P^\nu = \frac{1}{\mathrm{c}} \int_{\mathbb{R}^3} T^{0\nu}(x)\ \mathrm{d}\mathbf{x}, \tag{10.2.26}$$

and its components, the total energy and momentum are respectively

$$H = \mathrm{c} P^0 = \int_{\mathbb{R}^3} T^{0\nu}(x)\ \mathrm{d}\mathbf{x},\ P^j = \int_{\mathbb{R}^3} T^{0j}(x)\ \mathrm{d}\mathbf{x},\ j = 1,2,3. \tag{10.2.27}$$

Evidently, the formulas (10.2.26), (10.2.27) are particular cases of the formulas (10.2.16) for the special important choice of $\sigma = \{x = (x_0, \mathbf{x}) :\ \mathbf{x} \in \mathbb{R}^3\}$.

Importantly, *for closed systems the conserved total energy-momentum P^ν and $J^{\nu\gamma}$ angular momentum as defined by formulas (10.2.16) and (10.2.26) transform respectively as 4-vector and 4-tensor* under Lorentz transformation and that directly related to the conservations laws (10.2.17), [Moller, Section 6.2], [Jackson, Section 12.10 A]. But *for open (not closed) systems generally the total energy-momentum P^ν and $J^{\nu\gamma}$ angular momentum do not transform as respectively 4-vector and 4-tensor*, [Moller, Section 7.1, 7.2], [Jackson, Section 12.10 A, 16.4].

10.3 Noether's theorem

In this section following to [Goldstein, Section 13.7] we provide basic information on the Noether's theorem which relates symmetries to conservation laws based on the Lagrangian formalism. Suppose that the Lagrangian $\mathcal{L}$ as defined (10.2.1) does not depend explicitly on the field variable q^ℓ. Then the Euler-Lagrange equations (10.2.3) for that variable turns into

$$\partial_\mu J^\mu_\ell = 0,\ J^\mu_\ell = \frac{\partial \mathcal{L}}{\partial q^\ell_{,\mu}}, \tag{10.3.1}$$

which is a conservation law (continuity equation) for the four-"current" J^μ_ℓ. Noether theory interprets the above situation as certain invariance (symmetry) of the Lagrangian and provides its far reaching generalization allowing to obtain conservation laws based on the Lagrangian invariance (symmetry) with respect to a general Lie group of transformation. Symmetry under coordinate transformation refers to the effects of infinitesimal transformation of the form

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \tag{10.3.2}$$

where the infinitesimal change δx^μ may depend on other x^ν. The field transformations are assumed to be of the form

$$q^\ell(x) \to q'^\ell(x') = q^\ell(x) + \delta q^\ell(x), \tag{10.3.3}$$

where $\delta q^{\ell}(x)$ measures to *total* change of q^{ℓ} due to both x and q^{ℓ} and it can depend on other field variables q^{ℓ_1}. We also consider a *local* change $\bar{\delta} q^{\ell}(x)$ of $q^{\ell}(x)$ at a point x

$$q'^{\ell}(x) = q^{\ell}(x) + \bar{\delta} q^{\ell}(x) \tag{10.3.4}$$

We assume the following three conditions to hold for the transformations (10.3.2)-(10.3.3): (i) the 4-space (time and the space) is flat; (ii) the Lagrangian density $\mathcal{L}$ displays the same functional form in terms of the transformed quantities as it does of the original quantities, that is,

$$\mathcal{L}(\left\{q^{\ell}(x)\right\}, \left\{q^{\ell}_{,\mu}(x)\right\}, x) = \mathcal{L}(\left\{q'^{\ell}(x')\right\}, \left\{q'^{\ell}_{,\mu}(x')\right\}, x'); \tag{10.3.5}$$

(iii) The magnitude of the action integral is invariant under the transformation, that is

$$\int_{\nleq} \mathcal{L}(\left\{q^{\ell}(x)\right\}, \left\{q^{\ell}_{,\mu}(x)\right\}, x)\,\mathrm{d}x = \int_{\nleq'} \mathcal{L}(\left\{q'^{\ell}(x')\right\}, \left\{q'^{\ell}_{,\mu}(x')\right\}, x')\,\mathrm{d}x', \tag{10.3.6}$$

where $\nleq$ is 4-dimensional region bounded by two space-like 3-dimensional surfaces and $\mathrm{d}x = \sqrt{|\det g|}\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$.

If the three above conditions are satisfied the following conservation law holds

$$\partial_{\mu} J^{\mu} = 0,\ \ J^{\mu} = \sum\nolimits_{\ell} \frac{\partial \mathcal{L}}{\partial q^{\ell}_{,\mu}} \bar{\delta} q^{\ell}(x) + \mathcal{L}\delta x^{\mu}. \tag{10.3.7}$$

The above conservation law can be made more specific if we introduce infinitesimal parameters ξ_r related to Lie group of transformations and represent δx^{μ} and δq^{ℓ} in their terms, namely

$$\delta x^{\mu} = \sum\nolimits_r X^{\mu}_r \xi_r,\ \ \delta q^{\ell} = \sum\nolimits_r Q^{\ell}_r \xi_r, \tag{10.3.8}$$

where the functions X^{μ}_r and Q^{ℓ}_r may depend upon the other coordinates and field variables, respectively. Then we for r the following conservation law holds for the respective Noether's current J^{μ}_r:

$$\partial_{\mu} J^{\mu}_r = 0,\ \ J^{\mu}_r = \left[\sum\nolimits_{\ell} \frac{\partial \mathcal{L}}{\partial q^{\ell}_{,\mu}} q^{\ell}_{,\nu} - \mathcal{L}\delta^{\mu}_{\nu}\right] X^{\nu}_r - \sum\nolimits_{\ell} \frac{\partial \mathcal{L}}{\partial q^{\ell}_{,\mu}} Q^{\ell}_r. \tag{10.3.9}$$

For instance, in the case of the group of inhomogeneous Lorentz transformations defined by (10.1.10) there are ten parameters a^{μ} and $\xi^{\mu\nu}$ and consequently there are ten corresponding conserved quantities $P^{\nu}$and $J^{\nu\gamma} = -J^{\gamma\nu}$ defined by (10.2.16). Another important example is the group of gauge transformation of the first kind defined by (10.5.7) below. For this group there is a conserved current $J^{\ell\nu}$ for every ℓ defined by (10.5.14).

10.4 Electromagnetic fields and the Maxwell equations

We consider the Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\varrho, \qquad \nabla \cdot \mathbf{B} = 0 \tag{10.4.1}$$

$$\nabla \times \mathbf{E} - \frac{1}{\mathrm{c}}\partial_t \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \frac{1}{\mathrm{c}}\partial_t \mathbf{E} = \frac{4\pi}{\mathrm{c}}\mathbf{J}. \tag{10.4.2}$$

for the EM fields and their covariant form following to [Jackson, Section 11.9], [LandauLif EM, Sections 23, 30], [Griffiths, Sections 7.4, 11.2], in CGS units. To represent Maxwell equations in a manifestly Lorentz invariant form it is common to introduce a four-vector potential A^μ and a four-vector current density J^ν

$$A^\mu = (\varphi, \mathbf{A}), \; J^\mu = (\mathrm{c}\varrho, \mathbf{J}), \; \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{\mathrm{c}}\partial_t, \nabla\right), \; \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{\mathrm{c}}\partial_t, -\nabla\right), \tag{10.4.3}$$

and, then, an antisymmetric second-rank tensor, the "field strength tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{10.4.4}$$

so that

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \; F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & -E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{10.4.5}$$

and

$$\mathbf{E} = -\nabla\varphi - \frac{1}{\mathrm{c}}\partial_t \mathbf{A}, \; \mathbf{B} = \nabla \times \mathbf{A}. \tag{10.4.6}$$

Then the two inhomogeneous equations and the two homogeneous equations from the four Maxwell equations (10.4.1) take respectively the form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{\mathrm{c}} J^\nu, \tag{10.4.7}$$

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \; \alpha, \beta, \gamma = 0, 1, 2, 3. \tag{10.4.8}$$

It follows from the asymmetry of $F^{\mu\nu}$, the Maxwell equation (10.4.7) and (10.4.3)-(10.4.4) that the four-vector current J^μ must satisfy the *continuity equation*

$$\partial_\mu J^\mu = 0 \text{ or } \partial_t \varrho + \nabla \cdot \mathbf{A} = 0. \tag{10.4.9}$$

The Maxwell equations (10.4.7) turn into the following equations for the four-vector potential A^μ

$$\Box A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{\mathrm{c}} J^\nu, \tag{10.4.10}$$

where

$$\Box = \partial_\mu \partial^\mu = \frac{1}{\mathrm{c}^2}\partial_t^2 - \nabla^2 \text{ (d'Alembertian operator).} \tag{10.4.11}$$

According to [Jackson, Section 11.10] the electric and magnetic fields are transformed from one frame to another one moving relatively with the velocity $\mathbf{v}$ by the following formulas

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1}(\boldsymbol{\beta} \cdot \mathbf{E})\boldsymbol{\beta}, \; \mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1}(\boldsymbol{\beta} \cdot \mathbf{B})\boldsymbol{\beta} \tag{10.4.12}$$

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{\mathrm{c}}, \; \beta = |\boldsymbol{\beta}|, \; \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{\mathrm{c}}\right)^2}}.$$

which also can be recast as, [Grainer EM, Section 22],

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}), \; \mathbf{E}'_\parallel = \mathbf{E}_\parallel, \; \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}), \; \mathbf{B}'_\parallel = \mathbf{B}_\parallel, \tag{10.4.13}$$

where subindices $\perp$ and $\parallel$ stand for vector components respectively parallel and perpendicular to $\mathbf{v}$. Observe that for $\beta \ll 1$ the formulas (10.4.12), (10.4.13) yield the following approximations with an error proportional to β^2 where J_μ is an external four-vector current.

$$\mathbf{E}'_\perp \cong \mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B},\ \mathbf{E}'_\parallel = \mathbf{E}_\parallel,\ \mathbf{B}'_\perp \cong \mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E},\ \mathbf{B}'_\parallel = \mathbf{B}_\parallel. \tag{10.4.14}$$

The EM field Maxwell Lagrangian is, [Jackson, Section 12.7], [Barut, Section IV.1]

$$L_{\mathrm{em}}(A^\mu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{\mathrm{c}} J_\mu A^\mu, \tag{10.4.15}$$

where J_μ is an external (impressed) current. Using (10.4.5), (10.4.6) and (10.4.3) we can recast (10.4.15) as

$$\begin{gathered} L_{\mathrm{em}}(A^\mu) = \frac{1}{8\pi}\left(\mathbf{E}^2 - \mathbf{B}^2\right) - \rho\varphi + \frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{A} \\ = \frac{1}{8\pi}\left[\left(\nabla\varphi + \frac{1}{\mathrm{c}}\partial_t \mathbf{A}\right)^2 - (\nabla\times\mathbf{A})^2\right] - \rho\varphi + \frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{A}. \end{gathered} \tag{10.4.16}$$

In particular, if there are no sources the above Lagrangians turn into

$$L_{\mathrm{em}}(A^\mu) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} = \frac{1}{8\pi}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \frac{1}{8\pi}\left[\left(\nabla\varphi + \frac{1}{\mathrm{c}}\partial_t \mathbf{A}\right)^2 - (\nabla\times\mathbf{A})^2\right]. \tag{10.4.17}$$

The canonical stress (power-momentum) tensor $\mathring{\Theta}^{\mu\nu}$ for the EM field is as follows, [Jackson, (12.104)], [Barut, Section III.4.D]

$$\mathring{\Theta}^{\mu\nu} = -\frac{F^{\mu\gamma}\partial^\nu A_\gamma}{4\pi} + g^{\mu\nu}\frac{F^{\xi\gamma}F_{\xi\gamma}}{16\pi}, \tag{10.4.18}$$

or, in particular,

$$\begin{gathered} \mathring{\Theta}^{00} = -\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} + \rho\varphi - \frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{A} - \frac{\partial_0 \mathbf{A}\cdot\mathbf{E}}{4\pi},\ \mathring{\Theta}^{0i} = -\frac{\partial_i \mathbf{A}\cdot\mathbf{E}}{4\pi}, \\ \mathring{\Theta}^{i0} = -\frac{E_i \partial_0 \varphi}{4\pi} + \frac{(\mathbf{B}\times\partial_0\mathbf{A})_i}{4\pi},\ i = 1,2,3, \\ \mathring{\Theta}^{ij} = -\frac{E_i\partial_j\varphi}{4\pi} + \frac{(\mathbf{B}\times\partial_j\mathbf{A})_i}{4\pi} + \frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\varphi + \frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{A},\ i,j = 1,2,3, \end{gathered} \tag{10.4.19}$$

whereas the symmetric one $\Theta^{\alpha\beta}$ for the EM field is, [Jackson, Section 12.10, (12.113)], [Barut, Section III.3]

$$\Theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu}F_{\mu\nu}F^{\nu\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right), \tag{10.4.20}$$

implying the following formulas for the field energy density w, the momentum $\mathrm{c}\mathbf{g}$ and the Maxwell stress tensor τ_{ij}:

$$w = \Theta^{00} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi},\ \mathrm{c}g_i = \Theta^{0i} = \Theta^{i0} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi},\ i = 1,2,3, \tag{10.4.21}$$

$$\Theta^{ij} = -\frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(\mathbf{E}^2 + \mathbf{B}^2\right)\right],\ i,j = 1,2,3, \tag{10.4.22}$$

$$\begin{aligned}\Theta^{\alpha\beta} &= \begin{bmatrix} w & \mathrm{c}\mathbf{g} \\ \mathrm{c}\mathbf{g} & -\tau_{ij}\end{bmatrix}, \ \Theta_{\alpha\beta} = \begin{bmatrix} w & -\mathrm{c}\mathbf{g} \\ -\mathrm{c}\mathbf{g} & -\tau_{ij}\end{bmatrix}, \\ \Theta^{\alpha}{}_{\beta} &= \begin{bmatrix} w & -\mathrm{c}\mathbf{g} \\ \mathrm{c}\mathbf{g} & -\tau_{ij}\end{bmatrix}, \ \Theta_{\alpha}{}^{\beta} = \begin{bmatrix} w & \mathrm{c}\mathbf{g} \\ -\mathrm{c}\mathbf{g} & -\tau_{ij}\end{bmatrix}.\end{aligned} \tag{10.4.23}$$

Note that in the special case when the vector potential $\mathbf{A}$ vanishes and the scalar potential φ does not depend on time using the expressions (10.4.6) we get the following representation for the canonical energy density defined by (10.4.19)

$$\mathring{\Theta}^{00} = -\frac{(\nabla\varphi)^2}{8\pi} + \rho\varphi \text{ for } \mathbf{A} = \mathbf{0} \text{ and } \partial_0\varphi = 0, \text{ whereas } \Theta^{00} = \frac{(\nabla\varphi)^2}{8\pi}. \tag{10.4.24}$$

It is instructive to observe a substantial difference between the above expressions $\mathring{\Theta}^{00}$, which is the Hamiltonian density of the EM field, and the energy density Θ^{00} defined by (10.4.21).

If there no external currents the with differential *conservation law* takes the form

$$\partial_\alpha \Theta^{\alpha\beta} = 0, \tag{10.4.25}$$

and, in particular, the energy conservation law

$$0 = \partial_\alpha \Theta^{\alpha\beta} = \frac{1}{\mathrm{c}}\left(\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S}\right), \text{ where } u \text{ is the energy density, and} \tag{10.4.26}$$
$$\mathbf{S} = \mathrm{c}^2\mathbf{g} = \frac{\mathrm{c}}{4\pi}\mathbf{E}\times\mathbf{B} \text{ is the Poynting vector.}$$

In the presence of external currents the conservation laws take the form, [Jackson, Section 12.10]

$$\partial_\alpha \Theta^{\alpha\beta} = -f^\beta, \ f^\beta = \frac{1}{\mathrm{c}} F^{\beta\nu} J_\nu, \tag{10.4.27}$$

and the time and space components of the equations (10.4.27) are the conservation of energy u and momentum which can be recast as

$$\frac{1}{\mathrm{c}}\left(\frac{\partial w}{\partial t} + \nabla\cdot\mathbf{S}\right) = -\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}, \tag{10.4.28}$$

$$\frac{\partial g_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x^j}\tau_{ij} = -\left[\rho E_i + \frac{1}{\mathrm{c}}(\mathbf{J}\times\mathbf{B})_i\right]. \tag{10.4.29}$$

The 4-vector f^β in the conservation law (10.4.27) is known as the *Lorentz force density*

$$f^\beta = \frac{1}{\mathrm{c}} F^{\beta\nu} J_\nu = \left(\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}, \rho\mathbf{E} + \frac{1}{\mathrm{c}}\mathbf{J}\times\mathbf{B}\right). \tag{10.4.30}$$

10.4.1 Green functions for the Maxwell equations

This section in an excerpt from [Jackson, Section 12.11]. The EM fields $F^{\mu\nu}$ arising from an external source J^ν satisfy the inhomogeneous Maxwell equations

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{\mathrm{c}} J^\nu, \ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{10.4.31}$$

which take the following form for the potentials A^ν

$$\Box A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{\mathrm{c}} J^\nu. \tag{10.4.32}$$

If the potentials satisfy the Lorentz condition, $\partial_\mu A^\mu = 0$, they are then solutions of the four-dimensional wave equation,

$$\Box A^\nu = \frac{4\pi}{\mathrm{c}} J^\nu \tag{10.4.33}$$

The solution of (10.4.33) is accomplished by finding a Green function $D(x, x')$ for the equation

$$\Box D(z) = \delta^{(4)}(z), \;\; D(x, x') = D(x - x'), \tag{10.4.34}$$

where $\delta^{(4)}(z) = \delta(z_0)\,\delta(\mathbf{z})$ is a four-dimensional delta function. One can introduce then the so-called *retarded or causal Green function* solving the above equation (10.4.34), namely

$$D_{\mathrm{r}}(x - x') = \frac{\theta(x_0 - x_0')\,\delta(x_0 - x_0' - R)}{4\pi R}, \;\; R = |\mathbf{x} - \mathbf{x}'|, \tag{10.4.35}$$

where $\theta(x_0)$ is the Heaviside step function. The name causal or retarded is justified by the fact that the source-point time x_0' is always earlier then the observation-point time x_0. Similarly one can introduce the *advanced Green function*

$$D_{\mathrm{a}}(x - x') = \frac{\theta[-(x_0 - x_0')]\,\delta(x_0 - x_0' + R)}{4\pi R}, \;\; R = |\mathbf{x} - \mathbf{x}'|. \tag{10.4.36}$$

These Green functions can be written in the following covariant form

$$\begin{aligned} D_{\mathrm{r}}(x - x') &= \frac{1}{2\pi}\theta(x_0 - x_0')\,\delta\left[(x - x')^2\right], \\ D_{\mathrm{a}}(x - x') &= \frac{1}{2\pi}\theta(x_0' - x_0)\,\delta\left[(x - x')^2\right], \end{aligned} \tag{10.4.37}$$

where $(x - x')^2 = (x_0 - x_0')^2 - |\mathbf{x} - \mathbf{x}'|^2$ and

$$\delta\left[(x - x')^2\right] = \frac{1}{2R}\left[\delta(x_0 - x_0' - R) + \delta(x_0 - x_0' + R)\right]. \tag{10.4.38}$$

The solution to the wave equation (10.4.33) can be written in terms of the Green functions

$$A^\nu(x) = A^\nu_{\mathrm{in}}(x) + \frac{4\pi}{\mathrm{c}}\int D_{\mathrm{r}}(x - x')\,J^\nu(x')\,\mathrm{d}x \tag{10.4.39}$$

or

$$A^\nu(x) = A^\nu_{\mathrm{out}}(x) + \frac{4\pi}{\mathrm{c}}\int D_{\mathrm{a}}(x - x')\,J^\nu(x')\,\mathrm{d}x \tag{10.4.40}$$

where $A^\nu_{\mathrm{in}}(x)$ and $A^\nu_{\mathrm{out}}(x)$ are solutions to the homogeneous wave equation. In (10.4.39) the retarded Green function is used. In the limit $x_0 \to -\infty$, the integral over the sources vanishes, assuming the sources are localized in space and time, because of the retarded nature of the Green function, and $A^\nu_{\mathrm{in}}(x)$ can be interpreted as "*incident*" or "*incoming*" potential, specified at $x_0 \to -\infty$. Similarly, in (10.4.40) with the advanced Green function, the homogeneous solution $A^\nu_{\mathrm{out}}(x)$ is the asymptotic "*outgoing*" potential, specified at $x_0 \to +\infty$. The radiation fields are defined as the difference between the "outgoing" and "incoming" fields, and their 4-vector potential is

$$A^\nu_{\mathrm{rad}}(x) = A^\nu_{\mathrm{out}}(x) - A^\nu_{\mathrm{in}}(x) = \frac{4\pi}{\mathrm{c}}\int D(x - x')\,J^\nu(x')\,\mathrm{d}x, \;\text{ where} \tag{10.4.41}$$

$$D(x - x') = D_{\mathrm{r}}(x - x') - D_{\mathrm{a}}(x - x').$$

10.5 Many charges interacting with the electromagnetic field

In this section we consider the Lagrange formalism for complex-valued fields ψ^{ℓ}, $\ell = 1, \ldots, N$ that describe charges following to [Pauli RFTh, Part I] and [Wentzel, Section I.3]. For every complex-valued ψ^{ℓ} we always assume the presence of its conjugates $\psi^{\ell *}$, so the product $\psi^{\ell}\psi^{\ell *}$ is real. The Lagrangian is assumed to be real valued and its general form is

$$\mathcal{L} = \mathcal{L}\left(\left\{\psi^{\ell}, \psi^{\ell}_{;\mu}, \psi^{\ell *}, \psi^{\ell *}_{;\mu}, \right\}, \left\{V^{g}, V^{g}_{,\mu}\right\}, x^{\mu}\right), \tag{10.5.1}$$

where $\{V^{g}\}$ are real-valued quantities. In the Lagrangian (10.5.1) the fields ψ^{ℓ} and their conjugates $\psi^{*\ell}$ are treated as independent and the corresponding Euler-Lagrange field equations are, [Morse Feshbach 1, Section 3.3], [Wentzel, Section II.3, (3.3)],

$$\frac{\partial \mathcal{L}}{\partial \psi^{\ell}} - \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial \psi^{\ell}_{,\mu}}\right) = 0,\ \frac{\partial \mathcal{L}}{\partial \psi^{\ell *}} - \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial \psi^{\ell *}_{,\mu}}\right) = 0,\ \frac{\partial \mathcal{L}}{\partial V^{g}} - \partial_{\mu}\left(\frac{\partial \mathcal{L}}{\partial V^{g}_{,\mu}}\right) = 0. \tag{10.5.2}$$

The canonical energy-momentum tensor for the Lagrangian (10.5.1) is similar to the general formula (10.2.5), namely

$$\mathring{\mathcal{T}}^{\mu\nu} = \sum_{\ell} \frac{\partial \mathcal{L}}{\partial \psi^{\ell}_{,\mu}} \psi^{\ell,\nu} + \frac{\partial \mathcal{L}}{\partial \psi^{\ell *}_{,\mu}} \psi^{\ell *,\nu} + \sum_{g} \frac{\partial \mathcal{L}}{\partial V^{g}_{,\mu}} V^{g,\nu} - g^{\mu\nu}\mathcal{L}. \tag{10.5.3}$$

In the case when the Lagrangian $\mathcal{L}$ depends on only complex-valued fields ψ^{ℓ} and $\psi^{\ell *}$ the canonical stress tensor is symmetric and is of the form, [Morse Feshbach 1, (3.3.23)], [Wentzel, (3.8)],

$$\mathcal{T}^{\mu\nu} = \mathring{\mathcal{T}}^{\mu\nu} = \sum_{\ell} \frac{\partial \mathcal{L}}{\partial \psi^{\ell}_{,\mu}} \psi^{\ell,\nu} + \frac{\partial \mathcal{L}}{\partial \psi^{\ell *}_{,\mu}} \psi^{\ell *,\nu} - g^{\mu\nu}\mathcal{L}. \tag{10.5.4}$$

An important for us special case of the Lagrangian (10.5.1) is when there are several charges described by complex valued fields ψ^{ℓ} and $\psi^{\ell *}$ interacting with the EM field described by the real-valued four-potential A^{μ}. For this case we introduce the Lagrangian of the form

$$\mathcal{L}\left(\left\{\psi^{\ell}, \psi^{\ell}_{;\mu}, \psi^{\ell *}, \psi^{\ell *}_{;\mu}, \right\}, A^{\mu}\right) = \sum_{\ell} L^{\ell}\left(\psi^{\ell}, \psi^{\ell}_{;\mu}, \psi^{\ell *}, \psi^{\ell *}_{;\mu}\right) - \frac{F^{\mu\nu}F_{\mu\nu}}{16\pi}, \tag{10.5.5}$$
$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}.$$

where we have introduced the so-called *covariant derivatives* $\psi^{\ell}_{;\mu}$ and $\psi^{\ell *}_{;\mu}$ by the following formulas

$$\psi^{\ell;\mu} = \tilde{\partial}^{\ell\mu}\psi^{\ell},\ \psi^{\ell;\mu *} = \tilde{\partial}^{\ell\mu *}\psi^{\ell},\ \tilde{\partial}^{\ell\mu} = \partial^{\mu} + \frac{\mathrm{i}q^{\ell}}{\chi \mathrm{c}}A^{\mu},\ \tilde{\partial}^{\ell\mu *} = \partial^{\mu} - \frac{\mathrm{i}q^{\ell}}{\chi \mathrm{c}}A^{\mu}. \tag{10.5.6}$$

In the above formula for every ℓ the real number q^{ℓ} is the charge of the ℓ-th elementary charge and $\tilde{\partial}^{\ell\mu}$ and $\tilde{\partial}^{\ell\mu *}$ are called the *covariant differentiation operators*. The particular forms (10.5.5)-(10.5.6) of the multiparticle Lagrangian $\mathcal{L}$ and its ℓ-the charge components L^{ℓ} originates from the condition of *gauge invariance*. More precisely, one introduces the *gauge transformation of the first or the second kind* (known also as respectively *global and local gauge transformation*) for the fields ψ^{ℓ} and $\psi^{\ell *}$. These transformations are described respectively by the following formulas namely, [Pauli RFTh, (17), (23a), (23b)], [Wentzel, Section 11, (11.4)],

$$\psi^{\ell} \rightarrow \mathrm{e}^{\mathrm{i}\gamma^{\ell}} \psi^{\ell}, \ \psi^{\ell *} \rightarrow \mathrm{e}^{-\mathrm{i}\gamma^{\ell}} \psi^{\ell *}, \text{ where } \gamma^{\ell} \text{ is any real constant,} \tag{10.5.7}$$

$$\psi^{\ell} \rightarrow \mathrm{e}^{-\frac{\mathrm{i}q^{\ell}\lambda(x)}{\chi \mathrm{c}}} \psi^{\ell}, \ \psi^{\ell *} \rightarrow \mathrm{e}^{\frac{\mathrm{i}q^{\ell}\lambda(x)}{\chi \mathrm{c}}} \psi^{\ell *}, \ A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\lambda, \tag{10.5.8}$$

and the Lagrangian is assumed to be invariant with respect to the all gauge transformations (10.5.7), (10.5.8). Notice that for the multi-charge Lagrangian $\mathcal{L}$ as defined by (10.5.5)-(10.5.6) the following is true: (i) every charge interacts with the EM field described by the four-potential A^{μ}; (ii) different charges don't interact directly with each other, but they interact only indirectly through the EM field.

We also introduce the following symmetry condition on charges Lagrangians L^{ℓ}:

$$\frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;\mu}} \psi^{\ell;\nu} + \frac{\partial L^{\ell}}{\partial \psi^{\ell *}_{;\mu}} \psi^{\ell;\nu *} = \frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;\nu}} \psi^{\ell;\mu} + \frac{\partial L^{\ell}}{\partial \psi^{\ell *}_{;\nu}} \psi^{\ell;\mu *}. \tag{10.5.9}$$

As we show below the symmetry condition (10.5.9) implies that energy-momentum assigned to every individual charge is symmetric and gauge invariant energy-momentum. A simple sufficient condition for the symmetry condition (10.5.9) is a requirement for the Lagrangians L^{ℓ} to depend on the field covariant derivatives only through the combination $\psi^{\ell;\mu}\psi^{\ell *}_{;\mu}$, in other words if there exist such functions $K^{\ell}\left(\psi^{\ell}, \psi^{\ell *}, b\right)$ that

$$L^{\ell}\left(\psi^{\ell}, \psi^{\ell}_{;\mu}, \psi^{\ell *}, \psi^{\ell *}_{;\mu}\right) = K^{\ell}\left(\psi^{\ell}, \psi^{\ell *}, \psi^{\ell;\mu}\psi^{\ell *}_{;\mu}\right). \tag{10.5.10}$$

Indeed, in this case

$$\frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;\mu}} \psi^{\ell;\nu} + \frac{\partial L^{\ell}}{\partial \psi^{\ell *}_{;\mu}} \psi^{\ell;\nu *} = \frac{\partial K^{\ell}}{\partial b}\left(\psi^{\ell;\mu *}\psi^{\ell;\nu} + \psi^{\ell;\mu}\psi^{\ell;\nu *}\right), \tag{10.5.11}$$

readily implying that the symmetry condition (10.5.9) does hold.

The field equations for the Lagrangian $\mathcal{L}$ defined by (10.5.5)-(10.5.6) are

$$\frac{\partial L^{\ell}}{\partial \psi^{\ell}} - \tilde{\partial}^{\ell *}_{\mu}\left[\frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;\mu}}\right] = 0, \ \frac{\partial L^{\ell}}{\partial \psi^{\ell *}} - \tilde{\partial}^{\ell}_{\mu}\left[\frac{\partial L^{\ell}}{\partial \psi^{\ell *}_{;\mu}}\right] = 0, \tag{10.5.12}$$

$$\partial_{\mu}F^{\mu\nu} = \frac{4\pi}{\mathrm{c}}J^{\nu}, \ F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \ J^{\nu} = \sum_{\ell} J^{\ell\nu}, \tag{10.5.13}$$

where $J^{\ell\nu}$ is the four-vector current related to the ℓ-th charge is defined as follows. Under the gauge invariance conditions (10.5.7), (10.5.8) for the Lagrangian $\mathcal{L}$ using the Noether's theorem and the formula (10.3.9) one can introduce for every charge ψ^{ℓ} the following *4-vector current*, [Pauli RFTh, (19)], [Wentzel, (3.11)-(3.13)]

$$J^{\ell\nu} = -\mathrm{i}\frac{q^{\ell}}{\chi}\left(\frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;\nu}}\psi^{\ell} - \frac{\partial L^{\ell}}{\partial \psi^{*\ell}_{;\nu}}\psi^{*\ell}\right), \tag{10.5.14}$$

or, since $J^{\nu} = (\mathrm{c}\rho, \mathbf{J})$,

$$\rho^{\ell} = -\mathrm{i}\frac{q^{\ell}}{\chi}\left(\frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;0}}\psi^{\ell} - \frac{\partial L^{\ell}}{\partial \psi^{*\ell}_{;0}}\psi^{*\ell}\right), \ \mathbf{J}^{\ell}_{j} = -\mathrm{i}\frac{q^{\ell}}{\chi}\left(\frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;j}}\psi^{\ell} - \frac{\partial L^{\ell}}{\partial \psi^{*\ell}_{;j}}\psi^{*\ell}\right), \ j = 1, 2, 3, \tag{10.5.15}$$

which satisfy for every ℓ the charge conservation/continuity equations

$$\partial_\nu J^{\ell\nu} = 0, \text{ or } \partial_t \rho^\ell + \nabla \cdot \mathbf{J}^\ell = 0, \ J^{\ell\nu} = \left(\mathrm{c}\rho^\ell, \mathbf{J}^\ell\right). \tag{10.5.16}$$

Notice that in view of the relations (10.5.5)-(10.5.6) the following alternative representation holds for the four-current $J^{\ell\nu}$ defined by (10.5.14)

$$J^{\ell\nu} = -\mathrm{i}\frac{q^\ell}{\chi}\left(\frac{\partial L^\ell}{\partial \psi^\ell_{;\nu}}\psi^\ell - \frac{\partial L^\ell}{\partial \psi^{*\ell}_{;\nu}}\psi^{*\ell}\right) = -\mathrm{c}\frac{\partial L^\ell}{\partial A_\nu}. \tag{10.5.17}$$

We would like to emphasize here the physical significance of identity (10.5.17) equating two complimentary views on the electric current: (i) as a source current in the Maxwell equations (10.5.13); (ii) as the gauge electric current (10.5.14) satisfying the continuity equation (10.5.16). Notice that the equality (10.5.17) originates from a particular form of the coupling between the EM field and charges in the Lagrangian (10.5.5), namely the coupling through the covariant derivatives (10.5.6). One may also view the electric currents identity (10.5.17) as a physical rational for introducing the coupling exactly as it is done in the expressions (10.5.5)-(10.5.6).

10.5.1 Gauge invariant and symmetric energy-momentum tensors

In this subsection we consider a Lagrangian defined by formulas (10.5.5)-(10.5.6) and assume it to be gauge invariant with respect to transformations if the first and the second type. To obtain an expression for the total symmetric energy-momentum tensor $\mathcal{T}^{\mu\nu}$ for such a Lagrangian we use the theory described in Section 10.2, formulas (10.2.11) and (10.5.4), namely

$$\mathcal{T}^{\mu\nu} = \mathring{\mathcal{T}}^{\mu\nu} - \partial_\gamma f^{\mu\gamma\nu}, \ \mathring{\mathcal{T}}^{\mu\nu} = \mathring{\Theta}^{\mu\nu} + \sum_\ell \mathring{T}^{\ell\mu\nu}, \tag{10.5.18}$$

where the canonical energy-momentum of EM field $\mathring{\Theta}^{\mu\nu}$ and the energy-momentum tensor $\mathring{T}^{\ell\mu\nu}$ of ℓ-th charge are represented as follows (see [Jackson, (12.104)], [Barut, Section III.4.D] for $\mathring{\Theta}^{\mu\nu}$ and (10.5.4) for $\mathring{T}^{\ell\mu\nu}$)

$$\mathring{T}^{\ell\mu\nu} = \frac{\partial L^\ell}{\partial \psi^\ell_{;\mu}}\psi^{\ell,\nu} + \frac{\partial L^\ell}{\partial \psi^{\ell*}_{;\mu}}\psi^{\ell*,\nu} - g^{\mu\nu}L^\ell, \tag{10.5.19}$$

$$\mathring{\Theta}^{\mu\nu} = -\frac{1}{4\pi}F^{\mu\gamma}\partial^\nu A_\gamma + g^{\mu\nu}\frac{F^{\xi\gamma}F_{\xi\gamma}}{16\pi}. \tag{10.5.20}$$

The above canonical energy-momenta tensors are neither gauge invariant nor symmetric. To find a representation for $f^{\mu\gamma\nu}$ we use the formulas (10.2.11) noticing that for the scalar fields ψ^ℓ and $\psi^{*\ell}$ we apply the formula (10.2.14), whereas for the vector field A^μ we apply the formula (10.2.15). This yields

$$f^{\mu\gamma\nu} = -\frac{1}{4\pi}F^{\mu\gamma}A^\nu, \tag{10.5.21}$$

and, consequently

$$-\partial_\gamma\frac{1}{4\pi}f^{\mu\gamma\nu} = \frac{1}{4\pi}\partial_\gamma\left(F^{\mu\gamma}\right)A^\nu + \frac{1}{4\pi}F^{\mu\gamma}\partial_\gamma A^\nu = -\frac{1}{\mathrm{c}}J^\mu A^\nu + \frac{1}{4\pi}F^{\mu\gamma}\partial_\gamma A^\nu, \tag{10.5.22}$$

where we used the Maxwell equations (10.5.13) producing a term with the current $J^{\mu} = \sum_{\ell} J^{\ell\mu}$. We introduce now the following energy-momentum tensors

$$T^{\ell\mu\nu} = \mathring{T}^{\ell\mu\nu} - \frac{1}{\mathrm{c}} J^{\mu} A^{\nu}, \ \Theta^{\mu\nu} = \mathring{\Theta}^{\mu\nu} + \frac{1}{4\pi} F^{\mu\gamma} \partial_{\gamma} A^{\nu}, \tag{10.5.23}$$

and find that they have the following representations

$$\Theta^{\mu\nu} = \frac{1}{4\pi} \left(g^{\mu\gamma} F_{\gamma\xi} F^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right), \ F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}, \tag{10.5.24}$$

$$T^{\ell\mu\nu} = \frac{\partial L^{\ell}}{\partial \psi^{\ell}_{;\mu}} \psi^{\ell;\nu} + \frac{\partial L^{\ell}}{\partial \psi^{\ell *}_{;\mu}} \psi^{\ell;\nu *} - g^{\mu\nu} L^{\ell}. \tag{10.5.25}$$

The the formula (10.5.24) is a well known representation (10.4.20) for the symmetric and gauge invariant energy-momentum tensor of the EM field (see [Jackson, Section 12.10], [Barut, III.3]). Notice also that each tensors $T^{\ell\mu\nu}$ defined by (10.5.25) is manifestly gauge invariant. In the case when symmetry condition (10.5.9) is satisfied $T^{\ell\mu\nu}$ is also symmetric.

Using (10.5.24) and (10.5.25) we define now the total energy-momentum tensor by

$$\mathcal{T}^{\mu\nu} = \Theta^{\mu\nu} + \sum_{\ell} T^{\ell\mu\nu}, \tag{10.5.26}$$

and that it is an admissible since it differs from the canonical one by the divergence $\partial_{\gamma} f^{\mu\gamma\nu}$.

In the case of the Lagrangian of the form (10.5.10) in view of (10.5.11) the energy-momentum expression (10.5.25) turns into

$$T^{\ell\mu\nu} = \frac{\partial K^{\ell}}{\partial b} \left(\psi^{\ell;\mu *} \psi^{\ell;\nu} + \psi^{\ell;\mu} \psi^{\ell;\nu *} \right) - g^{\mu\nu} K^{\ell}. \tag{10.5.27}$$

Consequently, as expected the energy conservation low for the total system takes the familiar form

$$\partial_{\mu} \mathcal{T}^{\mu\nu} = \sum_{\ell} \partial_{\mu} T^{\ell\mu\nu} + \partial_{\mu} \Theta^{\mu\nu} = 0. \tag{10.5.28}$$

10.5.2 Equations for the energy-momentum tensors

Notice that using the field equations (10.5.12)-(10.5.13) and the expression (10.5.24) for the energy-momentum $\Theta^{\mu\nu}$ of the EM field we get (see details of the derivation in [Jackson, Section 12.10C]) the following equation

$$\partial_{\mu} \Theta^{\mu\nu} = -\frac{1}{\mathrm{c}} J_{\mu} F^{\nu\mu}, \ \text{where } J_{\mu} = \sum_{\ell} J^{\ell}_{\mu}. \tag{10.5.29}$$

We show below that the above equation for the energy-momentum $\Theta^{\mu\nu}$ is complemented by the following equations for the energy-momenta $\mathrm{T}^{\ell\mu\nu}$ defined by (10.5.25) of individual charges

$$\partial_{\mu} T^{\ell\mu\nu} = \frac{1}{\mathrm{c}} J^{\ell}_{\mu} F^{\nu\mu}. \tag{10.5.30}$$

Observe now that in view of the representation (10.4.30) for the Lorentz force the following is true: (i) the right-hand side of the equation (10.5.30) is the Lorentz force exerted by the EM field on the ℓ-th charge; (ii) the right-hand side of the equation (10.5.29) is the force exerted by all the charges on the EM field and, as one can expect based on the Third Newton Law, "every action has an equal and opposite reaction", this force it is exactly the negative sum of all the Lorentz forces for involved charges. In fact, based on general consideration of the equations for energy-momenta as in relations (10.2.19)-(10.2.23) we can view the equations (10.5.29)-(10.5.30) with involved Lorentz forces as a continuum version of classical equations of motion. An important difference though of the equation (10.5.29)-(10.5.30) unlike the equations of motion for point particles do not by themselves determine the dynamics of all involved fields, and, in fact, they hold only under an assumption that the field equations (10.5.12)-(10.5.13) are satisfied.

To verify the identities (10.5.30) we, following to [Pauli RFTh, Part I, Section 2], introduce a useful computational tool for dealing with the covariant differentiation operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{\mu *}$ as defined in (10.5.6). Namely, let us consider a function $f\left(\psi, \psi_{;\mu}, \psi^{*}, \psi^{*}_{;\mu}\right)$ where

$$\psi_{;\mu} = \tilde{\partial}^{\mu}\psi, \;\; \psi^{*}_{;\mu} = \tilde{\partial}^{\mu *}\psi, \;\; \tilde{\partial}^{\mu} = \partial^{\mu} + \frac{\mathrm{i}q}{\chi \mathrm{c}} A^{\mu}, \;\; \tilde{\partial}^{\mu *} = \partial^{\mu} - \frac{\mathrm{i}q}{\chi \mathrm{c}} A^{\mu}, \tag{10.5.31}$$

which is invariant with respect to the gauge transformations of the first kind (global) as in (10.5.7):

$$\psi \to \mathrm{e}^{\mathrm{i}\gamma}\psi, \;\; \psi^{*} \to \mathrm{e}^{-\mathrm{i}\gamma}\psi^{*}, \text{ where } \gamma \text{ is any real constant.} \tag{10.5.32}$$

The invariance of f readily implies the following identity

$$\begin{aligned} &\frac{d}{d\gamma}\left(f\left(\mathrm{e}^{\mathrm{i}\gamma}\psi, \mathrm{e}^{\mathrm{i}\gamma}\psi_{;\mu}, \mathrm{e}^{-\mathrm{i}\gamma}\psi^{*}, \mathrm{e}^{-\mathrm{i}\gamma}\psi^{*}_{;\mu}\right)\right)\Big|_{\gamma=0} = \\ &= \frac{\partial f}{\partial \psi} + \frac{\partial f}{\partial \psi_{;\mu}}\tilde{\partial}_{\mu}\psi - \frac{\partial f}{\partial \psi^{*}} - \frac{\partial f}{\partial \psi^{*}_{;\mu}}\tilde{\partial}^{*}_{\mu}\psi^{*} = 0. \end{aligned} \tag{10.5.33}$$

Observe also that from the definition (10.5.6) of the covariant differentiation operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{\mu *}$ we have

$$\begin{aligned} \tilde{\partial}^{\mu}\tilde{\partial}^{\nu} - \tilde{\partial}^{\nu}\tilde{\partial}^{\mu} &= \frac{\mathrm{i}q}{\chi \mathrm{c}}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\partial^{\nu}\right) = \frac{\mathrm{i}q}{\chi \mathrm{c}}F^{\mu\nu} \\ \tilde{\partial}^{\mu *}\tilde{\partial}^{\nu *} - \tilde{\partial}^{\nu *}\tilde{\partial}^{\mu *} &= -\frac{\mathrm{i}q}{\chi \mathrm{c}}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\partial^{\nu}\right) = -\frac{\mathrm{i}q}{\chi \mathrm{c}}F^{\mu\nu} \end{aligned} \tag{10.5.34}$$

Now for a gauge invariant f we have

$$\begin{aligned} \partial^{\nu} f &= \frac{\partial f}{\partial \psi}\partial^{\nu}\psi + \frac{\partial f}{\partial \psi_{;\mu}}\partial^{\nu}\tilde{\partial}_{\mu}\psi + \frac{\partial f}{\partial \psi^{*}}\partial^{\nu}\psi^{*} + \frac{\partial f}{\partial \psi^{*}_{;\mu}}\partial^{\nu}\tilde{\partial}^{*}_{\mu}\psi^{*} = \\ &\frac{\partial f}{\partial \psi}\tilde{\partial}^{\nu}\psi + \frac{\partial f}{\partial \psi_{;\mu}}\tilde{\partial}^{\nu}\tilde{\partial}_{\mu}\psi + \frac{\partial f}{\partial \psi^{*}}\tilde{\partial}^{\nu *}\psi^{*} + \frac{\partial f}{\partial \psi^{*}_{;\mu}}\tilde{\partial}^{\nu *}\tilde{\partial}^{*}_{\mu}\psi^{*} \\ &+\frac{\mathrm{i}q}{\chi \mathrm{c}}A^{\nu}\left(-\frac{\partial f}{\partial \psi}\psi - \frac{\partial f}{\partial \psi_{;\mu}}\tilde{\partial}_{\mu}\psi + \frac{\partial f}{\partial \psi^{*}}\psi^{*} + \frac{\partial f}{\partial \psi^{*}_{;\mu}}\tilde{\partial}^{*}_{\mu}\psi^{*}\right) \end{aligned} \tag{10.5.35}$$

which together with (10.5.33) implies the following identity

$$\partial^{\nu} f = \frac{\partial f}{\partial \psi}\tilde{\partial}^{\nu}\psi + \frac{\partial f}{\partial \psi_{;\mu}}\tilde{\partial}^{\nu}\tilde{\partial}_{\mu}\psi + \frac{\partial f}{\partial \psi^{*}}\tilde{\partial}^{\nu *}\psi^{*} + \frac{\partial f}{\partial \psi^{*}_{;\mu}}\tilde{\partial}^{\nu *}\tilde{\partial}^{*}_{\mu}\psi^{*}. \tag{10.5.36}$$

With an argument similar to the above on can verify that if f and g^* are functions of ψ, ψ^*, $\tilde{\partial}_\mu\psi$, $\tilde{\partial}^*_\mu\psi^*$ which transform under the gauge transformations (10.5.32) as $\mathrm{e}^{\mathrm{i}\gamma}f$ and $\mathrm{e}^{-\mathrm{i}\gamma}g^*$ then the following identity holds

$$\partial^\nu (fg^*) = \left(\tilde{\partial}^\nu f\right) g^* + f\left(\tilde{\partial}^{\nu*} g^*\right). \tag{10.5.37}$$

One can verify that the function $\mathrm{T}^{\ell\mu\nu}$ defined by (10.5.25) is an expression for which the identities (10.5.36) and (10.5.37) can be applied. Now applying these identities to $\partial_\mu \mathrm{T}^{\ell\mu\nu}$ and using the field equations (10.5.12) together with identities (10.5.34) and the representation (10.5.17) for the current J^ℓ_ν we obtain

$$\partial_\mu T^{\ell\mu\nu} = \partial_\mu\left(\frac{\partial L^\ell}{\partial\psi^\ell_{;\mu}}\tilde{\partial}^{\ell\nu}\psi^\ell + \frac{\partial L^\ell}{\partial\psi^{\ell*}_{;\mu}}\tilde{\partial}^{*\ell\nu}\psi^{\ell*}\right) - \partial^\nu L^\ell = \tag{10.5.38}$$

$$= \left[\tilde{\partial}^{\ell*}_\mu\left(\frac{\partial L^\ell}{\partial\psi^\ell_{;\mu}}\right)\right]\tilde{\partial}^{\ell\nu}\psi^\ell + \frac{\partial L^\ell}{\partial\psi^\ell_{;\mu}}\tilde{\partial}^\ell_\mu\tilde{\partial}^{\ell\nu}\psi^\ell + \left[\tilde{\partial}^\ell_\mu\left(\frac{\partial L^\ell}{\partial\psi^{\ell*}_{;\mu}}\right)\right]\tilde{\partial}^{\ell\nu*}\psi^{\ell*} + \frac{\partial L^\ell}{\partial\psi^{\ell*}_{;\mu}}\tilde{\partial}^{\ell*}_\mu\tilde{\partial}^{\ell\nu*}\psi^{\ell*}$$

$$-\left(\frac{\partial L^\ell}{\partial\psi^\ell}\tilde{\partial}^{\ell\nu}\psi^\ell + \frac{\partial L^\ell}{\partial\psi^\ell_{;\mu}}\tilde{\partial}^{\ell\nu}\tilde{\partial}^\ell_\mu\psi^\ell + \frac{\partial L^\ell}{\partial\psi^{\ell*}}\tilde{\partial}^{\ell\nu*}\psi^{\ell*} + \frac{\partial L^\ell}{\partial\psi^{\ell*}_{;\mu}}\tilde{\partial}^{\ell\nu*}\tilde{\partial}^{\ell*}_\mu\psi^{\ell*}\right) =$$

$$= \left(\frac{\partial L^\ell}{\partial\psi^\ell_{;\mu}}\psi^\ell - \frac{\partial L^\ell}{\partial\psi^{\ell*}_{;\mu}}\psi^{\ell*}\right)\frac{\mathrm{i}q}{\chi\mathrm{c}}F_\mu{}^\nu = -\frac{1}{\mathrm{c}}J^\ell_\mu F^{\mu\nu} = \frac{1}{\mathrm{c}}J^\ell_\mu F^{\nu\mu},$$

which is the desired equation (10.5.30).

Observe that the equation (10.5.29)-(10.5.30) for the energy-momenta are evidently consistent with the total energy conservation (10.5.28).

10.5.3 Gauge invariant and partially symmetric energy-momentum tensors

In our studies, in particular of non relativistic approximations, we have Lagrangians which are gauge invariant and invariant with respect space and time translations but they might not be invariant with respect to the entire Lorentz group of transformations. This subsection is devoted to this kind of Lagrangians with the main point that essentially all important results of the subsections 10.5.1 and 10.5.2 apply to them with the only difference that the energy-momentum tensor is not fully symmetric but commonly its space part, the stress tensor, is symmetric.

As in the previous subsection we assume the Lagrangian to be of the form described by formulas (10.5.5)-(10.5.6) and assume it to be gauge invariant with respect to transformations if the first and the second type and invariant with respect space and time translations. A careful analysis of the arguments in subsections 10.5.1 and 10.5.2 which produced the expressions (10.5.24) and (10.5.25) for respectively energy-momentum $\Theta^{\mu\nu}$ of the EM field and energy-momenta $T^{\ell\mu\nu}$ of charges show the same expressions hold for gauge and translation invariant Lagrangian even if it is not invariant with respect to the entire Lorentz group of transformation.

We notice first the field equations and expressions for *conserved currents* are provided by (10.5.12), (10.5.13), (10.5.14), (10.5.15), (10.5.17), namely

$$\frac{\partial L^\ell}{\partial\psi^\ell} - \tilde{\partial}^{\ell*}_\mu\left[\frac{\partial L^\ell}{\partial\psi^\ell_{;\mu}}\right] = 0,\ \frac{\partial L^\ell}{\partial\psi^{\ell*}} - \tilde{\partial}^\ell_\mu\left[\frac{\partial L^\ell}{\partial\psi^{\ell*}_{;\mu}}\right] = 0, \tag{10.5.39}$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{\mathrm{c}} J^\nu,\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,\ J^\nu = \sum_\ell J^{\ell\nu}, \tag{10.5.40}$$

where $J^{\ell\nu}$ is the four-vector current related to the ℓ-th charge is defined by

$$J^{\ell\nu} = -\mathrm{i}\frac{q^\ell}{\chi}\left(\frac{\partial L^\ell}{\partial \psi^\ell_{;\nu}}\psi^\ell - \frac{\partial L^\ell}{\partial \psi^{*\ell}_{;\nu}}\psi^{*\ell}\right) = -\mathrm{c}\frac{\partial L^\ell}{\partial A_\nu}, \tag{10.5.41}$$

or, since $J^\nu = (\mathrm{c}\rho, \mathbf{J})$,

$$\rho^\ell = -\mathrm{i}\frac{q^\ell}{\chi}\left(\frac{\partial L^\ell}{\partial \psi^\ell_{;0}}\psi^\ell - \frac{\partial L^\ell}{\partial \psi^{*\ell}_{;0}}\psi^{*\ell}\right),\ \mathbf{J}^\ell_j = -\mathrm{i}\frac{q^\ell}{\chi}\left(\frac{\partial L^\ell}{\partial \psi^\ell_{;j}}\psi^\ell - \frac{\partial L^\ell}{\partial \psi^{*\ell}_{;j}}\psi^{*\ell}\right),\ j = 1, 2, 3. \tag{10.5.42}$$

Then *we assign* to the energy-momenta of the EM field $\Theta^{\mu\nu}$ and the ℓ-th charge $T^{\ell\mu\nu}$ respectively expressions (10.5.24) and (10.5.25), namely

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(g^{\mu\gamma}F_{\gamma\xi}F^{\xi\nu} + \frac{1}{4}g^{\mu\nu}F_{\gamma\xi}F^{\gamma\xi}\right), \tag{10.5.43}$$

$$T^{\ell\mu\nu} = \frac{\partial L^\ell}{\partial \psi^\ell_{;\mu}}\psi^{\ell;\nu} + \frac{\partial L^\ell}{\partial \psi^{\ell*}_{;\mu}}\psi^{\ell;\nu*} - g^{\mu\nu}L^\ell, \tag{10.5.44}$$

The above expressions for the energy-momenta are manifestly gauge invariant.

Looking at the arguments in subsections 10.5.1 and 10.5.2 we compare the above expressions of the energy-momenta of the EM field $\Theta^{\mu\nu}$ and the ℓ-th charge $T^{\ell\mu\nu}$ with their canonical expression and observe that

$$T^{\ell\mu\nu} = \mathring{T}^{\ell\mu\nu} - \frac{1}{\mathrm{c}}J^{\ell\mu}A^\nu,\ \Theta^{\mu\nu} = \mathring{\Theta}^{\mu\nu} + \frac{1}{4\pi}F^{\mu\gamma}\partial_\gamma A^\nu. \tag{10.5.45}$$

It remains to verify that the difference between the total energy-momentum and its canonical value is a 4-divergence. Indeed it follows from (10.5.45) that

$$\begin{aligned}\sum_\ell\left(T^{\ell\mu\nu} - \mathring{T}^{\ell\mu\nu}\right) + \left(\Theta^{\mu\nu} - \mathring{\Theta}^{\mu\nu}\right) &= \frac{1}{4\pi}F^{\mu\gamma}\partial_\gamma A^\nu - \sum_\ell \frac{1}{\mathrm{c}}J^{\ell\mu}A^\nu \\ &= \frac{1}{4\pi}F^{\mu\gamma}\partial_\gamma A^\nu - \frac{1}{\mathrm{c}}J^\mu A^\nu = -\partial_\gamma\frac{1}{4\pi}f^{\mu\gamma\nu},\ \text{where } f^{\mu\gamma\nu} = -\frac{1}{4\pi}F^{\mu\gamma}A^\nu.\end{aligned} \tag{10.5.46}$$

Using the arguments of the subsections 10.5.2 we also find that the relations (10.5.29) and (10.5.30) hold here, namely

$$\partial_\mu\Theta^{\mu\nu} = -\frac{1}{\mathrm{c}}J_\mu F^{\nu\mu},\ \text{where } J_\mu = \sum_\ell J^\ell_\mu, \tag{10.5.47}$$

$$\partial_\mu \mathrm{T}^{\ell\mu\nu} = \frac{1}{\mathrm{c}}J^\ell_\mu F^{\nu\mu}, \tag{10.5.48}$$

where once again we recognize in the right-hand sides of equalities (10.5.47)-(10.5.48) the relevant Lorentz force densities. Consequently, as expected the energy conservation low for the total system takes the familiar form

$$\partial_\mu\mathcal{T}^{\mu\nu} = \sum_\ell \partial_\mu T^{\ell\mu\nu} + \partial_\mu\Theta^{\mu\nu} = 0. \tag{10.5.49}$$

The equations (10.5.47)-(10.5.49) reconfirm that our assignment (10.5.43)-(10.5.44) of energy-momenta to the EM field and individual charges is physically sound.

We would like to notice now that even if a Lagrangian of the form (10.5.5)-(10.5.6) is not invariant with respect the entire Lorentz group of transformations it often satisfies a reduced version of the symmetry condition (10.5.9) which holds for the space indices only, namely

$$\frac{\partial L^{\ell}}{\partial \psi_{;i}^{\ell}}\psi^{\ell;j}+\frac{\partial L^{\ell}}{\partial \psi_{;i}^{\ell*}}\psi^{\ell;j*}=\frac{\partial L^{\ell}}{\partial \psi_{;j}^{\ell}}\psi^{\ell;i}+\frac{\partial L^{\ell}}{\partial \psi_{;j}^{\ell*}}\psi^{\ell;i*},\ i,j=1,2,3. \tag{10.5.50}$$

Under the reduced symmetry condition (10.5.50) the space part of the energy-momenta $\mathrm{T}^{\ell\mu\nu}$, known as the stress tensor, is symmetric, namely

$$\mathrm{T}^{\ell ij}=\mathrm{T}^{\ell ji},\ i,j=1,2,3. \tag{10.5.51}$$

We remind that the symmetry of the stress tensor is a very important property equivalent to the space angular momentum conservation, see Section 10.2 and, for instance, [Moller, Section 6.1, 6.2].

10.6 A single free charge

A single free charge interacting with the EM field is evidently a particular case of considered above system of many charges in Section 10.5 and the Lagrangian (10.5.5) takes the form

$$\mathcal{L}_0=L_0\left(\psi,\psi_{;\mu},\psi^*,\psi_{;\mu}^*\right)-\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi},\ F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}, \tag{10.6.1}$$

where $\psi_{;\mu}$ and $\psi_{;\mu}^*$ are the covariant derivatives with respect to the covariant differential operators $\tilde{\partial}^{\mu}$ and $\tilde{\partial}^{\mu*}$ defined by

$$\psi_{;\mu}=\tilde{\partial}^{\mu}\psi,\ \psi_{;\mu}^*=\tilde{\partial}^{\mu*}\psi^*,\ \tilde{\partial}^{\mu}=\partial^{\mu}+\frac{\mathrm{i}q}{\chi\mathrm{c}}A^{\mu},\ \tilde{\partial}^{\mu*}=\partial^{\mu}-\frac{\mathrm{i}q}{\chi\mathrm{c}}A^{\mu}. \tag{10.6.2}$$

The Lagrangian is assumed to be Lorentz and gauge invariant with respect to the gauge transformations of the first and the second type (10.5.7)-(10.5.8). The field equations are

$$\frac{\partial L_0}{\partial \psi}-\tilde{\partial}_{\mu}^*\left[\frac{\partial L_0}{\partial \psi_{;\mu}}\right]=0,\ \frac{\partial L_0}{\partial \psi^*}-\tilde{\partial}_{\mu}\left[\frac{\partial L_0}{\partial \psi_{;\mu}^*}\right]=0, \tag{10.6.3}$$

$$\partial_{\mu}F^{\mu\nu}=\frac{4\pi}{\mathrm{c}}J^{\nu},\ F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}, \tag{10.6.4}$$

where J^{μ} is the four-vector micro-current related to the charge is defined by

$$J^{\nu}=-\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_0}{\partial \psi_{;\nu}}\psi-\frac{\partial L_0}{\partial \psi_{;\nu}^*}\psi^*\right)=-\mathrm{c}\frac{\partial L_0}{\partial A_{\nu}}. \tag{10.6.5}$$

or, since $J^{\nu}=(\mathrm{c}\rho,\mathbf{J})$,

$$\rho=-\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_0}{\partial \psi_{;0}}\psi-\frac{\partial L_0}{\partial \psi_{;0}^*}\psi^*\right),\ \mathbf{J}_j=-\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_0}{\partial \psi_{;j}}\psi-\frac{\partial L_0}{\partial \psi_{;j}^*}\psi^*\right),\ j=1,2,3, \tag{10.6.6}$$

which satisfy the charge conservation/continuity equations

$$\partial_\nu J^\nu = 0,\ \text{or}\ \partial_t \rho + \nabla \cdot \mathbf{J} = 0,\ J^\nu = (\mathrm{c}\rho, \mathbf{J}). \tag{10.6.7}$$

The energy-momentum of the charge and the EM field according to the formulas (10.5.24)-(10.5.25) are respectively as follows

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial L_0}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L_0}{\partial \psi^*_{;\mu}} \psi^{;\nu *} - g^{\mu\nu} L_0, && (10.6.8) \\ \Theta^{\mu\nu} &= \frac{1}{4\pi} \left(g^{\mu\gamma} F_{\gamma\xi} F^{\xi\nu} + \frac{1}{4} g^{\mu\nu} F_{\gamma\xi} F^{\gamma\xi} \right),\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, && (10.6.9) \end{aligned}$$

and the energy conservation equations (10.5.29)-(10.5.30) turn here into

$$\partial_\mu T^{\mu\nu} = \frac{1}{\mathrm{c}} J_\mu F^{\nu\mu},\ \partial_\mu \Theta^{\mu\nu} = -\frac{1}{\mathrm{c}} J_\mu F^{\nu\mu}. \tag{10.6.10}$$

10.7 A single charge in an external electromagnetic field

Here we consider a single dressed charge in an external EM field. The very presence of external forces turns the dressed charge into an open system and that brings up subtleties in the set up of gauge invariant expressions for the energy-momentum tensor. One can find signs of those subtleties already in a simple case of a point charge in an external EM. Indeed for the point charge model the canonical momentum and force are not gauge invariant as discussed briefly in Section 10.1. The principle source of the problems lies is the openness of the system with consequent uncertainty of the energy and the momentum as system changes under action of external forces. That can be seen, in particular, based on the relativity grounds, [Moller, Section 7.1, 7.2], when seemingly well defined 4-momenta for a number of open systems do not transform as 4-vectors, that, in general, can be taken as a proof of openness of a system.

Coming back to our case we want, first of all, to define a Lagrangian for a dressed charge in EM field based on (i) our studies in Section 10.5 of a closed system of many dressed charges and (ii) the Lagrangian of a singe free charge considered in Section 10.6. We do that by altering the EM potential A^μ in the expressions (10.2.1)-(10.2.3) for the Lagrangian of the free single dressed charge with $\bar{A}^\mu = A^\mu + A^\mu_{\text{ex}}$, where A^μ_{ex} is the 4-potential of an the external EM field. Namely, we set

$$L_0 = L_0\left(\psi, \psi_{;\mu}, \psi^*, \psi^*_{;\mu}\right) - \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi},\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{10.7.1}$$

where $\psi_{;\mu}$ and $\psi^*_{;\mu}$ are the covariant derivatives with respect to the covariant differential operators $\tilde{\partial}^\mu$ and $\tilde{\partial}^{\mu *}$ defined by

$$\begin{gathered} \psi_{;\mu} = \tilde{\partial}^\mu \psi,\ \psi^*_{;\mu} = \tilde{\partial}^{\mu *} \psi^*, \qquad (10.7.2) \\ \tilde{\partial}^\mu = \partial^\mu + \frac{\mathrm{i}q}{\chi \mathrm{c}} \bar{A}^\mu,\ \tilde{\partial}^{\mu *} = \partial^\mu - \frac{\mathrm{i}q}{\chi \mathrm{c}} \bar{A}^\mu,\ \bar{A}^\mu = A^\mu + A^\mu_{\text{ex}}. \end{gathered}$$

To justify the expressions (10.7.1)-(10.7.2) for the Lagrangian let us look at a closed system of many charges studied in Section 10.5. We find there, in particular, that every individual charge with an index ℓ has a conserved current $J^{\ell\nu}$ and the total current is $J_\mu = \sum_\ell J^{\ell\nu}$. Hence, based on the linearity of Maxwell equation (10.5.13) we can introduce individual EM potentials $A^{\ell\mu}$ and the corresponding EM fields $F^{\ell\mu\nu}$ as the causal solutions to the following Maxwell equations

$$\partial_\mu F^{\ell\mu\nu} = \frac{4\pi}{\mathrm{c}} J^{\ell\nu}, \tag{10.7.3}$$

implying

$$A^\mu = \sum_\ell A^{\ell\mu},\ F^{\mu\nu} = \sum_\ell F^{\ell\mu\nu}. \tag{10.7.4}$$

Notice that every individual charge satisfies its field equation (10.5.12) with the EM field entering it via the potential A^μ in the covariant derivatives (10.5.31), and we can always represent it as

$$A^\mu = A^{\ell\mu} + A^{\ell\mu}_{\mathrm{ex}},\ A^{\ell\mu}_{\mathrm{ex}} = \sum_{\ell'\neq\ell} A^{\ell'\mu}. \tag{10.7.5}$$

This representation indicates that we can account for the interaction of the ℓ-th charge with remaining charges via an external field as in (10.7.5) justifying the expressions (10.7.1)-(10.7.2) for the Lagrangian L_0.

The Lagrangian (10.7.1)-(10.7.2) is assumed to be invariant with respect to the first and the second type gauge transformations (10.5.7), (10.5.8), which in this case take the form

$$\psi \to \mathrm{e}^{\mathrm{i}\gamma}\psi^\ell,\ \psi^* \to \mathrm{e}^{-\mathrm{i}\gamma}\psi^*,\ \text{where } \gamma \text{ is any real constant}, \tag{10.7.6}$$

$$\psi \to \mathrm{e}^{-\frac{\mathrm{i}e\lambda(x)}{\chi\mathrm{c}}}\psi,\ A^\mu \to A^\mu + \partial^\mu\lambda. \tag{10.7.7}$$

Similarly to the case of many charges we also assume the charge Lagrangian L_0 to satisfy the following symmetry condition

$$\frac{\partial L_0}{\partial\psi_{;\mu}}\psi^{;\nu} + \frac{\partial L_0}{\partial\psi^*_{;\mu}}\psi^{;\nu*} = \frac{\partial L_0}{\partial\psi_{;\nu}}\psi^{;\mu} + \frac{\partial L_0}{\partial\psi^*_{;\nu}}\psi^{;\mu*}. \tag{10.7.8}$$

As in already considered case of many charges there is a simple sufficient condition for the symmetry condition (10.7.8) to hold. It is when the Lagrangians L depends on the field covariant derivatives only through the combination $\psi^{\ell;\mu}\psi^{\ell*}_{;\mu}$, in other words if there exist such functions $K(\psi,\psi^*,b)$ that

$$L_0\left(\psi,\psi^\ell_{;\mu},\psi^*,\psi^{\ell*}_{;\mu}\right) = K\left(\psi,\psi^*,\psi^{;\mu}\psi^*_{;\mu}\right). \tag{10.7.9}$$

The field equations for the Lagrangian L_0 defined by (10.7.1)-(10.7.2) are

$$\frac{\partial L_0}{\partial\psi} - \tilde{\partial}^*_\mu\left[\frac{\partial L_0}{\partial\psi_{;\mu}}\right] = 0,\ \frac{\partial L_0}{\partial\psi^*} - \tilde{\partial}_\mu\left[\frac{\partial L_0}{\partial\psi^*_{;\mu}}\right] = 0, \tag{10.7.10}$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{\mathrm{c}} J^\nu,\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{10.7.11}$$

where J^μ is the four-vector current related to the charge is defined by manifestly gauge invariant expression

$$J^\nu = -\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_0}{\partial\psi_{;\nu}}\psi - \frac{\partial L_0}{\partial\psi^*_{;\nu}}\psi^*\right), \tag{10.7.12}$$

or, since $J^{\nu}=(\mathrm{c}\rho,\mathbf{J})$, the

$$\rho=-\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_{0}}{\partial\psi_{;0}}\psi-\frac{\partial L_{0}}{\partial\psi_{;0}^{*}}\psi^{*}\right),\ \mathbf{J}_{j}=-\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_{0}}{\partial\psi_{;j}}\psi-\frac{\partial L_{0}}{\partial\psi_{;j}^{*}}\psi^{*}\right),\ j=1,2,3, \tag{10.7.13}$$

which satisfy the charge conservation/continuity equations

$$\partial_{\nu}J^{\nu}=0,\ \text{or}\ \partial_{t}\rho+\nabla\cdot\mathbf{J}=0,\ J^{\nu}=(\mathrm{c}\rho,\mathbf{J})\,. \tag{10.7.14}$$

Notice as in the case of many charges the relations (10.7.1)-(10.7.2) imply the following alternative representation for the four-current J^{ν} defined by (10.7.12)

$$J^{\nu}=-\mathrm{i}\frac{q}{\chi}\left(\frac{\partial L_{0}}{\partial\psi_{;\nu}}\psi-\frac{\partial L_{0}}{\partial\psi_{;\nu}^{*}}\psi^{*}\right)=-\mathrm{c}\frac{\partial L_{0}}{\partial A_{\nu}}=-\mathrm{c}\frac{\partial L_{0}}{\partial\bar{A}_{\nu}}=-\mathrm{c}\frac{\partial L_{0}}{\partial A_{\mathrm{ex}\nu}}. \tag{10.7.15}$$

Now we would like to construct gauge invariant energy-momentum tensors for the single charge and its EM field. For that we start with their canonical expressions (10.5.4)-(10.4.18) obtained via Noether's theorem

$$\mathring{T}^{\mu\nu}=\frac{\partial L_{0}}{\partial\psi_{;\mu}}\psi^{,\nu}+\frac{\partial L_{0}}{\partial\psi_{;\mu}^{*}}\psi^{;\nu*}-g^{\mu\nu}L_{0}, \tag{10.7.16}$$

$$\mathring{\Theta}^{\mu\nu}=-\frac{F^{\mu\gamma}\partial^{\nu}A_{\gamma}}{4\pi}+g^{\mu\nu}\frac{F^{\xi\gamma}F_{\xi\gamma}}{16\pi},\ F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}. \tag{10.7.17}$$

The conservation law for the total energy-momentum tensor $\mathring{T}^{\mu\nu}+\mathring{\Theta}^{\mu\nu}$ in view of the general conservation law (10.2.6) and the current representation (10.7.15) take the form

$$\partial_{\mu}\left(\mathring{T}^{\mu\nu}+\mathring{\Theta}^{\mu\nu}\right)=-\frac{\partial L_{0}}{\partial x_{\nu}}=-\frac{\partial L_{0}}{\partial A_{\mathrm{ex}\mu}}\partial^{\nu}A_{\mu\mathrm{ex}}=\frac{1}{\mathrm{c}}J^{\mu}\partial^{\nu}A_{\mu\mathrm{ex}}. \tag{10.7.18}$$

Observe now that the both canonical expressions (10.7.16), (10.7.17) as well as the density of the generalized force $\frac{1}{\mathrm{c}}J^{\mu}\partial^{\nu}A_{\mu\mathrm{ex}}$ in (10.7.18) are evidently not gauge invariant, and this is very similar to what we already observed for the point charge model in Section 10.1. We recall that there is a general way to alter the canonical energy-momentum tensor as in the relation (10.2.8), namely

$$T^{\mu\nu}+\Theta^{\mu\nu}=\mathring{T}^{\mu\nu}+\mathring{\Theta}^{\mu\nu}-\partial_{\gamma}f^{\mu\gamma\nu},\ f^{\mu\gamma\nu}=-f^{\gamma\mu\nu}. \tag{10.7.19}$$

But any such alteration alone can not be satisfactory since by its very construction it would keep unchanged the not gauge invariant density of generalized force $\frac{1}{\mathrm{c}}J^{\mu}\partial^{\nu}A_{\mu\mathrm{ex}}$ in the right-hand side of (10.7.18). Therefore, a more profound alteration of the energy-momenta is required that would change the expression for the force density in the right-hand side of (10.7.18) so that it becomes be gauge invariant. More than that we expect it to produce exactly the density of the Lorentz force associated with the external EM potential A_{ex}^{μ}.

The results of Section 10.5.1 suggest a satisfactory choice for gauge invariant energy-momentum tensors and it is as in formulas (10.5.19)-(10.5.20), namely we set

$$T^{\mu\nu}=\frac{\partial L_{0}}{\partial\psi_{;\mu}}\psi^{;\nu}+\frac{\partial L_{0}}{\partial\psi_{;\mu}^{*}}\psi^{;\nu*}-g^{\mu\nu}L_{0}, \tag{10.7.20}$$

$$\Theta^{\mu\nu}=\frac{1}{4\pi}\left(g^{\mu\gamma}F_{\gamma\xi}F^{\xi\nu}+\frac{1}{4}g^{\mu\nu}F_{\gamma\xi}F^{\gamma\xi}\right),\ F^{\mu\nu}=\partial^{\mu}A^{\nu}-\partial^{\nu}A^{\mu}. \tag{10.7.21}$$

Observe that the EM energy-momentum $\Theta^{\mu\nu}$ is also manifestly symmetric and the charge energy-momentum $T^{\mu\nu}$ is symmetric if the the symmetry condition (10.7.8) is satisfied. The energy-momenta conservation laws here take the form

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu \frac{1}{4\pi}\left(g^{\mu\gamma}F_{\gamma\xi}F^{\xi\nu} + \frac{1}{4}g^{\mu\nu}F_{\gamma\xi}F^{\gamma\xi}\right) = -\frac{1}{\mathrm{c}}J_\mu F^{\nu\mu}, \tag{10.7.22}$$

$$\partial_\mu T^{\mu\nu} = \frac{1}{\mathrm{c}}J_\mu \bar{F}^{\nu\mu},\ \ \bar{F}^{\nu\mu} = \partial^\mu \bar{A}^\nu - \partial^\nu \bar{A}^\mu. \tag{10.7.23}$$

Indeed as in the case of derivation of the similar formula (10.5.30) we observe that the $T^{\mu\nu}$ defined by (10.7.20) has an expression for which the identities (10.5.36) and (10.5.37) can be applied. Now we literally repeat the calculation (10.5.38). Namely applying the mentioned identities to $\partial_\mu T^{\mu\nu}$ and using the field equations (10.7.10) together with identities (10.5.34) and the representation (10.5.17) for the current J_ν we obtain

$$\begin{gathered}\partial_\mu T^{\mu\nu} = \partial_\mu\left(\frac{\partial L_0}{\partial\psi_{;\mu}}\tilde{\partial}^\nu\psi + \frac{\partial L_0}{\partial\psi^*_{;\mu}}\tilde{\partial}^{*\nu}\psi^*\right) - \partial^\nu L_0 = \\ = \left[\tilde{\partial}^*_\mu\left(\frac{\partial L_0}{\partial\psi_{;\mu}}\right)\right]\tilde{\partial}^\nu\psi + \frac{\partial L}{\partial\psi_{;\mu}}\tilde{\partial}_\mu\tilde{\partial}^\nu\psi + \left[\tilde{\partial}_\mu\left(\frac{\partial L_0}{\partial\psi^*_{;\mu}}\right)\right]\tilde{\partial}^{\nu*}\psi^* + \frac{\partial L_0}{\partial\psi^*_{;\mu}}\tilde{\partial}^*_\mu\tilde{\partial}^{\nu*}\psi^* \\ -\left(\frac{\partial L_0}{\partial\psi}\tilde{\partial}^\nu\psi + \frac{\partial L_0}{\partial\psi_{;\mu}}\tilde{\partial}^\nu\tilde{\partial}_\mu\psi + \frac{\partial L_0}{\partial\psi^*}\tilde{\partial}^{\nu*}\psi^* + \frac{\partial L_0}{\partial\psi^*_{;\mu}}\tilde{\partial}^{\nu*}\tilde{\partial}^*_\mu\psi^*\right) = \\ = \left(\frac{\partial L_0}{\partial\psi_{;\mu}}\psi - \frac{\partial L_0}{\partial\psi^{\ell*}_{;\mu}}\psi^*\right)\frac{\mathrm{i}q}{\chi\mathrm{c}}\bar{F}_\mu{}^\nu = -\frac{1}{\mathrm{c}}J_\mu\bar{F}^{\mu\nu} = \frac{1}{\mathrm{c}}J_\mu\bar{F}^{\nu\mu},\end{gathered} \tag{10.7.24}$$

implying the desired relation (10.7.23).

Now adding up the equalities (10.7.22) and (10.7.23) we get the conservation law for the total energy-momentum $\mathcal{T}^{\mu\nu} = T^{\mu\nu} + \Theta^{\mu\nu}$, i.e.

$$\partial_\mu \mathcal{T}^{\mu\nu} = \partial_\mu\left(T_0^{\mu\nu} + \Theta^{\mu\nu}\right) = \frac{1}{\mathrm{c}}J_\nu F_{\mathrm{ex}}^{\nu\mu},\ \ F_{\mathrm{ex}}^{\nu\mu} = \partial^\mu A_{\mathrm{ex}}^\nu - \partial^\nu A_{\mathrm{ex}}^\mu. \tag{10.7.25}$$

Notice as we expected we have the density of the Lorentz force $\frac{1}{\mathrm{c}}J_\nu F_{\mathrm{ex}}^{\nu\mu}$ in the right-hand side of the conservation laws (10.7.25) and (10.7.23).

10.8 Energy partition for static and time harmonic fields

Let us consider the Lagrangian of the form

$$\mathcal{L} = \mathcal{L}\left(\left\{\psi^\ell, \psi^\ell_{,\mu}, \psi^{\ell*}, \psi^{\ell*}_{,\mu}, V^g, V^g_{,\mu}\right\}\right), \tag{10.8.1}$$

where $\{V^g\}$ are real-valued quantities. The corresponding Euler-Lagrange field equations are

$$\frac{\partial\mathcal{L}}{\partial\psi^\ell} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^\ell_{,\mu}}\right) = 0,\ \frac{\partial\mathcal{L}}{\partial\psi^{\ell*}} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\psi^{\ell*}_{,\mu}}\right) = 0,\ \frac{\partial\mathcal{L}}{\partial V^g} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial V^g_{,\mu}}\right) = 0. \tag{10.8.2}$$

Static regime is characterized as one when the fields $\left\{\psi^\ell\right\}$, $\left\{\psi^{\ell*}\right\}$ and $\{V^g\}$ are time independent and hence depend only on the space variable, and we will use the following abbreviated notation for it

$$\mathrm{stat} \equiv \partial_t\psi^\ell = 0,\ \partial_t\psi^{\ell*} = 0,\ \partial_t V^g = 0. \tag{10.8.3}$$

Then using the canonical energy-momentum $\mathring{\mathcal{T}}^{\mu\nu}$ as defined by (10.5.3) we readily obtain the following formulas for the total energy $\mathcal{E}_{\mathrm{stat}}$ of static field

$$\mathcal{E}_{\mathrm{stat}}=\int_{\mathbb{R}^3}\mathring{\mathcal{U}}\,\mathrm{d}x,\ \text{ where } \mathring{\mathcal{U}}=\mathring{\mathcal{T}}^{00}\left(\left\{\psi^{\ell},\psi^{\ell}_{,\mu},\psi^{\ell*},\psi^{\ell*}_{,\mu},V^{g},V^{g}_{,\mu}\right\}\right)\Big|_{\mathrm{stat}}. \tag{10.8.4}$$

The energy $\mathcal{E}_{\mathrm{stat}}$ in a static regime can be identified with a potential energy. Notice that in view of the formula (10.5.3) for the canonical energy-momentum $\mathring{\mathcal{T}}^{\mu\nu}$ the corresponding energy density $\mathring{\mathcal{T}}^{00}\Big|_{\mathrm{stat}}$ is represented by

$$\mathring{\mathcal{U}}=\mathring{\mathcal{T}}^{00}\Big|_{\mathrm{stat}}=-\,\mathcal{L}\left(\left\{\psi^{\ell},\psi^{\ell}_{,\mu},\psi^{\ell*},\psi^{\ell*}_{,\mu},V^{g},V^{g}_{,\mu}\right\}\right)\big|_{\mathrm{stat}}. \tag{10.8.5}$$

Consequently, we have the following representation for the potential energy $\mathcal{E}_{\mathrm{stat}}$

$$\mathcal{E}_{\mathrm{stat}}=\int_{\mathbb{R}^3}-\,\mathcal{L}\left(\left\{\psi^{\ell},\psi^{\ell}_{,\mu},\psi^{\ell*},\psi^{\ell*}_{,\mu},V^{g},V^{g}_{,\mu}\right\}\right)\big|_{\mathrm{stat}}\,\mathrm{d}x. \tag{10.8.6}$$

which we use to establish the following variational principle. Based (10.8.5) we can conclude that a static solution $\left\{\psi^{\ell},\psi^{\ell*},V^{g}\right\}$ to the Euler-Lagrange field equations (10.8.2) evidently transforms into a solution to the equation

$$\frac{\partial\mathring{\mathcal{U}}}{\partial\psi^{\ell}}-\sum_{j=1,2,3}\partial_{x_j}\left(\frac{\partial\mathring{\mathcal{U}}}{\partial\psi^{\ell}_{,j}}\right)=0,\ \frac{\partial\mathring{\mathcal{U}}}{\partial\psi^{\ell*}}-\sum_{j=1,2,3}\partial_{x_j}\left(\frac{\partial\mathring{\mathcal{U}}}{\partial\psi^{\ell*}_{,j}}\right)=0, \tag{10.8.7}$$
$$\frac{\partial\mathring{\mathcal{U}}}{\partial V^{g}}-\sum_{j=1,2,3}\partial_{x_j}\left(\frac{\partial\mathring{\mathcal{U}}}{\partial V^{g}_{,j}}\right)=0,$$

and, hence, in view of the representation (10.8.6) it is a stationary point of the static energy functional $\mathcal{E}_{\mathrm{stat}}$ in the complete agreement with the *principle of virtual work* for the state of equilibrium, [Lanczos VPM, Section III.1], [Sommerfeld M, Section II.8].

Let us expand now the potential energy density $\mathring{\mathcal{U}}$ defined by (10.8.5) into the series with respect to the derivatives $\nabla\psi^{\ell}$, $\nabla\psi^{\ell*}$, ∇V^{g}, namely

$$\mathring{\mathcal{U}}=\sum_{n=0}^{\infty}\mathring{\mathcal{U}}^{(n)},\ \text{ where} \tag{10.8.8}$$
$$\mathring{\mathcal{U}}^{(n)}=\sum_{\sum n_{\ell}+n^{*}_{\ell}+n_{g}=n}\mathring{\mathcal{U}}_{\left\{n_{\ell},n^{*}_{\ell},n_{g}\right\}}\left(\left\{\psi^{\ell},\psi^{\ell*},V^{g}\right\}\right)\prod_{\ell,g}\partial^{n_{\ell}}_{j_{\ell}}\psi^{\ell}\partial^{n^{*}_{\ell}}_{j^{*}_{\ell}}\psi^{*\ell}\partial^{n_{g}}_{i_{g}}V^{g}.$$

This expansion via the representation (10.8.6) for the potential energy $\mathcal{E}_{\mathrm{stat}}$ readily implies the the corresponding expansion for $\mathcal{E}_{\mathrm{stat}}$:

$$\mathcal{E}_{\mathrm{stat}}=\sum_{n=0}^{\infty}\mathcal{E}^{(n)}_{\mathrm{stat}},\ \text{ where } \mathcal{E}^{(n)}_{\mathrm{stat}}=\int_{\mathbb{R}^3}\mathring{\mathcal{U}}^{(n)}\,\mathrm{d}x. \tag{10.8.9}$$

Now being given a static solution $\left\{\psi^{\ell},\psi^{\ell*},V^{g}\right\}$ we use its established above property to be a stationary point of the functional $\mathcal{E}_{\mathrm{stat}}$ as defined by formula (10.8.6) and (10.8.9). Namely, we introduce the following family of fields

$$\psi^{\ell}_{\xi}(x)=\psi^{\ell}(\xi x),\ \psi^{*\ell}_{\xi}(x)=\psi^{*\ell}(\xi x),\ V^{g}_{\xi}(x)=V^{g}(\xi x)\ \text{ where } \xi \text{ is real,} \tag{10.8.10}$$

and observe that since $\{\psi^\ell, \psi^{\ell*}, V^g\}$ is a stationary point of the functional $\mathcal{E}_{\mathrm{stat}}$ we have

$$\frac{d}{d\xi}\mathcal{E}_{\mathrm{stat}}\left(\{\psi^\ell_\xi, \psi^{\ell*}_\xi, V^g_\xi, \nabla\psi^\ell_\xi, \nabla\psi^{\ell*}_\xi, \nabla V^g_\xi\}\right)\Big|_{\xi=1} = \sum_{n=0}^{\infty}\xi^{n-3}\mathcal{E}^{(n)}_{\mathrm{stat}}\Big|_{\xi=1} \tag{10.8.11}$$
$$= \sum_{n=0}^{\infty}(n-3)\,\mathcal{E}^{(n)}_{\mathrm{stat}}\left(\{\psi^\ell, \psi^{\ell*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell*}, \nabla V^g\}\right) = 0$$

In other words, for a static solution its energy components $\mathcal{E}^{(n)}_{\mathrm{stat}}$ always satisfy the identity

$$\sum_{n=0}^{\infty}(n-3)\,\mathcal{E}^{(n)}_{\mathrm{stat}} = 0. \tag{10.8.12}$$

Very often the density $\mathring{\mathcal{U}}$ of the potential energy depends on the field derivatives so that

$$\mathring{\mathcal{U}}\left(\{\psi^\ell, \psi^{\ell*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell*}, \nabla V^g\}\right) = \tag{10.8.13}$$
$$= \mathring{\mathcal{U}}^{(2)}\left(\{\psi^\ell, \psi^{\ell*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell*}, \nabla V^g\}\right) + \mathring{\mathcal{U}}^{(0)}\left(\{\psi^\ell, \psi^{\ell*}, V^g\}\right).$$

where $U^{(2)}$ satisfies the following identity for any real θ

$$\mathring{\mathcal{U}}^{(2)}\left(\{\psi^\ell, \psi^{\ell*}, V^g, \theta\nabla\psi^\ell, \theta\nabla\psi^{\ell*}, \theta\nabla V^g\}\right) = \theta^2\mathring{\mathcal{U}}^{(2)}\left(\{\psi^\ell, \psi^{\ell*}, V^g, \nabla\psi^\ell, \nabla\psi^{\ell*}, \nabla V^g\}\right) \tag{10.8.14}$$

In this case the identity (10.8.12) turns into the following important identity for the two constituting components $\mathcal{E}^{(2)}_{\mathrm{stat}}$ and $\mathcal{E}^{(0)}_{\mathrm{stat}}$ of the total potential energy $\mathcal{E}_{\mathrm{stat}}$:

$$\mathcal{E}_{\mathrm{stat}} = \mathcal{E}^{(2)}_{\mathrm{stat}} + \mathcal{E}^{(0)}_{\mathrm{stat}},\ \ \mathcal{E}^{(0)}_{\mathrm{stat}} = -\frac{1}{3}\mathcal{E}^{(2)}_{\mathrm{stat}} \text{ implying } \mathcal{E}_{\mathrm{stat}} = \frac{2}{3}\mathcal{E}^{(2)}_{\mathrm{stat}}. \tag{10.8.15}$$

The significance of the above identity for our goals is that in the cases of interest the energy component $\mathcal{E}^{(0)}_{\mathrm{stat}}$ accounts for the energy of nonlinear self-interactions and the formula $\mathcal{E}_{\mathrm{stat}} = \frac{2}{3}\mathcal{E}^{(2)}_{\mathrm{stat}}$ shows the *total energy has a representation that does not depend explicitly on the nonlinear self-interactions. This is one among other properties allowing us to characterize the introduced nonlinear self-interactions as stealthy.*

The identity (10.8.15) for a single field is known as the *Pokhozhaev-Derrick identity*, [Pokhozhaev], [Derrick] (see also [Kapitanskii] and [Coleman 1, Section 2.4]). It was often used to prove the nonexistence of nonzero solutions to the corresponding field equations in situations when a priory the both energies $\mathcal{E}^{(2)}_{\mathrm{stat}}$ and $\mathcal{E}^{(0)}_{\mathrm{stat}}$ are nonnegative and vanish for the zero field. Indeed if the nonnegativity of the energy components is combined with the identity (10.8.15) the both energies $\mathcal{E}^{(2)}_{\mathrm{stat}}$ and $\mathcal{E}^{(0)}_{\mathrm{stat}}$ must vanish implying that the field must vanish as well.

10.8.1 Time-harmonic fields

The above statements for static fields can be generalized for the case when complex valued fields ψ^ℓ are time harmonic, namely when

$$\psi^\ell = \mathrm{e}^{-\mathrm{i}\omega^\ell t}\tilde{\psi}^\ell,\ \ \psi^{\ell*} = \mathrm{e}^{\mathrm{i}\omega^\ell t}\tilde{\psi}^{\ell*}, \tag{10.8.16}$$

where $\tilde{\psi}^{\ell}$ and $\tilde{\psi}^{\ell*}$ are static, i.e. time independent. We provide such a generalization for the Lagrangian of the form (10.5.5)-(10.5.6) describing many charges coupled with the EM field $F^{\mu\nu}$, i.e.

$$\mathcal{L}\left(\left\{\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right\},A^{\mu}\right)=\sum_{\ell}L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right)-\frac{F^{\mu\nu}F_{\mu\nu}}{16\pi},\tag{10.8.17}$$

with an additional assumption on the charge Lagrangians L^{ℓ} to be of the form

$$L^{\ell}\left(\psi^{\ell},\psi^{\ell}_{;\mu},\psi^{\ell*},\psi^{\ell*}_{;\mu}\right)=K^{\ell}\left(\psi^{\ell*}\psi^{\ell},\psi^{\ell*}_{;\mu}\psi^{\ell;\mu}\right),\tag{10.8.18}$$

where $K^{\ell}(a,b)$ is a function of real variables a and b. In the case of interest represented by the Lagrangian (10.8.17)-(10.8.18) we add to the assumption (10.8.16) an assumption that the EM field is static namely

$$\partial_t\varphi=0,\ \ \mathbf{A}=\mathbf{0}.\tag{10.8.19}$$

Treating the equalities (10.8.16) as variables change let us recast the charges Lagrangians in the new variable $\tilde{\psi}^{\ell}$ and $\tilde{\psi}^{\ell*}$. Notice first that

$$\psi^{\ell*}\psi^{\ell}=\tilde{\psi}^{\ell*}\tilde{\psi}^{\ell}.\tag{10.8.20}$$

Then using (10.1.1), (10.1.2) and (10.1.5) and (10.4.3) we obtain

$$\begin{gathered}\psi^{\ell*}_{;0}=\tilde{\partial}^{\ell*}_{\mu}\psi^{\ell*}=\mathrm{e}^{\mathrm{i}\omega^{\ell}t}\left(\frac{\partial_t}{\mathrm{c}}+\mathrm{i}\frac{\omega^{\ell}}{\mathrm{c}}-\frac{\mathrm{i}q^{\ell}\varphi}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell*}=\mathrm{e}^{\mathrm{i}\omega^{\ell}t}\left(\partial_0+\mathrm{i}\frac{\omega^{\ell}}{\mathrm{c}}-\frac{\mathrm{i}q^{\ell}\varphi}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell*},\\ \psi^{\ell;0}=\tilde{\partial}^{\ell\mu}\psi^{\ell}=\mathrm{e}^{-\mathrm{i}\omega^{\ell}t}\left(\frac{\partial_t}{\mathrm{c}}-\mathrm{i}\frac{\omega^{\ell}}{\mathrm{c}}+\frac{\mathrm{i}q^{\ell}\varphi}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell}=\mathrm{e}^{-\mathrm{i}\omega^{\ell}t}\left(\partial_0-\mathrm{i}\frac{\omega^{\ell}}{\mathrm{c}}+\frac{\mathrm{i}q^{\ell}\varphi}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell},\\ \psi^{\ell*}_{;j}=\tilde{\partial}^{\ell*}_{j}\psi^{\ell*}=\mathrm{e}^{\mathrm{i}\omega^{\ell}t}\left(\partial_j-\frac{\mathrm{i}q^{\ell}A_j}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell*},\ \ \psi^{\ell;j}=\tilde{\partial}^{\ell\mu}\psi^{\ell}=\mathrm{e}^{-\mathrm{i}\omega^{\ell}t}\left(\partial^j+\frac{\mathrm{i}q^{\ell}A_j}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell}.\end{gathered}\tag{10.8.21}$$

Observe that in the case when there is just a single charge the expressions (10.8.21) show that the time derivatives $\psi^{\ell*}_{;0}$ and $\psi^{\ell;0}$ are modified so as the potential φ is added a constant, namely

$$\varphi\rightarrow\varphi-\frac{\chi\omega}{q}.\tag{10.8.22}$$

Substituting (10.8.20) and (10.8.21) into the Lagrangian L^{ℓ} we get the Lagrangian which denote $L^{\omega\ell}$ as a function of the variables $\tilde{\psi}^{\ell}$ and $\tilde{\psi}^{\ell*}$ and we obtain

$$L^{\ell}_{\omega}=K^{\ell}\left(\tilde{\psi}^{\ell*}\tilde{\psi}^{\ell},\psi^{\ell*}_{;\mu}\psi^{\ell;\mu}\right),\ \text{where}\ \psi^{\ell*}_{;\mu}\psi^{\ell;\mu}=\tag{10.8.23}$$

$$=\left(\frac{\partial_t}{\mathrm{c}}+\frac{\mathrm{i}\omega^{\ell}}{\mathrm{c}}-\frac{\mathrm{i}q^{\ell}\varphi}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell*}\left(\frac{\partial_t}{\mathrm{c}}-\frac{\mathrm{i}\omega^{\ell}}{\mathrm{c}}+\frac{\mathrm{i}q^{\ell}\varphi}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell}-\left(\nabla+\frac{\mathrm{i}q^{\ell}\mathbf{A}}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell*}\cdot\left(\nabla-\frac{\mathrm{i}q^{\ell}\mathbf{A}}{\chi\mathrm{c}}\right)\tilde{\psi}^{\ell}.\tag{10.8.24}$$

We can apply now to the Lagrangian (10.8.17), (10.8.24) as a function of the fields $\tilde{\psi}^{\ell}$, $\tilde{\psi}^{\ell*}$ and A^{μ} the obtained above results for the static fields taking into account also the assumption (10.8.19) for the EM field to static. In this case the static regime is characterized by the time independence of the charge fields $\tilde{\psi}^{\ell}$, $\tilde{\psi}^{\ell *}$ and the assumption (10.8.19) on the potential A^{μ} and these conditions which are abbreviated by the symbol stat:

$$\mathrm{stat} \equiv \partial_t \tilde{\psi}^{\ell} = 0,\ \partial_t \tilde{\psi}^{\ell *} = 0,\ \partial_t \varphi = 0,\ \mathbf{A} = \mathbf{0},\ A^{\mu} = (\varphi, \mathbf{A}) . \tag{10.8.25}$$

Hence the Lagrangian of interest now is

$$\mathcal{L}_{\omega}\left(\left\{\psi^{\ell}, \psi^{\ell}_{;\mu}, \psi^{\ell *}, \psi^{\ell *}_{;\mu}\right\}, A^{\mu}\right) = \sum_{\ell} L^{\ell}_{\omega}\left(\psi^{\ell}, \psi^{\ell}_{;\mu}, \psi^{\ell *}, \psi^{\ell *}_{;\mu}\right) - \frac{F^{\mu\nu} F_{\mu\nu}}{16\pi}. \tag{10.8.26}$$

Now applying the formula (10.8.5) to the Lagrangian $\mathcal{L}_{\omega}$, as defined by (10.8.26), (10.8.18), (10.8.23)-(10.8.24), and the formula (10.4.17) for the Lagrangian of EM field we obtain the following expression for the energy density $\mathring{\mathcal{U}}_{\omega\,\mathrm{stat}}$ of the system of charges and EM field:

$$\mathring{\mathcal{U}}_{\omega\,\mathrm{stat}} = \mathring{\mathcal{T}}^{00}_{\omega}\Big|_{\mathrm{stat}} \tag{10.8.27}$$
$$= -\frac{(\nabla\varphi)^2}{8\pi} - \sum_{\ell} K^{\ell}\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}, \left(\frac{\omega^{\ell}}{\mathrm{c}} - \frac{q^{\ell}\varphi}{\chi \mathrm{c}}\right)^2 \tilde{\psi}^{\ell *}\tilde{\psi}^{\ell} - \nabla\tilde{\psi}^{\ell *}\cdot\nabla\tilde{\psi}^{\ell}\right),$$

Let us take now the function $K^{\ell}(a,b)$ to be of a more special form

$$K^{\ell}(a,b) = k^{\ell}_2(a)\, b + k^{\ell}_0(a),\ \text{implying} \tag{10.8.28}$$
$$L^{\ell}_{\omega} = K^{\ell}\left(\psi^{\ell *}\psi^{\ell}, \psi^{\ell *}_{;\mu}\psi^{\ell;\mu}\right) = k^{\ell}_2\left(\psi^{\ell *}\psi^{\ell}\right)\psi^{\ell *}_{;\mu}\psi^{\ell;\mu} + k^{\ell}_0\left(\psi^{\ell *}\psi^{\ell}\right). \tag{10.8.29}$$

Notice that the term $k^{\ell}_0(a)$ in the cases of interest contains the nonlinear self-interaction. In the special case (10.8.28) the expression (10.8.27) for the total energy density of the charges and the EM field takes the form

$$\mathring{\mathcal{U}}_{\omega\,\mathrm{stat}} = \mathring{\mathcal{U}}^{(2)}_{\omega\,\mathrm{stat}} + \mathring{\mathcal{U}}^{(0)}_{\omega\,\mathrm{stat}},\ \text{where} \tag{10.8.30}$$
$$\mathring{\mathcal{U}}^{(2)}_{\omega\,\mathrm{stat}} = -\frac{(\nabla\varphi)^2}{8\pi} + \sum_{\ell} k^{\ell}_2\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}\right)\nabla\tilde{\psi}^{\ell *}\cdot\nabla\tilde{\psi}^{\ell},$$
$$\mathring{\mathcal{U}}^{(0)}_{\omega\,\mathrm{stat}} = -\sum_{\ell}\left[k^{\ell}_2\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}\right)\left(\frac{\omega^{\ell}}{\mathrm{c}} - \frac{q^{\ell}\varphi}{\chi \mathrm{c}}\right)^2 \tilde{\psi}^{\ell *}\tilde{\psi}^{\ell} + k^{\ell}_0\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}\right)\right].$$

The corresponding expression for the total energy is

$$\mathcal{E}_{\omega\,\mathrm{stat}} = \mathcal{E}^{(2)}_{\omega\,\mathrm{stat}} + \mathcal{E}^{(0)}_{\omega\,\mathrm{stat}},\ \text{where} \tag{10.8.31}$$
$$\mathcal{E}^{(2)}_{\omega\,\mathrm{stat}} = \int_{\mathbb{R}^3}\left[-\frac{(\nabla\varphi)^2}{8\pi} + \sum_{\ell} k^{\ell}_2\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}\right)\nabla\tilde{\psi}^{\ell *}\cdot\nabla\tilde{\psi}^{\ell}\right]\mathrm{d}x, \tag{10.8.32}$$
$$\mathcal{E}^{(0)}_{\omega\,\mathrm{stat}} = -\sum_{\ell}\int_{\mathbb{R}^3}\left[k^{\ell}_2\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}\right)\left(\frac{\omega^{\ell}}{\mathrm{c}} - \frac{q^{\ell}}{\chi \mathrm{c}}\varphi\right)^2 \tilde{\psi}^{\ell *}\tilde{\psi}^{\ell} + k^{\ell}_0\left(\tilde{\psi}^{\ell *}\tilde{\psi}^{\ell}\right)\right]\mathrm{d}x. \tag{10.8.33}$$

Applying now the formula (10.8.15) we get

$$\mathcal{E}^{(0)}_{\omega\,\mathrm{stat}} = -\frac{1}{3}\mathcal{E}^{(2)}_{\omega\,\mathrm{stat}}. \tag{10.8.34}$$

implying the following representation for the total system energy

$$\mathcal{E}_{\omega\,\mathrm{stat}}=\frac{2}{3}\mathcal{E}_{\omega\,\mathrm{stat}}^{(2)}=\frac{2}{3}\int_{\mathbb{R}^3}\left[-\frac{(\nabla\varphi)^2}{8\pi}+\sum_\ell k_2^\ell\left(\psi^{\ell*}\psi^\ell\right)\nabla\tilde{\psi}^{\ell*}\cdot\nabla\tilde{\psi}^\ell\right]\mathrm{d}x. \tag{10.8.35}$$

In the case of a single charge the above formula turns into

$$\mathcal{E}_{\omega\,\mathrm{stat}}=\frac{2}{3}\mathcal{E}_{\omega\,\mathrm{stat}}^{(2)}=\frac{2}{3}\int_{\mathbb{R}^3}\left[-\frac{(\nabla\varphi)^2}{8\pi}+k_2\left(\psi^*\psi\right)\nabla\tilde{\psi}^*\cdot\nabla\tilde{\psi}\right]\mathrm{d}x. \tag{10.8.36}$$

We want to emphasize once more the importance of the representation (10.8.35) in comparison with the original formula (10.8.31)-(10.8.33), which shows that the *total energy of the system of charges interacting with EM field does not explicitly depend on the terms $k_0^\ell\left(\psi^{\ell*}\psi^\ell\right)$ which include the nonlinear self-interactions.*

We would like to point out now that so far we carried computation for the energy computation for the Lagrangian $\mathcal{L}_\omega$, as defined by (10.8.26), (10.8.18), (10.8.23)-(10.8.24). But, in fact, what we really need is the energy for fields of the form (10.8.16) under static conditions (10.8.25) for the initial Lagrangian $\mathcal{L}$ as defined (10.8.17)-(10.8.18). In turns out, as one may expect, the difference between the two is just the sum of the rest energies. Indeed, using once more the formula (10.5.3) for the canonical energy $\mathring{\mathcal{T}}^{00}$ under static conditions (10.8.25), the formulas (10.8.5), (10.8.27) together with the formulas (10.5.14), (3.0.10) for microcharge density ρ^ℓ we obtain

$$\begin{aligned}&\mathring{\mathcal{U}}\left(\mathrm{e}^{-\mathrm{i}\omega^\ell t}\tilde{\psi}^\ell,\ \mathrm{e}^{\mathrm{i}\omega^\ell t}\tilde{\psi}^{\ell*},\varphi\right)= \qquad (10.8.37)\\ &=\sum_\ell\frac{\partial\mathcal{L}}{\partial\psi^\ell_{,0}}\left(-\mathrm{i}\frac{\omega^\ell}{\mathrm{c}}\tilde{\psi}^\ell\right)+\frac{\partial\mathcal{L}}{\partial\psi^{\ell*}_{,\mu}}\left(\mathrm{i}\frac{\omega^\ell}{\mathrm{c}}\tilde{\psi}^{\ell*}\right)-\mathcal{L}\left(\mathrm{e}^{-\mathrm{i}\omega^\ell t}\tilde{\psi}^\ell,\ \mathrm{e}^{\mathrm{i}\omega^\ell t}\tilde{\psi}^{\ell*},\varphi\right)\\ &=\sum_\ell-\mathrm{i}\frac{\omega^\ell}{\mathrm{c}}\left(\frac{\partial\mathcal{L}}{\partial\psi^\ell_{,0}}\tilde{\psi}^\ell-\frac{\partial\mathcal{L}}{\partial\psi^{\ell*}_{,\mu}}\tilde{\psi}^{\ell*}\right)+\left.\mathring{\mathcal{T}}^{00}_\omega\right|_{\mathrm{stat}}=\\ &=\sum_\ell\frac{\omega^\ell\chi}{q}\rho^\ell+\mathring{\mathcal{U}}_{\omega\,\mathrm{stat}}=\sum_\ell\frac{m^\ell\mathrm{c}^2}{q}\rho^\ell+\mathring{\mathcal{U}}_{\omega\,\mathrm{stat}}\end{aligned}$$

Now integrating the above density over the entire space and using the micro-charge normalization condition (2.0.18) obtain

$$\mathcal{E}\left(\mathrm{e}^{-\mathrm{i}\omega^\ell t}\tilde{\psi}^\ell,\ \mathrm{e}^{\mathrm{i}\omega^\ell t}\tilde{\psi}^{\ell*},\varphi\right)=\sum_\ell m^\ell\mathrm{c}^2+\mathcal{E}_{\omega\,\mathrm{stat}}. \tag{10.8.38}$$

For the special case (10.8.28) combining the last formula with formulas (10.8.35), (10.8.36) we obtain the following important formulas for respectively many charges and a single charge

$$\mathcal{E}\left(\mathrm{e}^{-\mathrm{i}\omega^\ell t}\tilde{\psi}^\ell,\ \mathrm{e}^{\mathrm{i}\omega^\ell t}\tilde{\psi}^{\ell*},\varphi\right)=\sum_\ell m^\ell\mathrm{c}^2+\frac{2}{3}\int_{\mathbb{R}^3}\left[-\frac{(\nabla\varphi)^2}{8\pi}+\sum_\ell k_2^\ell\left(\psi^{\ell*}\psi^\ell\right)\nabla\tilde{\psi}^{\ell*}\cdot\nabla\tilde{\psi}^\ell\right]\mathrm{d}x \tag{10.8.39}$$

$$\mathcal{E}\left(\mathrm{e}^{-\mathrm{i}\omega t}\tilde{\psi},\ \mathrm{e}^{\mathrm{i}\omega t}\tilde{\psi}^*,\varphi\right)=mc^2+\frac{2}{3}\int_{\mathbb{R}^3}\left[-\frac{(\nabla\varphi)^2}{8\pi}+k_2\left(\psi^*\psi\right)\nabla\tilde{\psi}^*\cdot\nabla\tilde{\psi}\right]\mathrm{d}x. \tag{10.8.40}$$

The formulas (10.8.39) and (10.8.40) give important representation for the energy of time harmonic fields which does not explicitly involve the nonlinear self-interactions.

10.9 Compressional waves in nonviscous compressible fluid

In this section following to [Morse Feshbach 1, Section 3.3] and [Morse Ingard, Section 6.2] we consider here compressional waves in nonviscous and compressible fluid which are described by the pressure field p and velocity field $\mathbf{v}$ and governed by the following system of equations

$$\rho\partial_t\mathbf{v} = -\nabla p,\ \kappa\partial_t p = -\nabla\cdot\mathbf{v},\ \mathrm{c}^2 = \frac{1}{\rho\kappa} \tag{10.9.1}$$

where ρ and κ are respectively uniform constant mass density and compressibility (adiabatic) of the fluid at equilibrium and c is the velocity of wave propagation. We also have

$$\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} \text{ is the kinetic energy and } \frac{1}{2}\kappa p^2 \text{ is the potential energy} \tag{10.9.2}$$

Then we if introduce the velocity potential ψ so that

$$p = \rho\partial_t\psi,\ \mathbf{v} = -\nabla\psi, \tag{10.9.3}$$

it immediately follows from (10.9.3) that ψ satisfies the classical wave equation

$$\frac{1}{\mathrm{c}^2}\partial_t^2\psi - \nabla^2\psi = 0. \tag{10.9.4}$$

The compressional waves have the following Lagrangian density

$$L = \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} - \frac{1}{2}\kappa p^2 = \frac{1}{2}\rho\left[\frac{1}{\mathrm{c}^2}(\partial_t\psi)^2 - (\nabla\psi)^2\right] \tag{10.9.5}$$

and the following canonical energy-momentum tensor

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & \rho\partial_0\psi\partial_1\psi & \rho\partial_0\psi\partial_2\psi & \rho\partial_0\psi\partial_3\psi \\ \rho\partial_1\psi\partial_0\psi & T^{11} & -\rho\partial_1\psi\partial_2\psi & -\rho\partial_1\psi\partial_3\psi \\ \rho\partial_2\psi\partial_0\psi & -\rho\partial_2\psi\partial_1\psi & T^{22} & -\rho\partial_2\psi\partial_3\psi \\ \rho\partial_3\psi\partial_0\psi & -\rho\partial_3\psi\partial_1\psi & -\rho\partial_3\psi\partial_2\psi & T^{33} \end{bmatrix},\ \partial_0 = \frac{1}{\mathrm{c}}\partial_t, \tag{10.9.6}$$

$$T^{00} = \frac{\rho}{2}\left[(\partial_0\psi)^2 + (\nabla\psi)^2\right],\ T^{jj} = \frac{\rho}{2}\left[(\nabla\psi)^2 - 2(\partial_j\psi)^2 - (\partial_0\psi)^2\right].$$

10.10 Klein-Gordon equation and Yukawa potential

Klein-Gordon equation is well known model for a free charge, [Pauli PWM, Section 18]. In particular, its certain modification describes a charge interacting with an external EM field, [Schwabl, Section 8.1]. Here we follow to [Martin, Section 1.5.2]. If the spin is neglected a freely propagating particle X of the rest mass m_X is described by a complex-valued wave function $\varphi(\mathbf{r})$ satisfying the *Klein-Gordon equation*

$$-\frac{1}{\mathrm{c}^2}\partial_t^2\varphi = \left\{-\Delta + \left(\frac{m_X\mathrm{c}}{\hbar}\right)^2\right\}\varphi. \tag{10.10.1}$$

This equation is obtained from the fundamental relativistic mass-energy relation

$$\frac{E^2}{\mathrm{c}^2} = \mathbf{p}^2 + m_X\mathrm{c}^2, \tag{10.10.2}$$

where E is the particle energy, $\mathbf{p}$ is the three-dimensional space momentum, by the substitution $E = \hbar\partial_t$ and $\mathbf{p} = -\hbar\nabla_{\mathbf{r}}$. A static solution V to the Klein-Gordon equation (10.10.1) with a δ-function source, i.e.

$$\left\{-\Delta + \mu^2\right\} V = -g^2\delta\left(\mathbf{x}\right), \tag{10.10.3}$$

is called the *Yukawa potential*

$$V\left(|\mathbf{x}|\right) = -\frac{g^2}{4\pi}\frac{\mathrm{e}^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} = -\left(\mu^2 - \Delta\right)^{-1} g^2\delta\left(\mathbf{x}\right), \ \mu = \frac{m_X \mathrm{c}}{\hbar}, \ . \tag{10.10.4}$$

The quantity $\mu^{-1} = \frac{\hbar}{mc}$ is called the *range of the potential* V, is and it is also known as the *Compton wavelength* of the relativistic particle of the mass m_X. The constant g is a so-called coupling constant representing the basic strength of the interaction.

There is an interpretation of Klein-Gordon equation as a flexible string with additional stiffness forces provided by the medium surrounding it. Namely, if the string is embedded in a thin sheet of rubber or if it is along the axis of a cylinder of rubber whose outside surface kept fixed, [Morse Feshbach 1, Section 2.1].

10.11 Schrödinger Equation

The Schrödinger equation with the potential V is

$$\hbar\mathrm{i}\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi. \tag{10.11.1}$$

It is the Euler-Lagrange field equation (together with its conjugate) for the following Lagrangian, [Morse Feshbach 1, (3.3.20)]

$$L = \mathrm{i}\frac{\hbar}{2}\left(\psi^*\partial_t\psi - \partial_t\psi^*\psi\right) - \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi - \psi^* V\psi. \tag{10.11.2}$$

The stress-tensor here

$$\mathring{T}^{\mu\nu} = \frac{\partial L}{\partial\psi_{,\mu}}\psi^{,\nu} + \frac{\partial L}{\partial\psi^*_{,\mu}}\psi^{*,\nu} - \delta^{\mu\nu}L, \tag{10.11.3}$$

implying the following formula for the energy density

$$H = \mathring{T}^{00} = -\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + \psi^* V\psi. \tag{10.11.4}$$

The energy flow vector $\mathbf{S}$, the momentum density vector $\mathbf{P}$ and the current density vector $\mathbf{J}$ for the Schrödinger equation (10.11.1) are respectively, [Morse Feshbach 1, (3.3.25), (3.3.26)],

$$\mathbf{S} = -\frac{\hbar^2}{2m}\left[\partial_t\psi^*\cdot\nabla\psi + \partial_t\psi\cdot\nabla\psi^*\right], \ \mathbf{P} = \mathrm{i}\frac{\hbar}{2}\left[\psi^*\cdot\nabla\psi - \psi\cdot\nabla\psi^*\right], \ \mathbf{J} = -\frac{q}{m}\mathbf{P}, \tag{10.11.5}$$

with the equation of continuity $\partial_t H + \nabla\cdot\mathbf{S} = 0$.

Quantum mechanical charged particle in an external EM field with the 4-potential $A^\mu = (\varphi, \mathbf{A})$ is described by the following Schrödinger equation, [Morse Feshbach 1, (2.6.47)]

$$\hbar\mathrm{i}\partial_t\psi = \frac{1}{2m}\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{\mathrm{c}}\mathbf{A}\right)\cdot\left(\frac{\hbar}{\mathrm{i}}\nabla - \frac{q}{\mathrm{c}}\mathbf{A}\right)\psi + q\varphi\psi \tag{10.11.6}$$

or

$$\hbar\mathrm{i}\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \mathrm{i}\frac{q\hbar}{2mc}\mathbf{A}\cdot\nabla\psi + \left[\frac{q^2\left|\mathbf{A}\right|^2}{2mc^2} + q\varphi\right]\psi, \tag{10.11.7}$$

with the charge density $\rho = q\psi\psi^*$ and the current $\mathbf{J}$ as follows, [Morse Feshbach 1, (2.6.46)]

$$\rho = q\psi\psi^*,\ \mathbf{J} = \mathrm{i}\frac{q\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right). \tag{10.11.8}$$

The quantities ρ and $\mathbf{J}$ satisfy the continuity equation

$$\partial_t\rho + \nabla\cdot\mathbf{J} = 0. \tag{10.11.9}$$

11 Appendix: Fourier transforms and Green functions

The polar coordinates representation of the Laplace operator, [Taylor 1, (4.56)], is

$$\Delta = \Delta_r + \frac{1}{r^2}\Delta_s = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_s,\ \mathbf{x}\in\mathbb{R}^3,\ r = \left|\mathbf{x}\right|, \tag{11.0.10}$$

where Δ_s is the Laplace operator on the unit sphere $\mathbb{S}^2$. We also have $n = 3$, [Taylor 1, (5,53)-(5,59)],

$$\left(\kappa^2 - \Delta\right)^{-1}\delta\left(\mathbf{x}\right) = \frac{\mathrm{e}^{-\kappa\left|\mathbf{x}\right|}}{4\pi\left|\mathbf{x}\right|},\ \kappa\geq 0. \tag{11.0.11}$$

Notice that the action of the operator Δ on radial functions $g\left(r\right)$, i.e. functions depending on $r = |x|$, is reduced to the action of Δ_r only for smooth functions, i.e.

$$\Delta g\left(r\right) = \Delta_r g\left(r\right)\ \text{ if } g\left(r\right)\ \text{ is continuos and smooth for } r\geq 0. \tag{11.0.12}$$

Indeed, in view of (11.0.11)

$$\Delta_r\frac{1}{r} = 0,\ \text{whereas}\ \Delta\frac{1}{r} = -4\pi\delta\left(\mathbf{x}\right). \tag{11.0.13}$$

Let us consider the Fourier transform of radial functions following to [Taylor 1, Section 3.6]:

$$\hat{f}\left(\mathbf{k}\right) = \hat{f}\left(\left|\mathbf{k}\right|\right) = \frac{1}{\left(2\pi\right)^{\frac{3}{2}}}\int_0^\infty f\left(r\right)\psi_d\left(r\left|\mathbf{k}\right|\right)r^{n-1}\,\mathrm{d}r,\ \psi_d\left(\left|\mathbf{k}\right|\right) = \int_{\left|\mathbf{x}\right|=1}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}s. \tag{11.0.14}$$

Then the following identity holds

$$\hat{f}\left(\mathbf{k}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\left|\mathbf{k}\right|}\int_0^\infty f\left(r\right)\sin\left(r\left|\mathbf{k}\right|\right)\,\mathrm{d}r. \tag{11.0.15}$$

Let $w\left(\mathbf{x}\right)$, $\mathbf{x}\in\mathbb{R}^3$ be a real function satisfying

$$0\leq w\left(\mathbf{x}\right)\leq w_\infty < \infty. \tag{11.0.16}$$

Then the Green function $G\left(\mathbf{x},\mathbf{y}\right) = \left(-\Delta + w\right)^{-1}\left(\mathbf{x},\mathbf{y}\right)$ defined as a fundamental solution to the equation

$$\left(-\Delta + w\right)G\left(\mathbf{x},\mathbf{y}\right) = \delta\left(\mathbf{x}-\mathbf{y}\right), \tag{11.0.17}$$

satisfies the following inequalities

$$\frac{\mathrm{e}^{-\sqrt{w_\infty}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \leq (-\Delta+w_\infty)^{-1}(\mathbf{x},\mathbf{y}) \leq (-\Delta+w)^{-1}(\mathbf{x},\mathbf{y}) \leq \\ \leq (-\Delta)^{-1}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \tag{11.0.18}$$

which follow from the Feynman-Kac formula for the heat kernel, [Oksendal, Section 8.2] , applied to the operator $-\Delta+w$.

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