Some New Makovian Models for Traffic and Performance Evaluation in Telecommunication Networks

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1 Introduction

A number of models are being developed for the traffic, service and queuing in the emerging telecommunication networks. The task is rather difficult because the traffic is bursty and correlated. These models include the compound Poisson process (CPP), the Markov modulated Poisson process (MMPP) and self-similar traffic models such as the Fractional Brownian Motion (FBM). The CPP and the $\sum_{k=1}^{K} CPP_k$ traffic models often give a good representation of the burstiness (batch size distribution) of the traffic from one or more sources, but not the auto-correlations of the inter-arrival time observed in real traffic. Conversely, the MMPP models can capture the auto-correlations but not the burstiness. Self-similar traffic models such as the FBM can represent both burstiness and auto-correlations, but they are analytically intractable in a queuing context. The traffic arriving to a node

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is often the superposition of traffic from a number of sources (homogeneous or heterogeneous), which further complicates the analysis of the system.

The MMCPP/GE/c queue and the MMCPP/GE/c/L G-queue serve as models for communication nodes with bursty arrivals and batch services, in the Markovian framework. In these queues, the steady state balance equations are highly complicated and hard to solve. The method of *successive simultaneous weighted subtractions* (SSS) was used on the balance equations in order to transform them to a computable form (the QBD-M –Quasi simultaneous-bounded-multiple Births and simultaneous-bounded-multiple Deaths)

Recently, we have introduced and proposed a new queuing model the MM $\sum_{k=1}^{K}$CPP$_k$/ GE/c/L G-queue with homogenous servers (the *sigma* queue) and with heterogenous servers (the *het+sigma* queue). We also developed some transformations which, when applied on the steady state balance equations, result in QBD-M type computable form. These models, in the Markovian framework, do provide a large flexibility to accommodate geometric as well as non-geometric batch sizes in both arrivals and services, and hence are capable of emerging as generalized Markovian node models.

We present the application of the *sigma* queue to the performance evaluation of some problems in telecommunications networks. We propose the use of negative customers, in an innovative way, to accommodate in the queuing model, the phenomenon due to the limitations of technology.

We also provide new ideas (concerning the *BMAP+sigma* and *gBMAP+sigma*queue), in this area, that we are actively pursuing. Moreover, some proposals for the application of the new queues will be provided at the presentation of this tutorial.

2 The *sigma* queue

In this section we present a methodology to analyze the MM $\sum_{k=1}^{K} CPP_k$/GE/c/L G-queue (which we name as the *sigma* queue).

2.1 The arrival process

The arrival and service processes are modulated by the same continuous time, irreducible Markov phase process with N states. Let Q be the generator matrix of this process, given by

$$Q = \begin{bmatrix} -q_1 & q_{1,2} & \cdots & q_{1,N} \\ q_{2,1} & -q_2 & \cdots & q_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N,1} & q_{N,2} & \cdots & -q_N \end{bmatrix},$$

where $q_{i,k}(i \neq k)$ is the instantaneous transition rate from phase i to phase k, and $q_i = \sum_{j=1}^{N} q_{i,j}, \quad (i = 1, \ldots, N)$. Let $\mathbf{r} = (r_1, r_2, \ldots, r_N)$ be the vector of equilibrium probabilities of the modulating phases. Then, $\mathbf{r}$ is uniquely determined by the equations, $\mathbf{r}Q = 0 \quad ; \quad \mathbf{r}\mathbf{e}_N = 1$, where $\mathbf{e}_N$ stands for the column vector with N elements, each of which is unity.

The arrival process, in any given modulating phase, is the superposition of K independent CPP arrival streams of customers (or packets, in packet-switched networks) and an independent CPP of negative customers. Customers of different arrival streams are not distinguishable. The parameters of the GE inter-arrival time distribution of the k^{th} $(1 \leq k \leq K)$ customer arrival stream in phase i are $(\sigma_{i,k}, \theta_{i,k})$, and (ρ_i, δ_i) are those of the negative customers. That is, the inter-arrival time probability distribution function is $1 - (1 - \theta_{i,k})e^{-\sigma_{i,k}t}$, in phase i, for the k^{th} stream of customers and $1 - (1 - \delta_i)e^{-\rho_i t}$ for the negative customers. Thus, all the $K+1$ arrival *point*-processes are Poisson, with batches arriving at each point having geometric size distribution. Specifically, in phase i, the probability that a batch is of size s is $(1 - \theta_{i,k})\theta_{i,k}^{s-1}$ for the k^{th} stream of customers, and $(1 - \delta_i)\delta_i^{s-1}$ for the negative customers.

Let $\sigma_{i,.}, \overline{\sigma_{i,.}}$ be the average arrival rate of customer batches and customers in phase i respectively. Let $\sigma, \overline{\sigma}$ be the overall average arrival rate of batches and customers respectively. Then,

$$\sigma_{i,.} = \sum_{k=1}^{K} \sigma_{i,k}, \qquad \overline{\sigma_{i,.}} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{(1 - \theta_{i,k})}, \qquad \sigma = \sum_{i=1}^{N} \sigma_{i,.} r_i, \qquad \overline{\sigma} = \sum_{i=1}^{N} \overline{\sigma_{i,.}} r_i \tag{1}$$

2.2 The GE multi-server

The service facility has c homogeneous servers in parallel, each with GE-distributed service times with parameters (μ_i, ϕ_i) in phase i. The service discipline is FCFS and each server serves at most one positive customer at any given time. Negative customers neither wait in the queue, nor are served. The operation of the GE server is similar to that described for the CPP arrival processes above. L denotes the queuing capacity, in all phases, including the customers in service. L can be finite or infinite. We assume, when the number of customers is j and the arriving batch size of positive customers is greater than $L-j$ (assuming finite L), that only $L-j$ customers are taken in and the rest are rejected.

However, the batch size associated with a service completion is bounded by one more than the number of customers waiting to commence service at the departure instant. For queues of length $c \leq j < L+1$ (including any customers in service), the maximum batch size at a departure instant is $j-c+1$, only one server being able to complete a service period at any one instant under the assumption of exponentially distributed batch-service times. Thus, the probability that a departing batch has size s is $(1-\phi_i)\phi_i^{s-1}$ for $1 \leq s \leq j-c$ and ϕ_i^{j-c} for $s = j-c+1$. In particular, when $j = c$, the departing batch has size 1 with probability one, and this is also the case for all $1 \leq j \leq c$ since each customer is already engaged by a server and there are then no customers waiting to commence service.

It is assumed that the first positive customer in a batch arriving at an instant when the queue length is less than c (so that at least one server is free) *never* skips service, i.e. always has an exponentially distributed service time. However, even without this assumption the methodology described in this paper is still applicable.

2.3 Negative customer semantics

A negative customer removes a positive customer in the queue, according to a specified *killing discipline*. We consider here a variant of the RCE killing discipline (removal of customers from the end of the queue), where the most recent positive arrival is removed, but which does *not* allow a customer actually in service to be removed: a negative customer that arrives when there are no positive customers waiting to start service has no effect. We may say that customers in service are immune to negative customers or that the service itself is *immune servicing*. Such a killing discipline is suitable for modeling e.g. load balancing, where work is transferred from overloaded queues but never work, that is, actually in progress.

When a batch of negative customers of size $l\,(1 \leq l < j-c)$ arrives, l positive customers are removed from the end of the queue leaving the remaining $j-l$ positive customers in the system. If $l \geq j-c \geq 1$, then $j-c$ positive customers are removed, leaving none waiting to commence service (queue length equal to c). If $j \leq c$, the negative arrivals have no effect.

$\overline{\rho_i}$, the average arrival rate of negative customers in phase i and $\overline{\rho}$, the overall average arrival rate of negative customers are given by,

$$\overline{\rho_i} = \frac{\rho_i}{1-\delta_i} \ ; \quad \overline{\rho} = \sum_{i=1}^{N} r_i \overline{\rho_i} \tag{2}$$

2.4 Condition for stability

When L is finite, the system is ergodic since the representing Markov process is irreducible. Otherwise, i.e. when $L = \infty$, the overall average departure rate increases with the queue length, and its maximum (the overall average departure rate when the queue length tends to ∞) can be determined as,

$$\overline{\mu} = c \sum_{i=1}^{N} \frac{r_i \mu_i}{1-\phi_i}. \tag{3}$$

Hence, we conjecture that the necessary and sufficient condition for the existence of steady state probabilities is,

$$\overline{\sigma} < \overline{\rho} + \overline{\mu} \ . \tag{4}$$

2.5 The steady state balance equations

The state of the system at any time t can be specified completely by two integer-valued random variables, $I(t)$ and $J(t)$. $I(t)$ varies from 1 to N, representing the phase of the modulating Markov chain, and $0 \leq J(t) < L+1$ represents the number of positive customers in the system at time t, including any in service. The system is now modeled by a continuous time discrete state Markov process, $\overline{Y}$ (Y if L is infinite), on a rectangular lattice strip. Let $I(t)$, the phase, vary in the horizontal direction and $J(t)$, the queue length or *level*, in the vertical direction. We denote the steady state probabilities by $\{p_{i,j}\}$, where $p_{i,j} = \lim_{t\to\infty} Prob(I(t) = i, J(t) = j)$, and let $\mathbf{v}_j = (p_{1,j}, \ldots, p_{N,j})$.

The process $\overline{Y}$ evolves due to the following instantaneous transition rates:

(a) $q_{i,k}$ – purely lateral transition rate – from state (i,j) to state (k,j), for all $j \geq 0$ and $1 \leq i,k \leq N$ $(i \neq k)$, caused by a phase transition in the Markov chain governing the arrival phase process ($q_{i,i} = 0$);

(b) $B_{i,j,j+s}$ – s-step upward transition rate – from state (i,j) to state $(i,j+s)$, for all phases i, caused by a new batch arrival of size s positive customers. For a given j, s can be seen as bounded when L is finite and unbounded when L is infinite;

(c) $C_{i,j,j-s}$ – s-step downward transition rate – from state (i,j) to state $(i,j-s)$, $(j-s \geq c+1)$ for all phases i, caused by either a batch service completion of size s or a batch arrival of negative customers of size s;

(d) $C_{i,c+s,c}$ – s-step downward transition rate – from state $(i,c+s)$ to state (i,c), for all phases i, caused by a batch arrival of negative customers of size $\geq s$ or a batch service completion of size $s\,(1 \leq s \leq L-c)$;

(e) $C_{i,c-1+s,c-1}$ – s-step downward transition rate, from state $(i,c-1+s)$ to state $(i,c-1)$, for all phases i, caused by a batch departure of size $s\,(1 \leq s \leq L-c+1)$;

(f) $C_{i,j+1,j}$ – 1-step downward transition rate, from state $(i,j+1)$ to state (i,j), $(c \geq 2\,;\,0 \leq j \leq c-2)$, for all phases i, caused by a single departure;

where

$$\begin{aligned}
B_{i,j-s,j} &= \sum_{k=1}^{K}(1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} \quad (\forall i;\ 0 \leq j-s \leq L-2\ ;\ j-s < j < L);\\
B_{i,j,L} &= \sum_{k=1}^{K}\sum_{s=L-j}^{\infty}(1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} = \sum_{k=1}^{K}\theta_{i,k}^{L-j-1}\sigma_{i,k} \quad (\forall i;\ j \leq L-1);
\end{aligned}$$

$$\begin{aligned}
C_{i,j+s,j} &= (1-\phi_i)\phi_i^{s-1}c\mu_i + (1-\delta_i)\delta_i^{s-1}\rho_i\\
&\quad (\forall i;\ \ c+1 \leq j \leq L-1;\ 1 \leq s \leq L-j);\\
&= (1-\phi_i)\phi_i^{s-1}c\mu_i + \delta_i^{s-1}\rho_i \quad (\forall i;\ j=c;\ 1 \leq s \leq L-c);\\
&= \phi_i^{s-1}c\mu_i \quad (\forall i;\ j=c-1;\ 1 \leq s \leq L-c+1);\\
&= 0 \quad (\forall i;\ c \geq 2;\ 0 \leq j \leq c-2;\ s \geq 2);\\
&= (j+1)\mu_i \quad (\forall i;\ c \geq 2;\ 0 \leq j \leq c-2;\ s=1).
\end{aligned}$$

Define,

$$\begin{aligned}
B_{j-s,j} &= \text{Diag}\left[B_{1,j-s,j}, B_{2,j-s,j}, \ldots, B_{N,j-s,j}\right] \quad (j-s<j\leq L);\\
B_s &= B_{j-s,j} \quad (j<L)\\
&= \text{Diag}\left[\ldots, \sum_{k=1}^{K}\sigma_{i,k}(1-\theta_{i,k})\theta_{i,k}^{s-1}, \ldots\right];\\
\Sigma_k &= \text{Diag}\left[\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{N,k}\right] \quad (k=1,2,\ldots,K);\\
\Theta_k &= \text{Diag}\left[\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}\right] \quad (k=1,2,\ldots,K);\\
\Sigma &= \sum_{k=1}^{K}\Sigma_k;\ R=\text{Diag}\left[\rho_1,\rho_2,\ldots,\rho_N\right];\ \Delta=\text{Diag}\left[\delta_1,\delta_2,\ldots,\delta_N\right];\\
M &= \text{Diag}\left[\mu_1,\mu_2,\ldots,\mu_N\right];\ \Phi=\text{Diag}\left[\phi_1,\phi_2,\ldots,\phi_N\right];\\
C_j &= jM \quad (0\leq j\leq c);\\
&= cM=C \quad (j\geq c);\\
C_{j+s,j} &= \text{Diag}\left[C_{1,j+s,j}, C_{2,j+s,j}, \ldots, C_{N,j+s,j}\right];\ E=\text{Diag}(\mathbf{e}_N^{'}).
\end{aligned}$$

Then, we get,

$$\begin{aligned}
B_s &= \sum_{k=1}^{K}\Theta_k^{s-1}(E-\Theta_k)\Sigma_k;\ B_1=B=\sum_{k=1}^{K}(E-\Theta_k)\Sigma_k;\\
B_{L-s,L} &= \sum_{k=1}^{K}\Theta_k^{s-1}\Sigma_k;
\end{aligned}$$

$$\begin{aligned}
C_{j+s,j} &= C(E-\Phi)\Phi^{s-1}+R(E-\Delta)\Delta^{s-1}\\
&\quad (c+1\leq j\leq L-1;\ s=1,2,\ldots,L-j);\\
&= C(E-\Phi)\Phi^{s-1}+R\Delta^{s-1} \quad (j=c;\ s=1,2,\ldots,L-c);\\
&= C\Phi^{s-1} \quad (j=c-1;\ s=1,2,\ldots,L-c+1);\\
&= 0 \quad (c\geq 2;\ 0\leq j\leq c-2;\ s\geq 2);\\
&= C_{j+1} \quad (c\geq 2;\ 0\leq j\leq c-2;\ s=1).
\end{aligned}$$

The steady state balance equations are,

$$\sum_{s=1}^{L}\mathbf{v}_{L-s}B_{L-s,L}+\mathbf{v}_L\left[Q-C-R\right]=0; \tag{5}$$

$$\sum_{s=1}^{j} \mathbf{v}_{j-s} B_s + \mathbf{v}_j \left[Q - \Sigma - C_j - R I_{j>c}\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0; \quad (0 \le j \le L-1) \tag{6}$$

$$\sum_{j=0}^{L} \mathbf{v}_j \mathbf{e}_N = 1. \tag{7}$$

where, $I_{j>c} = 1$ if $j > c$ else 0, and $\mathbf{e}_N$ is a column vector of size N with all ones.

2.6 Solution methodology and technique

When $L < c + K + 4$, then the number of the states of the Markov process $\overline{Y}$ is not large and can be solved by traditional methods [10]. However, when L is large, computationally more efficient other methods are available.

In this section the balance equations are essentially transformed into a computable form of QBD-M type. The necessary mathematical transformations are based on profound theoretical analysis and proofs [5]. Moreover, they are convenient for software implementation.

Let the balance equations for level j be denoted by $< j >$. Hence, $< 0 >$, $< 1 >$, …, $< j >$, …, $< L >$ are the balance equations for the levels $0, 1, \ldots, j, \ldots, L$ respectively. Substituting $B_{L-s,L} = \sum_{k=1}^{K} \Theta_k^{s-1} \Sigma_k$ and $B_s = \sum_{k=1}^{K} \Theta_{.k}^{s-1} (E - \Theta_k) \Sigma_k$ in (5, 6), we get the balance equations for level L and for all the other levels as:

$< L >$:

$$\sum_{s=1}^{L} \sum_{k=1}^{K} \mathbf{v}_{L-s} \Theta_k^{s-1} \Sigma_k + \mathbf{v}_L \left[Q - C - R\right] = 0 \tag{8}$$

$$\vdots$$

$< j >$:

$$\begin{gathered} \sum_{s=1}^{j} \sum_{k=1}^{K} \mathbf{v}_{j-s} \Theta_k^{s-1} (E - \Theta_k) \Sigma_k + \\ \mathbf{v}_j \left[Q - \Sigma - C_j - R I_{j>c}\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \\ (j = L-1, \ldots, 0). \end{gathered} \tag{9}$$

Define the functions, $F_{k,l}$, $(k = 1, 2, \ldots, K \;;\; l = 1, 2, \ldots, k)$, recursively as,

$$\begin{aligned}
F_{k,0} &= E \;,\; F_{k,k} = \prod_{n=1}^{k} \Theta_n \;\; (k = 1, 2, \ldots, K); \\
F_{k,l} &= 0 \quad (k = 1, 2, \ldots, K;\; l < 0); \\
F_{k,l} &= 0 \quad (k = 1, 2, \ldots, K;\; l > k); \\
F_{1,0} &= E \;\;;\;\; F_{1,1} = \Theta_1 \;; \\
F_{k,l} &= F_{k-1,l} + \Theta_k F_{k-1,l-1} \quad (2 \le k \le K \;,\; 1 \le l \le k-1) \,. \qquad (10)
\end{aligned}$$

Transformation 1 *Modify simultaneously the balance equations for levels j $(L-1 \ge j \ge K)$, by the transformation:*

$$\begin{aligned}
< j >^{(1)} \quad &\longleftarrow < j > + \sum_{l=1}^{K} (-1)^l < j-l > F_{K,l} \quad (K \le j \le L-1); \\
< j >^{(1)} \quad &\longleftarrow < j > \quad (j = L \; or \; j < K).
\end{aligned}$$

Apply the following transformation to the resulting equations.

Transformation 2 *Modify simultaneously the balance equations for levels j $(L-3 \ge j \ge c+K+1)$, by the transformation:*

$$\begin{aligned}
< j >^{(2)} \quad &\longleftarrow < j >^{(1)} - < j+1 >^{(1)} (\Phi + \Delta) + < j+2 >^{(1)} \Phi\Delta \\
&\quad (c+K+1 \le j \le L-3); \\
< j >^{(2)} \quad &\longleftarrow < j >^{(1)} \quad (j > L-3 \; or \; j < c+K+1).
\end{aligned}$$

After these two transformations, the transformed balance $(< j >^{(2)})$ equations for the rows $(c+K+1 \le j \le L-3)$, will be of the form (refer to [5] for the proof):

$$\begin{aligned}
&\mathbf{v}_{j-K} Q_0 + \mathbf{v}_{j-K+1} Q_1 + \ldots + \mathbf{v}_{j+2} Q_{K+2} = 0 \\
&\qquad (j = L-3, L-4, \ldots, c+K+1), \qquad (11)
\end{aligned}$$

where, $Q_0, Q_1, \ldots, Q_{K+2}$ are $K+3$ number of *j-independent* matrices and can be easily derived and computed (refer to [5]).

Thus, the resulting equations (11) corresponding to the rows from $c+K+1$ to $L-3$, are of the same form as those of the QBD-M processes and hence have an efficient solution by several alternative methods such as the spectral expansion method, Bini-Meini's method or the matrix-geometric method with folding or block size enlargement [7].

The set of equations (11) concerning the levels $c+K+1$ to $L-3$ have the coefficient matrices $Q_0, Q_1, \ldots, Q_{K+2}$ that are independent of j and hence have an efficient solution by the spectral expansion method [9]. These Q_l's ($l = 0, 1, \ldots, K+2$) can be obtained quite easily following the computational procedure in [5].

2.7 Other killing disciplines

Apart from the killing discipline that was used above, there are two other popular killing disciplines, the RCE-*inimmune servicing* and the RCH killing disciplines. The applicability of the killing disciplines rather depends on the situation and the purpose, and hence depending on these, many more killing disciplines are theoretically possible. Our methodology can easily be extended to many other killing disciplines also, this is explained briefly in this section.

2.7.1 The RCE-*inimmune discipline*

In the RCE-*inimmune* servicing, the negative customer removes the most recent positive arrival regardless of whether it is in service or waiting; thus a negative arrival has no effect only when it encounters an empty queue and all servers idle. This is the traditional killing discipline, suited to the modelling of killing signals in speculative parallelism, for example. It can also be used to model cell losses caused by the arrival of a corrupted cell or one encountering a full buffer, when the preceding cells of a packet would be discarded. In this case, the later and the upward transitions remain as before, but the downward transition rates become [6]:

$$
\begin{aligned}
C_{j+s,j} &= C(E-\Phi)\Phi^{s-1} + R(E-\Delta)\Delta^{s-1} \\
&\qquad (c \le j \le L-1 \ ; \ s = 1, 2, \ldots, L-j) \\
&= C\Phi^{s-1} + R(E-\Delta)\Delta^{s-1} \qquad (c > 1 \ ; \ j = c-1 \ ; \ 1 \le s \le L-c+1) \\
&= C\Phi^{s-1} + R\Delta^{s-1} \qquad (c = 1 \ ; \ j = c-1 = 0 \ ; 1 \le s \le L) \\
&= C_{j+1} + R(E-\Delta) \qquad (1 \le j \le c-2 \ ; \ s = 1) \\
&= R(E-\Delta)\Delta^{s-1} \qquad (1 \le j \le c-2 \ ; \ 2 \le s \le L-j) \\
&= R\Delta^{s-1} \qquad (c > 1 \ ; \ j = 0 \ ; \ 2 \le s \le L) \\
&= C_1 + R \qquad (j = 0 \ ; \ s = 1)
\end{aligned}
$$

Thus, after obtaining the required transition rates, e.g. lateral, upward and downward, and the balance equations in this case, the same procedure as in Section 2.6 can be used, perhaps with slight appropriate modifications, to transform the balance equations to a suitable form (QBD-M type equations) for solving by the existing methods.

2.7.2 The RCH discipline

Another popular killing discipline is the RCH (Removal of customers from the head of the queue) killing discipline. This is appropriate for modelling server breakdowns, where a customer in service will be lost for sure and may be also a portion of queue of waiting customers. Here too, the A matrices and the B matrices remain unchanged, however the C matrices would be different and can be determined. For example, to obtain $C_{j+s,j}$, consider the system (say, a single server system with $c = 1$) in state $(i, j+s)$, where $j+s \le L$. The rate at which a batch of l negative customers arrives is $(1-\delta_i)\delta_i^{l-1}\rho_i \ (l = 1, 2, \ldots)$. If $l \ge j+s$ all the jobs will be removed by the negative customers. If $l < j+s$, then the job in service, then the job *in service* plus $l-1$ jobs waiting for service would be removed, leaving only $j+s-l$ jobs. However, due to the nature of the generalised exponential service times, a certain number of customers may be 'leaked out', i.e. serviced *instantaneously* just after the killing takes place. Alternatively, if we redefine the operation of the system such that this leakage does not occur, i.e. immediately after a negative arrival, the next customer in the queue (if any) *cannot* skip service, then the equilibrium state probabilities would be same as in the case of the RCE-inimmune servicing.

Assuming leakage, the rate matrices $C_{j+s,j}$ can be derived as [6],

$$\begin{aligned} C_{j+s,j} &= (E-\Phi)\Phi^{s-1}M + (E-\Delta)R(E-\Phi)\{\Phi\Delta\}_{s-1} \\ &\qquad (1 \le j \le L-1\ ;\ s \ge 1)\ ; \\ &= \Phi^{s-1}M + (E-\Delta)R\{\Phi\Delta\}_{s-1} + R\Delta^{s} \quad (j=0\ ;\ s \ge 2)\ ; \\ &= M + R \qquad (j=0\ ;\ s=1)\ ; \end{aligned}$$

where,

$$\begin{aligned} \{\Phi\Delta\}_{s-1} &= \sum_{k=0}^{s-1} \Phi^{s-1-k}\Delta^{k} \qquad (s \ge 2)\ ; \\ &= E \qquad (s=1)\ . \end{aligned}$$

This case also can be modelled following the previous procedures, perhaps with minor modifications.

2.8 Application

In this section the application of the proposed queue is demonstrated by carrying out the performance analysis of an OBS multiplexer and storage.

OBS core node architecture typically includes structures with multiplexers and FDL's and Turner [11] suggests the analysis of a rather simplified model of the same in order to predict the performance boundaries. In what follows, a similar OBS multiplexer that assigns incoming bursts to one of c wavelengths that are available for transmitting data bursts into the outgoing link and an optical storage that comprises $L-c$ storage locations for bursts, are considered. All storage locations are assumed to be implemented with FDL's of equal size. The OBS multiplexer receives traffic from K sources, where a source may represent traffic coming from a group of subscribers.

Our model assumes the superposition of traffic from all sources following the $MM\sum_{k=1}^{K} CPP_k$ arrivals. The service times of data bursts are assumed to be of GE distribution. The resulting queue, which is an approximate model, is then used to predict the performance of the multiplexer and the storage locations, where the servers represent the wavelengths.

It is clearly discussed by Yoo et al. [12] that due to the explained properties and technology limitations of FDL's structure, the OBS system cannot be accurately modeled as a classical queue. We now introduce the use of the

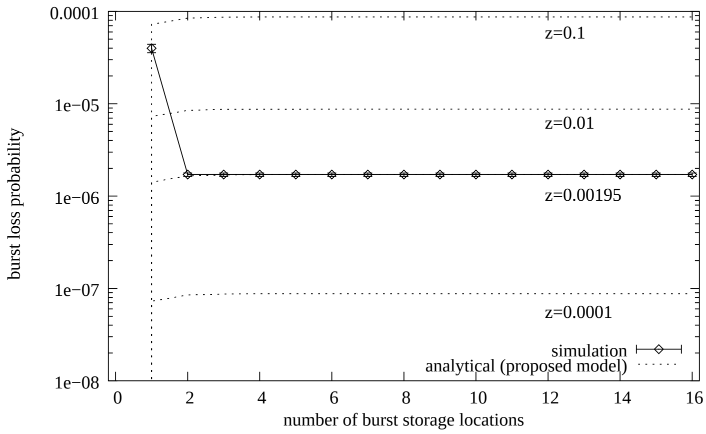

Figure 1: Simulation and analytical results with negative customers: loss of bursts due to FDL-technology constraints

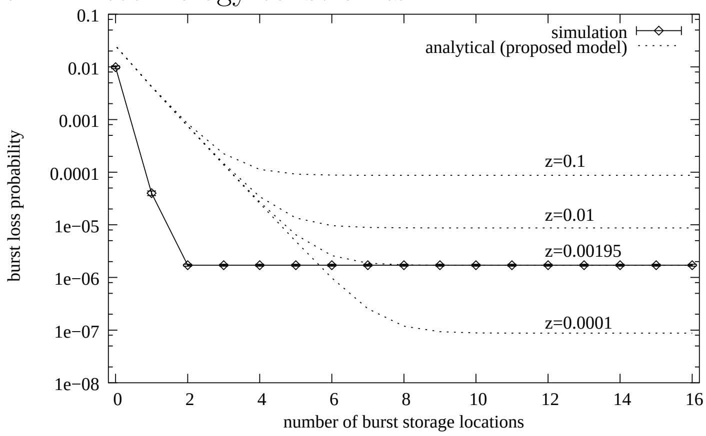

Figure 2: Simulation and analytical results with negative customers: total loss which also includes buffer-overflow

negative customers to account for the loss of bursts due the technology limitations of the FDL's, which is specific to the optical storage solution using FDL's.

Figure 1 demonstrates that the loss of bursts caused by the effect of FDL's technology constraints versus number of FDL's. This is fairly well accounted in the queuing model by the use of negative customers. Figure 2 depicts the total loss which also includes the loss of bursts due to buffer-overflow. Our model estimates are very good for larger values of the number of FDL's used (for more details in the used parameters value see [5]). However, for smaller values of FDL's, the total loss estimates are rather higher, never lower.

3 The *het+sigma* queue

3.1 The extension of the *sigma* queue

The *het-sigma* queue is the extension of the *sigma* queue, where the service facility has c *heterogeneous* servers in parallel, numbered just as their service priorities, *i.e.* $1, 2, \ldots, c$. Their respective GE-distributed service time parameters are $\mu_{i,n}, \phi_{i,n}$ $(n = 1, 2, \ldots, c)$ in phase i.

The service discipline is FCFS and each server serves at most one positive customer at any given time. When the number of customers j is less than c, only the first j servers (*i.e.* servers numbered $1, 2, \ldots, j$) are occupied and the rest are idle. Negative customers neither wait in the queue, nor are served. The operation of the GE server is similar to that described for the CPP arrival processes above. However, the batch size associated with a service completion is bounded by one more than the number of customers waiting to commence service at the departure instant. For queues of length $c \leq j < L + 1$ (including any customers in service), the maximum batch size at a departure instant is $j - c + 1$, only one server being able to complete a service period at any one instant under the assumption of exponentially distributed batch-service times. Thus, the probability that a departing batch is of size s is $\sum_{n=1}^{c} \frac{\mu_{i,n}(1-\phi_{i,n})\phi_{i,n}^{s-1}}{\mu_{i.}}$ for $1 \leq s \leq j - c$ and $\sum_{n=1}^{c} \frac{\mu_{i,n}\phi_{i,n}^{j-c}}{\mu_{i.}}$ for $s = j - c + 1$, where $\mu_{i.} = \sum_{n=1}^{c} \mu_{i,n}$. In particular, when $1 \leq j \leq c$, the departing batch has size 1 with probability 1.0 since each customer is already engaged by a server and there are then no customers waiting to commence service.

It is assumed that the first positive customer in a batch arriving at an instant when the queue length is less than c (so that at least one server is free) *never* skips service, i.e. always has an exponentially distributed service time. However, even without this assumption the methodology described in this paper is still applicable.

3.2 Condition for stability

When L is finite, the system is ergodic since the representing Markov process is irreducible. Otherwise, i.e. when the queueing capacity is unbounded, the overall average departure rate increases with the queue length, and its maximum (the overall average departure rate when the queue length tends to ∞) can be determined as,

$$\overline{\mu} = \sum_{n=1}^{c} \sum_{i=1}^{N} \frac{r_i \mu_{i,n}}{1 - \phi_{i,n}}. \tag{12}$$

Hence, we conjecture the necessary and sufficient condition for the existence of steady state probabilities is

$$\overline{\sigma} < \overline{\rho} + \overline{\mu}. \tag{13}$$

3.3 The steady state balance equations

The instantaneous transition rates of the process can be written as follows:

$$\begin{aligned}
B_{i,j-s,j} &= \sum_{k=1}^{K} (1 - \theta_{i,k}) \theta_{i,k}^{s-1} \sigma_{i,k} \quad (\forall i \ ; \ 0 \leq j - s \leq L - 2 \ ; \ j - s < j < L) \ ; \\
B_{i,j,L} &= \sum_{k=1}^{K} \sum_{s=L-j}^{\infty} (1 - \theta_{i,k}) \theta_{i,k}^{s-1} \sigma_{i,k} = \sum_{k=1}^{K} \theta_{i,k}^{L-j-1} \sigma_{i,k} \quad (\forall i \ ; \ j \leq L - 1) \ ;
\end{aligned}$$

$$
\begin{aligned}
C_{i,j+s,j} &= \sum_{n=1}^{c} \mu_{i,n}(1-\phi_{i,n})\phi_{i,n}^{s-1} + (1-\delta_i)\delta_i^{s-1}\rho_i \\
&\qquad\qquad (\forall i\ ;\ c+1 \le j \le L-1\ ;\ 1 \le s \le L-j) \\
&= \sum_{n=1}^{c} \mu_{i,n}(1-\phi_{i,n})\phi_{i,n}^{s-1} + \delta_i^{s-1}\rho_i \quad (\forall i\ ;\ j=c\ ;\ 1 \le s \le L-c) \\
&= \sum_{n=1}^{c} \phi_{i,n}^{s-1}\mu_{i,n} \qquad (\forall i\ ;\ j=c-1\ ;\ 1 \le s \le L-c+1) \\
&= 0 \qquad (\forall i\ ;\ c \ge 2\ ;\ 0 \le j \le c-2\ ;\ s \ge 2) \\
&= \sum_{n=1}^{j+1} \mu_{i,n} \qquad (\forall i\ ;\ c \ge 2\ ;\ 0 \le j \le c-2\ ;\ s=1)\ ;
\end{aligned}
$$

Define,

$$
\begin{aligned}
B_{j-s,j} &= \mathrm{Diag}\left[B_{1,j-s,j}, B_{2,j-s,j}, \ldots, B_{N,j-s,j}\right] \qquad (j-s < j \le L)\ ; \\
B_s &= B_{j-s,j} \qquad (j < L) \\
&= \mathrm{Diag}\left[\sum_{k=1}^{K} \sigma_{1,k}(1-\theta_{1,k})\theta_{1,k}^{s-1}, \ldots, \sum_{k=1}^{K} \sigma_{N,k}(1-\theta_{N,k})\theta_{N,k}^{s-1}\right]\ ; \\
\Sigma_k &= \mathrm{Diag}\left[\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{N,k}\right] \qquad (k=1,2,\ldots,K)\ ; \\
\Theta_k &= \mathrm{Diag}\left[\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}\right] \qquad (k=1,2,\ldots,K)\ ; \\
\Sigma &= \sum_{k=1}^{K} \Sigma_k\ ; \\
R &= \mathrm{Diag}\left[\rho_1, \rho_2, \ldots, \rho_N\right]\ ;
\end{aligned}
$$

$$
\begin{aligned}
\Delta &= \mathrm{Diag}\left[\delta_1, \delta_2, \ldots, \delta_N\right]\ ; \\
M_n &= \mathrm{Diag}\left[\mu_{1,n}, \mu_{2,n}, \ldots, \mu_{N,n}\right] \qquad (n=1,2,\ldots,c)\ ; \\
\Phi_n &= \mathrm{Diag}\left[\phi_{1,n}, \phi_{2,n}, \ldots, \phi_{N,n}\right] \qquad (n=1,2,\ldots,c)\ ; \\
C_j &= \sum_{n=1}^{j} M_n \qquad (1 \le j \le c)\ ; \\
&= \sum_{n=1}^{c} M_n = C \qquad (j \ge c)\ ; \\
C_{j+s,j} &= \mathrm{Diag}\left[C_{1,j+s,j}, C_{2,j+s,j}, \ldots, C_{N,j+s,j}\right]\ ; \\
E &= \mathrm{Diag}(\mathbf{e}_N')\ .
\end{aligned}
$$

Then, we get,

$$\begin{aligned} B_s &= \sum_{k=1}^{K} \Theta_k^{s-1}(E-\Theta_k)\Sigma_k \ ; \ B_1 = B = \sum_{k=1}^{K}(E-\Theta_k)\Sigma_k \ ; \\ B_{L-s,L} &= \sum_{k=1}^{K} \Theta_k^{s-1}\Sigma_k \ ; \end{aligned}$$

$$\begin{aligned} C_{j+s,j} &= \sum_{n=1}^{c} M_n(E-\Phi_n)\Phi_n^{s-1} + R(E-\Delta)\Delta^{s-1} \\ &\qquad (c+1 \le j \le L-1 \ ; \ s = 1,2,\ldots,L-j) \ ; \\ &= \sum_{n=1}^{c} M_n(E-\Phi_n)\Phi_n^{s-1} + R\Delta^{s-1} \qquad (j = c \ ; \ s = 1,2,\ldots,L-c) \ ; \\ &= \sum_{n=1}^{c} M_n\Phi_n^{s-1} \qquad (j = c-1 \ ; \ s = 1,2,\ldots,L-c+1) \ ; \\ &= 0 \qquad (c \ge 2 \ ; \ 0 \le j \le c-2 \ ; \ s \ge 2) \ ; \\ &= C_{j+1} \qquad (c \ge 2 \ ; \ 0 \le j \le c-2 \ ; \ s = 1) \ . \end{aligned}$$

The steady state balance equations are,

(1) For the L^{th} row or level:

$$\sum_{s=1}^{L} \mathbf{v}_{L-s}B_{L-s,L} + \mathbf{v}_L\left[Q - C - R\right] = 0 \ ; \tag{14}$$

(2) For the j^{th} row or level:

$$\sum_{s=1}^{j} \mathbf{v}_{j-s}B_s + \mathbf{v}_j\left[Q - \Sigma - C_j - RI_{j>c}\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s}C_{j+s,j} = 0 \quad (0 \le j \le L-1) \ ; \tag{15}$$

(3) Normalisation

$$\sum_{j=0}^{L} \mathbf{v}_j\mathbf{e}_N = 1 \ . \tag{16}$$

where, $I_{j>c} = 1$ if $j > c$ else 0, and $\mathbf{e}_N$ is a column vector of size N with all ones.

Define the functions $H_{c,n}\,(n = 1, 2, \ldots, c)$ recursively as,

$$\begin{aligned}
H_{m,0} &= E \;\;,\;\; H_{m,m} = \prod_{i=1}^{m} \Phi_i \;\; (m = 1, 2, \ldots, c); \\
H_{m,n} &= 0 \; (m = 1, 2, \ldots, c;\; n < 0) \\
H_{m,n} &= 0 \; (m = 1, 2, \ldots, c;\; n > m) \\
H_{1,0} &= E \;\;;\;\; H_{1,1} = \Phi_1 \;; \\
H_{m,n} &= H_{m-1,n} + \Phi_m H_{m-1,n-1} \;\; (2 \le m \le c \;,\; 1 \le n \le m-1) \;. \quad (17)
\end{aligned}$$

Transformation 3 *Modify simultaneously the balance equations for levels $j\,(L-2-c \ge j \ge c+K+1)$, by the transformation:*

$$\begin{aligned}
<\mathbf{j}>^{(\mathbf{1})} \quad &\longleftarrow <\mathbf{j}> + \sum_{l=1}^{K} (-1)^l < \mathbf{j} - \mathbf{l} > F_{K,l} (c+K+1 \le j \le L-2-c) \\
<\mathbf{j}>^{(\mathbf{1})} \quad &\longleftarrow <\mathbf{j}> \qquad\qquad\qquad\qquad (j > L-2-c \; or \; j < c+K+1)
\end{aligned}$$

Apply the second transformation to the resulting equations.

Transformation 4 *Modify simultaneously the balance equations for levels $j\,(L-2-c \ge j \ge c+K+1)$, by the transformation:*

$$\begin{aligned}
<\mathbf{j}>^{(\mathbf{2})} \quad &\longleftarrow <\mathbf{j}>^{(\mathbf{1})} + \sum_{n=1}^{c} (-1)^n < \mathbf{j} + \mathbf{n} >^{(\mathbf{1})} H_{c,n} \\
&\qquad (c+K+1 \le j \le L-2-c) \\
<\mathbf{j}>^{(\mathbf{2})} \quad &\longleftarrow <\mathbf{j}>^{(\mathbf{1})} \qquad (j > L-2-c \; or \; j < c+K+1)
\end{aligned}$$

Apply the third and final transformation to the resulting equations.

Transformation 5 *Modify simultaneously the balance equations for levels $j\,(L-2-c \ge j \ge c+K+1)$, by the transformation:*

$$\begin{aligned}
<\mathbf{j}>^{(\mathbf{3})} \quad &\longleftarrow <\mathbf{j}>^{(\mathbf{2})} - <\mathbf{j} + \mathbf{1}>^{(\mathbf{2})} \Delta \qquad (c+K+1 \le j \le L-2-c) \\
<\mathbf{j}>^{(\mathbf{3})} \quad &\longleftarrow <\mathbf{j}>^{(\mathbf{2})} \qquad (j > L-2-c \; or \; j < c+K+1) \;;
\end{aligned}$$

With these above three transformations, the transformed balance equation, $< \mathbf{j} >^{(\mathbf{3})}$'s, for the rows $(c + K + 1 \leq j \leq L - 2 - c)$, will be of the form:

$$\mathbf{v}_{j-K}Q_0 + \mathbf{v}_{j-K+1}Q_1 + \ldots + \mathbf{v}_{j+c+1}Q_{K+c+1} = 0$$
$$(j = L - 2 - c, L - 1 - c, \ldots, c + K + 1) \tag{18}$$

where, $Q_0, Q_1, \ldots, Q_{K+c+1}$ are $K + c + 2$ number of j-independent matrices which can be easily derived algebraically from the system parameters by following the above mentioned transformation procedures in [3].

Thus, the resulting equations (18) corresponding to the rows from $c+K+1$ to $L - 2 - c$, are of the same form as those of the QBD-M processes [5] and hence have an efficient solution by several methods such as the spectral expansion method, Bini-Meini's method [1], the matrix-geometric method with folding or block size enlargement [7], the logarithmic reduction method [8].

4 The $BMAP{+}sigma$ queue

4.1 The customer arrival process

The arrival and service processes are modulated by the same continuous time, irreducible Markov phase process with N states. The arrival process is the superposition of an arrival stream associated with the phase transitions, K independent CPP arrival streams of (positive) customers and an independent CPP of negative customers.

Essentially when there is a transition in the modulating process from phase i to phase k, during the transition (at the transition epoch) a batch of customers is released. The size of that batch varies from 0 (minimum size) to m (maximum size). $u_{i,k}(l)$ $(i, k = 1, 2, ..., N$ and $l = 0, 1, 2, ..., m)$ is the transition rate from phase i to phase k generating a batch customers of batch size l. The total transition rate from phase i to phase k is $\sum_{l=0}^{m} u_{i,k}(l)$.

Matrices U_l $(l = 0, 1, 2, ..., m)$ are given by $U_l = [u_{i,k}(l)]$ $(N * N$ square matrices) for $l = 0, 1, 2, ..., m$. For the convenience define $U_l = 0$ for $l > m$.

The positive customers of different K arrival streams are not distinguishable. The parameters of the GE inter-arrival time distribution of the k^{th} $(1 \leq k \leq K)$ positive customer arrival stream in phase i are $(\sigma_{i,k}, \theta_{i,k})$, and (ρ_i, δ_i) are those of the negative customers. That is, the inter-arrival time probability distribution function is $1 - (1 - \theta_{i,k})e^{-\sigma_{i,k}t}$, in phase i, for the k^{th}

stream of positive customers and $1-(1-\delta_i)e^{-\rho_i t}$ for the negative customers. Thus, all the $K+1$ arrival *point*-processes are Poisson, with batches arriving at each point having geometric size distribution. Specifically, the probability that a batch is of size s is $(1-\theta_{i,k})\theta_{i,k}^{s-1}$, in phase i, for the k^{th} stream of positive customers, and $(1-\delta_i)\delta_i^{s-1}$ for the negative customers. Thus, in phase i, the average batch size for the k^{th} stream of customers, is $\frac{1}{1-\theta_{i,k}}$, and $\frac{1}{1-\rho_I}$ for the negative customer stream.

Let $\sigma_{i,.}, \overline{\sigma_{i,.}}$ be the average arrival rate of customer batches and customers in phase i respectively. Let $\sigma, \overline{\sigma}$ be the overall average arrival rate of batches and customers respectively. Then,

$$\sigma_{i,.} = \sum_{k=1}^{K} \sigma_{i,k} \quad ; \quad \overline{\sigma_{i,.}} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{(1-\theta_{i,k})} \tag{19}$$

$$\sigma = \sum_{i=1}^{N} \sigma_{i,.} r_i + \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=0}^{m} u_{i,k}(l) \quad ; \quad \overline{\sigma} = \sum_{i=1}^{N} \overline{\sigma_{i,.}} r_i + \sum_{i=1}^{N}\sum_{k=1}^{N}\sum_{l=0}^{m} u_{i,k}(l) l$$

4.2 The steady state balance equations

The state of the system at any time t can be specified completely by two integer-valued random variables, $I(t)$ and $J(t)$. $I(t)$ varies from 1 to N, representing the phase of the modulating Markov chain, and $0 \leq J(t) < L+1$ represents the number of positive customers in the system at time t, including any in service. The system is now modelled by a continuous time discrete state Markov process, $\overline{Y}$ (Y if L is infinite), on a rectangular lattice strip. Let $I(t)$, the phase, vary in the horizontal direction and $J(t)$, the queue length or *level*, in the vertical direction. We denote the steady state probabilities by $\{p_{i,j}\}$, where $p_{i,j} = \lim_{t\to\infty} Prob(I(t)=i, J(t)=j)$, and let $\mathbf{v}_j = (p_{1,j}, \ldots, p_{N,j})$.

The process $\overline{Y}$ evolves due to the following instantaneous transition rates:

(a) $u_{i,k}(0)$ – purely lateral transition rate – from state (i,j) to state (k,j), for all $j \geq 0$ and $1 \leq i,k \leq N$ $(i \neq k)$, caused by a phase transition in the Markov chain governing the arrival phase process;

(b) $BM_{i,k,j,j+l}$ – l-step upward transition rate – from state (i,j) to state $(k, j+l)$ caused by a new batch arrival of size l associated by the phase transitions

(c) $B_{i,j,j+s}$ – s-step upward transition rate – from state (i,j) to state $(i,j+s)$, for all phases i, caused by a new batch arrival of size s positive customers. For a given j, s can be seen as bounded when L is finite and unbounded when L is infinite;

(d) $C_{i,j,j-s}$ – s-step downward transition rate – from state (i,j) to state $(i,j-s)$, $(j-s \geq c+1)$ for all phases i, caused by either a batch service completion of size s or a batch arrival of negative customers of size s;

(e) $C_{i,c+s,c}$ – s-step downward transition rate – from state $(i,c+s)$ to state (i,c), for all phases i, caused by a batch arrival of negative customers of size $\geq s$ or a batch service completion of size $s\,(1 \leq s \leq L-c)$;

(f) $C_{i,c-1+s,c-1}$ – s-step downward transition rate, from state $(i,c-1+s)$ to state $(i,c-1)$, for all phases i, caused by a batch departure of size $s\,(1 \leq s \leq L-c+1)$;

(g) $C_{i,j+1,j}$ – 1-step downward transition rate, from state $(i,j+1)$ to state (i,j), $(c \geq 2\,;\, 0 \leq j \leq c-2)$, for all phases i, caused by a single departure;

where

$$\begin{aligned}
BM_{i,k,j-l,j} &= u_{i,k}(l) \;\; (\forall i,k \;;\; 0 \leq j-l \leq L-2 \;;\; j-l < j < L) \;; \\
BM_{i,k,j,L} &= \sum_{l=L-j}^{m} u_{i,k}(l) \quad (\forall i,k \;;\; j \leq L-1) \;;
\end{aligned}$$

$$\begin{aligned}
B_{i,j-s,j} &= \sum_{k=1}^{K} (1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} \quad (\forall i \;;\; 0 \leq j-s \leq L-2 \;;\; j-s < j < L) \;; \\
B_{i,j,L} &= \sum_{k=1}^{K} \sum_{s=L-j}^{\infty} (1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} = \sum_{k=1}^{K} \theta_{i,k}^{L-j-1}\sigma_{i,k} \quad (\forall i \;;\; j \leq L-1) \;;
\end{aligned}$$

$$
\begin{aligned}
C_{i,j+s,j} &= (1-\phi_i)\phi_i^{s-1}c\mu_i + (1-\delta_i)\delta_i^{s-1}\rho_i \\
&\qquad\qquad (\forall i\ ;\ c+1 \le j \le L-1\ ;\ 1 \le s \le L-j)\ ; \\
&= (1-\phi_i)\phi_i^{s-1}c\mu_i + \delta_i^{s-1}\rho_i \\
&\qquad\qquad (\forall i\ ;\ j = c\ ;\ 1 \le s \le L-c)\ ; \\
&= \phi_i^{s-1}c\mu_i \qquad (\forall i\ ;\ j = c-1\ ;\ 1 \le s \le L-c+1)\ ; \\
&= 0 \qquad (\forall i\ ;\ c \ge 2\ ;\ 0 \le j \le c-2\ ;\ s \ge 2)\ ; \\
&= (j+1)\mu_i \qquad (\forall i\ ;\ c \ge 2\ ;\ 0 \le j \le c-2\ ;\ s = 1)\ .
\end{aligned}
$$

Define,

$$
\begin{aligned}
B_{j-s,j} &= \mathrm{Diag}\left[B_{1,j-s,j}, B_{2,j-s,j}, \ldots, B_{N,j-s,j}\right] \\
&\qquad\qquad\qquad\qquad (j-s < j \le L)\ ; \\
B_s &= B_{j-s,j} \qquad (j < L) \\
&= \mathrm{Diag}\left[\sum_{k=1}^{K} \sigma_{1,k}(1-\theta_{1,k})\theta_{1,k}^{s-1}, \ldots, \sum_{k=1}^{K} \sigma_{N,k}(1-\theta_{N,k})\theta_{N,k}^{s-1}\right]\ ; \\
BM_{j-s,j} &= \left[BM_{i,k,j-s,j}\right] \qquad (j-s < j \le L)\ ;
\end{aligned}
$$

$$
\begin{aligned}
BM_s &= BM_{j-s,j} \qquad (j < L) \\
&= U_s; \\
\Sigma_k &= \mathrm{Diag}\left[\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{N,k}\right] \qquad (k = 1, 2, \ldots, K)\ ; \\
\Theta_k &= \mathrm{Diag}\left[\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}\right] \qquad (k = 1, 2, \ldots, K)\ ; \\
\Sigma &= \sum_{k=1}^{K} \Sigma_k\ ; \\
R &= \mathrm{Diag}\left[\rho_1, \rho_2, \ldots, \rho_N\right]\ ; \\
\Delta &= \mathrm{Diag}\left[\delta_1, \delta_2, \ldots, \delta_N\right]\ ; \\
M &= \mathrm{Diag}\left[\mu_1, \mu_2, \ldots, \mu_N\right]\ ; \\
\Phi &= \mathrm{Diag}\left[\phi_1, \phi_2, \ldots, \phi_N\right]\ ; \\
C_j &= jM \qquad (1 \le j \le c)\ ; \\
&= cM = C \qquad (j \ge c)\ ; \\
C_{j+s,j} &= \mathrm{Diag}\left[C_{1,j+s,j}, C_{2,j+s,j}, \ldots, C_{N,j+s,j}\right]\ ; \\
E &= \mathrm{Diag}(\mathbf{e}_N^{'})\ .
\end{aligned}
$$

Then, we get,

$$
\begin{aligned}
B_s &= \sum_{k=1}^{K} \Theta_k^{s-1}(E-\Theta_k)\Sigma_k \;; \\
B_1 &= B = \sum_{k=1}^{K} (E-\Theta_k)\Sigma_k \;; \\
B_{L-s,L} &= \sum_{k=1}^{K} \Theta_k^{s-1}\Sigma_k \;; \\
C_{j+s,j} &= C(E-\Phi)\Phi^{s-1} + R(E-\Delta)\Delta^{s-1} \\
&\qquad (c+1 \le j \le L-1 \;;\; s=1,2,\ldots,L-j) \;; \\
&= C(E-\Phi)\Phi^{s-1} + R\Delta^{s-1} \qquad (j=c \;;\; s=1,2,\ldots,L-c) \;; \\
&= C\Phi^{s-1} \qquad (j=c-1 \;;\; s=1,2,\ldots,L-c+1) \;; \\
&= 0 \qquad (c \ge 2 \;;\; 0 \le j \le c-2 \;;\; s \ge 2) \;; \\
&= C_{j+1} \qquad (c \ge 2 \;;\; 0 \le j \le c-2 \;;\; s=1) \;.
\end{aligned}
$$

The steady state balance equations are,

(1) For the L^{th} row or level:

$$\sum_{s=1}^{L} \mathbf{v}_{L-s}[B_{L-s,L} + U_s] + \mathbf{v}_L \left[U_0 - D_0^U - C - R\right] = 0 \;; \qquad (20)$$

(2) For the j^{th} row or level:

$$\sum_{s=1}^{j} \mathbf{v}_{j-s}[B_s + U_s] + \mathbf{v}_j \left[U_0 - D_0^U - \sum_{s=1}^{m} D_s^U - \Sigma - C_j - RI_{j>c}\right] + \sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \quad (0 \le j \le L-1) \;; \qquad (21)$$

(3) Normalisation

$$\sum_{j=0}^{L} \mathbf{v}_j \mathbf{e}_N = 1 \;. \qquad (22)$$

where, $I_{j>c} = 1$ if $j > c$ else 0, and $\mathbf{e}_N$ is a column vector of size N with all ones. D_s^U is the diagonal matrix whose diagonal element is the sum of the elements in the corresponding row in U_s.

In what follows we present a solution method that faster and its computational complexity *does not* depend on L. First we *transform* this system of equations by certain transformations to the QBD-M (Quasi simultaneous-multiple births and simultaneous-multiple deaths) form. The resulting equations can then be solved using known efficient methods.

Transformation 6 *Modify simultaneously the balance equations for levels $j\,(L-1 \geq j \geq K)$, by the transformation:*

$$
\begin{aligned}
<\mathbf{j}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j}-\mathbf{l}> F_{K,l} \qquad (K \leq j \leq L-1) \\
<\mathbf{j}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}> \qquad\qquad\qquad\qquad (j = L \;or\; j < K)
\end{aligned}
$$

Apply the second transformation to the resulting equations.

Transformation 7 *Modify simultaneously the balance equations for levels $j\,(L-3 \geq j \geq c+K+1)$, by the transformation:*

$$
\begin{aligned}
<\mathbf{j}>^{(\mathbf{2})} &\longleftarrow <\mathbf{j}>^{(\mathbf{1})} - <\mathbf{j}+\mathbf{1}>^{(\mathbf{1})}(\Phi+\Delta) + <\mathbf{j}+\mathbf{2}>^{(\mathbf{1})}\Phi\Delta \\
&\qquad\qquad\qquad (c+K+1 \leq j \leq L-3) \\
<\mathbf{j}>^{(\mathbf{2})} &\longleftarrow <\mathbf{j}>^{(\mathbf{1})} \qquad (j > L-3 \;or\; j < c+K+1)
\end{aligned}
$$

With these above transformations, the transformed balance equation, $<\mathbf{j}>^{(\mathbf{2})}$'s, for the rows $(c+K+1 \leq j \leq L-3)$, will be of the form:

$$
\begin{aligned}
&\mathbf{v}_{j-K-m}Q_0 + \mathbf{v}_{j-K+1}Q_1 + \ldots + \mathbf{v}_{j+2}Q_{K+m+2} = 0 \\
&\qquad (j = L-2-m, L-1-m, \ldots, m+K+1)
\end{aligned} \tag{23}
$$

where, $Q_0, Q_1, \ldots, Q_{K+m+2}$ are $K+m+3$ number of j-independent matrices which can be easily derived algebraically from the system parameters by following the above mentioned transformation procedures in [2]

5 The *gBMAP+sigma* queue

5.1 The customer arrival process

The arrival process is the superposition of an arrival stream associated with the phase transitions, K independent CPP arrival streams of (positive) customers and an independent CPP of negative customers.

- Essentially when there is a transition in the modulating process from phase i to phase k, during the transition (at the transition epoch) a batch of customers is generated. The size of batches follows the modified geometric distribution with parameter $\upsilon_{i,k}$ $(i \neq k; i = 1, 2, \ldots, N; j = 1, 2, \ldots, N;\ \upsilon_{i,i} = 0)$. That is, the probability that a batch is of size s $(s \geq 0$ is $(1 - \upsilon_{i,k})\upsilon_{i,k}^{s}$. Define the $N * N$ matrix Υ as $\Upsilon(i, j) = q_{i,j}\upsilon_{i,j}$ and $N * N$ matrices Ω_s $(s = 0, 1, \ldots))$ as $\Omega_s(i, k) = q_{i,k}(1 - \upsilon_{i,k})\upsilon_{i,k}^{s}$.
- The positive customers of the K arrival CPPs and those that arrive at the modulating phase transition epoch (gBMAP) are not distinguishable.

Let $\sigma_{i,.}, \overline{\sigma_{i,.}}$ be the average arrival rate of customer batches and customers in phase i respectively. Let $\sigma, \overline{\sigma}$ be the overall average arrival rate of batches and customers respectively. Then,

$$\sigma_{i,.} = \sum_{k=1}^{K} \sigma_{i,k} \quad ; \quad \overline{\sigma_{i,.}} = \sum_{k=1}^{K} \frac{\sigma_{i,k}}{(1 - \theta_{i,k})} \tag{24}$$

$$\sigma = \sum_{i=1}^{N} \sigma_{i,.} r_i + \sum_{i=1}^{N} r_i \sum_{k=1}^{N} q_{i,k} \quad ; \quad \overline{\sigma} = \sum_{i=1}^{N} \overline{\sigma_{i,.}} r_i + \sum_{i=1}^{N} r_i \sum_{k=1}^{N} \frac{q_{i,k}\upsilon_{i,k}}{1 - \upsilon_{i,k}}$$

5.2 The steady state balance equations

The state of the system at any time t can be specified completely by two integer-valued random variables, $I(t)$ and $J(t)$. $I(t)$ varies from 1 to N, representing the phase of the modulating Markov chain, and $0 \leq J(t) < L + 1$ represents the number of positive customers in the system at time t, including any in service. The system is now modelled by a continuous time discrete state Markov process, $\overline{Y}$ (Y if L is infinite), on a rectangular lattice strip. Let $I(t)$, the phase, vary in the horizontal direction and $J(t)$, the queue length or *level*, in the vertical direction. We denote the steady state probabilities by $\{p_{i,j}\}$, where $p_{i,j} = \lim_{t\to\infty} Prob(I(t) = i, J(t) = j)$, and let $\mathbf{v}_j = (p_{1,j}, \ldots, p_{N,j})$.

The process $\overline{Y}$ evolves due to the following instantaneous transition rates:

(a) A_j – purely lateral transition rates – from state (i, j) to state (k, j), for all $j \geq 0$ and $1 \leq i, k \leq N \quad (i \neq k)$, caused by a phase transition in the Markov chain governing the arrival phase process (A_j are the transition matrices);

(b) $BM_{i,k,j,j+l}$ – l-step upward transition rate – from state (i, j) to state $(k, j+l)$ caused by a new batch arrival of size l associated by the phase transitions;

(c) $B_{i,j,j+s}$ – s-step upward transition rate – from state (i, j) to state $(i, j+s)$, for all phases i, caused by a new batch arrival of size s positive customers. For a given j, s can be seen as bounded when L is finite and unbounded when L is infinite;

(d) $C_{i,j,j-s}$ – s-step downward transition rate – from state (i, j) to state $(i, j-s)$, $(j - s \geq c + 1)$ for all phases i, caused by either a batch service completion of size s or a batch arrival of negative customers of size s;

(e) $C_{i,c+s,c}$ – s-step downward transition rate – from state $(i, c+s)$ to state (i, c), for all phases i, caused by a batch arrival of negative customers of size $\geq s$ or a batch service completion of size $s\,(1 \leq s \leq L - c)$;

(f) $C_{i,c-1+s,c-1}$ – s-step downward transition rate, from state $(i, c - 1 + s)$ to state $(i, c - 1)$, for all phases i, caused by a batch departure of size $s\,(1 \leq s \leq L - c + 1)$;

(g) $C_{i,j+1,j}$ – 1-step downward transition rate, from state $(i, j + 1)$ to state (i, j), $(c \geq 2\,;\, 0 \leq j \leq c - 2)$, for all phases i, caused by a single departure;

where

$$A_j(i, k) = q_{i,k}(1 - \upsilon_{i,k})$$

$$
\begin{aligned}
BM_{i,k,j-l,j} &= q_{i,k}(1-\upsilon_{i,k})\upsilon_{i,k}^{l} \;\; (\forall i,k\;;\; 0\le j-l\le L-2\;;\; j-l<j<L)\;;\\
BM_{i,k,j,L} &= \sum_{l=L-j}^{\infty} q_{i,k}(1-\upsilon_{i,k})\upsilon_{i,k}^{l} = q_{i,k}\upsilon_{i,k}^{L-j} \quad (\forall i,k\;;\; j\le L-1)\;;
\end{aligned}
$$

$$
\begin{aligned}
B_{i,j-s,j} &= \sum_{k=1}^{K}(1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} \quad (\forall i\;;\; 0\le j-s\le L-2\;;\; j-s<j<L)\;;\\
B_{i,j,L} &= \sum_{k=1}^{K}\sum_{s=L-j}^{\infty}(1-\theta_{i,k})\theta_{i,k}^{s-1}\sigma_{i,k} = \sum_{k=1}^{K}\theta_{i,k}^{L-j-1}\sigma_{i,k} \quad (\forall i\;;\; j\le L-1)\;;\\
C_{i,j+s,j} &= (1-\phi_i)\phi_i^{s-1}c\mu_i + (1-\delta_i)\delta_i^{s-1}\rho_i\\
&\qquad (\forall i\;;\; c+1\le j\le L-1\;;\; 1\le s\le L-j)\;;\\
&= (1-\phi_i)\phi_i^{s-1}c\mu_i + \delta_i^{s-1}\rho_i\\
&\qquad (\forall i\;;\; j=c\;;\; 1\le s\le L-c)\;;\\
&= \phi_i^{s-1}c\mu_i \qquad (\forall i\;;\; j=c-1\;;\; 1\le s\le L-c+1)\;;\\
&= 0 \qquad (\forall i\;;\; c\ge 2\;;\; 0\le j\le c-2\;;\; s\ge 2)\;;\\
&= (j+1)\mu_i \qquad (\forall i\;;\; c\ge 2\;;\; 0\le j\le c-2\;;\; s=1)\,.
\end{aligned}
$$

Define,

$$
\begin{aligned}
B_{j-s,j} &= \mathrm{Diag}\left[B_{1,j-s,j}, B_{2,j-s,j}, \ldots, B_{N,j-s,j}\right]\\
&\qquad\qquad\qquad (j-s<j\le L)\;;\\
B_s &= B_{j-s,j} \qquad (j<L)\\
&= \mathrm{Diag}\left[\sum_{k=1}^{K}\sigma_{1,k}(1-\theta_{1,k})\theta_{1,k}^{s-1}, \ldots, \sum_{k=1}^{K}\sigma_{N,k}(1-\theta_{N,k})\theta_{N,k}^{s-1}\right]\;;\\
BM_{j-s,j} &= \left[BM_{i,k,j-s,j}\right] \qquad (j-s<j\le L)\;;\\
BM_s &= BM_{j-s,j} \qquad (j<L)\\
\Sigma_k &= \mathrm{Diag}\left[\sigma_{1,k}, \sigma_{2,k}, \ldots, \sigma_{N,k}\right] \qquad (k=1,2,\ldots,K)\;;\\
\Theta_k &= \mathrm{Diag}\left[\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}\right] \qquad (k=1,2,\ldots,K)\;;
\end{aligned}
$$

$$\begin{aligned}
\Sigma &= \sum_{k=1}^{K} \Sigma_k \ ; \\
R &= \mathrm{Diag}\left[\rho_1, \rho_2, \ldots, \rho_N\right] \ ; \\
\Delta &= \mathrm{Diag}\left[\delta_1, \delta_2, \ldots, \delta_N\right] \ ; \\
M &= \mathrm{Diag}\left[\mu_1, \mu_2, \ldots, \mu_N\right] \ ; \\
\Phi &= \mathrm{Diag}\left[\phi_1, \phi_2, \ldots, \phi_N\right] \ ; \\
C_j &= jM \qquad (1 \le j \le c) \ ; \\
&= cM = C \qquad (j \ge c) \ ; \\
C_{j+s,j} &= \mathrm{Diag}\left[C_{1,j+s,j}, C_{2,j+s,j}, \ldots, C_{N,j+s,j}\right] \ ; \\
E &= \mathrm{Diag}(\mathbf{e}_N^{'}) \ .
\end{aligned}$$

Then, we get,

$$\begin{aligned}
B_s &= \sum_{k=1}^{K} \Theta_k^{s-1}(E - \Theta_k)\Sigma_k \ ; \\
B_1 &= B = \sum_{k=1}^{K} (E - \Theta_k)\Sigma_k \ ; \\
B_{L-s,L} &= \sum_{k=1}^{K} \Theta_k^{s-1}\Sigma_k \ ;
\end{aligned}$$

$$\begin{aligned}
C_{j+s,j} &= C(E-\Phi)\Phi^{s-1} + R(E-\Delta)\Delta^{s-1} \\
&\qquad (c+1 \le j \le L-1 \ ; \ s = 1, 2, \ldots, L-j) \ ; \\
&= C(E-\Phi)\Phi^{s-1} + R\Delta^{s-1} \qquad (j = c \ ; \ s = 1, 2, \ldots, L-c) \ ; \\
&= C\Phi^{s-1} \qquad (j = c-1 \ ; \ s = 1, 2, \ldots, L-c+1) \ ; \\
&= 0 \qquad (c \ge 2 \ ; \ 0 \le j \le c-2 \ ; \ s \ge 2) \ ; \\
&= C_{j+1} \qquad (c \ge 2 \ ; \ 0 \le j \le c-2 \ ; \ s = 1) \ .
\end{aligned}$$

The steady state balance equations are,

(1) For the L^{th} row or level:

$$\sum_{s=1}^{L} \mathbf{v}_{L-s} \left[B_{L-s,L} + \sum_{l=L-s}^{\infty} \Omega_l \right] + \mathbf{v}_L \left[\Omega_0 - D_0^{\Omega} - C - R\right] = 0 \ ; \qquad (25)$$

(2) For the j^{th} row or level:

$$\sum_{s=1}^{j} \mathbf{v}_{j-s}[B_s + \Omega_s] + \mathbf{v}_j \left[A_j - \sum_{s=0}^{\infty} D_s^{\Omega} - \Sigma - C_j - RI_{j>c} \right] + \tag{26}$$

$$\sum_{s=1}^{L-j} \mathbf{v}_{j+s} C_{j+s,j} = 0 \quad (0 \le j \le L-1)\ ; \tag{27}$$

(3) Normalisation

$$\sum_{j=0}^{L} \mathbf{v}_j \mathbf{e}_N = 1\ . \tag{28}$$

where, $I_{j>c} = 1$ if $j > c$ else 0, and $\mathbf{e}_N$ is a column vector of size N with all ones. D_s^U is the diagonal matrix whose diagonal element is the sum of the elements in the corresponding row in U_s.

Transformation 8 *Modify simultaneously the balance equations for levels $j\,(L-1 \ge j \ge K)$, by the transformation:*

$$\begin{aligned} <\mathbf{j}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}> + \sum_{l=1}^{K} (-1)^l <\mathbf{j-l}> F_{K,l} && (K \le j \le L-1) \\ <\mathbf{j}>^{(\mathbf{1})} &\longleftarrow <\mathbf{j}> && (j = L \textit{ or } j < K) \end{aligned}$$

Apply the second transformation to the resulting equations.

Transformation 9 *Modify simultaneously the balance equations for levels $j\,(L-3 \ge j \ge K+1)$, by the transformation:*

$$\begin{aligned} <\mathbf{j}>^{(\mathbf{2})} &\longleftarrow <\mathbf{j}>^{(\mathbf{1})} - <\mathbf{j+1}>^{(\mathbf{1})}(\Phi + \Delta) + <\mathbf{j+2}>^{(\mathbf{1})}\Phi\Delta \\ & \qquad\qquad (c+K+1 \le j \le L-3) \\ <\mathbf{j}>^{(\mathbf{2})} &\longleftarrow <\mathbf{j}>^{(\mathbf{1})} \qquad (j > L-3 \textit{ or } j < c+K+1) \end{aligned}$$

Transformation 10 *Modify simultaneously the balance equations for levels $j\,(L-3 \ge j \ge c+K+1)$, by the transformation:*

$$\begin{aligned} <\mathbf{j}>^{(\mathbf{3})} &\longleftarrow <\mathbf{j}>^{(\mathbf{2})} - <\mathbf{j-1}>^{(\mathbf{2})}\Upsilon \\ & \qquad\qquad (c+K+1 \le j \le L-3) \\ <\mathbf{j}>^{(\mathbf{3})} &\longleftarrow <\mathbf{j}>^{(\mathbf{2})} \qquad (j > L-3 \textit{ or } j < c+K+1) \end{aligned}$$

With these above transformations, the transformed balance equation, $<\mathbf{j}>^{(\mathbf{3})}$'s, for the rows $(K+1 \leq j \leq L-3)$, will be of the form:

$$\mathbf{v}_{j-K-1}Q_0 + \mathbf{v}_{j-K}Q_1 + \ldots + \mathbf{v}_{j+2}Q_{K+3} = 0 \quad (j = L-3, L-1, \ldots, K+1+c) \tag{29}$$

where, $Q_0, Q_1, \ldots, Q_{K+3}$ are $K+4$ number of j-independent matrices which can be easily derived algebraically from the system parameters by following the above mentioned transformation procedures. This is dealt with in [4].

Acknowledgement

The work of Tien Van Do is partially supported by the IKM (Ipar A Korszerű Mérnökképzésért Alapítvány) foundation and the Pannon Professor Scholarship from Pannon GSM mobile network operator.

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