# A bridge between Discontinuous Galerkin and Residual distibution schemes : preliminary report

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**Abstract**

The most recent developments on numerical schemes for hyperbolic problems deal with two classes of schemes : the now very popular Discontinuous Galerkin schemes (DG for short) [1, 2, 3, 4, 5, 6] and the less mature Residual Distribution schemes (RD for short)[7, 8, 9, 10, 11, 12]. Each have its advantages and drawbacks. For the DG schemes, the situation is quite clean when dealing with smooth problems, and $h-p$ refinement is theoreticaly relatively easy, as well as the handling of non conformal meshes. In the RD schemes, the non linear stabilisation for discontinuous solution is quite mature, but the handling of non conformal meshes does not seems to be an easy task. In the present paper, we developp a technique where we can use the non linear stabilisation mechanism within the DG framework, therefore curing one of its major drawbacks. The main advantage is that no ad–hoc tuning parameters are any more necessary, and the formal accuracy is retained even very close to discontinuities. The analysis is carried out for scalar linear and nonlinear problems but should be extended without problems to the system case following the method of [9].

We consider the scalar Cauchy problem

$$\begin{array}{ll} \dfrac{\partial u}{\partial t} + \mathrm{div}\mathbf{f}(u) = 0 & x \in \Omega \\ u(x, t=0) = u-0 & t=0, x \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega, t>0 \end{array} \tag{1}$$

and its steady version,

$$\begin{array}{ll} \mathrm{div}\mathbf{f}(u) = 0 & x \in \Omega \\ u(x,t) = g(x,t) & x \in \partial\Omega^-, t>0 \end{array} \tag{2}$$

Currently, a very popular class of methods to integrate (1) is the Discontinuous Galerkin method. It uses a finite element representation of the solution within each element of a triangulation of $\Omega$, and the approximation function is discontinuous accross the edges (or faces in 3D) of the mesh elements. The second ingredient is a weak formulation of (1) combined with a flux formulation for the contour integral. The method can be shown to be intrinsicaly stable for a very large clas of time discretisation. When handling discontinuous solutions, an additional stabilisation is introduced, in order to mimic the non oscillatory behaviour of the exact solution, all this in the spirit of the TVD schemes by A. Harten. The net effect of this is that in most cases the formal accuracy of the scheme is distroyed around the discontinuities of the solution (this is no surprise), but also around the extrema of the solution which is more annoying. Moreover, the width of the discontinuities is also affected by this. Of course, this picture can be improved [13, 14], but the optimal choice of limiters is by far not known up to now.

On the other hand, another class of scheme exists, the so–called Residual Distribution schemes. These schemes bear many similarities with the stabilised finite elements schemes such as the SUPG scheme [15, 16], but the shock capturing method is completely different. It is inspired from the MUSCL and TVD type of schemes.

In this paper, we reformulate the DG schemes so that they can be seen as RD schemes, and this enables to propose a new class of non linear stabilisation that never destrois the formal accuracy of the schemes. This opens an avenue toward non linear schemes with $h-p$ refinement capabilities having a parameter free non oscillatory behavior.

The paper is organised as follows. First we recall the standard DG schemes, then show how the RD technique can be introduces. In order to improve the stability, a blending between this RD-DG scheme and the standard DG scheme is introduced. This is possible thanks to the RD formulation of the standard DG scheme. Then numerical results are presented and a conclusion follows. In this paper, we illustrate the technique by a second order accurate scheme, the more general case will be considered elsewhere.

Throughout the paper, we consider a triangulation $\mathcal{T}$ which elements are triangles for now. A triangulation is $\{T_l\}_{l=1,n_t}$. We denote the vertices by $\{M_i\}_{i=1,n_s}$. The interpolation space is the broken space

$$V^h = \bigoplus_{i=1}^{n_t} \mathbb{P}^1(T_i).$$

## 1 Choice of the basis functions

Consider for now a single triangle $T$. There are several more or less natural basis that generates $\mathbb{P}^1(T)$. If $\{A_j\}_{j=1,3}$ denotes the set of vertices of $T$, the most natural one is the set of barycentric coordinates denoted by $\{\Lambda_{A_j}\}_{j=1,3}$. They verify $\Lambda_{A_j}(A_k) = \delta_j^k$. The main problem of this basis is that (i) since the elements of $V^h$ are not continuous, it is not that natural to use the vertices of $\mathcal{T}$ as degrees of freedom, (ii) these functions are not $L^2$–orthogonal. A classical choice in the DG litterature are the Dubliner polynomials, but they are not associated to any particular degree of freedom.

It is possible to construct a set of three polynomials of degree one such that there exists 3 vertices $a_1$, $a_2$, $a_3$ in $T$ such that the Lagrange interpolant associated to them, denoted by $\varphi_1$, $\varphi_2$ and $\varphi_3$, are orthogonal, i.e.

$$\int_T \varphi_i(x)\varphi_j(x)dx = \delta_i^j \int_T \varphi_i(x)^2 dx$$

Denoting by

$$\omega_i = \frac{\displaystyle\int_T \varphi_i(x)^2 dx}{|T|}$$

we have the quadrature formula

$$\int_T f(x)dx = |T|\Big(\sum_{j=1}^{2} \omega_j f(a_j)\Big) \tag{3}$$

The quadrature formula is second order accurate.

After calculations, the points $a_j$ are the midpoints of $[G, M_j]$ where $G$ is the centroid of $T$. By symmetry, $\omega_j = \frac{1}{3}$.

## 2 Construction

Using the weak form of (1), we get, for any $\varphi \in \mathbb{P}^1(T)$,

$$\int_T \varphi(x)\frac{\partial u}{\partial t}dx - \int_T \nabla\varphi \cdot \mathbf{f}(u)dx + \int_{\partial K} \varphi(x)\mathbf{f}(u)\cdot \vec{n}d\sigma \tag{4}$$

Using the DG method, and considering $\mathcal{F}$ a numerical flux, we get

$$\int_T \varphi(x)\frac{\partial u}{\partial t}dx - \int_T \nabla\varphi \cdot \mathbf{f}(u)dx + \int_{\partial K} \varphi(x)\mathcal{F}(u^+, u^-, \vec{n}))\cdot \vec{n}d\sigma. \tag{5}$$

Here, the integral over $\partial K$ is a sum over three edges. For any edge, see Figure 1, $T^+ =$ and $T^-$ is the element on the other side of $\Gamma$. Accordingly, $u^+ = u_{|T}$ and $u^- = u_{|T^-}$. We choose $\varphi = \varphi_i$ and

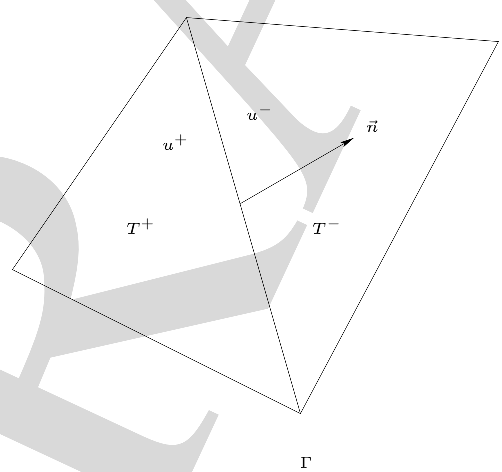


Figure 1:

get

$$|T|\omega_i \frac{du(a_i)}{dt} - \int_T \nabla\varphi\cdot\mathbf{f}(u)dx + \int_{\partial K}\varphi(x)\mathcal{F}(u^+, u^-, \vec{n})\cdot\vec{n}d\sigma \tag{6}$$

The equation (6) can be rewritten as

$$|T|\omega_i \frac{du(a_i)}{dt} + \Phi_i^{1,T} + \Phi_i^{2,T} + \Phi_i^{3,T} = 0$$

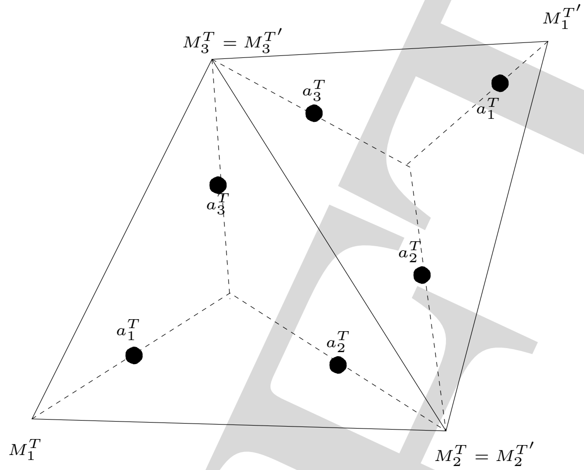


Figure 2:

where, following the notations of Figure 2

$$\Phi_i^{\ell,T} = -\int_{T_\ell} \nabla\varphi_i \cdot \mathbf{f}(u)dx + \int_{\partial T_\ell} \varphi_i(x)\mathcal{F}(u^+, u^-\vec{n})d\sigma$$

The residuals $\Phi_i^{\ell,T}$ are associated to the edge $\Gamma_\ell$. We have (using the consistency of the flux $\mathcal{F}$ and obvious notations)

$$\sum_{a_i^T} \Phi_i^{\ell,T} = \int_{\partial T_\ell} \mathcal{F}(u^+, u^-, \vec{n})d\sigma$$

because $\sum_{a_i} \varphi_i = 1$. Then, if one consider the opposite triangle (Figure 2), we have

$$\sum_{a_i^T} \Phi_i^{\ell,T} + \sum_{a_i^{T'}} \Phi_i^{\ell,T'} = \int_{\partial\left(T_\ell \cup T'_\ell\right)} \mathbf{f}(u) \cdot \vec{n} d\sigma =: \Phi_{\Gamma_\ell}$$

which has the right scaling $O(h^3)$ for a steady smooth solution. We have $3+3$ such sub–residuals.

Now we look at the same problem seen from the opposite way : define, for any edge $\Gamma_\ell$, a set of monotone residuals that sum up to $\Phi_{\Gamma_\ell}$. An example is the following adaptation of the Lax Friedrichs residual : take $b_i \in \{a_i^T, a_i^{T'}\}_{i=1,\cdot 3}$ and define

$$\Phi_{b_i}^{\Gamma_l} = \frac{1}{6}\left(\Phi_{\Gamma_j} + \alpha \sum_{b_k \neq b_i} (u(b_i) - u(b_j))\right)$$

for $\alpha$ large enough : this is a monotone scheme.

If we think in terms of flux, we might “define”

$$\int_{\Gamma_\ell} \mathcal{F} \cdot \vec{n} d\sigma = \Phi_{\Gamma_\ell} + \int_{T_\ell} \nabla\varphi_i \cdot \mathbf{f}(u)dx$$

The next step is to construct a high order monotonicity preserving scheme. We consider

$$x_{b_i}^{\Gamma_\ell} = \frac{\Phi_{b_i}}{\Phi_{\Gamma_\ell}}$$

and define

$$\beta_{b_i}^{\Gamma_\ell} = \frac{(x_{b_i})^+}{\sum\limits_{b_j} (x_{b_j})^+}$$

and

$$\Phi_{b_i}^{\Gamma_\ell, *} = \beta_{b_i}^{\Gamma_\ell} \Phi_{\Gamma_\ell}.$$

The scheme writes

$$|T| \omega_i \frac{du(a_i)}{dt} + \sum_{\Gamma_\ell \text{ edge of } T} \beta_{a_i}^{\Gamma_\ell} \Phi_{\Gamma_\ell}$$

Now, we consider one edge, say $\Gamma_i$. We have

$$\begin{aligned}
\Phi_{\Gamma_k}^T &:= \sum_{i=1}^3 \Phi_i^k \\
&= \Big( \sum_{l=1}^3 \omega_l \theta_l \big( \mathbf{f}(a_l) - \mathbf{f}(a_k) \big) \Big) \cdot \vec{n}_k \\
&\quad - \sum_{i=1}^3 \int_{\Gamma_k} \Big( \mathcal{F}(u^+, u^-, \vec{n}_k) - \mathbf{f}(a_i) \cdot \vec{n}_k \Big) \varphi_i d\sigma \\
&= \Big( \sum_{l=1}^3 \omega_l \theta_l \big( \mathbf{f}(a_l) - \mathbf{f}(a_k) \big) \Big) \cdot \vec{n}_k \\
&\quad - \int_{\Gamma_k} \Big( \mathcal{F}(u^+, u^-, \vec{n}_k - \mathbf{f}(u) \cdot \vec{n}_k \Big) d\sigma
\end{aligned} \tag{7}$$

Then, we have the residual

$$\begin{aligned}
\Phi_{\Gamma_k} &:= \Phi_{\Gamma_k}^T \Phi_{\Gamma_k}^{T'} \\
&= \Big( \sum_{l=1}^3 \omega_l^T \theta_l^T \big( \mathbf{f}(a_l^T) - \mathbf{f}(a_k^T) \big) - \sum_{l=1}^3 \omega_l^{T'} \theta_l^{T'} \big( \mathbf{f}(a_l^{T'}) - \mathbf{f}(a_k^{T'}) \big) \Big) \cdot \vec{n}_k \\
&\quad - \int_{\Gamma_k} [\![ \mathbf{f}(u) ]\!] \cdot \vec{n_k}
\end{aligned} \tag{8}$$

which enable to use the RDS technique.

The rest is quite standard. Given an edge, we have the 6 resisuals $\Phi_i^k$, and we define

$$x_i^k = \frac{\Phi_i^k}{\Phi_{\Gamma_k}}.$$

Clearly,

$$\sum_i x_i^k = 1$$

so that we can define (because $\sum_i (x_i^k)^+ = 1 - \sum_i (x_i^k)^- \geq 1$

$$\beta_i^k = \frac{(x_i^k)^+}{\sum_j (x_j^k)^+}$$

and then

$$(\Phi_i^k)^\star = \beta_i^k \Phi_{\Gamma_k}. \tag{9}$$

The resulting scheme, for the degree of freedom $\sigma \in T$ is then

$$\sum_{\Gamma,\text{edge of } T} (\Phi_i^k)^\star = 0. \tag{10}$$

As for standard residual scheme for conformal meshes, the scheme may suffer from iterative convergence problems. Hence, we modify $(\Phi_i^k)^\star$ as

$$(\Phi_i^k)^{\star\star} = \ell (\Phi_i^k)^\star + (1-\ell)(\Phi_i^k)^{\text{DG}} \tag{11}$$

where $\ell \simeq=$ in discontinuities and $\ell \simeq 1$ in smooth regions. In the simulations, we have made three choices

- $\ell = 1$ no blending,
- $\ell = 0$ pure DG,
- $$\ell = \frac{\left|\left(\overline{u}_+^k\right)^n - \left(\overline{u}_-^k\right)^n\right|}{\left|\left(\overline{u}_+^k\right)^n\right| + \left|\left(\overline{u}_-^k\right)^n\right|}$$

  with $n = 10$. The results are quite unsesitive to this parameter, provided it is large enough ($\geq 5$).

The scheme is then (10) with (9) replaced by (11). This is solved by an explicite forward Euler scheme.

# 3 Numerical results

We have tested the scheme on two examples, a convection problem and the Burgers equation. No particular care has been taken for the outflow boundary conditions.

## 3.1 Convection problem.

The problem is to solve

$$\begin{aligned} &-x\frac{\partial u}{\partial y} + y\frac{\partial (}{\partial u})x = 0 && (x,y) \in [0,1]\times[0,1], \\ &u(x,0) = \begin{cases} -\sin\left(\pi \dfrac{2x-0.2}{0.6}\right) & \text{if } 0.1 \leq x \leq 0.4 \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{12}$$

The solutions obtained by the different schemes are displayed in Figure 3. A cross–section (at $x = 0$) is displayed in Figure 4. We see that the resluts obtained for the DG and blended schemes are much more accurate than those of the RD scheme, and the blended one is non oscillatory.

### 3.2 Burgers problem.

Here, we have to solve

$$\begin{aligned} &\tfrac{1}{2}\frac{\partial u^2}{\partial x} + \frac{\partial u}{\partial y} = 0 && (x, y) \in ]0, 1[^2 \\ &u(x, y) = \begin{cases} 1.5 - 2x & x \in [0, 1], y = 0 \\ 1.5 & x = 0, y \in [0, 1] \\ 0.5 & x = 1, y \in [0, 1] \end{cases} \end{aligned} \tag{13}$$

Solutions are displayed in Figure 5 and zooms that indicate the width of the shock structure are shown in Figure 6.

We also show cuts in the fan, see Figure 7 where the oscillation free nature of the RD and blended schemes can be seen. A close inspection of the same figure shows that the DG and blended solutions are undistiguishable in the fan ($y = 0.1$).

## 4 Conclusions

In this paper, we have presented very preliminary results on a new scheme that is able to combine the advantages of both the DG schemes (compact stencil, $h - p$ refinement capabilities, accuracy) and those of the RD ones (compact stencil, accuracy, parameter free oscillation control around discontinuities). The results are presented for a formal second order schemes, and there should be no problem to extend the machinery to more than second order accuracy, as well as unsteady problems following [9, 17, 18].

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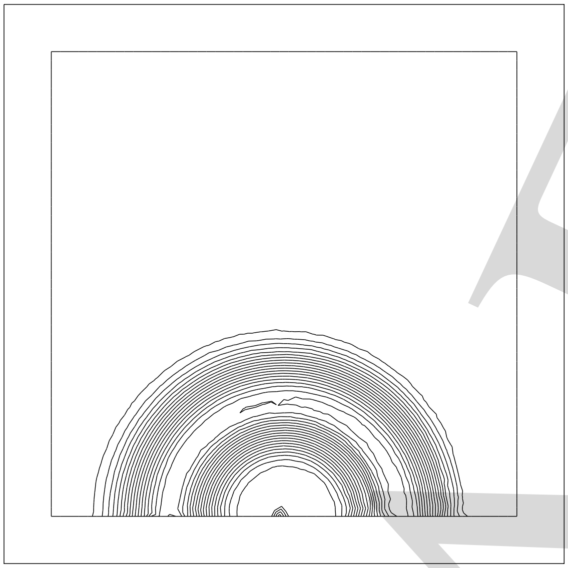
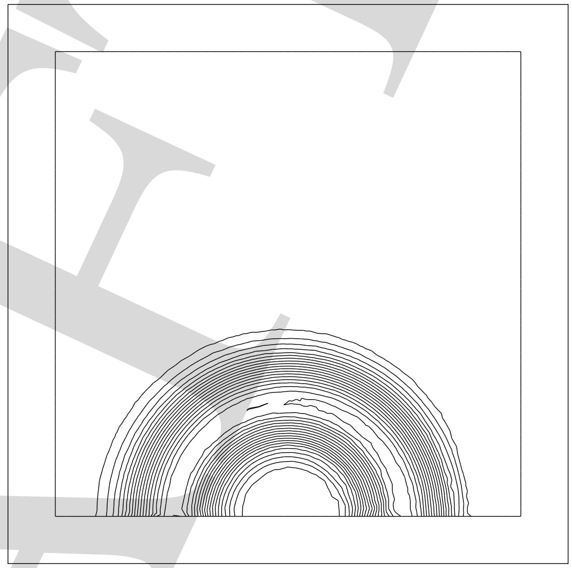
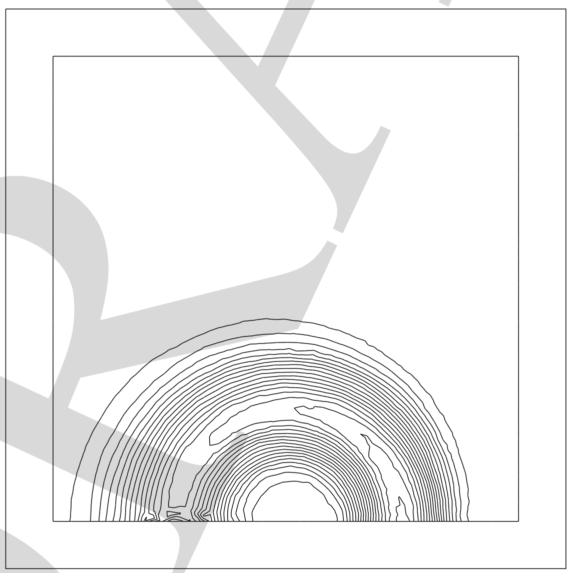


DG



Blend ($n = 10$)



scheme (9)

Figure 3: Solutions for (12).

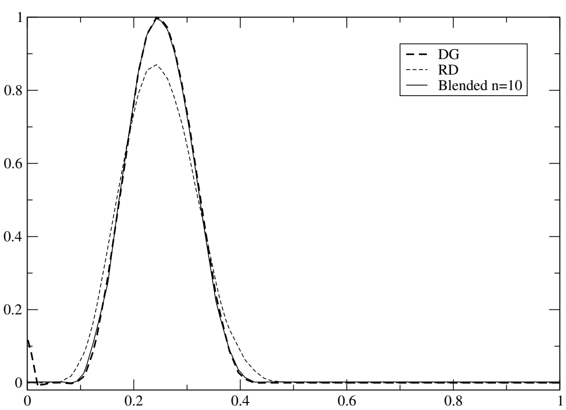
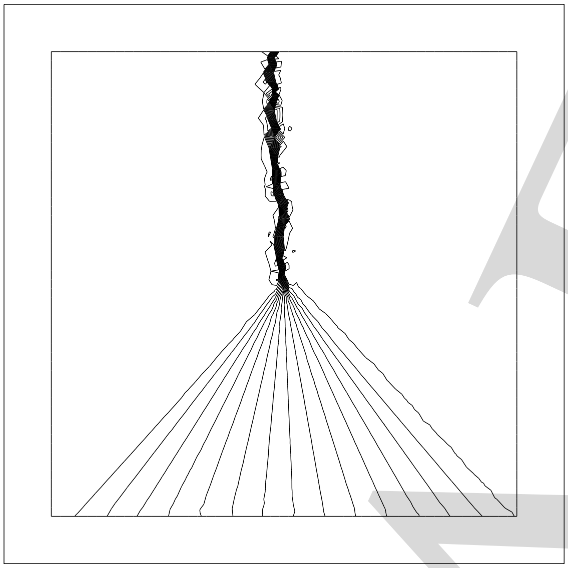
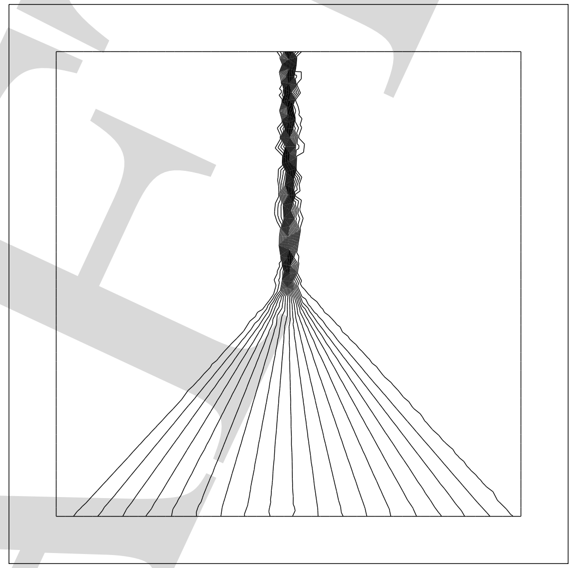


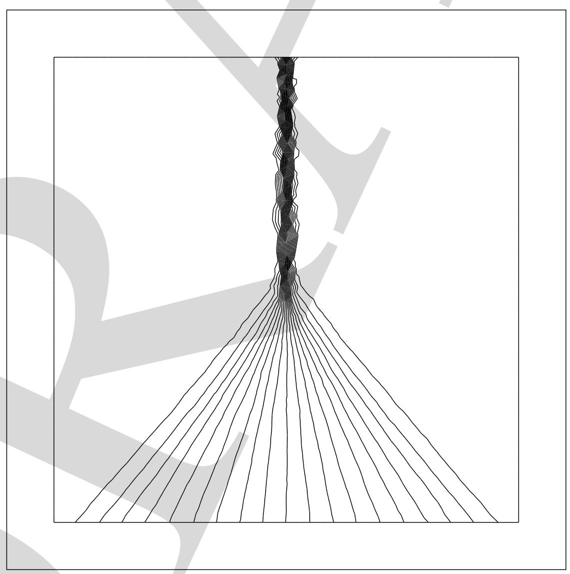

Figure 4: Cross–section of the solutions at $x = 0$.



DG

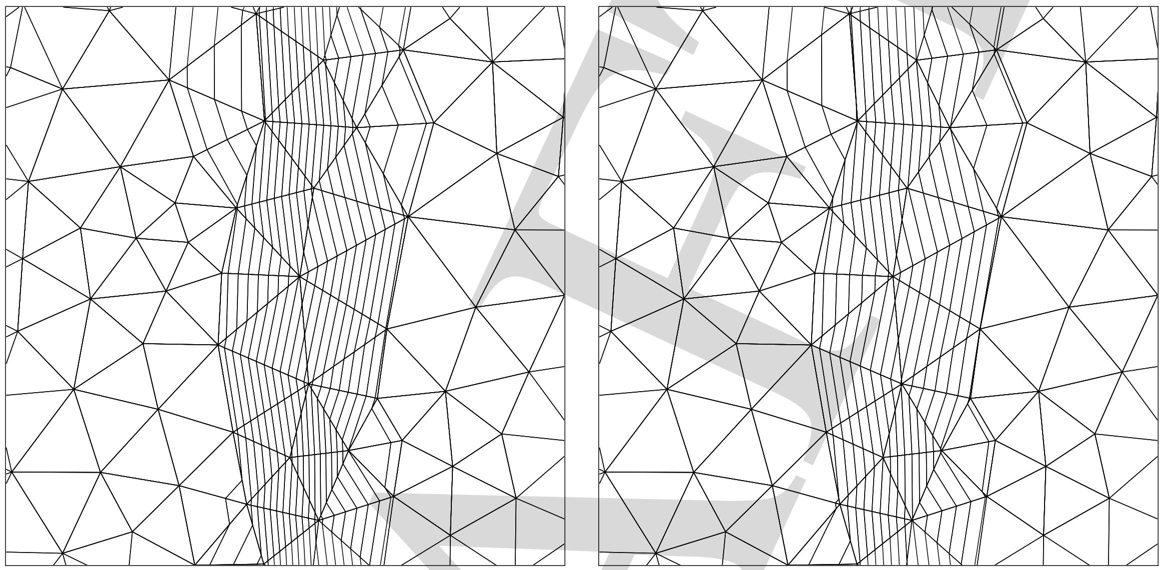


Blend ($n = 10$)

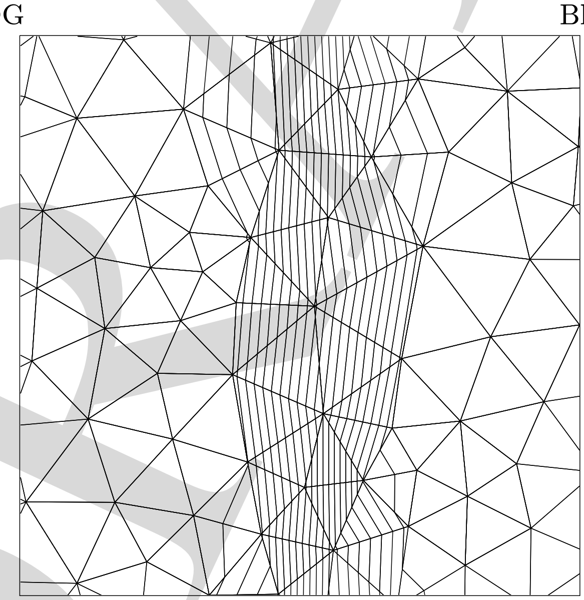


scheme (9)

Figure 5: Solutions for (13).



DG Blend



scheme (9)

Figure 6: Zoom for (13).

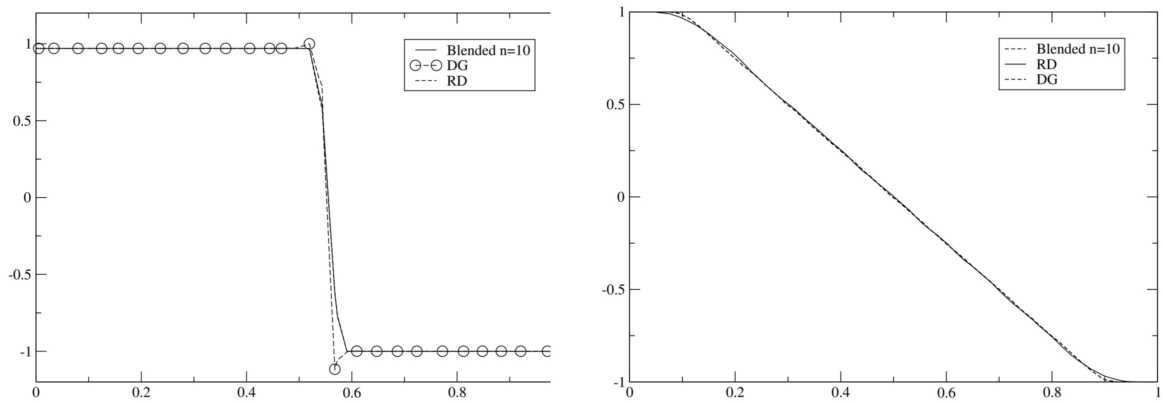



Figure 7: Cuts accross the shock and the fan