Ascending Price Vickrey Auctions Using Primal-Dual Algorithms*

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Abstract

The direct Vickrey-Clarke-Groves (VCG) mechanism requires solving underlying optimization problems of the main economy and of every "marginal" economy. We show that VCG outcome can be implemented using a competitive equilibrium (CE) of the main economy if and only if that CE price is also a CE price of every marginal economy. We call such a CE price a universal competitive equilibrium (UCE) price. We design an ascending price auction that implements the VCG outcome under any valuation profile of buyers using this UCE price concept. The auction searches for a UCE price, and calculates payments of buyers from the final UCE price and allocation. This approach is particularly useful in settings where a CE price supporting VCG payments may be absent (for example, when substitutes condition on valuations does not hold) and answers the question whether VCG outcome can be implemented using iterative ascending price auctions for generic valuation profiles. We relate our work to the recent work of de Vries et al. [13] and show that when buyers are substitutes, their auction converges to a UCE price. When buyers are not substitutes, their auction gives a starting price for our auction.

Keywords: combinatorial auctions; multi-item auctions; primal-dual algorithm; universal competitive equilibrium; Vickrey auctions

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1 Introduction

Primarily, auctions can be classified as: (i) sealed-bid auctions, or (ii) iterative auctions. Iterative auctions are preferred over their sealed-bid counterparts (having similar theoretical properties) for various reasons [11], including better revenue and efficiency properties, better privacy properties and ease of use. Also, iterative auctions can usefully mitigate preference elicitation costs, providing feedback to allow focused and incremental bidding [10, 25]. Contrast this with a sealed-bid auction, in which all buyers must submit complete bids, to cover all possible contingencies.

If a single item needs to be sold, the famous incentive compatible mechanism is Vickrey's second-price sealed-bid auction [28]. Implemented in an iterative manner, it gives rise to the English or Japanese auction. The English auction is well-known, where buyers overbid each other until there is only one buyer left. In the Japanese auction, the buyers indicate if they are interested in the item or not and accordingly the price is adjusted by the seller. Such auctions are also referred to as "clock auctions" [26, 3].

When generalized to the setting where a seller may have multiple items to sell and buyers may have valuations on bundles of items, the Vickrey auction gives the Vickrey-Clarke-Groves (VCG) mechanism [28, 9, 16]. In its natural form, a VCG mechanism is a sealed-bid auction that implements an efficient allocation and calculates payments for buyers such that their payoff is equal to their marginal product. Truthful bidding is a dominant strategy equilibrium.

1.1 Generalizations of English Auction

The English auction (and its clock auction counterpart) has been generalized to implement VCG outcome in various multi-item settings. One such setting is the assignment problem, where there are multiple items for sale but each buyer is interested in at most one item. Demange et al. [14] design ascending price auctions for such a setting that implement the VCG outcome. One of their auctions collects demand sets (items giving maximum payoff) from each buyer at every iteration and calculates a "minimally overdemanded" set of items based on this information. The prices on the minimally overdemanded set of items are increased, and the auction terminates when there are no overdemanded set of items.

This idea has been generalized to the setting when buyers can demand multiple items. The problem of finding an efficient allocation given valuation profiles of buyers can be formulated as a linear program [7]. de Vries et al. [13] apply primal-dual algorithm ideas on such formulations to design an ascending price auction. A primal-dual algorithm typically considers a restricted primal problem whose dual gives direction of change of dual variables (prices). de Vries et al. [13] introduce the concept of "undersupplied set of buyers", which generalizes the concept of overdemanded set of items. Using this concept, they design an ascending price auction that implements the VCG outcome if a submodularity condition on buyers is satisfied.

Similar ideas have been used for designing auctions for homogeneous units when buyers have non-increasing marginal values on units. Ausubel [1] designed an ascending price auction for such a setting that implements the VCG outcome. Bikhchandani and Ostroy [6] interpret Ausubel's auction as a primal-dual algorithm.

We should note that the primal-dual algorithm based auctions (discussed above) are some kind of clock auctions. The seller announces prices for every bundle (may be personalized for every buyer) and based on the quantity demanded (0 or 1) for each bundle, the prices are adjusted by the seller. Typically, the clock auction involves a "Walrasian" auctioneer who maintains linear and anonymous prices on items. In this strict sense, Japanese auction, the auction by Demange et al. [14] for unit demand case and the auction by Ausubel [1] for homogeneous units case may be viewed as true clock auctions. But the auction by de Vries et al. [13] is a stylized version of the clock auction.

There have been some other notable designs of ascending price Vickrey auctions. Demange et al. [14] propose another auction for the assignment problem in which buyers overbid each other until there is no bidding.[1] Parkes [22] and later Ausubel and Milgrom [4] have extended this auction to the case where buyers can demand multiple items. Under the buyer submodularity condition, these auctions implement the VCG outcome. de Vries et al. [13] interpret these auctions as *subgradient* algorithms.[2]

1.2 Negative Characterization in the Literature

Summarizing the literature above, there does not exist an iterative auction that implements the VCG outcome for all classes of valuation profiles. In particular, to be able to implement VCG outcome, all iterative auctions require the buyer submodularity condition to hold. In fact, de Vries et al. [13] prove that there exists *no* iterative ascending price auction that can implement the VCG outcome if there exists a buyer whose valuation fails to satisfy a condition (gross-substitutes) that precludes the existence of items that are complements for each other. Of course, this negative result depends on how an ascending price auction is defined. Using the traditional notion (formalized in [13]), an iterative auction should collect demand sets of buyers, adjust prices at an iteration using only current demand set and price information, and the final payments should be equal to the price seen in the final iteration of the auction. We relax only this final criteria, and allow the final payment to be adjusted downwards from final prices of the auction.

1.3 Our Results

Looking for iterative VCG auctions for general valuations, we introduce the concept of a "universal competitive equilibrium" (UCE) price. A price is a UCE price if it is a competitive

[1]This auction can be considered to be a natural generalization of English auction, whereas the auction where demand sets are collected can be considered as a generalization of the Japanese auction.

[2]Ausubel and Milgrom [4] call the bidding strategy of buyers in their auction "bidding the gradient".

equilibrium (CE) price in the main economy as well as in each of the economies where one of the buyers is absent (we call such economies *marginal economies*). We show that it is possible to determine payments in the VCG outcome, using only the CE in the main economy, if that CE price is also a CE price in every marginal economy. In Theorem 1, we show that if we have a UCE price and an efficient allocation, we can calculate VCG payments. This may involve calculation of "discounts" for every buyer at a UCE price. The discount for a buyer is his marginal contribution to the revenue of the seller at the UCE price. In Theorem 2, we show that if we can calculate VCG payments from a CE price and an efficient allocation, then it has to be a UCE price. We characterize some conditions under which these discounts are equal to zero (Proposition 4).

We define a generalized class of iterative price-based auctions in which we allow a price adjustment at the end of the auction. Our auctions are still ascending-price through the active bidding stage of the auction. It is only in the very last step, performed by the auctioneer after termination, that prices can be reduced. Given this class, we show that ascending price auctions that can guarantee VCG payments for any valuation profile must allow for both non-linear and non-anonymous prices (Theorem 3).

Our main result is a primal-dual auction that maintains non-linear and non-anonymous prices, and terminates with VCG payments. Our approach generalizes the approach proposed in [13]. While de Vries et al. [13] consider the main economy and design a primal-dual auction based on that, we consider not only the main economy, but also every marginal economy and design a primal-dual auction based on that. The generic idea behind a primal-dual auction is to formulate the efficient allocation problem of an economy as a linear program. Starting from a low price, where demand exceeds supply, we adjust prices (using the dual of a restricted primal problem) until supply meets demand. We need to adjust prices to balance supply and demand in the marginal economies, as well as in the main economy. The auction converges to a UCE price, and the final payments are calculated using discounts. Bidding truthfully (submitting true demand sets) is an *ex post* Nash equilibrium for buyers in such auctions.

In the special case when buyers are substitutes, we show that a primal-dual price adjustment defined for the main economy will also balance supply and demand in the marginal economies. This is already achieved in the auction in [13]. Thus, their auction can be used to implement the VCG outcome, using discounts defined in Theorem 1, if buyers are substitutes. If buyers are substitutes condition does not hold, then the final price from such an auction can still be used to start our auction and our auction always converges to a UCE price.

Our auction can be thought as a stylized version of clock auction. Buyers respond to non-anonymous prices on bundles with the quantity they demand (0 or 1) of every bundle and accordingly the seller adjusts the prices on these bundles. Our auction should not be confused with the auction in [2]. While Ausubel's auction (defined for gross substitutes preferences and maintains linear and anonymous prices) constructs $m + 1$ auctions (price paths) to discover VCG outcome information, our auction searches for a UCE price by maintaining a single (non-anonymous and non-linear) price vector.

A parallel stream of ascending price auctions in the literature are designed around an optimization paradigm called *subgradient algorithms* [5], [22], [4]. de Vries et al. [13] note that these auction are technically implementing a subgradient algorithm by building on linear programming duality. Both primal-dual and subgradient algorithms make use of linear programming duality to find price adjustment directions. While in primal-dual algorithms, a restricted primal problem is considered whose dual gives price adjustment direction, in subgradient algorithm the violated constraints of a primal problem give price adjustment direction. A related analysis, again based on the UCE price concept, can be used to generalize these subgradient algorithms (specifically *i*Bundle [22]) to always achieve the VCG outcome, again allowing a discount from final prices [24].

In Table 1, we categorize various auctions in the literature for different settings on when they can implement VCG outcome. We characterize ascending price auctions that are designed on the concept of primal-dual algorithms, or subgradient algorithms. The table clearly outlines our contribution to the ascending price Vickrey auction literature.

Settings	Primal-Dual Ascending Price Vickrey Auction	Subgradient Ascending Price Vickrey Auction	Discounts Used From Final Price	Primal-Dual Algorithm of
Single Item	English (Japanese) Auction	English Auction	0	Main Economy
		Multi-Item Auctions		
Heterogeneous Items Unit Demand	Demange et al.[14] (Exact Version)	Demange et al.[14] (Approximate Version) Bertsekas[5] Crawford and Knoer[12]	0	Main economy
Homogeneous Units Non-Increasing Marginal Values	Ausubel[1] Bikhchandani and Ostroy [6]	-	0	Main economy
Heterogeneous Items Buyers Are Submodular	de Vries et al.[13]	Parkes [22] Ausubel and Milgrom [4]	0	Main economy
Heterogeneous Items Buyers Are Substitutes	de Vries et al.[13] **modified in this paper**	Parkes and Ungar [24]	Marginal contribution to revenue	Main economy
Heterogeneous Items Generic Valuations	**This paper**	Parkes and Ungar [24]	Marginal contribution to revenue	Main economy and marginal economies

Table 1: Characteristics of Ascending Price Vickrey Auctions

The rest of the paper is organized as follows. In Section 2, we introduce our model and the UCE price concept. In Section 3, we formally define an iterative auction and show the necessity of non-linear and non-anonymous prices in these auctions. In Section 4, we design our ascending price auction, which always implements the VCG outcome. In Section 5, we introduce the concept of a *quasi-CE price* which helps us relate the auction in [13] to the UCE price concept and to our auction. Section 6 is devoted to discussions and Section 7 concludes with a summary and some directions for future research.

2 Universal Competitive Equilibrium

We define the combinatorial allocation problem and the concept of universal competitive equilibrium (UCE) prices. We prove that UCE price information is both necessary and sufficient, if an auction is able to use competitive equilibrium prices to determine the VCG outcome. Later, we will illustrate how this UCE concept can be used to design iterative ascending price auctions.

2.1 The Model

A seller has n heterogeneous items to sell. The set of items is denoted by $A = \{1, \ldots, j, \ldots, n\}$. There are m buyers, denoted by $B = \{1, \ldots, i, \ldots, m\}$. Let Ω denote the set of all bundles of items; $\Omega = \{S \subseteq A\}$. Naturally, $\emptyset \in \Omega$. Buyers have non-negative integer valuations on bundles in Ω. The valuation of buyer i on bundle S is denoted by $v_i(S)$. We impose the following restrictions on valuations of any buyer:

A1 *Private Valuations:* Each buyer exactly knows his own valuation and it does not depend on the valuations of other buyers.

A2 *Quasilinear Utility:* The utility or payoff of any buyer $i \in B$ on a bundle S is given by $u_i(S, p) = v_i(S) - p$, where p is the price paid by buyer i on bundle S. Also, if a buyer gets nothing and pays nothing, then his utility is zero: $v_i(\emptyset) = 0 \ \forall \ i \in B$.

A3 *Free Disposal (Monotonicity):* $v_i(S) \leq v_i(T) \ \forall \ S \subseteq T, \ \forall \ i \in B$.

A4 *Zero Seller Valuations:* The seller values the items at zero. Thus, his utility or payoff or revenue is the total payment he receives at a price.

Assumptions A1-A4 are standard in literature. Unless stated explicitly, we do not pose any restriction on the valuations of the buyers besides these four assumptions.

We define the combinatorial allocation problem [27]. We will denote a feasible assignment/allocation as $X = (X_1, \ldots, X_m)$, a vector of bundles on buyers such that $X_i \cap X_k = \emptyset$ for any $i \neq k$. Allocation X assigns bundle X_i to buyer i and the possibility of $X_i = \emptyset$ is allowed. We will denote the set of all feasible allocations as $\mathbb{X}$. An allocation X is *efficient* if there does not exist another allocation Y such that $\sum_{i \in B} v_i(Y_i) > \sum_{i \in B} v_i(X_i)$. From Assumption A3, every efficient allocation X should have $\cup_{i \in B} X_i = A$.

Let $B_{-i} = B \setminus \{i\}$ be the set of buyers without buyer i. Let $\mathbb{B} = \{B, B_{-1}, \ldots, B_{-m}\}$. We will denote the economy with buyers only from set $M \subseteq B$ as $E(M)$. Whenever, $M \neq B$ and $M \in \mathbb{B}$, we call such an economy a *marginal economy*. $E(B)$ is called the *main, or original economy*. The feasible allocation, efficient allocation and related notations can also be defined for every marginal economy. We will use the same notation irrespective of the economy (for example, $\mathbb{X}$ will denote the set of feasible allocations for economy $E(M)$ for any $M \subseteq B$) and the notations should be clear from the context of use.

2.2 Linear Programming Formulation

The problem of finding an efficient allocation can be formulated as a linear program. Specifically, we use an extended linear programming formulation, shown to be integral in Bikhchandani and Ostroy [7]. Let $y_i(S)$ be a variable which is assigned value 1 if a buyer $i \in B$ is allocated a bundle $S \in \Omega$ ($S = \emptyset$ is allowed) and assigned zero otherwise. Let $z(X)$ be a variable which is assigned 1 if allocation $X \in \mathbb{X}$ is selected. The efficient allocation problem of economy $E(M)$ (for any $M \subseteq B$) is as follows.

$$V(M) = \max_{y,z} \sum_{i \in M} \sum_{S \in \Omega} v_i(S) y_i(S).$$

$$\text{s.t.} \qquad (\mathbf{P}(M))$$

$$\sum_{S \in \Omega} y_i(S) = 1 \qquad \forall\, i \in M.$$

$$\sum_{X \in \mathbb{X}} z(X) = 1.$$

$$y_i(S) = \sum_{X : X_i = S} z(X) \qquad \forall\, S \in \Omega,\ \forall\, i \in M. \tag{1}$$

$$y_i(S) \geq 0 \qquad \forall\, i \in M,\ \forall\, S \in \Omega.$$

$$z(X) \geq 0 \qquad \forall\, X \in \mathbb{X}.$$

Given this, the dual of ($\mathbf{P}(M)$) is defined as:

$$V(M) = \min_{\pi, \pi^s, p} \pi^s + \sum_{i \in M} \pi_i.$$

$$\text{s.t.} \qquad (\mathbf{DP}(M))$$

$$\pi^s \geq \sum_{i \in M} p_i(X_i) \qquad \forall\, X \in \mathbb{X}.$$

$$\pi_i \geq v_i(S) - p_i(S) \qquad \forall\, i \in M,\ \forall\, S \in \Omega.$$

Dual variables $p_i(S)$ can be interpreted as the price on bundle S to buyer i, with π_i being the *maximal* payoff to buyer i at across all bundles and π^s being the *maximal* payoff to the seller across all allocations.

Although, the dual variables are otherwise unconstrained, we let $p_i(S) \geq 0\, \forall\, i \in M,\ \forall\, S \in \Omega$ and $p_i(\emptyset) = 0\ \forall\ i \in M$ and it can be verified that this (along with $v_i(\emptyset) = 0\ \forall\ i \in M$) makes π and π^s non-negative. If (y, z) is a feasible solution of ($\mathbf{P}(M)$) and $(\pi_1, \ldots, \pi_m, \pi^s)$ is a feasible solution of ($\mathbf{DP}(M)$), the complementary slackness (CS) conditions say that these are optimal solutions if and only if:

$$y_i(S)\Big[\pi_i - [v_i(S) - p_i(S)]\Big] = 0 \qquad \forall\ i \in M,\ \forall\ S \in \Omega. \tag{CS-1}$$

$$z(X)\Big[\pi^s - \sum_{i \in M} p_i(X_i)\Big] = 0 \qquad \forall\ X \in \mathbb{X}. \tag{CS-2}$$

Let the demand set of buyer i at price p be

$$D_i = \{S \in \Omega : v_i(S) - p_i(S) \geq v_i(T) - v_i(T)\ \forall\ T \in \Omega\} \tag{2}$$

and the supply set of the seller at price p in economy $E(M)$ be

$$L(M) = \{X \in \mathbb{X} : \sum_{i \in M} p_i(X_i) \geq \sum_{i \in M} p_i(Y_i)\ \forall\ Y \in \mathbb{X}\} \tag{3}$$

We will suppress the dependence of D_i and L on price whenever possible for simplicity. From CS-1, if $y_i(S) = 1$ then $S \in D_i$ and from CS-2, if $z(X) = 1$ then $X \in L(M)$. When we say payoff of a buyer at a price, it will refer to the payoff of that buyer on bundles in his demand set. Using this, we define a competitive equilibrium.

Definition 1 (Competitive Equilibrium) *A price (feasible solution of (**DP(M)**)) and an allocation (feasible solution of (**P(M)**)) tuple (p, X) is a **competitive equilibrium** (CE) of economy $E(M)$ for some $M \subseteq B$ if:*

- *For every buyer $i \in M$, $X_i \in D_i$.*
- *$X \in L(M)$.*

The price p is called a CE price.

Throughout the paper, price p is defined for every buyer $i \in B$ as $p \in \mathbb{R}^{\Omega \times B}$. But, when we talk about economy $E(M)$ for some $M \subseteq B$ the components of p associated with buyers in M only are considered.

Constraint (1) in (**P(M)**) is the "balance" of demand and supply constraint. In competitive equilibrium, supply and demand should balance in economy $E(M)$, given that seller and the buyers maximize their payoffs. Using this, we can define a concept called *universal competitive equilibrium* price.

Definition 2 (Universal Competitive Equilibrium Price) *A price p is a **universal competitive equilibrium** (UCE) price if p is a CE price of economy $E(M)$ for every $M \in \mathbb{B}$.*

Observe that the definition of UCE price does not make any statement about the allocations which support the UCE price in every marginal economy. In fact, it is very likely that different allocations (same buyer being allocated different bundles) will support a UCE price in different marginal economies.

A UCE price always exists in our model. Consider the price p with $p_i(S) = v_i(S)\ \forall\ i \in B,\ \forall\ S \in \Omega$. Clearly, this is a UCE price. We find this UCE price concept very useful in aligning efficiency and incentives using *iterative auctions*.

2.3 Vickrey Payments

The Vickrey-Clarke-Groves (VCG) mechanism [28, 9, 16] is an efficient, *individually rational* and *strategyproof* direct *revelation* mechanism. A direct revelation mechanism asks buyers to report their valuation profiles. If $\hat{v}$ is the reported valuation profile, then the VCG mechanism solves (**P**(M)) for every $M \in \mathbb{B}$. The final allocation X^* is the optimal solution of (**P**(M)) for $M = B$ and the payment for buyer i is calculated as $p_i^{vcg} = \hat{v}_i(X_i^*) - \Big[V(B) - V(B_{-i})\Big]$. Since, VCG mechanism is strategyproof, no buyer has any incentive to misreport his valuation. The payoff to a buyer in a VCG mechanism is often referred to as the Vickrey payoff and the payment as Vickrey payment.

Definition 3 (Individual Minimal CE Prices) *A price p is an* ***individual minimal CE price*** *of buyer i if it is a CE price of economy $E(B)$ and if the payoff to buyer i at p is maximum across all CE prices.*

Let π_k^{ind} denote the payoff of buyer k at an individual minimal CE price corresponding to k. This can be computed using the following linear program.

$$
\begin{aligned}
\pi_k^{ind} &= \max_{\pi,\pi^s,p} \pi_k. \\
\text{s.t.} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathbf{DP}(k)) \\
\pi^s + \sum_{i \in B} \pi_i &= V(B). \\
\pi^s &\geq \sum_{i \in B} p_i(X_i) \qquad \forall\, X \in \mathbb{X}. \\
\pi_i &\geq v_i(S) - p_i(S) \qquad \forall\, i \in B,\ \forall\, S \in \Omega. \\
p_i(S) &\geq 0\ \forall\, i \in M,\ \forall\, S \in \Omega.
\end{aligned}
$$

The objective function in formulation (**DP**(k)) maximizes the payoff of buyer k given that the total payoff of all buyers and the seller equal the optimal solution of (**DP**(M)) for $M = B$. This along with the other constraints ensures that the optimal solution of (**DP**(k)) is also an optimal solution of (**DP**(M)) for $M = B$. Put differently, out of multiple optimal solutions of (**DP**(M)) for $M = B$, we seek the one which maximizes the payoff of buyer k. The effect of maximizing only a particular buyer's payoff is the following proposition.

Proposition 1 (Parkes and Ungar[23]) *The value of optimal solution of (**DP**(k)) gives the Vickrey payoff to buyer k.*

A stronger condition is required for each buyer to receive its Vickrey payoff *simultaneously* in a CE price, or equivalently, for a single CE price to simultanesouly support the individual minimal CE price to each buyer.

Definition 4 (Buyers are Substitutes) *For every $K \subseteq B$, let $V(K)$ denote the optimal solution value of (**P(M)**) when $M = K$. We say **buyers are substitutes** if:*

$$V(B) - V(K) \geq \sum_{i \in B \setminus K} \Big[V(B) - V(B_{-i})\Big] \quad \forall\ K \subseteq B. \tag{4}$$

Definition 5 (Group Minimal CE Prices) *A price p is a **group minimal CE price** if it is a CE price of economy $E(B)$ and the total payoff to buyers is maximized across all CE prices.*

Let $(\pi_1^*, \ldots, \pi_m^*)$ denote the payoff of buyers at a group minimal CE price. This can be computed using the following formulation.

$$\begin{aligned} &(\pi_1^*, \ldots, \pi_m^*) = \max_{\pi, \pi^s, p} \sum_{i \in B} \pi_i. \\ &\text{s.t.} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\mathbf{DGP}) \\ &\quad \pi^s + \sum_{i \in B} \pi_i = V(B). \\ &\qquad\qquad \pi^s \geq \sum_{i \in B} p_i(X_i) \qquad \forall\ X \in \mathbb{X}. \\ &\qquad\qquad \pi_i \geq v_i(S) - p_i(S) \qquad \forall\ i \in B,\ \forall\ S \in \Omega. \\ &\qquad p_i(S) \geq 0\ \forall\ i \in B,\ \forall\ S \in \Omega. \end{aligned}$$

Formulations (**DGP**) and (**DP(k)**) have the same feasible regions but formulation (**DGP**) maximizes the payoffs of all buyers in stead of just one buyer.

Proposition 2 (Bikhchandani and Ostroy[7]) *Let (p, π^*, π^s) be the prices and payoffs in a group minimal CE price. π^* gives the Vickrey payoff to the buyers if and only if buyers are substitutes.*

Our contribution is to show that we can implement the VCG outcome from a CE price and an efficient allocation using discounts, defined as the marginal contribution to the revenue of the seller, if and only if the price is also a UCE price. Define $\pi^s(M) = \sum_{i \in M} p_i(X_i)$, where $X \in L(M)$ and is a feasible allocation in economy $E(M)$.

Theorem 1 (Sufficient) *If p is a UCE price, then for every buyer $i \in B$, the Vickrey payment of i can be calculated as $p_i^{vcg} = p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$, where (p, X) is a CE in $E(B)$.*

Proof: Consider a buyer $i \in B$. Let (p, Y) be a CE of economy $E(B_{-i})$. From the definition of Vickrey payment, we have:

$$
\begin{aligned}
p_i^{vcg} &= v_i(X_i) - \Big[V(B) - V(B_{-i})\Big]. \\
&= v_i(X_i) - \left[\sum_{k \in B}\Big[v_k(X_k) - p_k(X_k)\Big] + \sum_{k \in B} p_k(X_k)\right] \\
&+ \left[\sum_{k \in B_{-i}}\Big[v_k(Y_k) - p_k(Y_k)\Big] + \sum_{k \in B_{-i}} p_k(Y_k)\right]. \\
&= p_i(X_i) - \sum_{k \in B} p_k(X_k) + \sum_{k \in B_{-i}} p_k(Y_k) \\
&- \sum_{k \in B_{-i}}\Big[v_k(X_k) - p_k(X_k)\Big] + \sum_{k \in B_{-i}}\Big[v_k(Y_k) - p_k(Y_k)\Big] \text{ (Rearranging terms).}
\end{aligned} \tag{5}
$$

Since p is a CE price of economy $E(M)$ for every $M \in \mathbb{B}$, $X_k \in D_k$ and $Y_k \in D_k$ for every $k \in B$ (From CS-1 condition). This means $v_k(X_k) - p_k(X_k) = v_k(Y_k) - p_k(Y_k)$ for every $k \in B_{-k}$. This cancels terms in Equation 5 and transforms it as:

$$
\begin{aligned}
p_i^{vcg} &= p_i(X_i) - \Big[\sum_{k \in B} p_k(X_k) - \sum_{k \in B_{-i}} p_k(Y_k)\Big]. \\
&= p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big] \text{ (From CS-2 condition).}
\end{aligned}
$$

■

We will often refer to the term $\pi^s(B) - \pi^s(B_{-i})$ as the *discount* for buyer i and the term $p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$ as the *discounted price* of buyer i. Next, we show that if we can calculate Vickrey payments from a CE price using discounted price, then that CE price has to be a UCE price.

Proposition 3 *Let (p, X) be a CE. If $p_i^{vcg} = p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$ $\forall\, i \in B$, then p is a UCE price.*

Proof: Assume for contradiction that p is not a UCE price, so that there is some buyer $k \in B$ for which the prices are not CE for marginal economy $E(B_{-k})$. We show that the Vickrey payment to this buyer is less than the adjusted price. Consider economy $E(M)$ with $M = B_{-k}$, and let Y denote an allocation $Y \in L(M)$, that satisfies the CS-2 condition. Since p are not CE prices for economy $E(M)$, then $Y_i \notin D_i$ for some buyer $i \in M$ (Violating the CS-1 condition). Since (p, X) is a CE for the main economy, the CS-1 conditions give $X_i \in D_i$, and we have $v_i(X_i) - p_i(X_i) > v_i(Y_i) - p_i(Y_i)$. Then, applying the logic in Theorem 1 in the reverse direction, we get $p_k^{vcg} < p_k(X_k) - \Big[\pi^s(B) - \pi^s(B_{-k})\Big]$ for buyer k. This gives us a contradiction. ■

Notice that this requirement that prices are UCE continues to hold for restricted classes of valuations, and for anonymous or linear prices. The following Lemma shows that a buyer that will receive a discounted price from some CE price, always prefers that the prices are UCE prices. This will be useful in proving the equilibrium properties of our ascending Vickrey auction, and also leads to a useful corollary of Proposition 3.

Lemma 1 *Let (p, X) be a CE. Then, $p_i^{vcg} \leq p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-k})\Big]$ for every buyer $i \in B$.*

Proof: Consider some valuation v for which (p, X) is a CE. Since (p, X) is a CE,

$$V(B) = \sum_{k \in B} \pi_k + \pi^s(B). \tag{6}$$

Since $(p, \pi, \pi^s(B_{-i}))$ is a feasible dual solution of $(\mathbf{DP}(B_{-i}))$, we can also write,

$$V(B_{-i}) \leq \sum_{k \in B_{-i}} \pi_k + \pi^s(B_{-i}). \tag{7}$$

From Equations 6 and 7, $V(B) - V(B_{-i}) = \pi_i^{vcg} \geq \pi_i + \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$. Finally, we must have $p_i^{vcg} \leq p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$, since X is an efficient allocation. ■

Corollary 1 *Let (p, X) be a CE. If $p_i^{vcg} = p_i(X_i)\ \forall\ i \in B$, then p must be a UCE price.*

Proof: By Lemma 1, the discount $\pi^s(B) - \pi^s(B_{-i})$ must be zero for all buyers $i \in B$. Otherwise, we must have $p_i^{vcg} < p_i(X_i)$. Then, by Proposition 3 we have that the prices must be UCE. ■

Proposition 3 is not quite as general as we would like because it assumes that the discount method must be used to determine Vickrey payments from CE prices. We now relax this requirement by proving that we can restrict attention to the discounted price as a candidate for the Vickrey payment to a buyer. This leads to a more general statement about the necessity of UCE prices.

It is useful to clarify the precise information that is available to a seller about the valuation profile of buyers, given CE (p, X) and demand sets (utility maximizing bundles) of buyers. We say that valuation profile $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_m)$ is *consistent* with demand set profile $D = (D_1, \ldots, D_m)$ if constraints (2) are satisfied by tuple $< \hat{v}_i, D_i >$ for every $i \in B$. By definition, there can only be enough information to compute the Vickrey payments given CE (p, X) and demand sets when the Vickrey payments are *identical* for all consistent valuation profiles.

Lemma 2 *Let (p, X) be a CE. For every buyer $i \in B$, and for generic valuation profiles, there is some valuation profile $\hat{v}$ consistent with demand set profile D for which the Vickrey payment $\hat{p}_i^{vcg}$ is exactly $\hat{p}_i^{vcg} = p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$.*

Proof: We define a parameterized valuation profile, $\hat{v}^\epsilon$, that is consistent with (p, X). With this, we show that there can be no buyer i for which the Vickrey payment p_i^{vcg} is known to be less than the discounted price by constructing a $\hat{v}^\epsilon$ for which the Vickrey payment must be greater than p_i^{vcg}. Let $B^+ \subseteq B$ denote the set of buyers with positive payoff at the current prices. Buyers in B^+ do not have $\emptyset$ in their demand sets. For every buyer $i \in B^+$, construct valuation, $\hat{v}_i^\epsilon$, as

$$\hat{v}_i^\epsilon(S) = \begin{cases} p_i(S) + \epsilon & \forall\ S \in D_i \\ p_i(S) & \text{otherwise} \end{cases}$$

for some small $\epsilon > 0$. For every buyer $i \in B \setminus B^+$, construct valuation, $\hat{v}_i^\epsilon$, as

$$\hat{v}_i^\epsilon(S) = \begin{cases} p_i(S) & \forall\ S \in D_i \\ p_i(S) - \epsilon & \text{otherwise} \end{cases}$$

Valuation profile $\hat{v}^\epsilon$ is consistent with the demand set profile D, and every buyer $i \in B$ would submit D_i given valuation $\hat{v}_i^\epsilon$ and price p. Denote the value of the efficient allocation with respect to valuation profile $\hat{v}^\epsilon$ as $\hat{V}^\epsilon(\cdot)$. Since (p, X) is a CE price with respect to the valuation profile $\hat{v}^\epsilon$, and from the definition of $\hat{v}^\epsilon$, we can write

$$\hat{V}^\epsilon(B) = \pi^s(B) + |B^+|\epsilon \tag{8}$$

Consider some buyer $i \in B$. Let Y be a utility maximizing allocation of the seller at price p in economy $E(B_{-i})$. Using the definition of $\hat{v}^\epsilon$ we can write,

$$\pi^s(B_{-i}) = \sum_{k \in B_{-i}} p_k(Y_k) \leq \sum_{k \in B_{-i}} \hat{v}_k^\epsilon(Y_k) = \hat{V}^\epsilon(B_{-i}) \tag{9}$$

Now,

$$\begin{aligned} \hat{p}_i^{vcg} &= \hat{v}_i^\epsilon(X_i) - \Big[\hat{V}^\epsilon(B) - \hat{V}^\epsilon(B_{-i})\Big] \\ &\geq p_i(X_i) - \hat{V}^\epsilon(B) + \hat{V}^\epsilon(B_{-i}). \\ &\text{(Since } X_i \in D_i \text{ and from the definition of } \hat{v}^\epsilon\text{)} \\ &\geq p_i(X_i) - \pi^s(B) - |B^+|\epsilon + \pi^s(B_{-i}). \\ &\text{(From Equations 8 and 9)} \\ &= p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big] - |B^+|\epsilon. \end{aligned} \tag{10}$$

Assume for contradiction that there is enough information to claim that the Vickrey payment, for all valuations consistent with (p, X), is $p_i^{vcg} = p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big] - \epsilon'$, for some $\epsilon' > 0$. But, from Equation (10), we can choose some ϵ such that valuation $\hat{v}_i^\epsilon$ has Vickrey payment $\hat{p}_i^{vcg} \geq p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big] - |B^+|\epsilon > p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big] - \epsilon'$. This is a contradiction. ∎

Lemmas 1 and 2 show that the discounted price $p_i(X_i) - \left[\pi^s(B) - \pi^s(B_{-i})\right]$ is the *strongest upper-bound* that can be stated for the Vickrey payment of buyer i given CE prices (p, X) and demand sets. Any claim that the Vickrey payment to i is less than this cannot be valid because it must rule out some valuation profile that is consistent with the CE information. Given this, we can restrict attention to these discounted prices and show UCE prices are necessary to determine Vickrey payments from CE price information, for general valuations.

Theorem 2 (Necessary) *Let (p, X) be a CE. If there is enough information to determine the Vickrey payment for every buyer $i \in B$, and for generic valuation profiles, then p is also a UCE price.*

Proof: From Lemma 2, we can focus on the discounted prices as our candidate Vickrey payments. The result then follows from Proposition 3. ■

2.4 Examples

In this section, we will give some examples and some UCE prices in those examples. Clearly, the UCE price "price=value" is always present. We will give examples of some other UCE prices in these examples. Consider an example with two buyers and two items and valuations as shown in Table 2 below.

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
1	0	8	9	12
2	0	6	8	14

Table 2: Example when buyers substitutes

Buyers are substitutes in this example. A UCE price (p) in this example is $p_1(\emptyset) = p_2(\emptyset) = 0$, $p_1(\{1\}) = p_1^{vcg} = 6$, $p_1(\{2\} = 8$, $p_1(\{1,2\}) = p_2(\{1,2\}) = 10$, $p_2(\{1\}) = 2$, $p_2(\{2\}) = p_2^{vcg} = 4$. In the main economy, the efficient allocation (buyer 1 gets item 1 and buyer 2 gets item 2) is supported (i.e. the seller and buyers maximize their utility) at this price. In the marginal economy with only buyer 1, efficient allocation (buyer 1 gets both items) is supported at this price. Also, in the marginal economy with only buyer 2, efficient allocation (buyer 2 gets both items) is supported at this price. Also, observe that Vickrey payments are directly calculated (without discounts) at this UCE price.

Consider another example in Table 3.

Buyers are not substitutes in this example. A UCE price (p) is the following: $p_1(\emptyset) = p_2(\emptyset) = 0$, $p_1(\{1\}) = 2$, $p_1(\{2\}) = 0$, $p_1(\{1,2\}) = 2$, $p_2(\{1\}) = 0$, $p_2(\{2\}) = 4$, $p_2(\{1,2\}) = 4$, $p_3(\{1\}) = 0$, $p_3(\{2\}) = 2$, $p_3(\{1,2\}) = 4$. In the main economy and in the marginal economy with buyers 1 and 2 only, the efficient allocation (buyer 1 gets item 1, buyer 2 gets

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
1	0	3	0	3
2	0	0	6	6
3	0	0	2	4

Table 3: Example when buyers are not substitutes

item 2 and buyer 3 gets nothing) is supported at p. In the marginal economy with buyers 1 and 3 only, the efficient allocation (buyer 1 gets item 1 and buyer 3 gets item 2) is also supported at p. Similarly, in the marginal economy with buyers 2 and 3 only, the efficient allocation (buyer 2 gets item 2 and buyer 3 gets item 1) is supported at p. The Vickrey payments for buyers can be calculated as: $p_1^{vcg} = p_1(\{1\}) - [\pi^s(B) - \pi^s(B_{-1})] = 2 - [6-4] = 0$, $p_2^{vcg} = p_2(\{2\}) - [\pi^s(B) - \pi^s(B_{-2})] = 4 - [6-4] = 2$, $p_3^{vcg} = 0$.

3 Iterative Price-Based Auctions

Many known auctions for our model [22, 4] and de Vries et al. [13] maintain non-linear and non-anonymous prices (although prices are implicit in the auction of Ausubel and Milgrom [4]). Such a pricing scheme leads to maintaining (possibly) exponential number of prices in the auction. One wonders if it is possible to maintain simpler prices.

3.1 Defining Iterative Auctions

Several "orders" of prices have been described in Bikhchandani and Ostroy [7]. We explore if it is possible to design iterative Vickrey auctions in which we maintain prices which are (i) linear and anonymous ($p \in \mathbb{R}_+^A$) (ii) linear and non-anonymous ($p \in \mathbb{R}_+^{A \times B}$) (iii) non-linear and anonymous ($p \in \mathbb{R}_+^{\Omega}$). For this, we will define an iterative Vickrey Auction for our setting. We mainly follow de Vries et al. [13] and Gul and Stacchetti [17]. The only difference from their definitions and ours is that we do not require the final prices at the end of our auction to represent the actual payment. First, we define the notion of a price path.

Definition 6 *A **price path** is any of these four types of functions:*

- ***Linear and anonymous price path:*** $P : [0,1] \to \mathbb{R}_+^A$.
- ***Linear and non-anonymous price path:*** $P : [0,1] \to \mathbb{R}_+^{A \times B}$.
- ***Non-linear and anonymous price path:*** $P : [0,1] \to \mathbb{R}_+^{\Omega}$.
- ***Non-linear and non-anonymous price path:*** $P : [0,1] \to \mathbb{R}_+^{\Omega \times B}$.

$[0, 1]$ *denotes the possible "time" intervals with* $P(t)$ *denoting a price vector seen at time* t*. A price path is* ***ascending*** *if* P *is non-decreasing with time and* ***descending*** *if* P *is non-increasing with time.*

With the definition of price path, we define an iterative auction.

Definition 7 (Iterative Auction) *An* ***iterative auction*** *assigns to each profile of buyer valuations* $v \in \mathbb{R}_+^{\Omega \times B}$*, an ascending or a descending price path,* P^v *and a final allocation* X^v *such that:*

C1 *At every iteration* t*, buyers make claim about their demand set at price* $P^v(t)$*.*

C2 *Every buyer gets a bundle (possibly* $\emptyset$*) from his demand set at the end.* $X_i^v \in D_i(P^v(1)) \; \forall \, i \in B$*.*

C3 *The final payment of buyers in the auction is determined from final allocation (*X^v*) and final price (*$P^v(1)$*) only.*

C4 *For every iteration* $t \in [0, 1]$*, the price adjustment in iteration* t *is determined only by* $P^v(t)$ *and demand set information at price* $P^v(t)$*.*

An iterative price auction is an ***ascending price auction*** *if the price path is ascending and a* ***descending price auction*** *if the price path is descending.*

It is easy to verify that many auctions fall into the category of iterative auction as defined in Definition 7. Of all the iterative auctions, we are only concerned with those which implement an efficient outcome and in which truthful submission of demand set information is an *ex post Nash equilibrium*. We call such iterative auctions *efficient iterative auctions*.

Lemma 3 *Every efficient iterative auction terminates at a UCE price.*

Proof: From the revelation principle, the direct revelation mechanism of every efficient iterative auction should be strategyproof and efficient. But Groves mechanism is the only strategyproof and efficient direct mechanism [15]. This means the final payment in the efficient iterative auction is the VCG payment. From Definition 7, the final payment in an iterative auction is determined from the final price and allocation. The final price is a CE price and allocation an efficient allocation as buyers truthfully submit demand set information. From Theorem 2, the final price in an efficient iterative auction has to be a UCE price. ∎

3.2 Insufficiency of Simpler Prices

Now, we consider some special cases of UCE prices. We omit the subscript for buyer if a price is anonymous.

Proposition 4 (Anonymous UCE Prices) *If anonymous UCE prices exist, then buyers are substitutes.*

Proof: Let X be an efficient allocation of economy $E(B)$. If p is an anonymous UCE price, then $p(X_i) = p_i^{vcg}\ \forall\ i \in B$ (using Theorem 1). From Proposition 2, buyers are substitutes. ■

Consider the special case of anonymous and linear prices, with $p(S) = \sum_{j \in S} p(\{j\})\ \forall\ S \in \Omega$. If buyers demand at most one item, then Leonard [19] established that there exists a unique anonymous linear CE price p which gives every buyer his VCG payoff. From Corollary 1, price p is a UCE price. More generally, Kelso and Crawford [18] established that linear and anonymous CE prices exist when valuations of buyers satisfy a condition called gross substitutes.

Definition 8 (Gross Substitutes) *A valuation function v_i satisfies* ***gross substitutes*** *(GS) if, for all prices $p, p' \in \mathbb{R}_+^{\Omega \times B}$ such that*

$$p_i(S) = \sum_{j \in S} p_i(\{j\}) \leq p'_i(S) = \sum_{j \in S} p'_i(\{j\})\ \forall\ i \in B,\ \forall\ S \in \Omega.$$

and for all $S \in D_i(p)$, there exists $S' \in D_i(p')$ such that $\{j \in S : p_i(\{j\}) = p'_i(\{j\})\} \subseteq S'$.

But linear and anonymous UCE prices may not exist even if the valuation functions satisfy gross substitutes condition. In fact, both non-linear and non-anonymous prices can be required to support UCE prices. Consider the example in Table 2 from [17], where valuations of buyers satisfy the gross substitutes condition.

Proposition 5 *Even for gross-substitutes preferences both non-linear and non-anonymous prices are required to support a UCE price.*

Proof: Consider the example in Table 2. We show that no linear and non-anonymous or, non-linear and anonymous price is a UCE price in the example. Assume for contradiction that $p \in \mathbb{R}_+^{B \times A}$ is a UCE price for the example. It is easy to see that in the CE of $E(B)$, buyer 1 is assigned item 1 and in the CE of $E(B_{-2})$, buyer 1 is assigned bundle $\{1, 2\}$. This means, $8 - p_1(\{1\}) = 12 - [p_1(\{1\}) + p_1(\{2\})]$. This gives us $p_1(\{2\}) = 4$. Similarly, buyer 2 is assigned item 2 in CE of $E(B)$ and bundle $\{1, 2\}$ in CE of $E(B_{-1})$. This means, $8 - p_2(\{2\}) = 14 - [p_2(\{1\}) + p_2(\{2\})]$. This gives us, $p_2(\{1\}) = 6$.

Also, since buyer 1 is assigned item 1 in CE of $E(B)$, we have $8-p_1(\{1\}) \geq 9-p_1(\{2\}) = 5$. This gives us,

$$p_1(\{1\}) \leq 3. \tag{11}$$

Also, the seller should maximize his utility in the CE allocation of $E(B)$. This means, $p_1(\{1\}) + p_2(\{2\}) \geq p_2(\{1\}) + p_2(\{2\})$. This gives us,

$$p_1(\{1\}) \geq p_2(\{1\}) = 6. \tag{12}$$

Equations 11 and 12 give us a contradiction.

A similar argument shows that there does not exist $p \in \mathbb{R}_+^\Omega$, which is a UCE price in example in Table 2. ■

Theorem 3 *Every efficient iterative auction for generic valuation profiles must allow a non-linear and non-anonymous price path.*

Proof: From Lemma 3, every efficient iterative auction should terminate at a UCE price for every possible valuation profile. From Propositions 5, for the valuation profile in Table 2, linear and non-anonymous UCE price does not exist. Also, from Proposition 4, if buyers are not substitutes, then anonymous UCE price will not exist. So, no efficient iterative auction can have non-linear and anonymous price path or linear and non-anonymous price path. Gul and Stacchetti [17] have shown that no iterative auction which maintains linear and anonymous price can implement VCG outcome. Hence, every efficient iterative auction should have only non-linear and non-anonymous price path. ■

Thus, unless we maintain non-linear and non-anonymous prices, we cannot design an efficient iterative auction. Our ascending price auction maintains such a price path and converges to a UCE price.

4 Ascending Price Vickrey Auction

4.1 Outline of Methodology

In designing our auction, we follow primal-dual auction based ideas proposed in Bikhchandani et al.[8], Bikhchandani and Ostroy [6] and de Vries et al. [13]. But our approach departs from previous approaches in one significant way. All the previous work has considered primal-dual algorithms on the original economy and designed ascending price auctions based on that. In contrast, we consider the "intersection" of feasible regions of the main economy and every marginal economy and formulate a primal problem to search for a UCE price. The dual variables of such a formulation give price update directions that searches for a UCE price. To summarize our methodology:

- For every $M \in \mathbb{B}$, for economy $E(M)$, formulate the efficient allocation problem as a linear program.
- For every $M \in \mathbb{B}$, for economy $E(M)$, consider a *restricted primal* problem.
- Construct a single restricted primal problem by combining constraints from restricted primal problems for economy $E(M)$ for every $M \in \mathbb{B}$.
- Consider a dual optimal solution for this restricted primal that allows same price increase in all economies.

Together, this yields a design for an ascending-price Vickrey auction that searches for an optimal primal and dual solution in economy $E(M)$ for every $M \in \mathbb{B}$ and thus finds a UCE price.

It is worth noting that we are *not* solving optimization problems of all $m+1$ economies using $m+1$ primal-dual algorithms. The auction implements a *single* primal-dual algorithm, that solves the efficient allocation problem of economy $E(M)$ for every $M \in \mathbb{B}$ simultaneously. Our auction works for any valuation profile that meets the mild conditions A1-A4 introduced in Section 2.1.

4.2 The Universal Formulation

In this section, we formulate the problem of finding a UCE price as a linear program. For this, we define (and reiterate) some notations first. We suppress the dependence on price p for simplicity.

$$
\begin{aligned}
\pi_i &= \max_{S \in \Omega}[v_i(S) - p_i(S)] \qquad \forall\, i \in B. \\
\mathbb{X}(M) &= \text{Set of feasible allocations with buyers in the set } M \subseteq B. \\
\pi^s(M) &= \max_{X \in \mathbb{X}(M)} \sum_{i \in M} p_i(X_i) \qquad \forall\, M \subseteq B. \\
M^+ &= \{i \in M : \pi_i > 0\} \qquad \forall\, M \subseteq B. \\
B^+ &= \{i \in B : \pi_i > 0\}. \\
D_i &= \{S \in \Omega : v_i(S) - p_i(S) \geq v_i(T) - p_i(T) \;\forall\, T \in \Omega\} \qquad \forall\, i \in B. \\
L(M) &= \{X \in \mathbb{X}(M) : \pi^s(M) = \sum_{i \in M} p_i(X_i)\} \qquad \forall\, M \subseteq B. \\
L^*(M) &= \{X \in L(M) : X_i \in D_i \cup \{\emptyset\} \;\forall\, i \in M\} \qquad \forall\, M \subseteq B.
\end{aligned}
$$

Observe the difference between L and L^*. While L contains all the payoff maximizing allocations of the seller, L^* contains only those payoff maximizing allocations of the seller in which either a buyer is allocated a bundle from his demand set or he is allocated the empty bundle. Using primal feasibility, dual feasibility and CS conditions CS-1 and CS-2, we formulate the CE of economy $E(M)$ for any $M \subseteq B$ as follows (y, z variables indicate the same as in previous formulations; the superscripts indicate that the variables correspond to economy $E(M)$):

$$y_i^M(S) = \sum_{X \in L^*(M): X_i = S} z^M(X) \qquad \forall\, i \in M,\ \forall\, S \in D_i \cup \{\emptyset\}. \tag{$\mathbf{CE}(M)$}$$

$$\sum_{S \in D_i} y_i^M(S) = 1 \qquad \forall\, i \in M^+. \tag{13}$$

$$\sum_{\emptyset \neq S \in D_i} y_i^M(S) \leq 1 \qquad \forall\, i \in M \setminus M^+. \tag{14}$$

$$\sum_{X \in L^*(M)} z^M(X) = 1. \tag{15}$$

$$y_i^M(S) \geq 0 \qquad \forall\, i \in M,\ \forall\, S \in D_i \cup \{\emptyset\}.$$

$$z^M(X) \geq 0 \qquad \forall\, X \in L^*(M).$$

The first set of constraints in (**CE**(M)) enforce balance of supply and demand. Constraints 13 and 14 indicate the demand in the economy. They say that buyers should be allocated bundles from their demand set only. Constraint 15 indicate supply in the economy. It says that a seller should select an allocation which maximizes his payoff and which is compatible with the demand of the buyers. Any allocation in $L^*(M)$ is such an allocation. If (**CE**(M)) is feasible at price p, then p is a CE price of economy $E(M)$. If (**CE**(M)) is feasible at price p for every $M \in \mathbb{B}$, then p is a UCE price.

The restricted primal (in a primal-dual algorithm setting) considers only variables in the primal problem which are *admissible* and assigns zero value to rest of them. The admissible variables are identified using a dual feasible solution and CS conditions. The primal problem is thus transformed into a restricted primal problem where feasibility of admissible variables (and zero values to other variables) incorporate primal feasibility, dual feasibility and CS conditions in a single problem. To attain feasibility by using only admissible variables, the restricted primal problem introduces artificial variables into the problem and tries to minimize these artificial variables. If all artificial variables are zero, we have a feasible and thus an optimal primal and dual solution of the problem.

We should note that there are various ways to write the restricted primal (combining primal feasibility, dual feasibility and complementary slackness) for our problem. The formulation presented here is suited for designing ascending price auctions. To understand this further, realize that at a particular price, we may not find a feasible solution for (**CE**(M)) for some $M \in \mathbb{B}$. In that case, we introduce artificial variables (δ^M) in constraints 13 for (**CE**(M)) for every $M \in \mathbb{B}$. Using this, we define a concept called "quasi-CE price".

Definition 9 (Quasi-CE Price) *A price p is a **quasi-CE price** of economy $E(M)$ for any $M \subseteq B$, if there is a feasible solution to (**CE(M)**) with constraints 13 relaxed to:*

$$\sum_{S \in D_i} y_i^M(S) + \delta_i^M = 1 \qquad \forall\ i \in M^+.$$
$$\delta_i^M \geq 0 \qquad \forall\ i \in M^+. \tag{QUASI(M)}$$

Observe that if there is a feasible solution to (**QUASI(M)**) at a price p such that $\delta_i^M = 0\ \forall\ i \in M^+$, then p is a CE price of economy $E(M)$ and if such feasible solutions exist for every $M \in \mathbb{B}$, p is a UCE price.

Also, the feasibility of (**QUASI(M)**) is possible if $L^*(M)$ is non-empty (y^M, z^M, δ^M variables can be chosen appropriately to attain feasibility). This idea is central to the design of our ascending price auctions. Such quasi-CE prices are easy to find at low prices and thus an ascending price auction is possible to design.

Denote the feasible region of **QUASI(M)** as $U(M)$. Combining the feasible regions of (**QUASI(M)**) of economies $E(M)$ for every $M \in \mathbb{B}$, we write the following formulation for any $K \subseteq B^+$:

$$\Delta(K) = \max - \sum_{M \in \mathbb{B}} \sum_{i \in K \cap M^+} \delta_i^M.$$
$$\text{s.t. } (y^M, z^M, \delta^M) \in U(M) \qquad \forall\ M \in \mathbb{B}. \tag{RP}$$

The feasible region of (**RP**) is the intersection of feasible regions of (**QUASI(M)**) for all $M \in \mathbb{B}$. The objective function is defined for every $K \in B^+$. For any $K \in B^+$, the value of the objective function indicates the number of buyers from K that can be satisfied in the main economy as well as in every marginal economy. Observe that the feasible region of (**RP**) is independent of the choice of $K \subseteq B^+$. (**RP**) is feasible if the price is a quasi-CE price for economy $E(M)$ for every $M \in \mathbb{B}$.

We will show that our auction will start from a feasible solution to (**RP**) and adjust prices such that we get another quasi-CE price for economy $E(M)$ for every $M \in \mathbb{B}$. In other words, we will start from a price where there is "excess" demand in economy $E(M)$ and quasi-CE price exists in $E(M)$ for every $M \in \mathbb{B}$. We will adjust prices till there is no excess demand in the original economy and in any of the marginal economies. Thus, quasi-CE price in each economy becomes a CE price, giving us a desired UCE price. Since we search for a UCE price, our price adjustment strives for a "balance" of supply and demand (starting from excess demand) not only in the original economy but also in every marginal economy.

Definition 10 (Universal Overdemand) *We say* ***universal overdemand*** *holds at price* p *if (**RP**) is feasible and* $\Delta(B^+) < 0$ *at* p.

If universal overdemand holds, then there is excess demand either in the original economy or in some marginal economy. If p is a UCE price, there cannot be universal overdemand at p. If there is excess demand, then some buyers must have shortage in supply. This idea is captured in the following definition.

Definition 11 (Universally Undersupplied Buyers) *A set of buyers* $K \subseteq B^+$ *are* ***universally undersupplied*** *if universal overdemand holds and* $\Delta(K) < 0$. *A set of buyers* $K \subseteq B^+$ *is* ***minimally universally undersupplied*** *if it is universally undersupplied and there does not exist* $N \subset K$ *such that* N *is universally undersupplied.*

It is easy to see that given a universally undersupplied set of buyers K, it is sufficient that $\Delta(K_{-i}) = 0$ for every $i \in K$ for K to be minimally universally undersupplied.

Universally undersupplied buyers is a generalization of the idea of "undersupplied" buyers introduced in [13], who only consider the original economy.

4.3 Price Adjustment and the Auction Design

Formally, a primal-dual algorithm starts with a dual feasible solution (prices in this case). Then, it considers a restricted primal problem of the original primal problem. If an optimal solution to the restricted primal is zero (i.e. all artificial variables in the restricted primal have zero value), it stops with optimal primal and dual solutions. Else, it considers the dual of the restricted primal and adjusts the dual solution based on that.

In a similar vein, formulation (**RP**) is a restricted primal formulation of our problem. If its optimal solution is zero, we have found a UCE price and we stop. Else, we look for the dual of (**RP**) for price adjustment direction. Since, we maintain a single price vector, the important thing is to get price adjustment directions which are same across all marginal economies and the main economy. Expanding the restricted primal formulation (**RP**) for our problem, we get the following:

$$
\begin{aligned}
\Delta(K) = \max - \sum_{M \in \mathbb{B}} \sum_{i \in K \cap M^+} \delta_i^M \\
\text{s.t.} \\
y_i^M(S) = \sum_{X \in L^*(M): X_i = S} z^M(X) \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M,\ \forall\, S \in D_i \cup \{\emptyset\}. \\
\sum_{S \in D_i} y_i^M(S) + \delta_i^M = 1 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M^+. \\
\sum_{\emptyset \neq S \in D_i} y_i^M(S) \leq 1 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M \setminus M^+. \\
\sum_{X \in L^*(M)} z^M(X) = 1 \qquad & \forall\, M \in \mathbb{B}. \\
y_i^M(S) \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M,\ \forall\, S \in D_i \cup \{\emptyset\}. \\
z^M(X) \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, X \in L^*(M). \\
\delta_i^M \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M^+.
\end{aligned} \tag{\textbf{RP}}
$$

The dual of (**RP**) is the following formulation:

$$
\begin{aligned}
\Delta(K) = \min \sum_{M \in \mathbb{B}} \Big[\lambda^M + \sum_{i \in M} \theta_i^M\Big]. \\
\text{s.t.} \\
\theta_i^M + \rho_i^M(S) \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M,\ \forall\, S \in D_i. \\
\lambda^M - \rho_i^M(X_i) \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, X \in L^*(M). \\
\theta_i^M \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M \setminus (K \cap M^+). \\
\theta_i^M \geq -1 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in K \cap M^+. \\
\rho_i^M(\emptyset) \geq 0 \qquad & \forall\, M \in \mathbb{B},\ \forall\, i \in M^+.
\end{aligned} \tag{\textbf{DRP}}
$$

An optimal solution to (**DRP**) gives a direction of price (dual solutions) change in our auction.

Lemma 4 *If universal overdemand holds, then for any universally undersupplied set of buyers $K \subseteq B^+$, there is an optimal solution to (**DRP**) such that $\rho_i^M(S) = 1\ \forall\ M \in \mathbb{B},\ \forall\ i \in K \cap M^+,\ \forall\ S \in D_i$ and $\rho_i^M(S) = 0$ otherwise.*

Proof: For any $M \in \mathbb{B}$, we say a buyer $i \in M$ is *satisfied* in economy $E(M)$ if $\delta_i^M = 0$ in the optimal solution of (**RP**). Let $Q(M)$ denote the set of satisfied buyers from set $K \cap M^+$ in economy $E(M)$ for every $M \in \mathbb{B}$. This means at optimality of (**RP**),

$$
\Delta(K) = \sum_{M \in \mathbb{B}} \Big[|Q(M)| - |K \cap M^+|\Big]. \tag{16}
$$

$\Delta(K)$ represents the total number of unsatisfied buyers in K. Set $\theta_i^M = -1$ for all $i \in K \cap M^+$ and $\theta_i^M = 0$ otherwise for every $M \in \mathbb{B}$. Set $\lambda^M = |Q(M)|$ for every $M \in \mathbb{B}$. This is clearly feasible for (**DRP**) and optimal using Equation 16. ■

Firstly, observe that the price adjustment directions given in Lemma 4 is same across all the economies considered. This allows us to maintain a single price vector for all the economies and update the prices using Lemma 4.

Secondly, Lemma 4 gives us a lot of flexibility in chosing the set of buyers whose prices need to be adjusted. The largest universally undersupplied set of buyers is B^+ and the smallest universally undersupplied set of buyers is any minimally universally undersupplied set of buyers. In deed, it can be shown that starting from an appropriate price, if we adjust prices on all buyers with positive payoff (B^+) on bundles they demand till we find a UCE price, we will get an auction that terminates at a UCE price. But, this UCE price may be the "bad" UCE price where price equals value for buyers as adjusting prices on larger set of buyers, may drive the price too high. We want to avoid this situation whenever possible. We will adjust prices on the smallest possible set of buyers. This is helpful in two sense: (i) whenever possible, we want to keep the discounts in our auction zero, which is possible if we keep the UCE price low; (ii) fewer number of buyers participate in the auction resulting in good information revelation properties. So, we adjust prices on a minimally universally undersupplied set of buyers.

Definition 12 (Price Adjustment) *At price p, if universal overdemand holds:*

- *Identify a minimally universally undersupplied set of buyers $K \subseteq B^+$.*
- *For every $i \in B$ and $S \in D_i$:*
  - *If $i \in K$ and $S \in D_i$, increase $p_i(S)$ by 1 ($\rho_i^M(S) = 1 \;\forall\; M \in \mathbb{B}, \forall\, i \in K,\; \forall S \in D_i$).*
  - *Else do not change $p_i(S)$.*
- *For every $M \in \mathbb{B}$, increase $\pi^s(M)$ by λ^M ($= |Q(M)|$) for economy $E(M)$.*

This enables us to define our ascending price auction.

Definition 13 (Ascending Price Vickrey Auction) *The **ascending price Vickrey auction** is an iterative auction with the following steps:*

S0 Initialize prices such that it is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$ and all bundles which are not demanded have zero price. One such price is setting zero price on all bundles. Calculate $\pi^s(M) \;\forall\; M \in \mathbb{B}$ (these are zero if starting price is zero price).

S1 Collect demand set of every buyer at the current price.

S2 If universal overdemand holds, perform price adjustment using Definition 12. Go back to S1.

S3 If universal overdemand does not hold, the auction ends. Final allocation is any $X \in L^*(B)$ from the final iteration and every buyer $i \in B$ pays an amount equal to $p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$, where p is the final price.

Since price of all bundles in a demand set of a buyer in K is increased by one, the payoff of those buyers are decreased by one. Similarly, in economy $E(M)$ ($M \in \mathbb{B}$), buyers in $Q(M)$ will increase the payoff of the seller due to price increase. This gives us the following lemma.

Lemma 5 *Let K be a set of minimally universally undersupplied set of buyers at an iteration. Due to price adjustment:*

(i) If $i \in K$ the change in maximum payoff of i is -1 and zero otherwise.

(ii) Demand set of each buyer weakly increases after a price adjustment in Definition 12.

(iii) For any $M \in \mathbb{B}$, if for allocation $X \in L(M)$, $Q = \{i \in M : X_i \in D_i\}$, then there exists $Y \in L^*(M)$ such that $\{i \in M : Y_i \in D_i\} = Q$.

(iv) For every $M \in \mathbb{B}$, the change in maximum payoff to the seller due to price increase is λ^M.

Proof: Realize that starting from zero price, the price in the auction remains integral throughout.

(i) For every buyer $i \in K$, the price is increased by unity on bundles demanded by i. By integral prices, the change in maximum payoff is -1. Since prices of any buyer $i \notin K$ is unchanged, the change in maximum payoff is zero.

(ii) Since prices are integral, the claim follows from (ii).

(iii) From the starting price in Definition 13, the prices on bundles which are demanded by buyers are increased by unity in each iteration. From (ii), those bundles are demanded in every iteration. This means, the bundles which are not demanded have price of zero. Consider any $X \in L(M)$. This means $\sum_{i \in M} p_i(X_i) = \sum_{i \in Q} p_i(X_i)$. Let Y be an allocation with $Y_i = \emptyset$ if $i \notin Q$ and $Y_i = X_i$, otherwise. Clearly, $Y \in L^*(M)$.

(iv) Consider any $M \in \mathbb{B}$. Let $X \in L^*(M)$ be an allocation such that $z(X) = 1$ in the optimal solution of (**RP**). Since $Q(M)$ denotes the set of satisfied buyers from $K \cap M^+$ in economy $E(M)$, the payoff to seller from X is increased by $|Q(M)|$. Since K is minimally universally undersupplied, $|K \cap M^+| \geq |Q(M)| \geq |K \cap M^+| - 1$. Now consider any other feasible allocation $Y \neq X$ of economy $E(M)$. If $Y \notin L(M)$, then its payoff can increase by a maximum $|K \cap M^+|$. If $Y \in L(M)$, the maximum number of buyers that can be satisfied is $|Q(M)|$ (using (iii)). As prices are integral, the change in maximum payoff of the seller is $|Q(M)| = \lambda^M$. ■

This leads us to a result fundamental to the design of our auction.

Theorem 4 *Let universal overdemand hold at price p^t and let p^{t+1} be the price after price adjustment. Then, p^{t+1} is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$.*

Proof: Let K be a minimally universally undersupplied set of buyers at price p^t. Let (y^M, z^M, δ^M) denote the optimal solution of (**RP**) for set of buyers K for some economy $M \in \mathbb{B}$. Let $X \in L^*(M)$ be such that $z^M(X) = 1$. $\lambda^M = |Q(M)|$ is the number of buyers from M who were satisfied in the optimal solution. From Lemma 5, the increase in payoff of the seller in economy $E(M)$ is λ^M. This means, X remains payoff maximizing for the seller after price increase. Since demand sets of every buyer weakly increase (Lemma 5), $X \in L^*(M)$ after the increase. From Lemma 5, bundles demanded at price p^t by a buyer will also be demanded at price p^{t+1} by the same buyer. This means (y^M, z^M, δ^M) is a feasible solution for economy $E(M)$ at price p^{t+1} in (**RP**). This can be shown for every economy. This gives us a feasible solution of (**RP**) implying p^{t+1} is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$. ■

Theorem 4 helps us prove our main result.

Theorem 5 *The ascending price Vickrey auction implements the VCG outcome.*

Proof: Since starting price is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$, universal overdemand holds in the first iteration.

From Theorem 4, after the starting iteration, at every iteration either universal overdemand holds or $Z(B^+) = 0$ at the optimal solution of (**RP**). The latter is the terminating condition of the auction where we have converged to a UCE price and the final allocation is an efficient allocation. From Theorem 1 and the payment rule in Definition 13, the final payments of buyers are Vickrey payments. ■

4.4 Incentives

Although the VCG mechanism is strategyproof, it is well known that an iterative auction that implements VCG outcome may not support straightforward bidding in a dominant strategy equilibrium [17, 23, 4, 6, 13]. Instead, we can show that bidding truthfully is an *ex post* Nash equilibrium. The proof technique is similar to the one used in [17]. For the proof to work, we at least need the auction to terminate properly, i.e. every buyer must get an allocation from their demand set (may be untruthfully submitted) and the seller should maximize his payoff from such an allocation. Such termination can be ensured by imposing appropriate bidding rules in the auction (for example, demand sets of buyers should weakly grow, every buyer should demand the "grand" bundle A in the first iteration etc.). We call such restrictions on bidding strategies "consistency" requirements.[3]

[3] These consistency requirements are simple to implement and proxy agents may be used to enforce these rules [23].

Theorem 6 *Truthful bidding is an* ex post *Nash equilibrium in the ascending price Vickrey auction if proper consistency requirements are imposed.*

Proof: Let p be the final price of the auction and X be the final allocation. Due to the consistency requirements, the seller should maximize his payoff at p. Consider a buyer i who is not following truthful bidding but buyers in B_{-i} follow truthful bidding. Thus, buyers in B_{-i} maximize their payoff in allocation X with respect to their true valuations. Buyer i gets X_i at the end of the auction. The proof is based on designing a valuation function for i such that (p, X) becomes a competitive equilibrium with respect to this (possibly untruthful) valuation profile. Consider the valuation function $\hat{v}_i$ defined as: $\hat{v}_i(S) = p_i(S)$ if $X_i \subseteq S$ and $\hat{v}_i(S) = 0$, otherwise. Clearly, (p, X) is a competitive equilibrium with respect to valuation profile $(\hat{v}_i, v_{-i})$, where v_{-i} is the true valuation profile of buyers in the set B_{-i}. This means, in a VCG mechanism with this valuation profile, X_i can be allocated to buyer i. Let $p_i^{vcg}(\hat{v}_i, X_i)$ be the resulting VCG payment. Let π_i be the actual payoff of buyer i from the auction.

$$\begin{aligned}
\pi_i^{vcg}(v_i) &\geq v_i(X_i) - p_i^{vcg}(\hat{v}_i, X_i) \quad \text{(From the revelation principle)} \\
&\geq v_i(X_i) - p_i(X_i) + \pi^s(B) - \pi^s(B_{-i}) \quad \text{(From Lemma 2)} \\
&= \pi_i \quad \text{(From payment rules in Definition 13)}
\end{aligned}$$

From Theorem 5, buyer i will have a payoff of $\pi_i^{vcg}(v_i)$ by following truthful bidding. Thus, truthful bidding is a Nash equilibrium for buyers. ■

4.5 An Example

Consider the example in Table 3 in which the buyers are substitutes condition fails. The progress of the auction for the example in Table 3 is shown in Table 4. The columns corresponding to buyers show prices on bundles. The bundles which have prices in $(\cdot)$ are in the demand set of the respective buyers. The seller's revenue in economy $E(M)$ for $M \in \{B, B_{-1}, B_{-2}, B_{-3}\}$ is shown in every iteration. We follow the price updates in Definition 12 throughout the auction. The auction converges to a UCE price in 8 iterations.

5 Special Case: Buyers are Substitutes

One wonders if it is possible to stop the auction immediately when we achieve a CE price of the original economy. For this, we define the following concept.

Definition 14 (Universal Quasi-CE Price) *A price p is a* ***universal quasi-CE price*** *if it is a CE price of economy $E(B)$ and quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$.*

#	Buyer 1			Buyer 2			Buyer 3			Seller's Revenues
	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\pi^s(\cdot)$
1	(0)	0	(0)	0	(0)	(0)	0	0	(0)	0,0,0,0
	A minimal universal undersupplied set of buyers: $\{1,3\}$									
2	(1)	0	(1)	0	(0)	(0)	0	0	(1)	1,1,1,1
	A minimal universal undersupplied set of buyers: $\{2\}$									
3	(1)	0	(1)	0	(1)	(1)	0	0	(1)	2,1,1,2
	A minimal universal undersupplied set of buyers: $\{3\}$									
4	(1)	0	(1)	0	(1)	(1)	0	(0)	(2)	2,2,2,2
	A minimal universal undersupplied set of buyers: $\{2\}$									
5	(1)	0	(1)	0	(2)	(2)	0	(0)	(2)	3,2,2,3
	A minimal universal undersupplied set of buyers: $\{1\}$									
6	(2)	0	(2)	0	(2)	(2)	0	(0)	(2)	4,2,2,4
	A minimal universal undersupplied set of buyers: $\{2,3\}$									
7	(2)	0	(2)	0	(3)	(3)	0	(1)	(3)	5,3,3,5
	A minimal universal undersupplied set of buyers: $\{2,3\}$									
8	(2)	0	(2)	0	(4)	(4)	(0)	(2)	(4)	6,4,4,6
	Universal overdemand does not hold. A UCE price is found									
	Final allocation: $\{\{1\},\{2\},\emptyset\}$									
	Payment: $p_1(\{1\}) = 2 - [6-4] = 0$, $p_2(\{2\}) = 4 - [6-4] = 2$, $p_3(\emptyset) = 0$									

Table 4: Progress of Auction in Definition 13

Since we can know VCG outcome information from a UCE price (Theorem 1), we can terminate our auction at a Universal Quasi-CE price if it is a UCE price. The following theorem gives a condition under which that is possible.

Theorem 7 *A universal quasi-CE price is a UCE price if buyers are substitutes.*

Proof: Let p be a universal quasi-CE price. Consider a buyer $i \in B$. Since p is a quasi-CE price in economy $E(B_{-i})$, consider $X \in L^*(B_{-i})$ and let $K = \{k \in B_{-i} : X_k \neq \emptyset\}$. This gives us,

$$\pi^s(B_{-i}) = \sum_{k \in K} p_k(X_k) = \sum_{k \in K} \Big[v_k(X_k) - \pi_k \Big] \leq V(K) - \sum_{k \in K} \pi_k. \tag{17}$$

Now,

$$\begin{aligned}
\pi_i + \pi^s(B) - \pi^s(B_{-i}) &\geq \pi_i + \pi^s(B) - V(K) + \sum_{k \in K} \pi_k \text{ (From Equation 17)} \\
&= \pi_i + V(B) - \sum_{k \in B} \pi_k - V(K) + \sum_{k \in K} \pi_k \text{ (Since } p \text{ is a CE)} \\
&= V(B) - V(K) - \sum_{k \in B \setminus (K \cup \{i\})} \pi_k \\
&\geq \sum_{k \in B \setminus K} \pi_k^{vcg} - \sum_{k \in B \setminus (K \cup \{i\})} \pi_k \text{ (Since buyers are substitutes)} \\
&\geq \sum_{k \in B \setminus K} \pi_k^{vcg} - \sum_{k \in B \setminus (K \cup \{i\})} \pi_k^{vcg} \\
&= \pi_i^{vcg}
\end{aligned}$$

The last inequality comes from a result in [7] which states that under buyers are substitutes condition the core payoff (maximum payoff of buyer from CE prices) vectors form a lattice and the unique maximum core payoff is the Vickrey payoff vector.

But from Lemma 2 $\pi_i + \pi^s(B) - \pi^s(B_{-i}) \leq \pi_i^{vcg}$. This means, under buyers are substitutes condition, $\pi_i^{vcg} = \pi_i + \pi^s(B) - \pi^s(B_{-i})$. This is true for every buyer $i \in B$. From Proposition 3, p is a UCE price. ■

We now define an auction for this special case of buyers are substitutes. The auction is modified from the auction in [13] to incorporate discounts from the final price, but is otherwise identical. Discounts allow the modified auction to implement the VCG outcome when buyers are substitutes, where the original auction in de Vries et al. [13] only implements the VCG outcome when buyer are *submodular*. The modified auction can also be used as a starting point for our main auction, which helps to make the connection between the work of de Vries et al. [13] and our work clear.

We interpret the auction of de Vries et al. [13] in our terms as follows. They define a concept called "overdemand", which means existence of a quasi-CE price for the main economy $E(B)$. They adjust prices by balancing supply and demand in the main economy only. At every iteration they identify a "minimal undersupplied set" of buyers (defined similar to minimal universal undersupplied set, but only considering constraints from economy $E(B)$ in formulation (**RP**)). We have the following modified auction, which need not explicitly consider the marginal economies during the price adjustment.

Definition 15 (Primal-Dual Auction for Main Economy) *A* ***primal-dual auction for the main economy*** *is an iterative auction with the following steps:*

S0 Initialize prices to zero.

S1 Collect demand sets of buyers at the current price.

S2 If overdemand holds, perform price adjustment as below:

- *Identify a minimal undersupplied set of buyers K.*
- *For every buyer $i \in K$ and $S \in D_i$, increase $p_i(S)$ by unity. Do not change prices of other bundles.*

Go back to S1.

S3 If overdemand does not hold, the auction ends. Final allocation is any $X \in L^*(B)$ from the final iteration and every buyer $i \in B$ pays an amount equal to $p_i(X_i) - \Big[\pi^s(B) - \pi^s(B_{-i})\Big]$, where p is the final price.

Consider the following submodularity condition on buyers.

Definition 16 (Buyers are Submodular) *We say that* ***buyers are submodular*** *if for every $M \subseteq K \subseteq B$ and for every $i \in B \setminus K$, we have*

$$V(K \cup \{i\}) - V(K) \leq V(M \cup \{i\}) - V(M). \tag{18}$$

If buyers are submodular, then [13] show that the discounts in the auction in Definition 15 is zero. The difference between the auction in [13] and the auction in Definition 15 is the payment scheme. The payment scheme in Definition 15 allows us to implement VCG outcome for a richer class of valuation profiles as shown in the following theorem.

Theorem 8 *The following claims hold for the auction in Definition 15:*

(i) At every iteration of the auction, the auction price is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$.

(ii) If buyers are substitutes, the auction terminates at a UCE price, and the auction implements the VCG outcome (after discounts).

(iii) If buyers are submodular, then the final CE prices implement the VCG payments and no discount is necessary.

Proof: (i) The starting price is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$. Prices are integral throughout the auction. Since prices of demanded bundles increase by a maximum of unity, the demand sets of buyers weakly grow from first iteration. Since prices of non-demanded bundle does not increase, the bundles whose prices are positive are in the demand set of the respective buyer in every iteration. For quasi-CE price to exist, we need to show that $L^*(M)$ is non-empty in every iteration for every $M \in \mathbb{B}$. Consider any $M \in \mathbb{B}$ and let $X \in L(M)$ (clearly, $L(M)$ cannot be empty). Let $Q = \{i \in M : X_i \in D_i\}$. From the argument given before, $p_i(X_i) = 0 \ \forall \ i \in M \setminus Q$. This means $\sum_{i \in M} p_i(X_i) = \sum_{i \in Q} p_i(X_i)$. Consider an allocation Y such that $Y_i = X_i$ if $i \in Q$ and $Y_i = \emptyset$ otherwise. $\sum_{i \in M} p_i(Y_i) =$

$\sum_{i \in Q} p_i(X_i) = \sum_{i \in M} p_i(X_i)$. This means, $Y \in L^*(M)$. So, $L^*(M)$ is non-empty and price in every iteration is a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$.

(ii) The auction terminates at a CE price from [13]. From the definition of universal quasi-CE price and (i), the auction terminates at a universal quasi-CE price. From Theorem 7, under buyers are substitutes condition, a universal quasi-CE price is a UCE price. From the payment rule in the auction and Theorem 1, the auction implements a VCG outcome.

(iii) Follows directly from Theorem 5 in [13]. ■

Theorem 8 tells us that we can design an auction implementing VCG outcome based on primal-dual algorithm of the original economy if buyers are substitutes condition holds. Such an auction always converges to a universal quasi-CE price which is a UCE price if buyers are substitutes.

Consider the following example from [4]: There are 4 items and 5 buyers, each "interested" in only one bundle. Buyer 1 is interested in bundle $\{1, 2\}$ with value 10. This means, $v_1(S) = 10$ if $\{1, 2\} \subseteq S, v_1(S) = 0$ otherwise. Similarly, buyer 2 is interested in bundle $\{3, 4\}$ with value 20, buyer 3 is interested in bundle $\{3, 4\}$ with value 25, buyer 4 is interested in bundle $\{2, 4\}$ with value 10 and buyer 5 is interested in bundle $\{1, 3\}$ with value 10. Verify that the VCG payment of buyer 1 is 0. Also buyers are substitutes but not submodular. Applying the auction in Definition 15, we can choose $\{1, 4\}$ as a minimal undersupplied set in the first iteration of the auction. This raises final price of buyer 1 above zero. Thus, it will not converge to a UCE price where its discount is zero. But it can be verified that the auction always converges to a UCE price (may be with positive discounts).

6 Discussion

6.1 Variations on the Price Adjustment Rules

The final price in the auction defined to implement a primal-dual algorithm for the main economy (Definition 15) is a universal quasi-CE price. As such, this auction can be *staged* with the ascending-price Vickrey auction to provide an auction that works for general valuations. In particular, we can combine the auctions as follows:

- Run the main-economy primal-dual auction (Definition 15).
- If a UCE price is found, STOP with the allocation and payment defined in Definition 15. Else, use the final price in this auction to start the ascending-price Vickrey auction.
- Run the ascending-price Vickrey auction.

We can consider the progress of this modified auction in Table 5 for the example in Table 3. The columns corresponding to buyers show prices on bundles. The bundles which have prices in $(\cdot)$ are in the demand set of the respective buyers. The seller's revenue in economy

#	Buyer 1			Buyer 2			Buyer 3			Seller's Revenues
	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\pi^s(\cdot)$
1	(0)	0	(0)	0	(0)	(0)	0	0	(0)	0,0,0,0
	A minimal undersupplied set of buyers: $\{1,3\}$									
2	(1)	0	(1)	0	(0)	(0)	0	0	(1)	1,1,1,1
	A minimal undersupplied set of buyers: $\{2,3\}$									
3	(1)	0	(1)	0	(1)	(1)	0	(0)	(2)	2,2,2,2
	A minimal undersupplied set of buyers: $\{1,3\}$									
4	(2)	0	(2)	0	(1)	(1)	0	(1)	(3)	3,3,3,3
	A minimal undersupplied set of buyers: $\{2,3\}$									
5	(2)	0	(2)	0	(2)	(2)	(0)	(2)	(4)	4,4,4,4
	CE of $E(B)$ is reached. $\{\{1\},\{2\},\emptyset\}$ is an efficient allocation									
	A minimal universal undersupplied set of buyers: $\{2\}$									
6	(2)	0	(2)	0	(3)	(3)	(0)	(2)	(4)	5,4,4,5
	A minimal universal undersupplied set of buyers: $\{2\}$									
7	(2)	0	(2)	0	(4)	(4)	(0)	(2)	(4)	6,4,4,6
	Universal overdemand does not hold. A UCE price is found									
	Final allocation: $\{\{1\},\{2\},\emptyset\}$									
	Payment: $p_1(\{1\}) = 2 - [6-4] = 0$, $p_2(\{2\}) = 4 - [6-4] = 2$, $p_3(\emptyset) = 0$									

Table 5: Progress of auction in Definition 13 using the auction in Definition 15 as a starting point

$E(M)$ for $M \in \{B, B_{-1}, B_{-2}, B_{-3}\}$ is shown in every iteration. Initially, we follow the price updates in [13], which gives us a starting point for our auction at the end of iteration 5. Finally, the auction converges to a UCE price.

More generally, this suggests a *sequential* approach to search for a UCE price. Starting from a quasi-CE price of economy $E(M)$ for every $M \in \mathbb{B}$, we can select a "pivot" economy $E(M)$ in which overdemand holds. From the pivot economy, select a minimal undersupplied set of buyers (as defined in [13]) and increase the price of those buyers on demanded bundles by unity. Using arguments similar to Theorem 8, it can be shown that the suggested price adjustment maintains quasi-CE price in economy $E(M)$ for every $M \in \mathbb{B}$. Once overdemand ceases to hold in a pivot economy after repeated price adjustments, the pivot economy is changed and the process repeats. The auction stops when overdemand does not hold in economy $E(M)$ for every $M \in \mathbb{B}$ and thus giving a UCE price. As a special case, this sequential auction can be started by first adjusting prices of minimally undersupplied buyers exclusively on the basis of the main economy, before switching to the marginal economies. The initial stage of this auction is equivalent to that of the staged auction, and in turn to the auction in de Vries et al. [13].

6.2 Subgradient Approaches

The price adjustment procedures in primal-dual algorithm based auctions seem computationally intensive. There has been another stream of ascending price auctions [5, 22, 4] which have computationally less intensive price adjustment step. de Vries et al. [13] interpret these auctions as a subgradient algorithm of the underlying optimization problem.

To understand a subgradient algorithm, consider the formulation (**P**(M)) for some $M \in \mathbb{B}$. In this formulation, all constraints except constraints 1 can be satisfied trivially. Constraint 1 is difficult to satisfy and the violated constraints corresponding to (buyer,bundle) tuples are subgradients (dual variable update direction) [4]. By choosing a quasi-CE price as a starting price (dual variable), we can get the same price adjustment direction as in [22]. These price adjustments can be applied sequentially to every economy in $M \in \mathbb{B}$ to discover a UCE price.

An analysis parallel to what has been done using primal-dual algorithm ideas can be done using subgradient algorithm ideas. In particular, it can be shown that the auction in Parkes [22] converges to a UCE price if buyers are substitutes. If buyers are not substitutes, we can terminate at a UCE price by considering feasible regions of the main economy as well as marginal economies and finding appropriate subgradients for price adjustment. These results appear in Parkes and Ungar [24] but they do not explicitly use subgradient ideas in their analysis.

7 Summary and Open Questions

In this research, we showed that ascending price auctions implementing VCG outcome can be designed for any valuation profile using a concept called universal competitive equilibrium price. UCE price along with an efficient allocation gives enough "information" to compute VCG payments. We also show that any CE price that can compute VCG payments from an efficient allocation is a UCE price. UCE price requires supply and demand to be balanced in main economy and in every marginal economy. At a lower price demand dominates supply in every economy. We adjust prices by considering a "grand" formulation which takes the intersection of feasible regions of restricted primals for every economy. Our auction always converges to a UCE price. Relating our work to the work of [13], we showed that their auction converges to a UCE price if buyers are substitutes. If buyers are not substitutes, their auction gives a (good) starting point for our auction.

The central idea of our work is the UCE price concept, which requires the computation of VCG payments using discounts to buyers. Traditional auctions may look different from our auction, but they can be viewed as a special case of our auction.

[4] This can be verified by considering the Lagrangian dual problem where constraints 1 are lifted to the objective function. The derivative of this objective function is a subgradient and these are non-zero where the constraints are violated.

7.1 Open Questions

There are still many open questions left. One wonders under what valuation profiles auctions with *linear* and non-anonymous prices, or with non-linear and *anonymous* prices, can implement the VCG outcome. It also seems that when buyers are not substitutes, it is possible that there are instances in which our auction terminates with trivial UCE prices where the price equals value for all buyers (and with full information revelation). Are there conditions (on the number of bundles and buyers) under which such a UCE price is the *only* UCE price? We have also designed *descending* price auctions using UCE price ideas, that implement the VCG outcome for special cases of the combinatorial allocation problem [21, 20]. We would like to understand the information revelation advantages of ascending price versus descending price auctions better.

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