

Class-preserving automorphisms and the normalizer property for Blackburn groups

Martin Hertweck and Eric Jespers

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Abstract. For a group G , let \mathcal{U} be the group of units of the integral group ring $\mathbb{Z}G$. The group G is said to have the normalizer property if $N_{\mathcal{U}}(G) = Z(\mathcal{U})G$. It is shown that Blackburn groups have the normalizer property. These are the groups which have non-normal finite subgroups, with the intersection of all of them being non-trivial. Groups G for which class-preserving automorphisms are inner automorphisms, $\text{Out}_c(G) = 1$, have the normalizer property. Recently, Herman and Li have shown that $\text{Out}_c(G) = 1$ for a finite Blackburn group G . We show that $\text{Out}_c(G) = 1$ for the members G of certain classes of metabelian groups, from which the Herman–Li result follows.

Together with recent work of Hertweck, Iwaki, Jespers and Juriaans, our main result implies that, for an arbitrary group G , the group $Z_{\infty}(\mathcal{U})$ of hypercentral units of \mathcal{U} is contained in $Z(\mathcal{U})G$.

1 Introduction

A group G is said to have the *normalizer property* if $N_{\mathcal{U}}(G) = Z(\mathcal{U})G$, where \mathcal{U} denotes the group of units of the integral group ring $\mathbb{Z}G$, $Z(\mathcal{U})$ denotes the center of \mathcal{U} and $N_{\mathcal{U}}(G)$ is the normalizer of G in \mathcal{U} . Informally speaking, this means that G is normalized only by units of \mathcal{U} which obviously do so. Sometimes this definition is slightly tightened to incorporate G -adapted coefficient rings: these are integral domains of characteristic zero in which a rational prime p is not invertible whenever G has an element of order p . Throughout, we let R denote a G -adapted ring, write \mathcal{U} for the group of units of RG , and if $N_{\mathcal{U}}(G) = Z(\mathcal{U})G$, we say that G has the normalizer property. For classes of groups which are known to have the normalizer property, we refer the reader to [6] (and references therein).

The normalizer $N_{\mathcal{U}}(G)$ appears, among other things, in the study of the group $Z_{\infty}(\mathcal{U})$ of hypercentral units of \mathcal{U} , defined by $Z_{\infty}(\mathcal{U}) = \bigcup_{n=1}^{\infty} Z_n(\mathcal{U})$, where $Z_n(\mathcal{U})$

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denotes the n th term of the upper central series of \mathcal{U} . The question whether in general $Z_\infty(\mathcal{U}) \leq N_{\mathcal{U}}(G)$ was finally answered in the affirmative in [8]. As a consequence of the present work we obtain that $Z_\infty(\mathcal{U}) \leq Z(\mathcal{U})G$. We remark that this is known to hold for G torsion-free [6, Corollary 4.3], and for groups G whose finite subgroups are all normal [6, Corollary 4.12]. If G has finite non-normal subgroups, then, following Blackburn [2], we denote by $R(G)$ the intersection of all of them and say that $R(G)$ is defined. Now, if $R(G)$ is defined and $R(G) = 1$, then $Z_\infty(\mathcal{U}) = Z(\mathcal{U})$ by [6, Propositions 4.1 and 4.5]. It remains to consider the *Blackburn groups*, i.e., the groups G for which $R(G)$ is defined and non-trivial. Here, it is shown that Blackburn groups have the normalizer property.

We write $\text{Aut}_c(G)$ for the group of class-preserving automorphisms of G , and set $\text{Out}_c(G) = \text{Aut}_c(G)/\text{Inn}(G)$. Any u in $N_{\mathcal{U}}(G)$ gives rise—via conjugation—to a class-preserving automorphism of G (see [6, Proposition 2.6]). Thus $\text{Out}_c(G) = 1$ is a sufficient condition for G to have the normalizer property. In practice, this has proven to be a valuable criterion. We must point out, however, that we do not know whether class-preserving automorphisms of infinite Blackburn groups are inner automorphisms.

The finite Blackburn groups were classified in [2]. Li, Sehgal and Parmenter [9, Theorem 1] have shown by inspection that the finite Blackburn groups have the normalizer property. In fact, $\text{Out}_c(G) = 1$ for a finite Blackburn group G . Herman and Li noted that the proof given in [5, 22.4 Proposition] contains a gap, and completed it in [3] (Lemma 2.8 below also fixes the gap).

In Section 2, we record that class-preserving automorphisms of finite abelian-by-cyclic groups are inner automorphisms (see Proposition 2.7). Together with an *ad hoc* observation (Lemma 2.8), the last-mentioned result again follows.

In Section 3, results from [6, §3] are applied to show that infinite Blackburn groups have the normalizer property (Theorem 3.3). In the proof, an essential feature will be that the structure of the finite normal subgroups of Blackburn groups is known since these are either Dedekind groups or again Blackburn groups.

Finally, let us recall Blackburn's classification [2]. By [2, Theorem 1], a finite p -group G which is a Blackburn group is a 2-group, and either a Q-group or of the form $Q_8 \times C_4 \times E_2$ or $Q_8 \times Q_8 \times E_2$, where Q_8 is the quaternion group, E_2 is an elementary abelian 2-group and C_n denotes the cyclic group of order n . (A group G is called a *Q-group* if G has an abelian subgroup A of index 2 which is not elementary abelian, and $G = \langle A, b \rangle$ for some $b \in G$ of order 4 with $x^b = x^{-1}$ for all $x \in A$.) A finite Blackburn group which is not a p -group belongs to one of five classes which are listed in [2, Theorem 2]. In particular, such a group is the direct product of groups which are abelian or possess an abelian normal subgroup with factor group cyclic or isomorphic to Klein's four group.

2 On class-preserving automorphisms

In this section, only finite groups are considered. We present classes of metabelian groups for which class-preserving automorphisms are inner automorphisms, and conclude by showing that the finite Blackburn groups belong to them.

We first prove that class-preserving automorphisms of finite abelian-by-cyclic groups are inner automorphisms. For the class of metacyclic groups, this is an elementary exercise. Also, the following example, used in the proof of Lemma 2.8, is easily verified (see [6, Example 3.1]).

Example 2.1. A group G with an abelian subgroup of index 2 satisfies $\text{Out}_c(G) = 1$.

Our proof will be based on the following lemma, which is of interest in its own right.

Lemma 2.2. *Let G be a finite abelian p -group, and let α and β be automorphisms of G of p -power order. Assume that $\alpha\beta = \beta\alpha$ and that for each $g \in G$, there is $n \in \mathbb{N}$ such that $g\beta = g\alpha^n$. Then β is a power of α .*

Proof. Assume that G is a counter-example, with the order of the semidirect product $G\langle\alpha\rangle$ being minimal. Let Z be a central subgroup of order p in $G\langle\alpha\rangle$ which is contained in G . Then β centralizes Z . Thus α and β induce automorphisms $\bar{\alpha}$ and $\bar{\beta}$ of $\bar{G} = G/Z$, and $\bar{\beta}$ is a power of $\bar{\alpha}$ by minimality of $G\langle\alpha\rangle$. Hence we can assume that $\bar{\beta}$ is the identity. Then the map $G \rightarrow Z$, defined by $g \mapsto g^{-1}(g\beta)$, is a surjective homomorphism, with kernel K of index p . Clearly $K = C_G(\beta)$ and α fixes K since α and β commute. For all $g \in G$, there is $n(g) \in \mathbb{N}$ such that $g\beta = g\alpha^{n(g)}$. Choose $h \in G \setminus K$ such that $\alpha^{n(h)}$ is of maximal order among the $\alpha^{n(g)}$, $g \in G \setminus K$. By minimality of $G\langle\alpha\rangle$, we have $\langle\alpha\rangle = \langle\alpha^{n(h)}\rangle$. Thus we can assume that $G = \langle K, h \rangle$ and $h\beta = h\alpha$. Note that $h\beta = zh$ for a central element z of order p in Z .

For all $k \in K$, we have

$$kzh = (k\beta)(h\beta) = (kh)\beta = (kh)\alpha^{n(kh)} = k\alpha^{n(kh)}z^{n(kh)}h.$$

So $\alpha^{n(kh)}$ is not the identity and $k\alpha^{n(kh)} \in kZ$. Thus if α^q is a power of α having order p , then α^q induces the identity on G/Z . As above, it follows that $H := C_G(\alpha^q)$ is an α -invariant subgroup of G of index p . Also, H is fixed by β since β commutes with α . Thus by minimality of $G\langle\alpha\rangle$, the automorphism β agrees on H with some power α^l of α . Suppose that $h \notin H$. Then $h \neq h\alpha^q = z^q h$, so $p \nmid q$ and α has order p . Consequently $H = C_G(\alpha) \leq C_G(\beta) = K$, so $G = \langle h, H \rangle$ and $\beta = \alpha$, contradicting our assumption that G is a counter-example. Thus $h \in H$. It follows that $zh = h\beta = h\alpha^l$, and $\langle\alpha^l\rangle = \langle\alpha\rangle$ since h is a fixed point under α^p .

Since β induces the identity on K , it follows that α^l , and hence also α , induces the identity on $H \cap K$. Take any $k \in K \setminus H$. Since $H \cap K$ is of index p^2 in G , it follows that $G = \langle h, k, H \cap K \rangle$. Since α induces the identity on G/H (a quotient of order p) and K is α -invariant, we have $k\alpha = xk$ for some $x \in H \cap K$, and $x \neq 1$ since $\alpha \neq \beta$. Let x^m be a power of x having order p . Suppose that $\langle x^m \rangle = \langle z \rangle$, i.e., $x^m = z^{-1}$ for some $t \in \mathbb{N}$. Then $(hk^m)\alpha = hk^m$ (this is the crucial place where we need that G is abelian) but $(hk^m)\beta \neq hk^m$, a contradiction. Thus $\langle x \rangle \cap \langle z \rangle = 1$.

Since $x\alpha = x = x\beta$, the automorphisms α and β induce automorphisms of $G/\langle x^m \rangle$, and by minimality of $G\langle\alpha\rangle$, the automorphism induced by β equals the automor-

phism induced by some power α^j of α . Then $z^{j-1} = h\alpha^j(h\beta)^{-1} \in \langle x^m \rangle$ and thus $j = 1 + ps$ for some $s \in \mathbb{Z}$. Furthermore, $x^j = x^j k k^{-1} = k\alpha^j(k\beta)^{-1} \in \langle x^m \rangle$. So $\langle x \rangle = \langle x^j \rangle \leq \langle x^m \rangle$ and x has order p . Because $\langle x \rangle \cap \langle z \rangle = 1$, it follows that $(hk)\alpha^i = z^i h x^i k \neq z h k = (hk)\beta$ for all $i \in \mathbb{N}$, and we have reached a final contradiction. \square

In the above lemma, the assumption that G is abelian is necessary. In fact, the proof lends itself to the construction of adequate examples. Such examples yield p -groups with non-inner, class-preserving automorphisms.

Example 2.3. For an odd prime p , we construct a group G of order p^{p+2} which has automorphisms α and β satisfying the assumptions, but not the conclusion of the above lemma (for $p = 2$, see [5, 14.3 Remark]). First, we have to exhibit a certain kind of group K which cannot appear in the above (abelian group) setting. Let

$$\kappa = \begin{pmatrix} 1 & p & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 \\ 0 & & & \ddots & 1 & 1 \\ -1 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \text{GL}_{p-1}(\mathbb{Z}/p^2\mathbb{Z}),$$

an invertible $(p-1) \times (p-1)$ matrix (of order p^2) over $\mathbb{Z}/p^2\mathbb{Z}$, which acts on

$$A = \mathbb{Z}/p^2\mathbb{Z} \oplus \underbrace{p\mathbb{Z}/p^2\mathbb{Z} \oplus \cdots \oplus p\mathbb{Z}/p^2\mathbb{Z}}_{p-2 \text{ times}} \cong C_{p^2} \times C_p \times \cdots \times C_p$$

as an automorphism of order p , and has the property that $a(1 + \kappa + \cdots + \kappa^{p-1}) = 0$ for all $a \in A$ (cf. [7, III.10.15]). For completeness, we give the argument: The integer polynomials

$$f(t) = t^{p-2} + 2t^{p-3} + \cdots + (p-2)t + (p-1) \quad \text{and} \quad c(t) = t^{p-1} + t^{p-2} + \cdots + 1$$

satisfy $(t-1)f(t) = c(t) - p$. So, modulo p , we have $(t-1)^2 f(t) \equiv t^p - 1 \equiv (t-1)^p$ and $f(t) \equiv (t-1)^{p-2}$. Let $a \in A$. Then $pa(\kappa - 1) = 0$, and so

$$a(\kappa - 1)f(\kappa) = a(\kappa - 1)^{p-1}$$

by the established congruence. Note that $(\kappa - 1)^{p-1}$ is $-p$ times the identity matrix. Altogether, we obtain

$$a(1 + \kappa + \cdots + \kappa^{p-1}) = ac(\kappa) = a(\kappa - 1)f(\kappa) + pa = 0.$$

From this, it is obvious that κ acts as an automorphism of order p on A . We are ready to present the group construction. Let k be an (abstract) generator of a cyclic group of order p , and form the semidirect product $K = A\langle k \rangle$, with k acting on A via κ . Let x be a generator of the first factor of A , and set $z = x^p$, a central element of order p in K . By the mentioned property of κ (with $a = x$), the element xk has order p , so that an automorphism α of K is defined by prescribing that α fixes A point-wise, and $k\alpha = xk$.

Now let h be another (abstract) generator of a cyclic group of order p , and set $G = K \times \langle h \rangle$. Extend α to an automorphism of G by setting $h\alpha = zh$. Let β be the automorphism of G which fixes K point-wise, and sends h to zh . Certainly, α has order p^2 , β has order p , and both automorphisms commute. Note that $H := C_G(\alpha^p) = A \times \langle h \rangle$. Let $g \in G \setminus K$. If $g \in H$, then $g\beta = g\alpha$. Otherwise $g \in Ak^i h^j$ with $1 \leq i, j \leq p-1$, and $g\beta = g\alpha^{ps}$ for $s \in \mathbb{N}$ with $is \equiv j \pmod{p}$. However, β is not a power of α .

It readily follows that the automorphism of the semidirect product $G\langle \alpha \rangle$ which fixes α and agrees on G with β is a non-inner, class-preserving automorphism.

We shall need the following well-known lemma for which we were unable to find a suitable reference in the literature.

Lemma 2.4. *If N is a p -group on which an abelian p' -group H acts then $C_H(n_0) = C_H(N)$ for some $n_0 \in N$.*

Proof. Let \bar{N} be the quotient of N by its Frattini subgroup. Then $C_H(\bar{N}) = C_H(N)$ (see [1, (24.1)]). So it suffices to find $n_0 \in N$ with $C_H(\bar{n}_0) = C_H(\bar{N})$. Thus we can assume that N is elementary abelian. Form the semidirect product $X = HN$. By Maschke's theorem, $N = M_1 \times \cdots \times M_r$ with minimal normal subgroups M_i of X . Choose $m_i \in M_i \setminus \{1\}$ for each index i . Note that for $h \in H$, each subgroup $C_{M_i}(h)$ is normal in X since H is abelian. So if $h \in H$ with $m_i^h = m_i$ for some i then $C_{M_i}(h) \neq 1$ and $C_{M_i}(h) = M_i$ as M_i is a minimal normal subgroup. In other words, $C_H(m_i) = C_H(M_i)$ for all i . Hence for $h \in C_H(m_1 \dots m_r)$ we have $h \in \bigcap_{i=1}^r C_H(m_i) = \bigcap_{i=1}^r C_H(M_i) = C_H(N)$, and the lemma holds with $n_0 = m_1 \dots m_r$. \square

For the reader's convenience, we recall the following easy but useful observation (see [4, Remark 1]).

Remark 2.5. Let σ be a class-preserving automorphism of G of order a power of p , and let $\gamma \in \text{Inn}(G)$. Then $(\gamma\sigma)^r$, where r denotes the p' -part of the order of $\gamma\sigma$, is again a class-preserving automorphism of G of p -power order a power of p which is a non-inner automorphism if and only if σ is a non-inner automorphism (since $\langle \sigma^r \rangle = \langle \sigma \rangle$ and $(\gamma\sigma)^r$ and σ^r differ only by an inner automorphism).

Thus if one wishes to show that such a σ is an inner automorphism, and U is a subset of G which is conjugate to $U\sigma$ in G , one can assume in addition that $U = U\sigma$.

Now we are in a position to prove:

Proposition 2.6. *Let G be a finite group having an abelian normal subgroup A such that the quotient G/A has a normal cyclic Sylow p -subgroup, for some prime p . Then each class-preserving automorphism of G of p -power order is an inner automorphism.*

Proof. Let σ be a class-preserving automorphism of G of p -power order; we have to show that σ is an inner automorphism. Let P be a Sylow p -subgroup of G . By [4, Corollary 5] (applied with $N = PA$) we can assume that G/A is a p -group. Then we can choose $x \in P$ such that $G = \langle x, A \rangle$. By Sylow's theorem, and Remark 2.5, we can assume that $P\sigma = P$. Set $S = P \cap A = O_p(A)$ and $T = O_{p'}(A)$, so that $P = \langle x, S \rangle$ and $A = S \times T$.

Let $\gamma \in \text{Aut}(G)$ be the inner automorphism given by conjugation with x and set $H = \langle \sigma|_T, \gamma|_T \rangle \leq \text{Aut}(T)$. Note that for each $t \in T$ we have $t\sigma|_T = t(\gamma|_T)^n$ for some $n \in \mathbb{N}$, and that $x\sigma \in xA \leq xC_G(T)$. Thus H is an abelian p -group, $C_H(t_0) = C_H(T) = 1$ for some $t_0 \in T$ by Lemma 2.4, and $\sigma|_T$ is a power of $\gamma|_T$. So Remark 2.5 allows us to assume that $t\sigma = t$ for all $t \in T$.

Let y be a generator of $C_{\langle x \rangle}(t_0)$, and let δ be the inner automorphism given by conjugation with y . For each $s \in S$ there is $n(s) \in \mathbb{N}$ such that

$$(s\sigma)t_0 = (st_0)\sigma = (st_0)\gamma^{n(s)} = (s\gamma^{n(s)})(t_0\gamma^{n(s)}),$$

meaning that $\gamma^{n(s)} \in \langle \delta \rangle$ and $s\sigma = s\delta^{m(s)}$ for some $m(s) \in \mathbb{N}$. As before one sees that $\sigma|_S$ commutes with δ . Thus $\sigma|_S$ is a power of δ by Lemma 2.2, and we can modify σ according to Lemma 2.4 such that the new σ fixes A element-wise. Clearly $x\sigma = x^s$ for some $s \in S$ since $x\sigma \in P$, and then σ is the inner automorphism given by conjugation with s . \square

From this we obtain at once:

Proposition 2.7. *Let G be a finite group having an abelian normal subgroup A with cyclic quotient G/A . Then class-preserving automorphisms of G are inner automorphisms.*

Proof. Let σ be a class-preserving automorphism of G of p -power order, for some prime p ; we have to show that σ is an inner automorphism. By [4, Corollary 5] we can assume that p divides the order of G/A , and then Proposition 2.6 applies. \square

In preparation for the final result we record:

Lemma 2.8. *Suppose that a finite group G is a semidirect product of an abelian group A and a generalized quaternion group Q , and that each element of Q acts on A by raising each element to some fixed power. Suppose further that a Sylow 2-subgroup of G has an abelian subgroup of index 2. Then $\text{Out}_c(G) = 1$.*

Proof. Let $\sigma \in \text{Aut}_c(G)$. We shall show that $\sigma \in \text{Inn}(G)$ by induction on the order of G . By Sylow's theorem, we can assume that $S\sigma = S$ for a Sylow 2-subgroup S of G containing Q . Then σ induces a class-preserving automorphism of S since S can be viewed as a homomorphic image of G . Thus, by Example 2.1, we can assume that σ fixes S element-wise. Since the conjugation action of Q on A leaves each cyclic subgroup invariant, and the automorphism group of a cyclic group is abelian, the commutator subgroup Q' centralizes A .

Suppose that $A = M \times N$ with non-trivial normal subgroups M and N of G . Then we can assume inductively that σ induces an inner automorphism of G/N , say conjugation with Ng (some $g \in G$). Write $g = xa$ with $x \in Q$ and $a \in A$. Then for any $y \in Q$, since $Q \leq S$ and σ fixes S element-wise, we have $Ay = A(y\sigma) = Ay^{xa} = Ay^x$, i.e., $y^x y^{-1} \in Q \cap A = 1$. So $x \in Z(Q) \leq Q' \leq C_G(A)$. It follows that for $m \in M$, we have $N(m\sigma) = Nm^{xa} = Nm^a = Nm$. Hence σ induces the identity on G/N . Likewise, we obtain that σ induces the identity on G/M , so σ is the identity.

Thus we can assume that A is cyclic p -group, and by Example 2.1, we can assume that p is an odd prime. Let a be a generator of A . Note that $\text{Aut}(A)$ is cyclic, so we can choose, since Q/Q' is a Klein's four-group, an element $y \in Q \setminus Q'$ which commutes with A . Then $(ay)\sigma = (ay)^x$ for some $x \in Q$, that is, $a\sigma = a^x$ and $y = y\sigma = y^x$. The latter implies that $\langle x, y \rangle$ is a proper subgroup of Q , so $x \in Q' \cup Q'y$ and $a\sigma = a^x = a$. Again, we have shown that σ is the identity, and the proof is complete. \square

Now we can go through Blackburn's list to obtain:

Proposition 2.9. *A class-preserving automorphism of a finite Blackburn group is an inner automorphism.*

Proof. Note that a class-preserving automorphism stabilizes every normal subgroup, and that $\text{Aut}_c(-)$ commutes with taking direct products.

Let G be a finite Blackburn group. If G is a p -group, then G contains an abelian subgroup of index 2. Hence, the statement follows from Example 2.1 (it is, of course, possible to give a more direct proof).

Suppose that G is not of prime-power order. Then G belongs to one of the classes (a)–(e) described in [2, Theorem 2]. If G is of type (a) or (d), then G is abelian-by-cyclic and Proposition 2.7 applies (of course, one can argue more directly). The groups of type (b) or (e) are now recognized as direct products of groups which have only inner class-preserving automorphisms. Thus it remains to deal with G of type (c), but such a group G is of the form described in Lemma 2.8 (note that the groups of type (a) have an abelian Sylow p -subgroup), so we are done. \square

3 Blackburn groups have the normalizer property

From now on, G will denote a Blackburn group, so $R(G)$ is defined and $R(G) \neq 1$. It is easily seen that $R(G)$ is a cyclic p -group for some prime p (see [5, 22.1 Lemma] for a proof). Note that if a finite subgroup S of G is not a Dedekind group, then

$R(G) \leq R(S)$; in particular, $R(S) \neq 1$. The following observation was also stated in [6, Lemma 4.10]. For completeness sake we include the proof.

Lemma 3.1. *Suppose that G has a finite non-normal subgroup, and that $R(G)$ is a non-trivial cyclic p -group. Let q be a prime different from p . Then the set T_q of q -elements of G forms a normal q -subgroup of G , all of whose subgroups are normal in G . (In particular, T_q for odd q is an abelian q -group.)*

Proof. Let x and y be q -elements in G . Then $\langle x \rangle$ is normal in G since otherwise we would have $R(G) = 1$. Thus xy is also a q -element, and it follows that T_q is a group. If a subgroup of T_q is non-normal in G , then T_q also contains a finite cyclic subgroup which is non-normal in G , which again contradicts $R(G) \neq 1$. That T_q for odd q is abelian holds by the classification of Dedekind groups. \square

We shall use the following information on finite normal subgroups of G . Rather than deducing it from Blackburn's list [2], we extract the proof directly from Blackburn's work.

Claim 3.2. *Suppose that G has a finite non-normal subgroup, and that $R(G)$ is a non-trivial cyclic p -group. Let N be a finite normal subgroup of G . Then N has a normal p -complement H which is a Dedekind group. Let S be a Sylow p -subgroup of N . Then one of the following holds:*

- (a) N is nilpotent;
- (b) $Z(S) \leq O_p(N)$, i.e., $[Z(S), H] = 1$;
- (c) S is abelian, and $S = \langle x \rangle \times T$, where $x \in S$ is such that $[x, H] \neq 1$, and $T \leq O_p(N)$. The order of $C_{\langle x \rangle}(H)$ is the exponent of $O_p(N)$. If some g in G centralizes $C_{\langle x \rangle}(H)$, then g centralizes T .

Proof. For a prime q different from p , a Sylow q -subgroup of N is a normal subgroup of N by Lemma 3.1. So N has a normal p -complement H . Furthermore, all subgroups of H are normal in G and thus H is a Dedekind group.

If N is a Dedekind group, then (a) holds. Hence we can assume that $R(N) \neq 1$, i.e., N is a finite Blackburn group with $R(G) \leq R(N)$ and $R(N)$ is a cyclic p -group. Let S be a Sylow p -subgroup of N .

Assume S is not a Dedekind group. Then, $R(G) \leq R(S)$. Thus, S is a Blackburn group and a p -group. So $p = 2$. If S is of the form $Q_8 \times C_4 \times E_2$ or $Q_8 \times Q_8 \times E_2$ (where E_2 is an elementary abelian 2-group), then S is generated by certain subgroups, none of which contains $R(S)$. Since $|R(S)| = 2$, we get that $R(S) = R(G)$. Hence, the mentioned subgroups generating S are all normal in G . So S is normal in G , and (a) holds. If S is a Q-group (possibly a non-abelian Dedekind group), then $Z(S)$ is elementary abelian, and (b) holds since elements of S of order 2 are contained in the center of N as $R(N) \neq 1$. If S is a non-abelian Dedekind group then the last argument also yields that (b) holds. Hence, we are left to deal with S an abelian group.

We can assume that $[S, H] \neq 1$ since otherwise (a) holds. Set $C = O_p(N) = C_S(H)$. We will follow [2, p. 35]. Note that if s is any element of S not lying in C , then $\langle s \rangle$ is not normal in N and $R(N) \leq \langle s \rangle$. Hence, $R(N) \leq \langle s^p \rangle$ since $R(N) \trianglelefteq N$.

Let $x, y \in S$ with $x \notin C$ and the order of x greater than or equal to the order of y . By the basis theorem for abelian groups, $\langle x, y \rangle = \langle x \rangle \times \langle z \rangle$ for some $z \in S$, and since $R(N) \leq \langle x \rangle$, we have $z \in C$. Thus S/C is a cyclic group, say of order p^r . Suppose $S = \langle C, x \rangle$ and that x is of order p^m . Then $m > r$ since $R(N) \leq \langle x \rangle$, and $x^{p^{m-1}} \in R(N)$.

We prove next that the exponent of C is p^{m-r} . If this is not so, then C contains an element y of order p^{m-r+1} , for x^{p^r} is an element of C of order p^{m-r} . Then $\langle x, y \rangle = \langle x \rangle \times \langle c \rangle$ for some $c = x^\lambda y^\mu$. If $c \notin C$, then $R(N) \leq \langle c \rangle$ and $x^{p^{m-1}} \in \langle c \rangle$, contrary to the properties of direct products. Hence $c \in C$, so $x^\lambda \in C$ and $\lambda \equiv 0 \pmod{p^r}$. Also μ is not divisible by p since otherwise $\langle x, y \rangle = \langle x, y^p \rangle$, showing that y is in $\langle x \rangle$ and in C , but the intersection of these groups has order p^{m-r} . So $c^{p^{m-r}} = y^{\mu p^{m-r}} \neq 1$. But $x^{p^{r-1}} c \notin C$, so $R(N) \leq \langle x^{p^r} c^p \rangle$, from above. Hence $x^{p^{m-1}} = x^{vp^r} c^{pv}$ for some v , and by the properties of direct products, $c^{pv} = 1$. Hence $v \equiv 0 \pmod{p^{m-r}}$, whence $x^{p^{m-1}} = 1$, a contradiction. The assertion is therefore proved.

Now $S = \langle x \rangle \times T$ for some $T \leq S$, by the basis theorem for abelian groups, and $T \leq C$, from above. Suppose that some g in G centralizes $C_{\langle x \rangle}(H) = \langle x^{p^r} \rangle$. Note that $C = \langle x^{p^r} \rangle \times T$ and $R(N) \leq \langle x^{p^r} \rangle$. Let $t \in T$. Then $\langle t \rangle \trianglelefteq G$, in particular $t^g \in \langle t \rangle$. Since the exponent of C is the order of x^{p^r} , there exists c in $\langle x^{p^r} \rangle$ of the same order as t . Then $\langle ct \rangle \trianglelefteq G$ since $R(N)$ is not contained in $\langle ct \rangle$. It follows that $\langle ct \rangle = \langle ct \rangle^g = \langle ct^g \rangle$ and $t = t^g$. Thus (c) holds, and the claim is proved. \square

Theorem 3.3. *Suppose that G has a finite non-normal subgroup, and that $R(G)$ is non-trivial. Then G has the normalizer property, $N_{\mathcal{U}}(G) = Z(\mathcal{U})G$.*

Proof. We already noted that $R(G)$ is a cyclic p -group for some prime p .

For any $u \in N_{\mathcal{U}}(G)$, there exists a finite normal subgroup N of G such that $ug^{-1} \in RN$ for all $g \in \text{supp}(u)$, by [8, Theorem 1.4]. Hence, it is enough to consider an element u in $N_{\mathcal{U}}(G)$ which is contained in RN for some finite normal subgroup N of G .

If N is a Dedekind group, then $u \in Z(RN)N$ since class-preserving automorphisms of Dedekind groups are inner automorphisms (cf. Example 2.1). If N is not a Dedekind group, then $R(N) \neq 1$, and $u \in Z(RN)N$ by Proposition 2.9. Thus we can assume that $u \in Z(RN)$. Let σ be the automorphism of G given by conjugation with u ; we have to show that σ is an inner automorphism.

Note that σ is of finite order. By Lemma 3.1, any finite normal subgroup of G has a normal Sylow q -subgroup for all primes q distinct from p . Thus by [6, Corollary 3.4], we can assume that σ is of order a power of p .

If $Z(S_0) \leq O_p(N)$ for a Sylow p -subgroup S_0 of N , then there is $n \in N$ such that $un^{-1} \in Z(\mathcal{U})$, by [6, Lemma 3.2]. Note that this condition is trivially satisfied if N is nilpotent. Hence we can assume that N has the structure described in case (c) of Claim 3.2; we proceed to choose a suitable Sylow p -subgroup S of N .

The group G acts on $\text{supp}(u)$ via $n \xrightarrow{g} g^{-1}ng^u$ for $n \in \text{supp}(u)$ and $g \in G$ (see [6, Lemma 2.1]). Note that $C_G(N)$ lies in the kernel of this action. Choose a subgroup P of G containing $C_G(N)$ such that $P/C_G(N)$ is a Sylow p -subgroup of the (finite) group $G/C_G(N)$. By the Ward–Coleman Lemma (see [6, Lemma 2.9]), there is $a \in \text{supp}(u) \leq N$ such that $[P, ua^{-1}] = 1$. Since u is of p -power order over the center of \mathcal{U} , and $[u, a] = 1$, there is an element b in $\langle a \rangle$ of p -power order such that $[P, ub^{-1}] = 1$. Let S_0 be a Sylow p -subgroup of N . Then there exists a fixed point under the multiplication action of S_0 on the set of left cosets of P in G , say gP . So $S := S_0^g \leq P \cap N$ and $[S, b^{-1}] = [S, ub^{-1}] \leq [P, ub^{-1}] = 1$. It follows that S is a Sylow p -subgroup of N containing b . In particular, $b \in P$, and as $[P, ub^{-1}] = 1$, the automorphism σ , which is conjugation with u , fixes P on which it is the inner automorphism given by conjugation with b .

As in Claim 3.2(c), let H be the normal p -complement of N , and write $S = \langle x \rangle \times T$, where $x \in S$ is such that $[x, H] \neq 1$, and $T \leq O_p(N)$. Then $b \in \langle x \rangle t$ for some $t \in T$. Set $c = bt^{-1} \in \langle x \rangle$, $v = ut^{-1} \in Z(RN)$, and let τ be the automorphism of G which is given by conjugation with v . This modified automorphism τ is still of order a power of p . Also, τ fixes P , on which it is the inner automorphism given by conjugation with c .

Let $g \in G$. Then $g^{-1}(g\tau) \in [g, G]$ since τ is a class-preserving automorphism of G . Since G acts as an abelian group on H (each cyclic subgroup of H is normal in G since $H \cap R(G) = 1$), it follows that

$$g^{-1}(g\tau) \in [g, G] \cap N \leq C_G(H) \cap N \leq HO_p(N).$$

Since τ is of p -power order and fixes N element-wise, we even have $g\tau \in gO_p(N)$. Thus τ induces the identity on both $O_p(N)$ and $G/O_p(N)$.

All this suggests the use of a 1-cohomology argument. For the moment, let $A = O_p(N)$. Note that A is abelian as S is abelian, according to Claim 3.2(c). It is well known (see, for example, [10, Chapter 2, (8.7)]) that the group of automorphisms of G inducing the identity on both A and G/A is isomorphic to the group of 1-cocycles $Z^1(G/A, A)$, with the subgroup formed by the inner automorphisms given by conjugation with elements of A corresponding to the subgroup $B^1(G/A, A)$ of 1-coboundaries. That this isomorphism is ‘compatible’ with restrictions will be used as follows. By construction, if τ corresponds to the 1-cocycle $\delta: G/A \rightarrow A$, then its restriction $\tau|_P$ (an automorphism of P) corresponds to the 1-cocycle obtained by restricting δ to P/A . Now, since P is of finite p' -index in G , and A is a p -group, restriction–corestriction in 1-cohomology (see [10, Theorem 7.26]) shows that restriction induces an injection $H^1(G/A, A) \hookrightarrow H^1(P/A, A)$ on 1-cohomology. What follows is that τ is an inner automorphism if $\tau|_P$ is an inner automorphism given by conjugation with an element of A .

After this consideration, $c \in O_p(N)$ would yield that τ (and hence also σ) is given by conjugation with some element of $O_p(N)$. Thus assume that $c \notin O_p(N)$. We will show that $[P, c] = 1$, i.e., that τ induces the identity on P . Then again, τ is given by conjugation with some element of $O_p(N)$, and we are done. Suppose, by way of con-

tradition, that there is $y \in P$ such that $d = y^{-1}(y\tau) = [y, c] \neq 1$. This will be used to gain additional information on the element v .

Since $v \in N_{\mathcal{H}}(G)$, we know (see again [6, Lemma 2.1]) that G acts on $\text{supp}(v)$ via $n \xrightarrow{g} g^{-1}ng^v$, for $n \in \text{supp}(v)$ and $g \in G$, and elements of an orbit under this action have the same coefficient in v (viewed as an R -linear combination of elements of N). The group $\langle y \rangle$ acts via $n \xrightarrow{y} n^y \cdot y^{-1}(y\tau) = n^y d$. As $v \in Z(RN)$, the group N acts just by conjugation.

According to Claim 3.2(c), $\langle x \rangle$ acts on the nilpotent group H , and $C_{\langle x \rangle}(H) \leq O_p(N)$. Since the lattice of subgroups of $\langle x \rangle$ is linearly ordered, it follows that there is a Sylow subgroup Q of H such that $C_{\langle x \rangle}(Q) \leq O_p(N)$, that is, if some power of x centralizes Q , it centralizes all other Sylow subgroups of H . Recall that all cyclic subgroups of Q are normal in N . If x is of odd order, then x centralizes the Sylow 2-subgroup of H since the automorphism group of a cyclic 2-group is a 2-group. As $[x, H] \neq 1$, it follows that Q is of odd order, and therefore abelian as it is a Dedekind group. It follows that x acts on Q by raising all elements of Q to some fixed power, and $C_Q(x^i) = 1$ for any power x^i of x which is not contained in $O_p(N)$. So if $n \in N$ is not contained in $HO_p(N)$, then $[n, Q] \neq 1$ and $C_Q(n) = 1$.

For an element n in N , let C_n denote its class sum in $\mathbb{Z}N$, i.e., the sum of its N -conjugates in $\mathbb{Z}N$. Note that v is an R -linear combination of such class sums. For a finite subset X of G we shall write \hat{X} for the sum of the elements of X in RG , and we abbreviate $\hat{d} = \widehat{\langle d \rangle}$.

Note that if $C_Q(n) = 1$, then $n^Q = Qn$, and the same holds for any other N -conjugate of n . Thus $C_n \in \hat{Q}(\mathbb{Z}N)$ for all n in N which are not contained in $HO_p(N)$.

We record further facts for later use. First, $C_{\langle x \rangle}(H) \leq \langle c^p \rangle$ as $c \in \langle x \rangle$ and $c \notin O_p(N)$. Next, note that $\langle x \rangle$, acting non-trivially on H , is not normal in G . So $R(G) \leq \langle x \rangle$. Since $T \cap \langle x \rangle = 1$, it follows that each subgroup of T is normal in G . Also note that $d = y^{-1}(y\tau) \in O_p(N)$ as τ induces the identity on $G/O_p(N)$.

We shall distinguish two cases, according to whether d lies in T or not.

First, suppose that $d \in T$, and, moreover, that $d^p = 1$. Then

$$(c^p)^y = (c^y)^p = (cd^{-1})^p = c^p,$$

that is, $[c^p, y] = 1$. Since $C_{\langle x \rangle}(H) \leq \langle c^p \rangle$, it follows that $[T, y] = 1$ by Claim 3.2(c) and $[C_{\langle x \rangle}(H), y] \leq [\langle c^p \rangle, y] = 1$. So $[O_p(N), y] = 1$. Thus on elements of $\text{supp}(v)$ of the form hk with $h \in H$ and $k \in O_p(N)$, the above action of y is given by $hk \xrightarrow{y} h^ykd$. It follows that for an orbit O of such an element under the action of $\langle y \rangle$ on $\text{supp}(v)$, we have

$$\hat{H}O \in \hat{H}(RN)\hat{d} = \hat{H}(RS)\hat{d}.$$

Recall that if some $n \in \text{supp}(v)$ is not contained in $HO_p(N)$, then its class sum C_n , which is the sum of the elements of its orbit under the action of N , lies in $\hat{Q}(\mathbb{Z}N)$, and therefore $\hat{H}C_n \in |Q|\hat{H}(\mathbb{Z}S)$. It follows that $\hat{H}v = |Q|\hat{H}\alpha + \hat{H}\beta\hat{d}$ for some $\alpha, \beta \in RS$. Let $\pi : RN \rightarrow RN/H$ denote the natural map. Then $\hat{H}v\pi = |H|(v\pi)$. Identifying N/H with S , we have $\hat{H}v\pi = |H|(|Q|\alpha + \beta\hat{d})$. Thus $|Q|\alpha + \beta\hat{d}$, as the image of

v under π , is a unit in RS . It follows that $|Q|$ is a unit in the quotient $RS/\hat{d}RS$ of RS . So $|Q|\gamma \in 1 + \hat{d}RS$ for some $\gamma \in RS$, and thus $|Q|(d-1)\gamma = d-1$. But, as R is G -adapted, the coefficients ± 1 of $d-1$ are not divisible by $|Q|$ in R . This yields a contradiction in case $d^p = 1$ (and $d \in T$, of course). Now if $d \in T$, but $d^p \neq 1$, set $\bar{G} = G/\langle d^p \rangle$. Then all the properties listed in Claim 3.2(c) for G, H, S, T and $\langle x \rangle$ obviously carry over to corresponding properties for their images in \bar{G} . Thus this situation is reduced to the previous case, resulting again in a contradiction.

Second, suppose that $d \notin T$. To handle this case, we start by remarking that it derives from the normal subgroup structure of N that $(x^i)^g \in \langle x^i \rangle TH$ for all $g \in G$ and $i \in \mathbb{N}$. Since $c \in \langle x \rangle$ and $c^y = cd^{-1} \in cO_p(N)$, it follows that $c^y \in cO_p(N) \cap \langle c \rangle TH \leq \langle c \rangle T$. Also $P/C_G(N)$ is a p -group, so y acts on $\langle c \rangle T$ as an automorphism of order a power of p . Since $d \notin T$, it follows that y acts nontrivially on $\langle c \rangle T/T$, and $\langle [y, c^p] \rangle T/T$ is a proper subgroup of $\langle [y, c] \rangle T/T$ (an easy to verify general fact for the action of a cyclic p -group on another cyclic p -group). Thus $d = [y, c] \notin \langle [y, c^p] \rangle T = [y, \langle c^p \rangle] T$. Set $L = [y, O_p(N)] T$. Since T is a normal subgroup of G , the starting remark shows that L is a normal subgroup of G . We noticed above that $C_{\langle x \rangle}(H) \leq \langle c^p \rangle$, so $O_p(N) \leq \langle c^p \rangle T$ and $L \leq [y, \langle c^p \rangle] T = [y, \langle c^p \rangle] T$. Consequently, $d \notin L$. Set $\bar{G} = G/L$. On elements of $\text{supp}(v)$ of the form hk with $h \in H$ and $k \in O_p(N)$, the above action of y is given by $hk \xrightarrow{y} h^y k [k, y] d$, and $\overline{h^y k [k, y] d} = \overline{h^y k d}$. Writing ω for the sum of the elements of the subgroup $\langle \bar{d} \rangle$ of \bar{G} , it follows as above that $\hat{H}\bar{v} = |Q|\hat{H}\alpha + \hat{H}\beta\omega$ for some $\alpha, \beta \in R\bar{S}$, and that $|Q|\alpha + \beta\omega$ is a unit in $R\bar{S}$. Hence $|Q|$ is a unit in the quotient $R\bar{S}/\omega(R\bar{S})$ of $R\bar{S}$, leading to the same contradiction as in the first case. The proof of the theorem is complete. \square

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Martin Hertweck, Universität Stuttgart, Fachbereich Mathematik, Institut für Geometrie und Topologie, Pfaffenwaldring 57, 70550 Stuttgart, Germany
E-mail: hertweck@mathematik.uni-stuttgart.de

Eric Jespers, Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium
E-mail: efjesper@vub.ac.be