Positional Accuracy of Spatial Object after a Geometric Transformation

Bayesian approach

Lubomír Soukup

Institute of Information Theory and Automation, Prague, Czech Republic,
soukup@utia.cas.cz

Abstract. Influence of a geometric transformation to positional accuracy of geographic objects is discussed in this contribution. Firstly, transformation parameters are estimated with the aid of ground control points (GCP) by Bayesian approach. Secondly, probability distribution of position of transformed object is derived. Uncertainty of input data in both steps are simultaneously processed during the proposed procedure. Therefore the procedure enables exact statistical assessment of the resulting spatial information. Finally, the procedure is illustrated on example of linear conform transformation that is frequently used in practice.

1 Introduction

Geometric transformation represents an important procedure in collecting spatial data from different sources. Input data of any kind (aerial photos, satellite images of various spectral bands, digitized maps, LIDAR data, existing GIS data, etc.) have limited positional accuracy which has to be properly processed to determine spatial quality of the outcoming spatial objects in a *required coordinate system*.

Position of the object in the *required coordinate system* is mostly determined in two steps. Firstly, coordinates of GCPs are used to estimate transformation parameters of some specified *transformation model*. Secondly, coordinates of the object in a *source coordinate system* are transformed by means of transformation equations determined in the previous step.

In the first step, transformation parameters are estimated with the aid of a set of GCPs by solving an overdetermined system of equations. Uncertainty in both coordinate systems has to be considered simultaneously. This requirement makes the estimation problem nonlinear even for linear transformation. Traditionally, the least-squares method is used for the estimation. Exact estimation of transformation parameters was designed in [1] for linear conform transformation (generalized Helmert transformation). Classical approach based on linearization was applied to generalized Helmert transformation in [2]. In this contribution, nonlinear Bayesian estimation is used. In the case of linear conform transformation, Bayesian estimate gives similar result as in [1]. In addition, quality of transformation is enabled, since Bayesian approach results in probability distribution of estimated transformation parameters.

Traditionally, quality of geometric transformation obtained in the first step is not fully considered in the second step. Usually, once a specified quality of transformation is attained, only positional accuracy of the object in the *source coordinate system* is utilized in the final assessment of spatial quality. Indeed, uncertainty of input data processed in both steps has to be considered to obtain correct estimate of spatial quality of the outcoming transformed objects. Computational procedure which respects that requirement is proposed in this contribution.

2 Formulation of the Problem

A geographic object is given by a set $\mathcal{X}^{\square}$ of points (e.g. on its border). Coordinates of points from the set $\mathcal{X}^{\square}$ are known in some coordinate system; let us name it *source coordinate system.* Position of the same object in some other coordinate system is required; let us name it *required coordinate system.* Relationship between those coordinate systems is given by vector function $\boldsymbol{t_q}$ called *transformation model* in the sequel.

$$\boldsymbol{t_q} : \mathrm{I\!R}^2 \to \mathrm{I\!R}^2 : \check{\boldsymbol{x}} \mapsto \boldsymbol{t_q}(\check{\boldsymbol{x}}) = \widehat{\boldsymbol{x}} \ , \tag{1}$$

where

$\widehat{\boldsymbol{x}}$ ... coordinates of a point in the *required coordinate system*,
$\check{\boldsymbol{x}}$ ... coordinates of the same point in the *source coordinate system*,
$\boldsymbol{q}$ ... unknown vector of transformation parameters, $\boldsymbol{q} = [q_1, \ldots, q_m]$,
m ... number of scalar transformation parameters q_j, $j \in \{1, \ldots, m\}$ in *transformation model* $\boldsymbol{t_q}$.

Transformation model $\boldsymbol{t_q}$ is member of class of transformations defined by mapping

$$\boldsymbol{t} : \mathrm{I\!R}^m \times \mathrm{I\!R}^2 \to \mathrm{I\!R}^2 : [\,\boldsymbol{q}\,,\,\check{\boldsymbol{x}}\,] \mapsto \boldsymbol{t}(\boldsymbol{q}\,,\,\check{\boldsymbol{x}}) = \boldsymbol{t_q}(\check{\boldsymbol{x}}) \ .$$

Transformation parameters $\boldsymbol{q}$ are not known in advance. They are expected to be estimated with the aid of given set $\mathcal{X}^{\odot}$ of GCPs. Coordinates of the GCPs was captured in both coordinate systems.

$\widehat{\boldsymbol{x}}_j^{\odot}$ ... coordinates of j-th GCP in the *required coordinate system*,
$\check{\boldsymbol{x}}_j^{\odot}$ ... coordinates of the j-th GCP in the *source coordinate system*,
j ... identifier of a GCP; $j \in \mathcal{X}^{\odot}$ for short.

Spatial quality of the geographic object can be objectively assessed on the basis of probability distribution of position of points from the set $\mathcal{X}^{\square}$. Therefore positional accuracy of the object is required in form of probability distribution of position of its points in *required coordinate system.* Similarly, positional accuracy of the object in *source coordinate system* is presupposed to be given in form of the probability distribution of random errors in position of its points in *source coordinate system*. Furthermore, probability distribution of random errors in coordinates of the GCPs in both coordinate systems is supposed to be given as well.

Probability distribution is always treated in form of probability density function (pdf).

The above problem formulation can be briefly summarized as follows.

Required Result:

h ... pdf of points from $\mathcal{X}^{\square}$ in *required coordinate system*.

Given Input Data:

$\mathcal{X}^{\square}$ ... set of points characterizing a geographic object,
$\mathcal{X}^{\odot}$ ... set of GCPs,
$\check{\boldsymbol{x}}^{\square}$ ... vector of source coordinates of object points; $\check{\boldsymbol{x}}^{\square} = [\check{\boldsymbol{x}}_j \mid j \in \mathcal{X}^{\square}]$,
$\check{\boldsymbol{x}}^{\odot}$ ... vector of source coordinates of GCPs; $\check{\boldsymbol{x}}^{\odot} = [\check{\boldsymbol{x}}_k^{\odot} \mid k \in \mathcal{X}^{\odot}]$,
$\widehat{\boldsymbol{x}}^{\odot}$ ... vector of required coordinates of GCPs; $\widehat{\boldsymbol{x}}^{\odot} = [\widehat{\boldsymbol{x}}_k^{\odot} \mid k \in \mathcal{X}^{\odot}]$,
$\check{f}^{\square}$ ... pdf of random errors of $\check{\boldsymbol{x}}^{\square}$,
$\check{f}^{\odot}$ ... pdf of random errors of $\check{\boldsymbol{x}}^{\odot}$,
$\hat{f}^{\odot}$ ... pdf of random errors of $\widehat{\boldsymbol{x}}^{\odot}$,
$\boldsymbol{t}_{\boldsymbol{q}}$ ... *transformation model* for some unknown parameter $\boldsymbol{q} \in \mathbb{R}^m$

3 Solution of the Problem

Firstly, Bayesian approach is applied to estimate the transformation parameters with the aid of coordinates of GCPs. The quality of the transformation is specified by a posterior probability distribution of the transformation parameters $\boldsymbol{q}$. This probability distribution is then joined to the probability distribution of points that have to be transformed. The resulting probability distribution of points of the outcoming spatial objects is obtained by means of standard statistical technique.

3.1 Bayesian Estimation of Transformation Parameters

Transformation model (1) has to be valid also for coordinates of GCPs, so that

$$\widehat{\boldsymbol{x}}_j^{\odot} = \boldsymbol{t}_{\boldsymbol{q}}(\check{\boldsymbol{x}}_j^{\odot}) \,, \quad \forall\, j \in \mathcal{X}^{\odot} \,. \tag{2}$$

Reparametrization. It is convenient to separate the unknown parameters $\boldsymbol{q}$ from given variables $\check{\boldsymbol{x}}_j^{\odot}$, $\widehat{\boldsymbol{x}}_j^{\odot}$ to respect uncertainty of position of GCP in both coordinate systems. The separation can be most generally achieved by changing parameters of the fundamental transformation equation (2).

$$\begin{bmatrix} \check{\boldsymbol{x}}_j^{\odot} \\ \widehat{\boldsymbol{x}}_j^{\odot} \end{bmatrix} = \boldsymbol{\tau}_j(\boldsymbol{r}) \; . \tag{3}$$

Here function

$$\boldsymbol{\tau}_j : \mathbb{R}^n \rightarrow \mathbb{R}^4 : \boldsymbol{r} \mapsto \boldsymbol{\tau}_j(\boldsymbol{r})$$

have to be defined for all $j \in \mathcal{X}^{\odot}$. Furthermore, inequality

$$n \geq m \tag{4}$$

has to be held.

The most apparent way of changing parameters is

$$\boldsymbol{r} = [\, \boldsymbol{q} \,,\, \widetilde{\boldsymbol{q}}_1,\, \widetilde{\boldsymbol{q}}_2,\, \ldots \,,\widetilde{\boldsymbol{q}}_m \,] \; ,$$

$$\boldsymbol{\tau}_j(\boldsymbol{r}) = \begin{bmatrix} \widetilde{\boldsymbol{q}}_j \\ \boldsymbol{t}_{\boldsymbol{q}}(\widetilde{\boldsymbol{q}}_j) \end{bmatrix} \; ,$$

but in a specified case of transformation some more convenient change of parameters is usually feasible. Example of changing parameters in the case of linear conform transformation is presented in the contribution.

Equation (3) can be written repeatedly for $\forall\, j \in \mathcal{X}^{\odot}$ and modified to have clear vector form that is suitable for estimation of parameters $\boldsymbol{r}$.

$$\boldsymbol{\varepsilon} + \boldsymbol{\eta} = \boldsymbol{\tau}(\boldsymbol{r}) \; , \tag{5}$$

where

$\boldsymbol{\varepsilon}$ …random errors of coordinates $[\check{\boldsymbol{x}}_j^{\odot},\, \widehat{\boldsymbol{x}}_j^{\odot}\,]$; $\boldsymbol{\varepsilon} = [\check{\boldsymbol{\varepsilon}},\, \hat{\boldsymbol{\varepsilon}}]$,
$\boldsymbol{\eta}$ …measured values of coordinates of GCPs; $\boldsymbol{\eta} = [\check{\boldsymbol{\eta}},\, \widehat{\boldsymbol{\eta}}]$,
$\check{\boldsymbol{\eta}} + \check{\boldsymbol{\varepsilon}} = \check{\boldsymbol{x}}_j^{\odot}$, $\widehat{\boldsymbol{\eta}} + \hat{\boldsymbol{\varepsilon}} = \widehat{\boldsymbol{x}}_j^{\odot}$,
$\boldsymbol{\tau}$ …vector function that collects all the functions $\boldsymbol{\tau}_j = [\check{\boldsymbol{\tau}}_j,\, \widehat{\boldsymbol{\tau}}_j]^T$;
$\boldsymbol{\tau} = [\check{\boldsymbol{\tau}},\, \widehat{\boldsymbol{\tau}}]$, $\check{\boldsymbol{\tau}} = [\check{\boldsymbol{\tau}}_j \,|\, j \in \mathcal{X}^{\odot}]$, $\widehat{\boldsymbol{\tau}} = [\widehat{\boldsymbol{\tau}}_j \,|\, j \in \mathcal{X}^{\odot}]$.

Bayesian Inference. Parameters $\boldsymbol{r}$ of *transformation model* in form (5) can now be easily estimated with the aid of Bayes theorem.

$$g(\boldsymbol{r} \,|\boldsymbol{\eta}) = \frac{f(\boldsymbol{\eta} - \boldsymbol{\tau}(\boldsymbol{r}))\, p(\boldsymbol{r})}{\int_{\mathcal{R}} f(\boldsymbol{\eta} - \boldsymbol{\tau}(\boldsymbol{z}))\, p(\boldsymbol{z})\, d\boldsymbol{z}} \; , \tag{6}$$

where

g ... a posterior pdf of transformation parameters $\boldsymbol{r}$,
f ... pdf of coordinate errors of GCPs,
p ... a prior pdf of transformation parameters $\boldsymbol{r}$,
$\mathcal{R}$ ... set of acceptable values of transformation parameters $\boldsymbol{r}$.

Random errors $\check{\boldsymbol{\varepsilon}}$, $\hat{\boldsymbol{\varepsilon}}$ are usually statistically independent in practice. Therefore pdf f can be evaluated as product of pdfs $\check{f}$, $\hat{f}$, i.e.

$$f(\boldsymbol{\varepsilon}) = \check{f}(\check{\boldsymbol{\varepsilon}})\,\hat{f}(\hat{\boldsymbol{\varepsilon}}) \;.$$

A prior pdf p is often not known in advance. Therefore it is assumed to be noninformative, i.e. constant over the whole range $\mathcal{R}$.

Using the two assumptions in (6) results in following simplified version of Bayes theorem.

$$g(\boldsymbol{r}\,|\boldsymbol{\eta}) = \frac{\check{f}(\check{\boldsymbol{\eta}} - \check{\boldsymbol{\tau}}(\boldsymbol{r}))\,\hat{f}(\widehat{\boldsymbol{\eta}} - \widehat{\boldsymbol{\tau}}(\boldsymbol{r}))}{\int\limits_{\mathcal{R}} \check{f}(\check{\boldsymbol{\eta}} - \check{\boldsymbol{\tau}}(\boldsymbol{z}))\,\hat{f}(\widehat{\boldsymbol{\eta}} - \widehat{\boldsymbol{\tau}}(\boldsymbol{z}))\,d\boldsymbol{z}} \;. \tag{7}$$

Return to the Original Transformation Parameters. Relationship between new parameters $\boldsymbol{r}$ and the original parameters $\boldsymbol{q}$ can be easily obtained after substitution (3) in (2) for sufficient number of points $j \in \mathcal{J} \subseteq \mathcal{X}^{\odot}$.

$$\widehat{\boldsymbol{\tau}}_j(\boldsymbol{r}) = \boldsymbol{t}_{\boldsymbol{q}}(\check{\boldsymbol{\tau}}_j(\boldsymbol{r}))\;, \quad \forall\, j \in \mathcal{J} \;. \tag{8}$$

Function

$$\boldsymbol{\omega} : \mathrm{I\!R}^{m+n} \to \mathrm{I\!R}^{2\,|\mathcal{X}^{\odot}|} : [\boldsymbol{q},\, \boldsymbol{r}] \mapsto \boldsymbol{\omega}(\boldsymbol{q},\, \boldsymbol{r})$$

such that

$$\boldsymbol{\omega}(\boldsymbol{q},\, \boldsymbol{r}) = [\widehat{\boldsymbol{\tau}}_j(\boldsymbol{r}) - \boldsymbol{t}_{\boldsymbol{q}}(\check{\boldsymbol{\tau}}_j(\boldsymbol{r}))\,|\, j \in \mathcal{J}]$$

implicitly defines another function

$$\boldsymbol{w} : \mathrm{I\!R}^{n} \to \mathrm{I\!R}^{m} : \boldsymbol{r} \mapsto \boldsymbol{w}(\boldsymbol{r})$$

which makes possible expressing the original parameters $\boldsymbol{q}$ by means of the new ones $\boldsymbol{r}$.

$$\boldsymbol{q} = \boldsymbol{w}(\boldsymbol{r}) \;. \tag{9}$$

Elements of vector function $\boldsymbol{\omega}$ must fulfill presumptions of theorem on implicit functions to define function $\boldsymbol{w}$ properly. These presumptions are usually held if parameters $\boldsymbol{r}$ and functions $\boldsymbol{\tau}_j$ were reasonably chosen.

Thanks to inequality (4), pdf of the original transformation parameters $\boldsymbol{q}$ can be derived from pdf g of parameters $\boldsymbol{r}$ by application of elementary formula of probability theory to function $\boldsymbol{w}$ defined by (9). That formula is much more simple if

$$n = m \;. \tag{10}$$

For the sake of clarity we will suppose (10). Then pdf $\tilde{g}$ of parameters $\boldsymbol{q}$ is (see e.g. [3]):

$$\tilde{g}(\boldsymbol{q}) = g(\boldsymbol{w}^{-1}(\boldsymbol{q})\,|\,\boldsymbol{\eta}) \,\cdot\, |\det\left[\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{r}}(\boldsymbol{w}^{-1}(\boldsymbol{q}))\right]^{-1}| \;. \tag{11}$$

3.2 Propagation of Spatial Uncertainty

Result (11) is utilized to determine pdf of random vector $\widehat{\boldsymbol{x}}^{\square}$. Principle of this phase will be shown for the object containing just one point here, because it is much more clear than the general case with an arbitrary number of points in the set $\mathcal{X}^{\square}$. The general, multi-point case is performed in the full paper. Thus $|\mathcal{X}^{\square}| = 1$, $\check{\boldsymbol{x}}^{\square} = \check{\boldsymbol{x}}$ for now.

Pdf of $\widehat{\boldsymbol{x}}$ can be easily obtained from pdf of $\check{\boldsymbol{x}}$ with the aid of the same formula which was applied to (9) to produce (11). Here the formula will be be applied to mapping (1). After then, pdf $\tilde{f}_{\boldsymbol{q}}$ of random vector $\widehat{\boldsymbol{x}}$ outcomes.

$$\tilde{f}_{\boldsymbol{q}}(\widehat{\boldsymbol{x}}) = \check{f}^{\square}(\boldsymbol{t}_{\boldsymbol{q}}^{-1}(\widehat{\boldsymbol{x}})) \cdot |\det \left[\frac{\partial \boldsymbol{t}_{\boldsymbol{q}}}{\partial \check{\boldsymbol{x}}}(\boldsymbol{t}_{\boldsymbol{q}}^{-1}(\widehat{\boldsymbol{x}})) \right]^{-1} | \; . \tag{12}$$

Pdf $\tilde{f}_{\boldsymbol{q}}$ expressed in form (12) does not respect uncertainty of transformation parameters $\boldsymbol{q}$ since their pdf g did not appear in (12). Transformation parameters $\boldsymbol{q}$ encapsulated in symbol $\boldsymbol{t}_{\boldsymbol{q}}$ are actually constant in equation (12). Therefore pdf $\tilde{f}_{\boldsymbol{q}}$ of random vector $\widehat{\boldsymbol{x}}^{\square}$ has to be interpreted as conditional pdf. It means that another pdf, say $\tilde{f}$, has to be incorporated such that

$$\tilde{f}_{\boldsymbol{q}}(\widehat{\boldsymbol{x}}) = \tilde{f}(\widehat{\boldsymbol{x}} \,|\, \boldsymbol{q}\,) \; . \tag{13}$$

Pdf $\tilde{f}$ is defined with the aid of joint pdf of vectors $\widehat{\boldsymbol{x}}^{\square}$, $\boldsymbol{q}$. Let us denote such a joint pdf by symbol $\bar{f}$. Then the conditional pdf $\tilde{f}$ is defined as follows (see e.g. [3], p. 145).

$$\tilde{f}(\widehat{\boldsymbol{x}} \,|\, \boldsymbol{q}\,) = \frac{\bar{f}(\,\widehat{\boldsymbol{x}},\, \boldsymbol{q}\,)}{\tilde{g}(\boldsymbol{q})} \; .$$

Hence joint pdf $\bar{f}$ that respects uncertainty of source coordinates $\check{\boldsymbol{x}}$ as well as transformation parameters $\boldsymbol{q}$ can be evaluated with the aid of known pdfs $\tilde{f}$, $\tilde{g}$ prepared in (13), (12), (11).

$$\bar{f}(\,\widehat{\boldsymbol{x}},\, \boldsymbol{q}\,) = \tilde{f}(\widehat{\boldsymbol{x}} \,|\, \boldsymbol{q}\,)\, \tilde{g}(\boldsymbol{q}) \; . \tag{14}$$

We are in fact not interested in joint pdf $\bar{f}$. The required pdf h is marginal pdf of $\bar{f}$

$$h(\widehat{\boldsymbol{x}}) = \int\limits_{\mathcal{Q}} \bar{f}(\,\widehat{\boldsymbol{x}},\, \boldsymbol{q}\,)\, d\boldsymbol{q} = \int\limits_{\mathcal{Q}} \tilde{f}(\widehat{\boldsymbol{x}} \,|\, \boldsymbol{q}\,)\, \tilde{g}(\boldsymbol{q})\, d\boldsymbol{q} \; .$$

Here $\mathcal{Q}$ stands for a set of permitted values of transformation parameters $\boldsymbol{q}$.

With substitution (13) we can finally determine the required pdf h.

$$h(\widehat{\boldsymbol{x}}) = \int\limits_{\mathcal{Q}} \tilde{f}_{\boldsymbol{q}}(\widehat{\boldsymbol{x}})\, \tilde{g}(\boldsymbol{q})\, d\boldsymbol{q} \; . \tag{15}$$

The general solution given by (15) cannot be directly applied to any transformation. Specific algorithm for evaluation of integrals in (7) and (15) has to be designed first. This problem is resolved for linear conform transformation in the contribution.

4 Summary

Result of the proposed procedure is in form of probability distribution of points characterizing a geographic object. Pdf of a single point is given in (15). This solution brings hard numerical problems in general since the underlying estimation model is essentially nonlinear. These computational problems are resolved under specific assumptions on probability distribution of input data. 2D linear conform transformation is used as an example. It is shown that the solution is analytically tractable when the all input data are normally distributed and when precisions of corresponding GCPs are equal. These conditions are usually fulfilled in most practical cases.

The proposed procedure can be easily generalized to cover problems of multisensor data when data from several sources are simultaneously processed. Spatial accuracy of objects reconstructed from various images could be assessed more reliably than it has been attempted until now (see e.g. [4]).

References

1. Teunissen, P.J.T.: The non-linear 2D symmetric Helmert transformation: an exact non-linear least-squares solution. Bulletin Géodésique **62** (1988) 1–15
2. Kubáček, L., Kubáčková, L., Ševčík, J.: Linear conform transformation: errors in both coordinate systems. Applications of Mathematics **47** (2002) 361–380
3. von R., M.: Mathematical Theory of Probability and Statistics. Academic Press, New York (1964)
4. Lee, Y.R.: A study on accuracy of georeferencing of multisensor remote sensing images. In: ISSDQ'03 Proceedings, Hong Kong (2003)