A Monte Carlo Study of the Strength of Unidirectional Fiber-Reinforced Composites

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INTRODUCTION

BRITTLE MATERIALS SUCH as ceramics are attractive from many design viewpoints. For example, they inherently have high specific strengths which are stable over a wide range of temperature and environmental conditions. One notable disadvantage, however, is their susceptibility to strength-weakening flaws. Due to the increased probability of finding a serious flaw in a large part, the bulk strengths of these materials are typically several orders of magnitude lower than that predicted by atomic theory, which is about a tenth of the Young's modulus. Moreover, because the size and shape of these flaws are random, the observed strength invariably varies over a wide range. This raises the reliability question in a structure made of such brittle material.

Fiber-reinforced composites are designed to offset these problems. The idea is that when a bundle of "brittle" whiskers are bonded together, the strength of an individual whisker approaches that predicted by the atomic theory due to the reduced likelihood of finding a serious flaw. Yet, as a group the whiskers support each other. In the unlikely event that a particular whisker fractures, the load it carried is redistributed among the remaining fibers by shear transfer. In fact, at distances much larger than the fiber-to-fiber spacing from such a site of fracture, the same failed fiber still shares a part of the applied load. The size effect and the symbiotic relationship among the fibers lead to greatly improved strength of composites, which sometimes exceeds that of high strength steel. The same factors, however, make the prediction of composite strength from the strengths of the constituents a formidable task.

This paper deals with making a statistical estimate of the strength of composite

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materials. Various statistical theories are first reviewed. A numerical method is then proposed for estimating the strength of unidirectional fiber-reinforced composites, and some numerical results given.

SOME FAILURE MODELS

Weakest-Link Model

A simple statistical model for predicting the strength of the whole from the strengths of the parts is the so-called weakest-link model. Also known as the series model, it assumes the whole to fail when the weakest part in it fails. A graphic example of this model is the stretching of a chain of n links whose strengths are randomly distributed. Clearly, the strength of the chain is identical with the strength of the weakest link. The probability distribution of the chain's strength, denoted, say, by Φ, is therefore derivable from the probability distribution, denoted as F, of the strength of the links. In probability theory, Φ is known as the extreme value distribution, and F, variously known as the parent, initial or underlying distribution.

The first application of the series model to the study of material strength is due to Weibull [1, 2].[1] (For other applications see, for example, [3].) He showed that in a stressed volume of material, the probability of failure occurring at a reference stress[2] $\sigma_a \leqslant s$ is

$$\Phi(s) = Pr\,[\sigma_a \leqslant s] = 1 - \exp - \int_V \left(\frac{\sigma(s) - \sigma_u}{\sigma_o}\right)^m dV \tag{1}$$

where σ_u, σ_o and m are parameters, and $\sigma(s)$ is the stress distribution in the stressed volume V. In a simple-tension test the reference stress can be taken as the nominal applied stress σ_1 of the specimen. Equation (1) then becomes

$$\Phi(\sigma_1) = 1 - \exp - \left[V\left(\frac{\sigma_1 - \sigma_u}{\sigma_o}\right)^m\right] \tag{2}$$

The series model does not apply to the strength of composite materials, for the obvious reason that when the weakest fiber fails the remaining fibers continue to carry the load. It is only when there is a sufficient number of fiber fractures – a sufficient degradation of strength – that composite rupture occurs. Hence, the goodness that some failure data are observed to fit the Weibull distribution [4] can only be considered as phenomenological.[3]

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[1] Numbers in brackets designate References at end of paper.

[2] Since the behavior of a brittle material is linearly elastic, any stress component at a convenient point can be used as the reference stress.

[3] It is pointed out in [5] that sometimes as many as 1000 experiments are necessary to distinguish between, say, a Weibull and a log-normal distribution. The observed data could perhaps be fitted to another distribution, with another set of parameters.

The Parallel Model

The parallel model is appropriate for describing the strength of a bundle of "independent" fibers. If traction does not exist between fibers, then the load carried by a fiber is equal to the load on the bundle divided by the number of surviving fibers. Thus, if $L(\delta)$ is the applied load on the bundle at extension δ, the load on one fiber is $L(\delta)/n$, where the number of surviving fibers n is a random variable that depends upon the load level and the strength of the fibers. The bundle strength is then defined as the maximum value that $L(\delta)$ attains as δ is increased monotonically from zero. For the trivial case where all fibers have identical strength s, the bundle strength is simply $n_o s$, where n_o is the original number of fibers. If m fibers have strength s, and n fibers have strength $s' > s$, then the bundle strength is simply max $\{(m + n)s, ns'\}$. Clearly, the maximum value of $L(\delta)$, denoted by L^*, is a random quantity whose distribution depends upon the underlying distribution of the individual fibers. The determination of the probability distribution of L^* is known as the restricted classic fiber bundle problem. The best-known treatment of this problem is that of Daniels [6], who derived the exact distribution of L^* for any n, and also its asymptotic form as $n \to \infty$. A generalized version of the fiber bundle problem was analyzed by Phoenix [7], who allowed the load-strain relation in the fibers to be a function of some random vector. In particular, Coleman [8] and Platt, et al., [9, 10] have computed the bundle strength when the underlying fiber strength distribution is of the Weibull type.

The parallel model does not accurately describe the behavior of a unidirectional fibrous composite because the presence of traction between fibers leads to a more complex stress system than that assumed.

Other Fracture Models

The so-called dispersed fracture model, first proposed by Gücer and Gurland [11] and later investigated by Rosen [12], and Scop and Argon [13–15] is an alternative to the two models discussed above. The composite is modeled as a chain of n links in series, and each link is a bundle of m fibers in parallel. The composite fails when one of the links fails. Thus, the dispersed fracture model is essentially a weakest-link model, with an underlying random variable generated by a parallel model. The use of a series-parallel model is an attempt to account for the interaction between the fibers when the composite is loaded to failure. However, even the complex modified version proposed by Scop and Argon [15] failed to account for the interactions of many fiber breaks.

The cumulative weakening model of Zweben [16] and Zweben and Rosen [17] is a further refinement in modeling the complex stochastic process governing the failure of a composite. Again, the composite is conceptually partitioned into an $m \times n$ matrix, and the determination of the failure probability proceeds in several steps. The first step involves the calculation of the probabilities for the zero-order breaks, which are assumed to be far apart and hence independent of each other. The second step involves the calculation of probabilities for the first-order breaks, which occur when, as a result of the zero-order breaks, the stress redistribution causes additional fractures in the neighboring fibers. This process continues and composite failure occurs where there are k or more contiguous fiber breaks at some zero-order site. (Zweben and Rosen suggested $k = 2$ for continuous fibers, and $k = 3$ for discontinuous fibers.)

The cumulative weakening model probably offers, to date, the best description of the failure process in composite materials. However, because of the complex branching process and interaction of stresses, an accurate assessment of the conditional probabilities is difficult. The cumulative weakening model is reminiscent of the self-avoiding walk process. The positions of the current breaks can be considered as components of a random vector, which, as the load is increased, take a random walk into the unbroken material. The walks are self-avoiding because one break cannot "walk" into another. The value that the random vector assumes depends upon the current stress-strength distribution and also on the extent of the unbroken region. Clearly, it is difficult, if at all possible, to enumerate and analyze the events in a general self-avoiding walk.[4] Moreover, in this particular problem the terminal state which characterizes the failure of the composite (Zweben and Rosen took it to be $k = 2, 3$) cannot be specified easily. The difficulty of calculating the failure probability in closed form is indeed insurmountable. Thus, a numerical solution is proposed in the next section.

A MONTE CARLO SOLUTION

Just as complex field problems are amenable to numerical methods such as finite element or finite difference procedures, complicated stochastic processes can be simulated by a numerical method generally known as the Monte Carlo method. This technique has been used extensively to analyze many physical processes of a statistical nature, including the self-avoiding walk model of polymer and bacteria growth phenomena. It has been used to simulate the behavior of unidirectional composites by Larder and Beadle [20] in conjunction with a three-dimensional finite element procedure. Unfortunately, their paper does not address the convergence of the method, namely, the effects of increasing the number of elements and the number of fibers. Also, the removal of a broken element (particularly in a coarse finite element model) is not consistent with the fact that a good part of the broken element still transfers some load. In this report the Monte Carlo procedure will be used in conjunction with the shear lag equation and a finite difference scheme. Attention will be given to the convergence of the method, and the effects of different parameters.

A Monte Carlo *experiment* refers to the practice of randomly assigning a value to

[4] An approximate solution to the one-dimensional walk is possible. See [18, 19].

an underlying random variable or vector, and observing the outcome of the process of interest. A Monte Carlo *procedure* is composed of n such independent experiments. When n is sufficiently large the observations will yield, by virtue of the laws of large numbers, a good assessment of the statistical characteristics of the process. For the present problem the underlying random vector is the strength of the reinforcing fibers, and the process of interest is the fracture of the composite. A typical Monte Carlo experiment involves the partitioning of each fiber into n elements. Random numbers are generated and assigned as the components of the random vector, i.e., the strengths of the fiber elements. For a specified applied load the stresses in the fiber elements can be found by solving the shear lag equation. Points of fracture are then determined by comparing the stress and the strength for each fiber element. From the number and arrangement of the fiber breaks the failure load of a given sample of fiber elements is then obtained. Thus, determining the failure load *a la* Monte Carlo amounts to testing a sample "analytically" – the advantage is that the entire testing procedure is automated, fast, and inexpensive.

In the next three sections the main ingredients of a Monte Carlo experiment, namely, the random number generation, the stress solution, and the failure criterion for the composite are discussed.

Random Number Generation

As mentioned earlier each fiber is partitioned into n elements. If there are k fibers in the composite, then $k \times n$ random numbers are generated to characterize the strength of a particular sample.[5] It is assumed that the probability of a fiber element having a strength less than or equal to σ is given by the "truncated" Weibull distribution

$$F(\sigma) = \left\{ \left\{ \begin{array}{l} 0 \\ 1 - \exp\left[-\left(\dfrac{\sigma - \sigma_u}{\sigma_o} \right)^m \Delta x \right] \\ 1 \end{array} \right\} \Big/ F_x \quad , \quad \begin{array}{l} \sigma_u > \sigma \\ \sigma_u \leqslant \sigma \leqslant \sigma_{max} \\ \sigma > \sigma_{max} \end{array} \right\} \tag{3}$$

where Δx is the size of the element,

$$F_x = 1 - \exp\left(-\left(\frac{\sigma_{max} - \sigma_u}{\sigma_o} \right)^m \Delta x \right)$$

and σ_{max} is the maximum strength possible (taken as $E/10$). The particular form (3) of the underlying distribution guarantees that the strength of a whole fiber is always Weibull-distributed, as it should be, regardless of the number of mesh points chosen.

[5] The same partitioning is used later for the finite difference solution of the stress field.

To generate a random number from the Weibull distribution, one first generates a random number, say x, from a normalized uniform distribution. (This can be done by following the method in [21].) The Weibull random number is then obtained from x by equating probabilities. That is, if $U(x)$ is the probability associated with x in the normalized uniform distribution, one sets $U(x) = F(\sigma)$, or

$$U(x) = \left\{ 1 - \exp\left[-\left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m \Delta x \right] \right\} \Big/ F_x \tag{4}$$

Since $U(x) = x$, one readily obtains from Equation (4) the Weibull random number

$$\sigma = (-\log(1 - F_x)\,\Delta x)^{\frac{1}{m}}\ \sigma_o + \sigma_u$$

Stress Solution

A two-dimensional model is used in the present study. (The three-dimensional case differs only in the scope of the governing differential equation, see [22, 23]; the simulation techniques should apply equally to both cases.) It is assumed that the fibers are spaced uniformly, as shown in Figure 1. Force equilibrium requires that [22]

$$EA\frac{d^2u_i}{dx^2} + \frac{Gh}{d}(u_{i-1} - 2u_i + u_{i+1}) = 0 \tag{5a}$$

where u_i $(1 < i < k)$ refers to the displacement of ith fiber, and EA and Gh are the stiffness coefficients in extension and shear, respectively. For the first and the last fibers the differential equations take a slightly different form, viz.,

$$EA\frac{d^2u_1}{dx^2} + \frac{Gh}{d}(u_2 - u_1) = 0 \tag{5b}$$

$$EA\frac{d^2u_k}{dx^2} + \frac{Gh}{d}(u_k - u_{k-1}) = 0 \tag{5c}$$

For a composite pulled in simple tension the boundary conditions are

$$u_i(0) = 0 \quad , \quad E\frac{du_i}{dx} = \sigma \quad , \quad i = 1, 2 \ldots k \tag{6}$$

where σ is the applied stress.

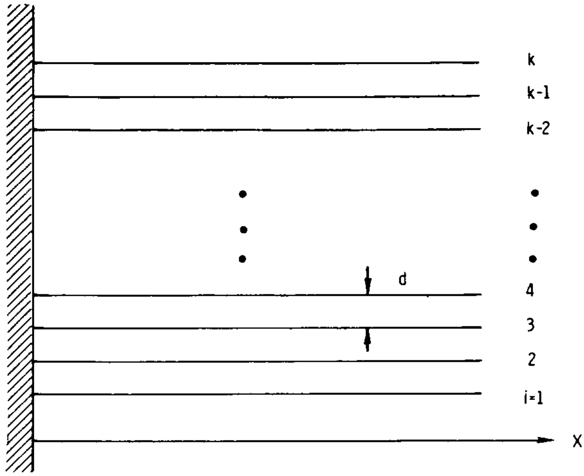

Figure 1. Schematic diagram of the unidirectional fiber composites.

If there are no breaks in the fibers it can be easily verified that $u_i = \delta/\ell\ x$, and $E\ du_i/dx = E\ \delta/\ell$, i.e., the fibers are in a state of constant stress. Fractures in the fibers, however, result in a nonuniform field. If the fracture locations were known *a priori*, the stress could be determined by the method given in [22]. In the present problem, the solution can be obtained by imposing on Equations (5) the duality condition that

either u_i *is single-valued and continuous (no break) and* $E\ du_i/dx\ (x) <$ $S_i\ (x)$, *where* $S_i\ (x)$ *is the strength of the ith fiber at x,*
or u_i *is double-valued and piecewise continuous, and* $du_i/dx = 0$ *(traction free) on either side of a break.*

The duality condition or constraint on u_i makes the solution of Equations (5) nonlinear with respect to the applied load. This type of nonlinearity also appears in certain free-boundary problems in lubrication [24], contact of deformable bodies [25], and elasto-plastic deformation of materials [26]. The Christopherson method [27] is known to be well suited for these problems, and is described next.

A finite difference method of uniform mesh size Δx is used to solve Equation (5). Thus, the second-order derivative is approximated by the difference form, viz.,

$$\frac{d^2 u_i}{dx^2} = \frac{u_i^{(j-1)} - 2u_i^{(j)} + u_i^{(j+1)}}{(\Delta x)^2} \tag{7}$$

where the superscripts $1 \leqslant j \leqslant n+1$ denote node numbers.

Substituting Equation (7) into Equation (5a) leads to

$$\frac{EA}{(\Delta x)^2}\left(u_i^{(j-1)} - 2u_i^{(j)} + u_i^{(j+1)}\right) + \frac{Gh}{d}\left(u_{i-1}^{(j)} - 2u_i^{(j)} + u_{i+1}^{(j)}\right) = 0$$

Or,

$$u_i^{(j)} = \frac{u_i^{(j-1)} + u_i^{(j+1)} + \dfrac{Gh}{EA}\dfrac{(\Delta x)^2}{d}\left(u_{i-1}^{(j)} + u_{i+1}^{(j)}\right)}{2\left(1 + \dfrac{Gh}{EA}\dfrac{(\Delta x)^2}{d}\right)} \tag{8}$$

Similar expressions can be obtained for the first and the last fibers. Equation (8) is the suitable form for successive over-relaxation (SOR). Thus, if $[u_i^{(j)}]^{\ell}$ denotes the approximate value of $u_i^{(j)}$ after ℓ iterations, the SOR algorithm states that

$$[u_i^{(j)}]^{\ell} = (1-\omega)\,[u_i^{(j)}]^{\ell-1} + \omega r \tag{9}$$

where $1 \leqslant \omega \leqslant 2$ is the relaxation factor and "r" is the right-hand side of Equation (8). Using the SOR algorithm one can start with an initial guess, say, $u = 0$, and, by repeating (9) over the ranges of i, j, obtain successively better approximations for the solution u. The Christopherson method states that if in Equation (9) $[u_i^{(j)}]^{\ell}$ violates the constraint, it is simply replaced by one that satisfies the constraint. Applied here the Christopherson method requires the testing of $du^{(j)}/dx_i$ each time a new estimate of $u_i^{(j)}$ is available. If $du^{(j)}/dx_i \geqslant S_i(x)$, (constraint violated) a double-valued $u_i^{(j)}$ is to be so chosen that $du^{(j)}/dx_i = 0$ on either side of the break.

The switching of the displacement from a single-valued to a double-valued function can be implemented conveniently through the use of "phantom points." In this method an extra node midway between $j-1$ and j is established whenever $u_i^{(j)}$ is found to violate the constraint. This midway node is used as the left-most point ($j-1$) when evaluating Equation (7) on the right side of the break, and as the right-most point ($j+1$) when evaluating the same on the left side of the break. Thus, e.g., on the left side of the break,

$$\frac{d^2u_i}{dx^2} = \frac{4}{3\Delta x}\left(\frac{u_i^{(j)} - u_i^{(j-1)}}{\Delta x} - \frac{u_i^{(j+½)} - u_i^{(j)}}{½\,\Delta x}\right)$$

Since the traction-free condition requires $u_i^{(j+½)} = u_i^{(j)}$, the last equation becomes

$$\frac{d^2u_i}{dx^2} = \frac{4}{3\Delta x^2}\left(u_i^{(j)} - u_i^{(j-1)}\right)$$

Similarly, when evaluating d^2u_i/dx^2 for the right side of the break, one obtains

$$\frac{d^2u_i}{dx^2} = \frac{4}{3\Delta x^2}\left(u_i^{(j+1)} - u_i^{(j)}\right)$$

The situation is depicted in Figure 2, where the phantom point is shown as a triangle. Note that on the left side of the break $u^{(\Delta)} = u^{(j-1)}$, whereas on the right side, $u^{(\Delta)} = u^{(j)}$. Thus, the displacement field is double-valued, but its finite difference representation (excluding the phantom nodes) are always single-valued. The use of phantom points is particularly convenient in computer implementation because the displacement array (memory locations for storing the nodal values of the displacement) is not cognizant of the presence of fiber breaks. The SOR method is used on the displacement array as it normally would; breaks in the fibers and the multivaluedness of the displacement are accounted for independently, through the judicious use of phantom points.

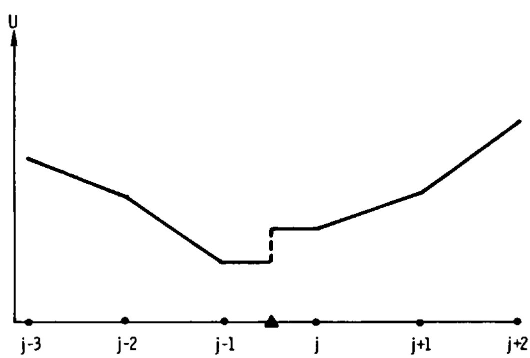

Figure 2. Schematic diagram showing the displacement field in a broken fiber. Note the use of the phantom point.

Fracture Criterion for the Composite

It was mentioned earlier that a criterion is required for determining whether a certain number and arrangement of fiber breaks will cause composite failure. A sufficient condition is that there are enough breaks so aligned as to form at least one cleavage plane perpendicular to the direction of loading. As shown in Figure 3, this condition disables the composite from supporting any load. From a numerical viewpoint, when the composite fails, the region in which Equation (5) are to be solved separates into two subregions. The left subregion, which is fixed to the ground, has the trivial solution $u \equiv 0$, and is independent of the boundary condition

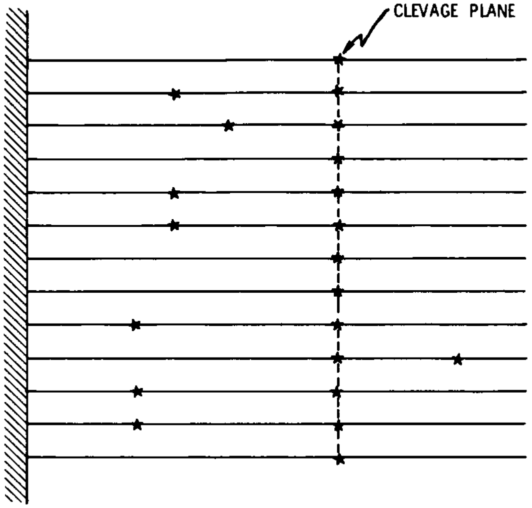

Figure 3. A possible failure mode.

at $x = \ell$. The right subregion has an indeterminate solution because the boundary conditions are both of the Neumann type. Clearly the use of SOR on Equations (5) in a fractured composite would be divergent.[6] Hence, the divergence of the solution scheme was used as the failure criterion. Indeed, in all the Monte Carlo experiments performed in the course of this study, the divergence of the SOR scheme is always associated with the formation of at least one cleavage plane.

NUMERICAL RESULTS

Results were obtained using the following set of parameters, typical of E-glass/epoxy composites:

Fiber tension modulus,	$82.6 \times 10^9\ N/m^2$
Matrix shear modulus,	$0.86 \times 10^9\ N/m^2$
Fiber diameter,	$0.25 \times 10^{-4}\ m$
Weibull parameters:	
location parameter,	0
scale parameter,	$2.09 \times 10^9\ N/m^2$
shape parameter,	4

The volume fraction and the specimen size are left as variables. The computer simulation consists of the three main ingredients described in the preceding

[6] The matrix corresponding to a separated region is reducible. Moreover, because of the boundary conditions in the right subregion, the matrix is singular. Hence, SOR will not converge [28].

sections. The loading, which appears as a boundary condition in Equation (6), starts at just below the strength value of the weakest element, and is incremented by 10% each load step. The matrix is considered to have no tensile modulus and does not contribute to the strength of the composite other than through the shear transfer mechanism. Hence, the value of σ just before the SOR scheme diverges is taken to be the strength of the composite.

To assess the convergence of the numerical procedure, Monte Carlo experiments were conducted using several mesh arrangements. The results are summarized in Figure 4. Each set of mean and variance represents the averages of 49 experiments. The convergence toward a mean of about 2.8×10^9 N/m^2 and variance of about 0.6×10^9 N/m^2 is evident as the number of mesh points increases. Similar convergence behavior is observed for $m = 8$. Note that the results using five fibers ($NF = 5$) are not significantly different from those using ten or twenty fibers, except when the number of mesh points is small. Thus, the "average" strength of a unidirectional composite can be simulated by as few as five fibers.

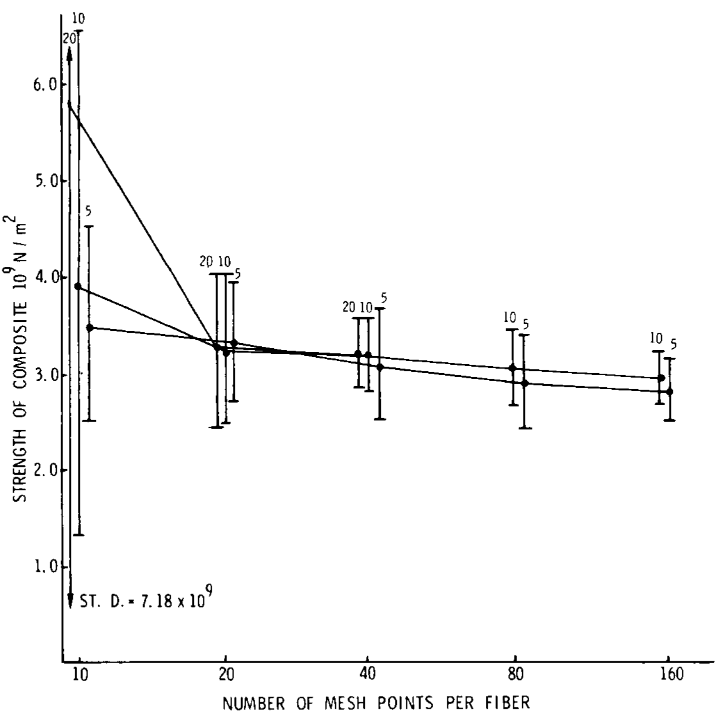

Figure 4. Convergence of the Monte Carlo/finite difference scheme.

The failure pattern of a typical sample is shown in the sequence of diagrams of Figure 5. The diagrams are output from the computer simulation, and each asterisk represents a point of fracture. It is interesting that the failure pattern resembles those observed in experiment [29].

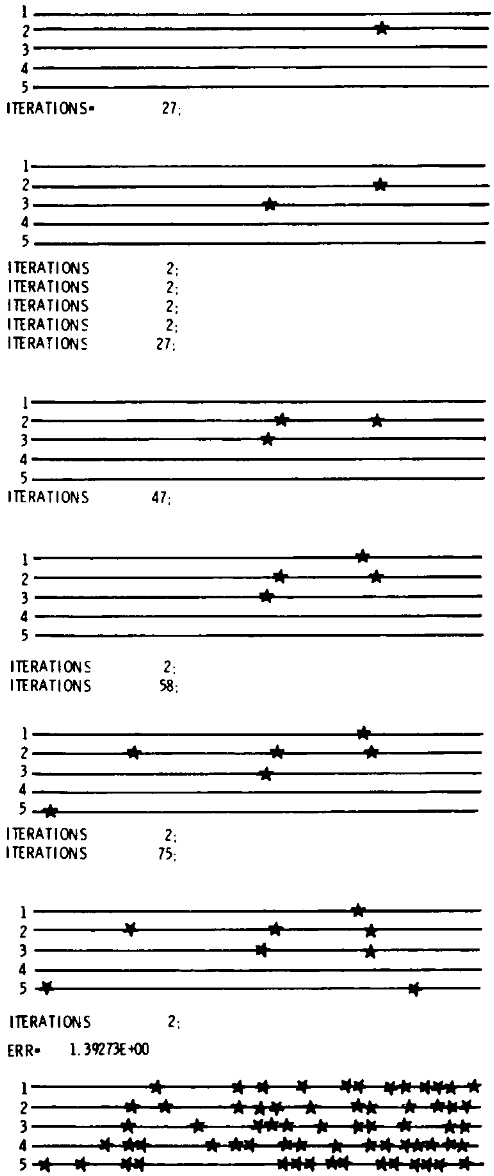

Figure 5. Development of the failure pattern of a typical sample.

The strength distributions of the composite and the fiber are shown in Figure 6 for a typical set of parameters (L = 0.025 m, V_f = 0.5). The probability of failure is approximated by i/n+1, where n is the total number of experiments, and i is the ranking of the observed strength, i.e., i = 1 corresponds to the smallest value, and i = n corresponds to the largest value. The strength data for the fibers fit as they should, a straight line on a Weibull probability paper (Figure 6a), but not on log-normal probability paper (Figure 6b). Similarly, the strength data for the composite appear to fit a straight line better on the Weibull scale than on the log-normal scale (Figure 6b). This indicates that, within the accuracy possible with 49 data points, the composite strength is probably Weibull-distributed. The parameters of the distribution are, however, different from those of the fibers – in particular, the shape parameter is larger, which indicates a smaller scatter in strength.

Effect of Volume Fraction

One of the important aspects in the analysis of composite strength is the effect of fiber volume fraction. There is evidence [30, 31] that the rule of mixtures is

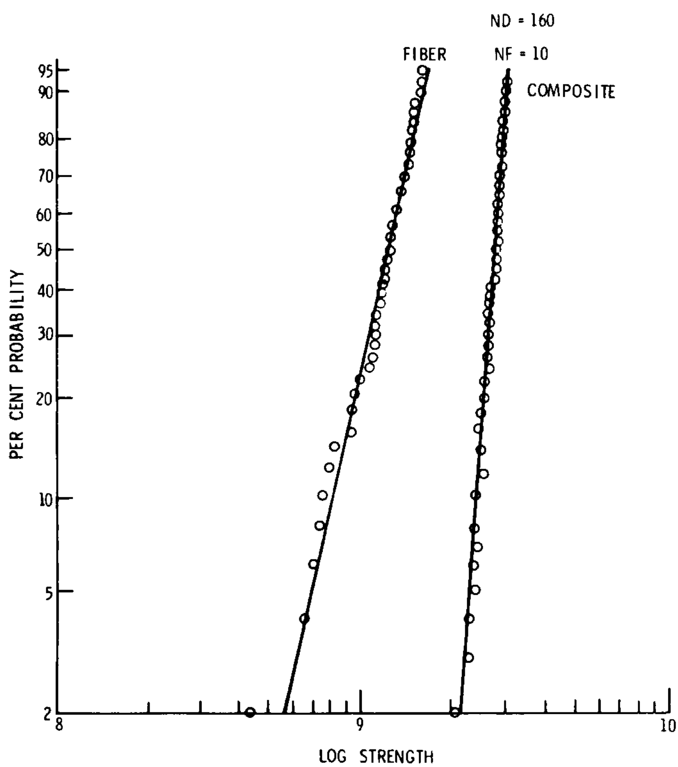

Figure 6a. Strength distribution, Weibull plot.

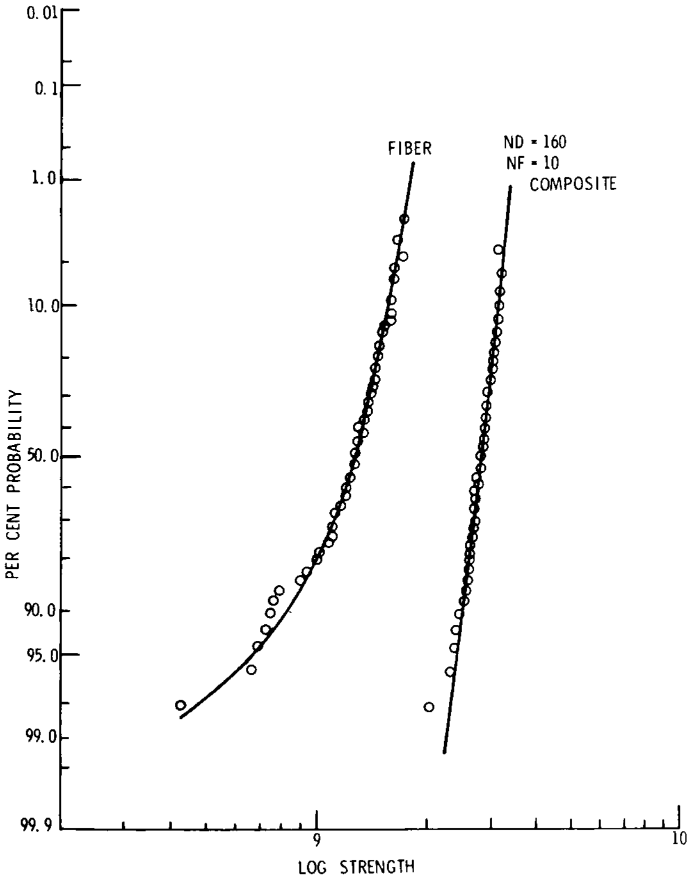

Figure 6b. Strength distribution, log normal plot.

generally valid in unidirectional fibrous composites. This behavior is observed here, as shown in Figure 7, where the composite strength is seen to increase linearly with volume fraction. This linear relationship can be explained as follows. In [22] it was shown that the fibers immediately adjacent to 'r' broken fibers are subjected to a stress concentration factor of $4 \cdot 6 \cdot 8 \ldots (2r + 2)/3 \cdot 5 \cdot 7 \ldots (2r + 1)$, independent of the parameters EA, Gh or 'd'. It follows that the events that lead to fracture and the forces on the fibers at failure must be independent of the volume fraction, the latter being inversely proportional to 'd'. Since the composite strength is given by $\sigma_c = \Sigma\sigma_f \cdot A_f/A_T$, where σ_f are the stresses in the fibers, A_f and A_T are the cross-sectional areas of the fibers and composite, respectively, it is clear that $\sigma_c = \sigma_f V_f$. It should be noted, however, that the strength of the fiber is not a well-defined quantity, it being dependent upon the gage length. Hence, it is probably more

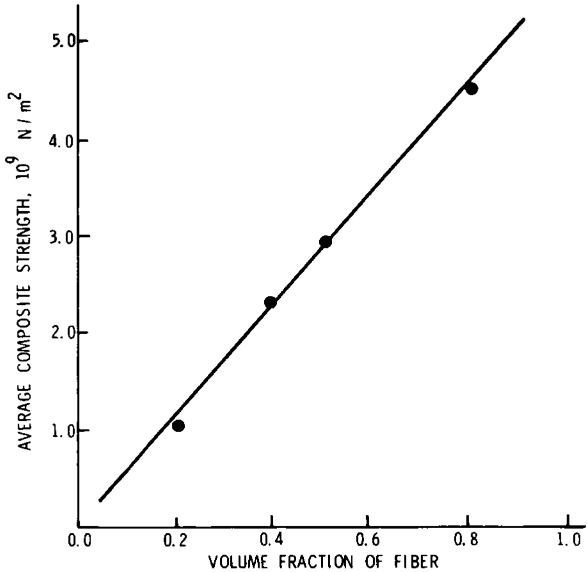

Figure 7. Linear relationship between strength and volume fraction.

appropriate to speak of a linear relationship between strength and volume fraction, rather than the rule of mixtures.

Size Effect

In brittle materials the probability of finding a severe flaw increases with the size of the structure. Hence, for equal loading a larger structure is more prone to fracture than a smaller one. While this size effect is an important aspect in the study of brittle fracture, there appears to be no previous study of the topic for composite materials. Monte Carlo experiments were therefore conducted for different sample sizes, while keeping the volume fraction fixed at 0.5. In these experiments the number of mesh points and the number of fibers are the same for all cases. It can be seen from Figure 8 that there is no apparent size effect over a one order-of-magnitude change in size.

DISCUSSION

In the foregoing analysis two critical assumptions were made, namely, the shear lag equation holds, and the fiber strength is Weibull-distributed. In reality the shear lag equation only provides a good approximation of the actual stress-strain distribution when the ratio Gh/dEA is not too small. For a soft matrix (Gh small) or small

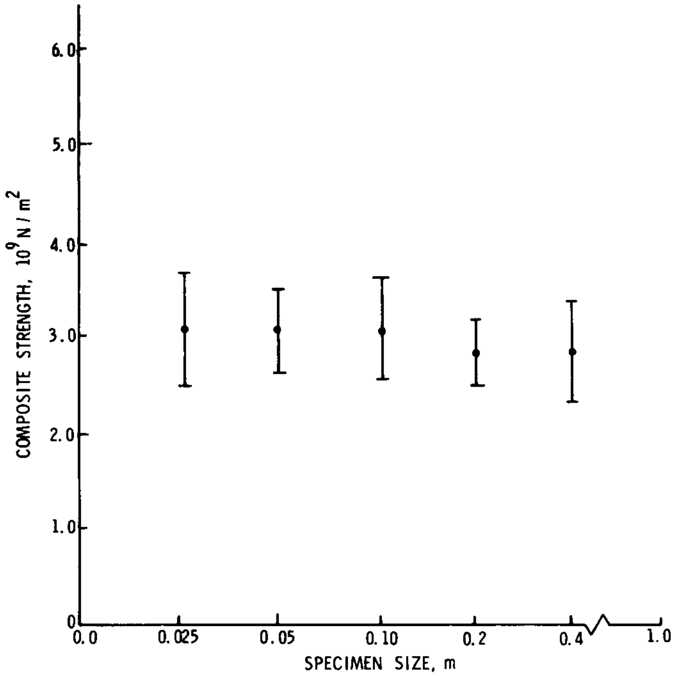

Figure 8. Size effect in composites.

fiber volume fraction (d large), the stress solution obtained from Equations (5a–5c) may no longer be accurate. In this case a more accurate stress solution is necessary, although this may increase the cost of computation beyond the practicality of the Monte Carlo procedure.

The actual distribution of fiber strength, which is generally non-Weibull as gage length diminishes, has been discussed in [32]. Whatever it might be, the actual distribution can be easily incorporated in the random number generator as described in a previous section. Since the composite strength depends to a great extent on the underlying fiber strength, it is important, from a predictive point of view, to determine what the actual strength distribution is.

There are other aspects that need consideration. For instance, while the extension to three-dimensional unidirectional composites is straightforward, the case of angle-ply or cross-ply laminates is much more difficult to analyze. There, it will be necessary to derive a new set of shear lag equations for fibers in adjacent laminae. The extension to chopped, unidirectional-fiber composites is relatively easy. The chopped fibers can simply be treated as continuous fibers with prescribed initial breaks. For randomly oriented fibers, however, the stress solution and the geometry of the problem is difficult, if indeed possible, to obtain and specify. In this case an entirely different approach is needed to analyze the problem. One such way is to treat the analytical solution of the problem as one involving random (elastic) coefficients. Preliminary indications are that the perturbation method is suitable for solving the governing differential equations, and is now under more detailed study.

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