FUZZY MULTIVALUED FUNCTIONS

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CONTENTS

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1. INTRODUCTION

A multivalued function is a set valued function. In last thirty years the theory of multivalued functions has advanced in a variety of ways. The theory of multivalued functions was first codified by Berge [ber]. Applications of this theory can be found in economic theory, viability theory, noncooperative games, artificial intelligence, medicine and existence of solutions for differential inclusions (see Castaing and Valadier [cas], Klein and Thompson [kle], Aubin and Frankowska [aub2], Aliprantis and Border [ali], Aubin and Ekeland [aub1], Cross [cro] and references therein). Arens [are], Nikodem [nik], Smajdor [sma] and Stromberg [str] have initiated the study of calculus of multivalued functions/relations and obtained many beautiful significant results in this area. The theory of fuzzy sets provides a framework for mathematical modelling

of those real world situations which involve an element of uncertainty / imprecision / vagueness in their description. Since its inception forty years ago by Zadeh [zad1], this theory has found wide applications in knowledge engineering, economics, information sciences, medicine, etc., for details we refer to Dubois and Prade [dub], Pedrycz and Gomide [ped], Li and Yen [li] and Zimmermann [zim]. Recently Heilpern [hei], Butnariu [but], Albrycht and Maltoka [alb], Papageorgiou [pap], Tsiporkova-Hristoskova, De Baets and Kerre[tsi1-4] and Beg [beg1-12] have started the study of fuzzy multivalued functions and investigated different fundamental properties. In this review article we present recently obtained results regarding fuzzy multivalued functions by the author [beg1-13]. In section 2, we define upper and lower inverse of a fuzzy multivalued function and then use these ideas to introduce and study the concept of hemicontinuity for fuzzy multivalued functions. The aim of section 3, is to study certain properties of fuzzy closed valued, fuzzy compact valued and fuzzy closed graph fuzzy multivalued functions defined on a fuzzy topological space. Fuzzy closed graph theorem for fuzzy multivalued functions is also obtained and an application to fixed point is shown.Section 4 is devoted to study basic operations on fuzzy multivalued functions defined between fuzzy topological spaces . Many interesting relations between basic operations and upper and lower hemicontinuity are obtained. As application we proved fuzzy maximal principle for fuzzy multivalued functions. In section 5, some of the properties of vector valued fuzzy multivalued functions are studied. The notion of sum fuzzy multivalued function, convex hull fuzzy multivalued function, closed convex hull fuzzy multivalued function and upper demicontinuous are given and some of the properties of these fuzzy multivalued functions are investigated.The purpose of section 6 is to define intersection convolution of fuzzy multivalued functions defined between groups and study their properties. As application we obtained extension of linear selector fuzzy multivalued func-

tion. In section 7, we proved the existence of selectors for fuzzy multivalued functions defined on fuzzy ordered sets under suitable conditions. A fixed point theorem for fuzzy order preserving fuzzy multivalued functions is also obtained. In Section 8, we introduced the notion of linear fuzzy multivalued operators and studied the algebra of linear fuzzy multivalued operators. We also obtained the laws governing addition and scalar multiplication of linear fuzzy multivalued operators combined with the operations of composition and inversion. In section 9, existence of fixed points of fuzzy multivalued mappings with values in fuzzy ordered sets under suitable
conditions is studied.

2. CONTINUITY OF FUZZY MULTIVALUED FUNCTIONS

The biggest difference between fuzzy functions and fuzzy multivalued functions has to do with the definition of an inverse image. For a fuzzy multivalued function there are two type of inverses. These two definitions of the inverse then leads to two definitions of continuity. In this section our purpose is two-fold. First, we define upper and lower inverse of a fuzzy multivalued function and study various properties of these. Next, we use these ideas to define the upper hemicontinuous and lower hemicontinuous fuzzy multivalued functions.

Let X be an arbitrary (nonempty) set. A fuzzy set (in X) is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, the function value $A(x)$ is called the *grade of membership* of x in A. The fuzzy set A^c defined by $A^c(x) = 1 - A(x)$ is called the *complement* of A. Let A and B be fuzzy sets in X. We write $A \subseteq B$ if $A(x) \leq B(x)$ for each $x \in X$. For any family $\{A_i\}_{i \in I}$ of fuzzy sets in X, we define

$$[\bigcap_{i \in I} A_i](x) = \inf_{i \in I} A_i(x)$$

and

$$[\underset{i\in I}{\cup} A_i](x) = \sup_{i\in I} A_i(x).$$

A family τ of fuzzy sets in X is called a *fuzzy topology* for X (and the pair (X,τ) a fuzzy topological space if : (i) $\chi_X \in \tau$ and $\chi_\phi \in \tau$; (ii) $\underset{i\in I}{\cup} A_i \in \tau$ whenever each $A_i \in \tau (i \in I)$; and (iii) $A \cap B \in \tau$ whenever $A, B \in \tau$. The elements of τ are called *open* and their complements *closed*. For details we refer to [cha , zim].

Definition 2.1. A *fuzzy multivalued function* f from a set X into a set Y assigns to each x in X a fuzzy subset $f(x)$ of Y. We denote it by $f : X \rightsquigarrow Y$. We can identify f with a fuzzy subset G_f of $X \times Y$ and

$$f(x)(y) = G_f(x,y).$$

If A is a fuzzy subset of X, then the fuzzy set $f(A)$ in Y is defined by

$$f(A)(y) = \sup_{x\in X} \left[G_f(x,y) \wedge A(x)\right].$$

The graph G_f of f is the fuzzy subset of $X \times Y$ associated with f,

$$G_f = \left\{(x,y) \in X \times Y : [f(x)](y) \neq 0\right\}.$$

Definition 2.2. The upper inverse f^u of a fuzzy multivalued function $f : X \rightsquigarrow Y$, in defined by

$$f^u(A)(x) = \inf_{y\in Y} \left[(1 - G_f(x,y)) \vee A(y)\right]$$

Definition 2.3. The lower inverse f^ℓ of a fuzzy multivalued function $f : X \rightsquigarrow Y$ is defined by

$$f^\ell(A)(x) = \sup_{y\in Y} \left[G_f(x,y) \wedge A(y)\right]$$

Theorem 2.4. [*Basic Identities*] . Let $f : X \rightsquigarrow Y$ be a fuzzy multivalued function, then:
(i) $f^u(A) = (f^\ell(A^c))^c$,

(ii) $f^{\ell}(A) = (f^{u}(A^{c}))^{c}$,
(iii) $f^{u}\left(\underset{i\in I}{\cap} A_i\right) = \underset{i\in I}{\cap} f^{u}(A_i)$, and
(iv) $f^{\ell}(\underset{i\in I}{\cup} A_i) = \underset{i\in I}{\cup} f^{\ell}(A_i)$

Proof. (i).

$$\begin{aligned}
\left(f^{\ell}(A^{c})\right)^{c}(x) &= 1 - f^{l}(A^{c})(x) \\
&= 1 - \sup_{y\in Y}\left[G_f(x,y) \wedge A^{c}(y)\right] \\
&= \inf_{y\in Y}\left[(1 - G_f(x,y)) \vee (1 - A^{c}(y))\right] \\
&= \inf_{y\in Y}\left[(1 - G_f(x,y)) \vee A(y)\right] \\
&= f^{u}(A)(x).
\end{aligned}$$

(ii).

$$\begin{aligned}
\left(f^{u}(A^{c})\right)^{c}(x) &= 1 - f^{u}(A^{c})(x) \\
&= 1 - \inf_{y\in Y}\left[(1 - G_f(x,y)) \vee A^{c}(y)\right] \\
&= \sup_{y\in Y}\left[(1 - (1 - G_f(x,y))) \wedge (1 - A^{c})(y)\right] \\
&= \sup_{y\in Y}\left[G_f(x,y) \wedge A(y)\right] \\
&= f^{\ell}(A)(x).
\end{aligned}$$

(iii).

$$\begin{aligned}
f^{u}(\underset{i\in I}{\cap} A_i)(x) &= \inf_{y\in Y}\left[(1 - G_f(x,y)) \vee (\underset{i\in I}{\cap} A_i)(y)\right] \\
&= \inf_{y\in Y}\inf_{i\in I}\left[(1 - G_f(x,y)) \vee A_i(y)\right] \\
&= \inf_{i\in I}\inf_{y\in Y}\left[(1 - G_f(x,y)) \vee A_i(y)\right] \\
&= \inf_{i\in I} f^{u}(A_i)(x).
\end{aligned}$$

(iv).

$$\begin{aligned} f^{\ell}(\underset{i\in I}{\cup} A_i)(x) &= \sup_{y\in Y}[G_f(x,y) \wedge (\underset{i\in I}{\cup} A_i)(y)] \\ &= \sup_{y\in Y} \sup_{i\in I} (G_f(x,y) \wedge A_i(y)) \\ &= \sup_{i\in I} \sup_{y\in y} (G_f(x,y) \wedge A_i(y)) \\ &= \sup_{i\in I} \left(f^{\ell}(A_i)(x)\right). \end{aligned}$$

Remark 2.5. Let $f : X \rightsquigarrow Y$ be a fuzzy multivalued function which has only nonempty values then $f^u(A)$ may not be a fuzzy subset of $f^{\ell}(A)$.

Example 2.6. Let $f : X \rightsquigarrow Y$ be defined as follow:

$$f(x) = \{ \begin{array}{ll} f(x_o), & x = x_0 \\ Y, & x \neq x_0 \end{array}$$

and $[f(x_0)](y) = \frac{\mathbf{1}}{\mathbf{3}}$, for every $y \in Y$. Then for arbitrary fuzzy subset A of Y,

$$f^u(A)(x_0) = \inf_{y\in Y} [(1 - G_f(x_0, y)) \vee A(y)] \geq \frac{2}{3},$$

and

$$f^l(A)(x_0) = \sup_{y\in Y} [G_f(x_0, y) \wedge A(y)] \leq \frac{1}{3}.$$

Therefore $f^u(A)(x_0) > f^l(A)(x_0)$. Hence $f^u(A)$ is not a fuzzy subset of $f^l(A)$.

Remark 2.7. For the inverse of a singleton, we introduce

$$f^{-1}(y) = \{x \in X : G_f(x, y) \neq 0\} = f^{\ell}\{y\}$$

Let X be a fuzzy topological space. A neighborhood of a fuzzy set $A \subset X$ is any fuzzy set B for which there is an open fuzzy set V satisfying $A \subset V \subset B$. Any open fuzzy set V that satisfies $A \subset V$ is called an open neighborhood of A.

Definition 2.8. A fuzzy multivalued function $f : X \rightsquigarrow Y$ between two fuzzy topological spaces X and Y is:

(a) upper hemicontinuous at the point x, if for every open neighborhood U of $f(x)$, $f^u(U)$ is a neighborhood of x in X. The fuzzy multivalued function f is upper hemicontinuous on X, if it is upper hemicontinuous at every point of X;

(b) lower hemicontinuous at x, if for every open fuzzy set U which intersects $f(x)$, $f^\ell(U)$ is a neighborhood of x. As above f is lower hemicontinuous on X if it is lower hemicontinuous at each point of X;

(c) continuous if it is both upper and lower hemicontinuous.

Throughout this section, if we assert that a fuzzy multivalued function is hemicontinuous, it should be understood that its domain and range space are fuzzy topological spaces.

For similar results we also refer to [tsi3-4]

Lemma 2.9. Let $f : X \rightsquigarrow Y$ be a fuzzy multivalued function. Then:

(i) if f is upper hemicontinuous then $f^u(\phi)$ is open ;

(ii) if f is lower hemicontinuous then $f^\ell(Y)$ is open.

Proof. (i) Let $f : X \rightsquigarrow Y$ be an upper hemicontinuous multivalued function, then for any open neighborhood U of $f(x)$, $f^u(U)$ is a neighborhood of x. Therefore there is an open fuzzy set V such that $\{x\} \subset V \subset \{x \in X : f(x) \subset U\}$. It further implies that if x is any point in $f^u(\phi)$ then $\{x\} \subset V \subset \{x \in X : f(x) = \phi\} = f^u(\phi)$. Hence $f^u(\phi)$ is open.

(ii). Let $f : X \rightsquigarrow Y$ be a lower hemicontinuous multivalued function. Then using Theorem 2.4 (i), we obtain $\left[f^\ell(Y)\right]^c = \left[f^\ell(\phi^c)\right]^c = f^u(\phi)$. Part (i) furhter implies that $f^\ell(Y)$ is open.

Remark 2.10. If f is a continuous multivalued function then $f^u(\phi) = \{x \in X : f(x) = \phi\}$ is a closed open fuzzy set.

Theorem 2.11. For $f : X \rightsquigarrow Y$, the following statements are equivalent:

1. $f^u(V)$ is open for each open fuzzy subset V of Y.
2. $f^\ell(W)$ is closed for each closed fuzzy subset W of Y.

Proof.(1)$\Longrightarrow$ (2): Let W be a closed fuzzy subset of Y. Then Theorem 2.4 (ii) implies, $f^\ell(W) = (f^u(W^c))^c$. Now (1) further implies that $f^l(W)$ is closed.

(2)$\Longrightarrow$ (1): Let V be an open fuzzy subset of Y. Then Theorem 2.4 (i) implies, $f^u(V) = \left(f^\ell(V^c)\right)^c$. Now (2) further implies that $f^u(V)$ is open.

Theorem 2.12. Let $f : X \rightsquigarrow Y$ and $f^u(V)$ be open for each open fuzzy subset V of Y then f is upper hemicontinuous.

Proof. Let $f^u(V)$ be open for each open fuzzy subset V of Y. Then $f^u(V)$ is a neighborhood of each of its point. Hence f is upper hemicontinuous.

Theorem 2.13. For $f : X \rightsquigarrow Y$, the following statements are equivalent:

1. $f^\ell(V)$ is open for each open fuzzy subset V of Y.
2. $f^u(W)$ is closed for each closed fuzzy subset W of Y.

Proof. (1) $\Longrightarrow$ (2): Let W be a closed fuzzy subset of Y. Then by Theorem 2.4 (i), we have, $f^u(W) = \left(f^\ell(W^c)\right)^c$. Now (1) further implies that $f^u(W)$ is closed.

(2) $\Longrightarrow$ (1): Let V be an open fuzzy subset of Y. Then by Theorem 2.4 (ii), we have $f^\ell(V) = (f^u(V^c))^c$. Now (2) further implies that $f^\ell(V)$ is open.

Theorem 2.14. Let $f : X \rightsquigarrow Y$ and $f^\ell(V)$ be open for each open fuzzy subset V of Y then f is lower hemicontinuous..

Proof. Let $f^\ell(V)$ be open for each open fuzzy subset V of Y. Then $f^\ell(V)$ is a neighborhood of each of its point. Hence f is lower hemicontinuous.

Recall that a fuzzy function $f : X \longrightarrow Y$ between two fuzzy topological spaces is:

(a) an open fuzzy mapping if $f(V)$ is open in Y for each open fuzzy subset V in X;

(b) a closed fuzzy mapping if $f(W)$ is closed in Y for each closed fuzzy subset W in X.

Next we characterize closed and open fuzzy mappings in terms of hemicontinuity of the inverse fuzzy multivalued function.

Theorem 2.15. Let $f : X \longrightarrow Y$ be a fuzzy function between fuzzy topological spaces and the inverse fuzzy multivalued function $f^{-1} : Y \rightsquigarrow X$ defined by the formula

$$\left[f^{-1}(y)\right](x) = G_f(x, y).$$

Then f is a closed fuzzy mapping if and only if f^{-1} is upper hemicontinuous fuzzy multivalued function.

Proof. Assume that f is a clossed fuzzy mapping. Fix $y \in Y$ and choose an open fuzzy subset V of X such that $f^{-1}(y) \subset V$. Put $E = [f(V^c)]^c$. Then E is an open neighborhood of y, satisfying $f^{-1}(z) \subset V$ for each $z \in E$. Thus $E \subset (f^{-1})^u(V)$ and so $(f^{-1})^u(V)$ is a neighborhood of y.

Conversely, suppose that f^{-1} is upper hemicontinuous. Let W be a closed fuzzy subset of X and pick $y \in [f(W)]^c$. Then $f^{-1}(y) \subset W^c$. So by the upper hemicontinuity of f^{-1} there exists an open neighborhood V of y such that $f^{-1}(z) \subset W^c$ for all $z \in V$. This implies $V \cap f(W) = \phi$ that is, $V \subset [f(W)]^c$. Hence $f(W)$ is closed.

Theorem 2.16. Let f and f^{-1} be as in Theorem 2.15. Then f is an open mapping if and only if f^{-1} is lower hemicontinuous.

Proof is similar to Theorem 2.15.

3. FUZZY CLOSED GRAPH FUZZY MULTIVALUED FUNCTIONS

In this section we investigate the properties of fuzzy closed valued, fuzzy compact valued and fuzzy closed graph fuzzy multivalued functions defined on a fuzzy topological space. Fuzzy closed graph theorem for fuzzy multivalued function is also proved. As application, we study the structure of the set of fixed points of the fuzzy multivalued functions.

A fuzzy set A in (X, τ) is called *fuzzy compact* if and only if each fuzzy open covering of A has a finite subcovering. Similarly we can define fuzzy Hausdorff and fuzzy regular spaces. For further details we refer to [low, pu].

Definition 3.1. The fuzzy multivalued function $f : X \rightsquigarrow Y$ is *fuzzy closed valued* if $f(x)$ is a closed fuzzy set for each x. The terms *fuzzy open valued, fuzzy compact valued* are defined similarly.

Theorem 3.2. The image of a fuzzy compact set under a fuzzy compact valued upper hemicontinuous multivalued function is fuzzy compact.

Proof. Let $f : X \rightsquigarrow Y$ be an upper hemicontinuous fuzzy multivalued function with fuzzy compact values, where X is fuzzy compact. Let $\{u_\alpha\}$ be an open covering of $f(X)$. Since each $f(x)$ is fuzzy compact, each can be covered by a finite subcover. Let V_x denote the union of such a finite subcover of $f(x)$. Then $W_x = f^u(V_x)$, is an open neighborhood of x, so $\{W_x : x \in X\}$ is an open covering of X. Since X is fuzzy compact therefore there is a finite subcover $\{W_{x_1}, W_{x_2}, ..., W_{x_n}\}$. The original u_α's corresponding to $V_{x_1}, V_{x_2}, ...V_{x_n}$ provide a finite cover of $f(x)$.

Definition 3.3. A fuzzy multivalued function $f : X \rightsquigarrow Y$ between fuzzy topological spaces is *fuzzy closed* or has *fuzzy closed graph* if its graph is a closed fuzzy subset of $X \times Y$.

Theorem 3.4. Every fuzzy closed fuzzy multivalued function is fuzzy closed valued.

Proof. Let $f : X \rightsquigarrow Y$ be a fuzzy closed fuzzy multivalued function and assume

that a net $\{y_\alpha\}$ in $f(x)$ satisfies $y_\alpha \to y$ in Y. Then the net $\{(x, y_\alpha)\} \subset G_f$ satisfies $(x, y_\alpha) \to (x, y)$ in $X \times Y$. Since G_f is closed, it follows that $(x, y) \in G_f$, that is, $y \in f(x)$. Thus, $f(x)$ is a closed fuzzy subset of Y.

Theorem 3.5. Let Y be a fuzzy regular space. Let $f : X \rightsquigarrow Y$ be a fuzzy closed valued upper hemicontinuous fuzzy multivalued function then f is fuzzy closed.

Proof. Let $f : X \rightsquigarrow Y$ be a fuzzy multivalued function between fuzzy topological spaces. Suppose $(x, y) \notin G_f$, that is $y \notin f(x)$. Since $f(x)$ is fuzzy closed and Y is fuzzy regular, therefore there are open neighborhoods V of y and W of $f(x)$ with $V \cap W = \phi$. Thus $U = f^u(W)$ is open and $U \times V$ is a neighborhood of (x, y) disjoint from G_f. Therefore G_f is closed. Hence f is fuzzy closed.

Lemma 3.6. Let K be a fuzzy compact subset of a fuzzy Hausdorff space X. If $x \notin K$ then there are disjoint open fuzzy sets U and V with $K \subset U$ and $y \in V$.

Proof. Since X is fuzzy Hausdorff, for each y in K, there are open neighborhoods U_y of y and V_y of x with $U_y \cap V_y = \phi$. The U_y's cover K, so there is a finite subfamily $U_{y_1}, U_{y_2}, ..., U_{y_n}$ covering K. The fuzzy sets $U = \bigcup_{i=1}^{n} U_{y_i}$ and $V = \bigcap_{i=1}^{n} V_{y_i}$ have the desired properties.

Theorem 3.7. Let Y be a fuzzy Hausdorff space. Let $f : X \rightsquigarrow Y$ be a fuzzy compact valued upper hemicontinuous fuzzy multivalued function then f is fuzzy closed fuzzy multivalued function.

Proof. Let $f : X \rightsquigarrow Y$ be a fuzzy multivalued function between two fuzzy topological spaces. Suppose $(x, y) \notin G_f$, that is, $y \notin f(x)$. Since Y is fuzzy

Hausdorff and $f(x)$ is compact, by Lemma 3.6 there are open neighborhoods V of y and W of $f(x)$ with $V \cap W = \phi$. Thus $U = f^u(W)$ is open and $U \times V$ is a neighborhood of (x, y) disjoint from G_f. Therefore G_f is closed. Hence f is fuzzy closed fuzzy multivalued function.

Theorem 3.9. (*Fuzzy Close Graph Theorem*) A fuzzy closed valued fuzzy multivalued function f with fuzzy Hausdorff range is fuzzy closed if and only if f is upper hemicontinuous.

Proof. The "if" part follows from Theorem 3.7. For the "only if" part, assume that $f : X \rightsquigarrow Y$ is a fuzzy closed fuzzy multivalued function and Y is a fuzzy compact Hausdorff space.

Suppose by way of contradiction that f is not upper hemicontinuous. Then there are some x and a fuzzy open set V containing $f(x)$ such that for every neighborhood U of x , there exist $x_U \in U$ and $y_U \in f(x_U)$ with $y_U \notin V$. Then $\{y_U\}$ is a directed net in V^c. Since Y is fuzzy compact, there is a convergent subnet, say to $y \in Y$. Since V^c is closed therefore $y \notin V$. But $x_U \to x$, $\{(x_U, y_U)\} \subset G_f$ and G_f is closed. It follows that $y \in f(x) \subset V$, a contradiction occurs.

Theorem 3.9. Let $f : X \rightsquigarrow Y$ be a fuzzy multivalued function. If G_f is fuzzy open then f is lower hemicontinuous.

Proof. Fuzzy open graph implies that f has fuzzy open lower section. It further implies that f is lower hemicontinuous.

We can use the proceeding results to study the structure of the set of fixed points of fuzzy multivalued functions. Detailed treatment for the existence of fixed points may be found in Heilpern [hei], Butnariu [but] and section 10.

Definition 3.10. Let $f : X \rightsquigarrow X$ be a fuzzy multivalued function. A *fixed point* of f is a point x satisfying $\{x\} \subset f(x)$.

Lemma 3.11. Let X be a fuzzy topological space and $f : X \rightsquigarrow X$ fuzzy closed graph fuzzy multivalued function. Then the set of fixed points of f is fuzzy closed.

Proof. The point x is a fixed point of f if and only if $(x, x) \in G_f$. Let $\{x_\alpha\}$ be a net of fixed points converging to x_0. Since $(x_\alpha, x_\alpha) \to (x_0, x_0)$, and G_f is closed. It follows that x_0 is a fixed point of f.

The set of fixed points of f may be empty.

Theorem 3.12. Let X be a fuzzy regular space and let $f : X \rightsquigarrow X$ be a fuzzy closed valued upper hemicontinuous fuzzy multivalued function. Then the set of fixed points of f is fuzzy closed.

Proof. Theorem 3.5 implies that f has fuzzy closed graph. Lemma 3.12 further implies that the set of fixed points of f is closed.

4. OPERATIONS ON FUZZY MULTIVALUED FUNCTIONS AND FUZZY MAXIMUM PRINCIPLE

In this section we introduce and study the basic operations of union, intersection and composition of fuzzy multivalued functions. Many interesting relations between basic operations and upper and lower hemicontinuity are obtained. As application we derive the fuzzy maximum principle for fuzzy multivalued functions.

A net $(x_\lambda)_{\lambda\in\Lambda}$ in a fuzzy topological space (X, τ) *converges* to a point x (denoted by $x_\lambda \to x$) : if given a neighborhood V of x there exists a $\lambda_0 \in \Lambda$ such that $x_\lambda \in V$ whenever $\lambda \geq \lambda_0$. A point x belongs to the closure of a fuzzy subset C of X if there is a net in C converging to x. In general a net in a fuzzy topological space may converge to several points but in a fuzzy Hausdorff space the convergence is unique. For further details we refer to [low, wei,

zim].

Definition 4.1. A fuzzy multivalued function $f : X \rightsquigarrow Y$ between two fuzzy topological spaces is *fuzzy closed* or has *fuzzy closed graph* if its graph is a closed fuzzy subset of $X \times Y$.

Let X and Y be two fuzzy topological spaces and $f : X \rightsquigarrow Y$ be a fuzzy multivalued function. The fuzzy multivalued function f is said to has *property (I):* If $x_\lambda \to x$ then for each $y \in f(x)$ there exists a subnet of indexes $\{\lambda_t\}_{t\in\Gamma}$ and elements $y_t \in f(x_{\lambda_t})$ for each $t \in \Gamma$ such that $y_t \to y$.

Theorem 4.2. Let X and Y be two fuzzy topological spaces and $f : X \rightsquigarrow Y$ be a lower hemicontinuous fuzzy multivalued function then f has property (I).

Proof. Assume that f is lower hemicontinuous at a point x. Let $\{x_\lambda\}_{\lambda\in A}$ be a net satisfying $x_\lambda \to x$. Fix $y \in f(x)$. Denote by $\aleph_x$ and $\aleph_y$ the neighborhood systems of x and y, respectively. Let $U \in \aleph_y$, then $y \in f(x) \cap U$. Lower hemicontinuity of f at x further implies that the set $f^l(U)$ is a neighborhood of x. Thus if $V \in \aleph_x$, then there exists some $\lambda_{u,v} \in A$ such that $x_\lambda \in V \cap f^l(U)$ for each $\lambda \geq \lambda_{u,v}$. Now consider the directed set $\Gamma = A \times \aleph_y \times \aleph_x$ and for each $t = (\lambda, U, V) \in \Gamma$ fix some index $\lambda_t \in A$ with $\lambda_t \geq \lambda$ and $\lambda_t \geq \lambda_{u,v}$. Then , $\{\lambda_t\}$ is a subnet of the net A of indexes. Also, if for each $t = (\lambda, U, V)$, we choose some $y_t \in f(x_{\lambda_t}) \cap U$ then $y_t \to y$. Hence f has property (I).

Definition 4.3. Given a family $\{f_i : i \in I\}$ of fuzzy multivalued functions from X into Y, we define the union and intersection of the family pointwise as follow:

The *union* $\underset{i\in I}{\cup} f_i : X \rightsquigarrow Y$ is defined by $(\underset{i\in I}{\cup} f_i)(x) = \underset{i\in I}{\cup} f_i(x)$.

The *intersection* $\underset{i\in I}{\cap} f_i : X \rightsquigarrow Y$ is defined by $(\underset{i\in I}{\cap} f_i)(x) = \underset{i\in I}{\cap} f_i(x)$.

Remark 4.4. The graph of $\underset{i\in I}{\cup} f_i$ is union of the graphs of f_i and the graph

of $\underset{i\in I}{\cap} f_i$ is the intersection of graphs of f_i.

Theorem 4.5. Let X and Y be fuzzy topological spaces and $f_i : X \rightsquigarrow Y$ $(i \in I)$ be a family of closed fuzzy multivalued functions then $\underset{i\in I}{\cap} f_i$ is a closed fuzzy multivalued function.

Proof. Since the intersection of a family of closed fuzzy sets is closed the intersection has closed graph.

Theorem 4.6. Let X and Y be two fuzzy topological spaces. Let $f : X \rightsquigarrow Y$ be a closed fuzzy multivalued function and $g : X \rightsquigarrow Y$ be an upper hemicontinuous fuzzy multivalued function with fuzzy compact values then $f \cap g$ is upper hemicontinuous fuzzy multivalued function.

Proof. Let W be a open fuzzy subset of Y and suppose $(f \cap g)(x) \subset W$. Let $K = g(x) \backslash W$, then K is fuzzy compact (possibly empty). If K is empty, then $g^u(W)$ is a neighborhood of x and $(f \cap g)(g^u(W)) \subset W$. In case, K is nonempty fuzzy set: Since $f(x) \cap g(x) \subset W$. Therefore for each y in K, we have $y \notin f(x)$ or $(x, y) \notin G_f$. Since graph of f is closed, for each y in K, there are neighborhoods U_y of x and V_y of y such that $(U_y \times V_y) \cap G_f = \phi$. Thus there is a finite subset $\{y_1, y_2, ..., y_n\}$ of K such that $V_{y_1}, V_{y_2}, ..., V_{y_n}$ cover K. Put

$$V = \bigcup_{i=1}^{n} V_{y_i} \text{ and } U = g^u(W \cup V).$$

The fuzzy set $U \cap (\bigcap_{i=1}^{n} U_{y_i})$ is an open neighborhood of x. Indeed, since

$g(x) \subset [g(x) \backslash W] \cup W \subset V \cup W$, therefore $U \cap (\bigcap_{i=1}^{n} U_{y_i})$ is open.

Now for each $z \in U \cap (\underset{i=1}{\overset{n}{\cap}} U_{y_i})$, $g(z) \subset W \cup V$ and $g(z) \cap V = \phi$. Thus

$$(f \cap g)(U \cap (\underset{i=1}{\overset{n}{\cap}} U_{y_i})) \subset W.$$

Hence $f \cap g$ is upper hemicontinuous fuzzy multivalued function.

Theorem 4.7. Let X and Y be two fuzzy topological spaces and $f_i : X \rightsquigarrow Y$ $(i \in I)$ be a family of lower hemicontinuous fuzzy multivalued functions. Then $\underset{i\in I}{\cup} f_i$ is a lower hemicontinuous fuzzy multivalued function.

Proof. Let U be an open neighborhood of $(\underset{i\in I}{\cup} f_i)(x)$. Then

$$\begin{aligned}
[(\underset{i\in I}{\cup} f_i)^l(U)](x) &= \sup_{y\in Y}[G_{\underset{i\in I}{\cup} f_i}(x,y) \wedge U(y)] \\
&= \sup_{y\in Y}[\{\sup_{i\in I} G_{f_i}(x,y) \wedge U(y)] \\
&= \sup_{y\in Y} \sup_{i\in I} G_{f_i}(x,y) \wedge U(y) \\
&= \sup_{i\in I} \sup_{y\in Y} G_{f_i}(x,y) \wedge U(y) \\
&= \sup_{i\in I}[(f_i)^l(U)](x) \\
&= [\underset{i\in I}{\cup}(f_i)^l(U)](x).
\end{aligned}$$

Hence $\underset{i\in I}{U} f_i$ is a lower hemicontinuous fuzzy multivalued function.

Theorem 4.8. Let X and Y be two fuzzy topological spaces and $f : X \rightsquigarrow Y$, $g : X \rightsquigarrow Y$ be two upper hemicontinuous fuzzy multivalued functions then $f \cup g$ is upper hemicontinuous.

Proof. Let U be an open neighborhood of $(f \cup g)(x)$. Then

$$\begin{aligned}
[(f \cup g)^u(U)](x) &= \inf_{y \in Y}[(1 - G_{f \cup g}(x, y)) \vee U(y)] \\
&= \inf_{y \in Y}[(1 - \{G_f(x, y) \vee G_g(x, y)\}) \vee U(y)] \\
&= \inf_{y \in Y}[\{(1 - G_f(x, y) \wedge (1 - G_g(x, y)\} \vee U(y)] \\
&= \inf_{y \in Y}[((1 - G_f(x, y)) \vee U(y)) \wedge ((1 - G_g(x, y)) \vee U(y))] \\
&= \inf_{y \in Y}\{(1 - G_f(x, y)) \vee U(y)\} \wedge \{(1 - G(x, y)) \vee U(y)\} \\
&= [f^u(U)](x) \wedge [g^u(U)](x).
\end{aligned}$$

Hence $f \cup g$ is upper hemicntinuous fuzzy multivalued function.

Theorem 4.9. Let X be a fuzzy topological space and Y be a fuzzy regular space. Let $f_i : X \rightsquigarrow Y$ $(i \in I)$ be a family of closed valued upper hemicontinuous fuzzy multivalued functions and $g : X \rightsquigarrow Y$ be a compact valued upper hemicontinuous fuzzy multivalued function. Then $g \cap (\underset{i \in I}{\cap} f_i)$ is upper hemicontinuous fuzzy multivalued function.

Proof. Theorem 3.5 and Theorem 3.8 imply that g and f_i are closed. Remark 4.4 implies that $(\underset{i \in I}{\cap} f_i)$ is closed. Theorem 4.5 further implies that $g \cap (\underset{i \in I}{\cap} f_i)$ is upper hemicontinuous fuzzy multivalued function.

Definition 4.10. Let $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ be two fuzzy multivalued functions. The *composition* $gof : X \rightsquigarrow Z$ is defined by $(gof)(x) = \cup\{g(y) : [f(x)](y) \neq 0\}$.

Theorem 4.11. Let X, Y and Z be three fuzzy topological spaces. Let $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ be two fuzzy multivalued functions. Then

(i). $(gof)^u(A) = f^u(g^u(A))$

and

(ii) $(gof)^l(A) = f^l(g^l(A))$,

where A is an open fuzzy subset of Z.

Proof. (i). Let A be an open neighborhood of $(gof)(x)$. Then

$$\begin{aligned}
[(gof)^u(A)](x) &= \inf_{z\in Z}[(1-G_{gof}(x,y))\vee A(z)] \\
&= \inf_{z\in Z}[(1-((gof)(x))(z))\vee A(z)] \\
&= \inf_{z\in Z}[(1-\{\underset{y\in f(x)}{\cup} g(y)\}(z))\vee A(z)] \\
&= \inf_{z\in Z}[(1-\sup_{y\in f(x)}(\{g(y)\}(z)))\vee A(z)] \\
&= \inf_{z\in Z}[(\inf_{y\in f(x)}(1-\{g(y)\}(z)))\vee A(z)] \\
&= \inf_{z\in Z}[\inf_{y\in f(x)}\{(1-(g(y))(z))\vee A(z)] \\
&= \inf_{y\in f(x)}[\inf_{z\in Z}\{(1-\{g(y)\}(z))\vee A(z)\}] \\
&= \inf_{y\in f(x)}[g^u(A)](y) \\
&= \inf_{y\in Y}[(1-G_f(x,y))\vee(g^u(A))(y)] \\
&= [f^u(g^u(A)](x).
\end{aligned}$$

Hence $(gof)^u(A) = f^u(g^u(A))$.

(ii). Part (i) together with Theorem 2.4 imply,

$$\begin{aligned}
(gof)^l(A) &= ((gof)^u(A^c))^c = (f^u(g^u(A^c)))^c = f^l(gu(A^c))^c \\
&= f^l(g^l(A^c)^c) = f^l(g^l(A)).
\end{aligned}$$

Hence $(gof)^l(A) = f^l(g^l(A))$.

Corollary 4.12. The composition of upper hemicontinuous fuzzy multivalued functions is upper hemicontinuous.

Corollary 4.13. The composition of lower hemicontinuous fuzzy multivalued functions is lower hemicontinuous.

The fuzzy maximum principle states that the set of solutions to a well behaved constrained maximization problem is upper hemicontinuous fuzzy multivalued function in its parameters and the value function is fuzzy continuous.

Definition 4.14. An extended real valued fuzzy function $h : X \to R \cup \{-\infty, +\infty\}$ on a fuzzy topological space X is:

(i) *lower semicontinuous* if for each $t \in R$, the fuzzy set $\{x \in X : h(X) \leq t\}$; is closed (or $\{x \in X : h(x) > t\}$ is open);

(ii) *upper semicontinuous* if for each $t \in R$, the set $\{x \in X : h(x) \geq t\}$ is closed (or $\{x \in X : h(x) < t\}$ is open);

(iii) *continuous* if and only if h is both upper and lower semicontinuous.

The following result is a fuzzy analogue of the generalized Weierstrass theorem on the extreme values of continuous functions.

Theorem 4.15. A real valued upper semicontinuous fuzzy function on a fuzzy compact space attains a maximum value and the nonempty set of all maximizer is fuzzy compact.

Proof. Let X be a fuzzy compact space and $h : X \to R$ be an upper semicontinuous fuzzy function. For each $t \in \text{range}(h)$, put $H_t = \{x \in X : h(x) \geq t\}$. Since h is upper semicontinuous , each H_t is closed. Furthermore, $\{H_t : t \in \text{range}(h)\}$ has finite intersection property. Since X is compact it further implies that $\underset{t\in\text{range}(h)}{\cap} H_t$ is nonempty and fuzzy compact. Also $\underset{t\in\text{range}(h)}{\cap} H_t$ is the set of maximizer of h.

Theorem 4.16. Let X and Y be fuzzy topological spaces and $f : X \rightsquigarrow Y$ be a lower hemicontinuous fuzzy multivalued function with nonempty values. Let $g : X \times Y \to R$ be a lower semicontinuous fuzzy function . Define the extended real valued fuzzy function $h : X \to R \cup \{-\infty, +\infty\}$ by $h(x) = \sup\{g(x, y) : y \in f(x)\}$. Then the fuzzy function h is lower semicontinuous.

Proof. Let $h(x) > \lambda$. Then $g(x_0, y_0) > \lambda$ for some $y_0 \in f(x_0)$. Since g is lower semicontinuous , therefore $W = \{(x, y) : g(x, y) > \lambda\}$ is an open neighborhood of (x_0, y_0). Thus there are open neighborhoods U of x_0 and V of y_0 such that $U \times V \subset W$. Then $U \cap f^l(V)$ is a neighborhood of x_0. For each $x \in U \cap f^l(V)$,

there is some $y \in f(x) \cap V$, so that $(x, y) \in U \times V \subset W$. Consequently $g(x, y) > \lambda$, so $h(x) > \lambda$. Thus $f(x) \cap V \subset \{x \in X : h(x) > \lambda\}$. Therefore $\{x \in X : h(x) > \lambda\}$ is open for every λ. Hence h is lower semicontinuous.

Theorem 4.17. Let X and Y be fuzzy topological spaces and $f : X \rightsquigarrow Y$ be an upper hemicontinuous fuzzy multivalued function with compact values and let $g : X \times Y \to R$ be an upper semicontinuous fuzzy function. Define $h : X \to R$ by $h(x) = \max\{g(x, y) : y \in f(x)\}$. Then the fuzzy function h is upper semicontinuous.

Proof. Theorem 4.15. implies that the maximum exists. Let $m(x_0) < \lambda$ and $W = \{(x, y) : g(x, y) < \lambda\}$. Thus for each $y \in f(x_0)$, $(x_0, y) \in W$. Since g is upper semicontinuous fuzzy function, W is open. Thus for each $y \in f(x_0)$ there is an open set $U_y \times V_y \subset W$ where U_y is an open neighborhood of x_0 and V_y is an open neighborhood of y. Obviously, $\{V_y : y \in f(x_0)\}$ is a open cover of $f(x_0)$. Let $\{V_{y_1}, V_{y_2}, ..., V_{y_n}\}$ be a finite subcover of $f(x_0)$.Then $(\bigcap_{i=1}^{n} U_{y_i}) \times (\bigcup_{i=1}^{n} V_{y_i}) \subset W$, and so $(\bigcap_{i=1}^{n} U_{y_i}) \cap f^u(\bigcup_{i=1}^{n} V_{y_i})$ is an open neighborhood of x_0. Now for each x in $(\bigcap_{i=1}^{n} U_{y_i}) \cap f^u(\bigcup_{i=1}^{n} V_{y_i})$; if $y \in f(x)$, then $(x, y) \in (\bigcap_{i=1}^{n} U_{y_i}) \times (\bigcup_{i=1}^{n} V_{y_i})$, so $g(x, y) < \lambda$. In particular $h(x) < \lambda$. Thus $\{x \in X : h(x) < \lambda\}$ is open fuzzy set. Hence h is upper semicontinuous .

Theorem 4.18. *[Fuzzy Maximum Theorem].* Let X and Y be fuzzy topological spaces and $f : X \rightsquigarrow Y$ be a continuous fuzzy multivalued function with nonempty fuzzy compact values and suppose $g : X \times Y \rightsquigarrow R$

is continuous fuzzy function. Define the value fuzzy function $h : X \to R$ by $h(x) = \max\{g(x,y) : y \in f(x)\}$ and the fuzzy multivalued function $m : X \rightsquigarrow Y$ of maximizers by $m(x) = \{y \in f(x) : g(x,y) = h(x)\}$. Then the fuzzy function h is continuous and fuzzy multivalued function m is upper hemicontinuous with compact values.

Proof. From Theorems 4.16. and 4.17., h is continuous. The fuzzy multivalued function $p : X \rightsquigarrow Y$ defined by $p(x) = \{y \in Y : g(x,y) = h(x)\}$ has closed graph and $m = f \cap p$. Thus by Theorem 4.6 m is upper hemicontinuous with fuzzy compact values.

5. VECTOR VALUED FUZZY MULTIVALUED FUNCTIONS

The aim of this section is to study properties of vector valued fuzzy multivalued functions. The notion of sum fuzzy multivalued function, convex hull fuzzy multivalued function, closed convex hull fuzzy multivalued function and upper demicontinuous fuzzy multivalued function are given and some of the properties of these fuzzy multivalued functions are investigated.

A single valued map f from a fuzzy topological space X to a fuzzy topological space Y is called *continuous* at some $x \in X$ if $f^{-1}(V)$ is a neighborhood of x for each neighborhood V of $f(x)$ (Here $f^{-1}(V)$ is the fuzzy set in X defined by $[(f^{-1}(V))(x) = V(f(x))]$. For details we refer to [ber, low1-2, wei].

Let E be a vector space over K, where K denotes either the real or the complex numbers. Let $A_1, A_2, \cdots, A_n$ be fuzzy subsets of E and denote by $A_1 \times A_2 \times A_3 \times \cdots \times A_n$ the fuzzy subset A in E^n defined by

$$A(x_1, x_2, \cdots, x_n) = min\{A_1(x_1), A_2(x_2), \cdots, A_n(x_n)\}$$

If $f : E^n \to E$ is defined by $f(x_1, x_2; \cdots x_n) = x_1 + x_2 + \cdots + x_n$ then the fuzzy set $f(A)$ in E is called the sum of the fuzzy sets $A_1, A_2, \cdots, A_n$ and it is denoted by $A_1 + A_2 + \cdots + A_n$. For a fuzzy subset A of E and t a scalar, we denote tA the image of A under the map $g : E \to E,\ g(x) = tx$. If α is a fuzzy set in K and A a fuzzy set in E, then the image in E of the fuzzy set $\alpha \times A$, a fuzzy subset of $K \times E\,[(\alpha \times A)(t, x) = min\,\{\alpha(t),\ A(x)\}]$ under the map $h : K \times E \to E,\ h(t, x) = tx$, is denoted by αA. A fuzzy set A in E is called *convex* if for each $t \in [0, 1],\ [tA + (1 - t)A](x) \leq A(x)$. The *convex hull* of a fuzzy set B is smallest convex fuzzy set containing B and is denoted by $c_0(B)$.

Given a toplogical space (X, τ), the collection $w(\tau)$ of all fuzzy sets in X, which are lower semicontinuous, as function from X to $[0, 1]$ equipped with the usual topology, is a fuzzy topology on X. The fuzzy topology $w(\tau)$ is called the fuzzy topology generated by the usual topology τ. The fuzzy usual topology on K is the fuzzy topology generated by the topology of K.

Definition 5.1. A *fuzzy linear topology* on a vector space E over K is a fuzzy topology τ on E such that the two mappings:

$$f : E \times E \to E,\ f(x, y) = x + y,$$

$$h : K \times E \to E, h(t, x) = tx,$$

are continuous when K has the usual fuzzy topology and $K \times E,\ E \times E$ the corresponding product fuzzy topologies.

A linear space with a fuzzy linear topology is called a *fuzzy topological vector space.*

Lemma 5.2 In a fuzzy topological vector space X, the algebraic sum of a compact fuzzy set and a closed fuzzy set is closed fuzzy set.

Proof. Let A be a compact fuzzy subset and B be a closed fuzzy subset of X. Let a net $\{x_\lambda + y_\lambda\}$ in $A + B$ satisfy $x_\lambda + y_\lambda \to z$. Since A is compact fuzzy

set, we can assume (by passing to a subnet) that $x_\lambda \to x \in A$. The continuity of the algebraic operations imply

$$y_\lambda = (x_\lambda + y_\lambda) - x_\lambda \to z - x = y.$$

Since B is closed fuzzy subsets. Therefore $y \in B$. So $z = x + y \in A + B$. Hence $A + B$ is a closed fuzzy set.

Lemma 5.3. In a fuzzy topological vector space X, the algebraic sum of two compact fuzzy sets is a compact fuzzy set.

Proof. Similar to lemma 5.2

Theorem 5.4. Let K be a compact fuzzy subset of a fuzzy topological vector space X. Suppose $K \subset U$, where U is open fuzzy subset. Then there is a neighborhood W of origin such that $K + W \subset U$.

Proof. For each $x \in K$, there is a neighborhood V_x of origin such that $x + V_x \subset U$. Choose an open neighborhood W_x of origin so that $W_x + W_x \subset V_x$ for each x. Since K is compact fuzzy set, there is a finite set $\{x_1, x_2, \cdots, x_n\}$ of points with $K \subseteq \bigcup_{i=1}^{n} (x_i + W_{x_i})$. Set $W = \bigcap_{i=1}^{n} W_{x_i}$. For every $x \in K$ there is some x_i satisfying $x \in x_i + W_{x_i}$. For this x_i,

$$x + W = (x - x_i) + (x_i + W) \subset W_{x_i} + x_i + W_{x_i}$$

$$\subset x_i + V_{x_i} \subset U.$$

Hence $K + W \subset U$.

Theorem 5.5. Let X be a fuzzy topological vector space. If each $A_i (i = 1, 2, \cdots n)$ is compact then $c_0(\bigcup_{i=1}^{n} A_i)$ is compact fuzzy set.

Proof. Since continuous image of a compact fuzzy set is compact fuzzy set and the

$$c_0 \left(\bigcup_{i=1}^{n} A_i \right) = f \left(K \times A_1 \times A_2 \times \cdots \times A_n \right).$$

Hence $C_0 \left(\bigcup_{i=1}^{n} A_i \right)$ is a compact fuzzy set.

Definition 5.6 A fuzzy topological vector space E is called *locally convex* if it has a base at origin of convex fuzzy sets.

For basic concepts and more details regarding fuzzy topological vector spaces we refer to [kat1-3, low1-2].

When the range space of a fuzzy multivalued function is a vector space, then there are additional natural operations on fuzzy multivalued functions.

Definition 5.7. If $f, g : X \rightsquigarrow Y$ are two fuzzy multivalued functions, where Y is a vector space, then we define:

(1) The *sum fuzzy multivalued function* $f + g$ by

$$(f + g)(x) = f(x) + g(x) = \{y + z : y \in f(x) \text{ and } z \in g(x)\}.$$

(2) The *convex hull fuzzy multivalued function* $c_0(f)$ of f by

$$(c_0(f))(x) = c_0(f(x)).$$

(3) If Y is a fuzzy topological vector space, the *closed convex hull fuzzy multivalued function* $c\ell(c_0(f))$ of f by

$$(c\ell(c_0(f)))(x) = c\ell\,(c_0(f(x))).$$

Lemma 5.2 and Lemma 5.3 imply the following theorem.

Theorem 5.8. Let $f, g : X \rightsquigarrow Y$ be two fuzzy multivalued functions from a fuzzy topological space X into a fuzzy topological vector space Y:
(1) If f is closed valued and g is compact-valued, then $f + g$ is closed value.
(2) If f and g are compact valued, then $f + g$ is compact valued.

Theorem 5.9. Let $f, g : X \rightsquigarrow Y$ be two fuzzy multivalued functions from a fuzzy topological space X into a fuzzy topological vector space Y. If f and g are compact valued and upper hemicontinuous at a point x_0 then $f + g$ is

upper hemicontinuous at x_0.

Proof. Let f and g be upper hemicontinuous fuzzy multivalued functions at the point x_0. Suppose $f(x_0) + g(x_0) \subset G$, where G is an open fuzzy subset of Y. By Theorem 5.4, there is a neighborhood V of origin such that $f(x_0) + g(x_0) + V \subset G$. Pick an open neighborhood W of origin with $W + W \subset V$. Since $f(x_0) \subset f(x_0) + W$ and $f(x_0) + W$ is open, the upper hemicontinuity of f at x_0 guarantees the existence of an open neighborhood N_1 of x_0 such that $f(N_1) \subset f(x_0) + W$. Similarly, there exists an open neighborhood N_2 of x_0 with $g(N_2) \subset g(x_0) + W$. Let $N = N_1 \cap N_2$, then N is an open neighborhood of x_0 and

$$(f + g)(N) \subset f(N_1) + g(N_2) \subset f(x_0) + W + g(x_0) + W \subset G,$$

It further implies that $f + g$ is upper hemicontinuous fuzzy multivalued functions at x_0.

Theorem 5.10. Let $f, g : X \rightsquigarrow Y$ be two fuzzy multivalued functions from a fuzzy topological space X into a fuzzy topological vector space Y. If f and g are lower hemicontinuous fuzzy multivalued functions at a point x_0, then $f+g$ is also lower hemicontinuous fuzzy multivalued functions at x_0.

Proof. Suppose $[f(x_0) + g(x_0)] \cap U \neq \phi$, where U is open fuzzy subset. Then there are y in $f(x_0)$ and $z \in g(x_0)$ with $y + z \in U$. Thus there is an open neighborhood V of origin such that $y+V+z+V \subset U$. Since $y \in f(x_0) \cap (y+V)$ and f is lower hemicontinuous fuzzy multivalued function at x_0, $f^\ell(y + V)$ is a neighborhood of x_0.

Similarly, $g^\ell(z + V)$ is a neighborhood of x_0. Hence if $x \in f^\ell(y + V) \cap g^\ell(z + V)$ then $[f(z) + g(x)] \cap U \neq \phi$.

Theorem 5.11. Let $f_i : X \to Y$ $(i = 1, 2, \cdots, n)$ be (single valued) fuzzy functions from a fuzzy topological space X into a fuzzy topological vector space Y and the fuzzy multivalued function $f : X \rightsquigarrow Y$ be given by $f(x) =$

$\{f_1(x),\ f_2(x), \cdots, f_n(x)\}$. If each f_i is continuous at a point $x_0 \in X$ then the fuzzy multivalued function f is continuous at x_0.

Proof. Let $f(x_0) = \{f_1(x_0), f_2(x_0), \cdots, f_n(x_0)\} \subset U$, where U is an open fuzzy subset of Y. Then

$$V = \bigcap_{i=1}^{n} f_i^{-1}(U)$$

is an open neighborhood of x_0 such that $x \in V$ implies $f(x) \subset U$. It further implies that the fuzzy multivalued function f is upper hemicontinuous.

Next, suppose $f(x_0) \cap W \neq \phi$ for some open fuzzy subset W of Y. If $f_n(x_0) \in W$, then $P = f_n^{-1}(W)$ is a neighborhood of x_0 and $x \in P$ implies $f(P) \cap W \neq \phi$. It further implies that the fuzzy multivalued function f is lower semicontinuous. Hence the fuzzy multivalued function f is continuous.

Theorem 5.12. Let $f_i : X \to Y$ $(i = 1, 2, 3, \cdots, n)$ be (single valued) fuzzy function from a fuzzy topological space X into a locally convex fuzzy topological vector space Y and $f : X \rightsquigarrow Y$ be given by $f(x) = \{f_1(x), f_2(x), \cdots, f_n(x)\}$. If each f_i is continuous at some point x_0 then the convex hull fuzzy multivalued function $c_0(f)$ is continuous at x_0.

Proof. Suppose that $(c_0(f))(x) \subset U$, where U is an open fuzzy subset of the locally convex fuzzy topological vector space Y. By theorem 5.5 $(c_0(f))(x)$ is compact. Theorem 5.4, further implies that there exists an open convex neighborhood W of origin satisfying $(c_0(f))(x_0) + W \subset U$. From $f(x_0) \subset f(x_0) + W$ and the upper hemicontinuity of f at x_0 (Theorem 5.11), there exists a neighborhood V of x_0 such that $f(x) \subset f(x_0) + W$ for each $x \in V$. So, if $x \in V$, then $(c_0(f))(x) \subset (c_0(f))(x_0) + W \subset U$. It implies that $c_0(f)$ is upper hemicontinuous fuzzy multivalued function at x_0.

Next let $(c_0(f))(x_0) \cap U \neq \phi$ for some open fuzzy subset U. Pick $\lambda_i \geq 0$ $(i = 1, 2, \cdots, n)$ with

$$\sum_{i=1}^{n} \lambda_i = 1 \text{ and } \sum_{i=1}^{n} \lambda_i f_i(x_0) \in U.$$

The fuzzy function $g : X \to Y$ defined by $g(x) = \sum_{i=1}^{n} \lambda_i f_i(x)$ is continuous at x_0 (by definition of fuzzy topological vector spaces). It implies that there exists a neighborhood V of x_0 such that $x \in V$ implies $\sum_{i=1}^{n} \lambda_i f_i(x) \in U$.

Therefore, $((c_0)(f))(x) \cap U \neq \phi$ for each $x \in V$. Thus $c_0(f)$ is a lower hemicontinuous fuzzy multivalued function at x_0. Hence $c_0(f)$ is a continuous fuzzy multivalued function at x_0

Theorem 5.13. Let X be a fuzzy topological space and Y be a locally convex fuzzy topological vector space. Let $f : X \rightsquigarrow Y$ be an upper hemicontinuous fuzzy multivalued function at x. If $(c\ell(c_0(f)))(x)$ is compact, then $c\ell(c_0(f))$ is upper hemicontinuous fuzzy multivalued function at x.

Proof. Let $(c\ell(c_0(f)))(x) \subset P$ for some open fuzzy set P. If $(c\ell(c_0(f)))(x)$ is compact, then there is a convex neighborhood V of origin with $(c\ell(c_0(f)))(x) + V + V \subset P$ (by definition of local convexity and Theorem 5.4). Lemma 5.2 further implies that $(c\ell(c_0(f)))(x) + c\ell(V)$ is a closed convex fuzzy set. Since f is upper hemicontinuous fuzzy multivalued function at x, $f^u(f(x) + V)$ is a neighborhood of x. If $z \in f^u(f(x) + V)$ then $f(z) \subset f(x) + V$, so

$$\begin{aligned}(c\ell(c_0(f)))(z) &\subset (c\ell(c_0(f)))(x) + c\ell(V) \\ &\subset (c\ell(c_0(f)))(x) + V + V \subset P.\end{aligned}$$

Therefore $(c\ell(c_0(f)))^u(P)$ includes $f^u(f(x) + V)$. Hence $c\ell(c_0(f))$ is upper hemicontinuous fuzzy multivalued function at x.

Definition 5.14. A fuzzy multivalued function $f : X \rightsquigarrow Y$ from a fuzzy topological space X into a fuzzy topological vector space Y is *upper demicontinuous* if $f^u(\{y \in Y : h(y) < \alpha$ where h is a continuous linear fuzzy single valued function from Y into $K\})$ is open fuzzy subset of X.

Theorem 5.15. A compact valued fuzzy multivalued 9999999function $f : X \rightsquigarrow Y$ from a fuzzy topological space X into a fuzzy topological vector space Y is upper demicontinuous if and only if $c\ell(c_0(f))$ is upper demicontinuous.

Proof. Let $H \subset Y$ be an open half fuzzy space of the form $H = \{y \in Y : \ell(y) < \alpha\}$ where $\ell : Y \to K$ is a continuous linear fuzzy single valued function. Since ℓ is a linear and continuous fuzzy function,

$$max\{\ell(y) : y \in f(x)\} = max\{\ell(y) : y \in (c\ell(c_0(f)))(x)\}.$$

It implies $f(x) \subset H$ if and only if $(c\ell(c_0(f)))(x) \subset H$. Hence f is upper demicontinuous if and only if $c\ell(c_0(f))$ is.

6. CONVOLUTION OF FUZZY MULTIVALUED FUNCTIONS AND APPLICATIONS

Convolutions of integrable functions are of fundamental importance and have several interesting applications in applied mathematics and probability theory (see Balakrishnan [bal] and Khan [kha]). The aim of this section is to define the intersection convolution of fuzzy multivalued functions defined from a group X into another group Y, and then to study their basic properties. As application we prove existence of an extension of linear selector fuzzy multivalued function from a vector subspace under suitable conditions.

We recall that a fuzzy multivalued function F from a set X into a set Y assigns to each x in X a fuzzy subset $F(x)$ of Y. We denote it by $F : X \rightsquigarrow Y$. We can also identify F with a fuzzy subset F of $X \times Y$ with $F(x, y) = [F(x)](y)$.

If A is a fuzzy subset of X, then the fuzzy set $F(A)$ in Y is defined by

$$[F(A)](y) = \sup_{x \in X} [F(x, y) \wedge A(x)].$$

The set $D_F = \{x \in X : F(x) \text{ is nonempty}\}$ is called the *domain* of F. If $D_F = X$, then F is called a fuzzy multivalued function from X into Y. If Domain$(F) \neq X$, then F is called a fuzzy multivalued function between X and Y.

Let F and T be two fuzzy multivalued functions from X into Y such that $T \subset F$, or equivalently $T(x) \subset F(x)$ for all $x \in X$, then T is a *selector fuzzy multivalued function* of F. The selector fuzzy multivalued function of the restriction $F|_z = F \cap (Z \times Y)$ of F to a subset Z of X is called *partial fuzzy selector* of F.

Let X be an additive group (with "$+$" as binary operation) and A, B fuzzy subsets of X then,

$$A + B = \{x + y : x \ \in A, \, y \in B\},$$

with ,

$$(A + B)(z) = \sup_{z=x+y} \inf\{A(x), B(y)\}.$$

Also,

$$-A = \{-x : A(x) \neq 0\},$$

with,

$$-A(-x) = A(x),$$

and,

$$A - B = A + (-B).$$

If X is a vector space (over Γ) then we write $\lambda A = \{\lambda x : A(x) \neq 0\}$ for all $\lambda \in \Gamma$ with $\lambda A(\lambda x) = A(x)$.

A fuzzy multivalued function F between groups X and Y is called *superadditive (resp. subadditive)* if,

$$F(x) + F(y) \subset F(x + y) \ \ (resp.\ F(x + y) \subset F(x) + F(y)),$$

for all x, y in X, and *additive* if it is both superadditive and subadditive. Moreover, F is said to be *odd* if $F(-x) = -F(x)$ for all x in X.

Remark 6.1. (i) F is odd if and only if $-F(x) \subset F(-x)$ (or equivalently $F(-x) \subset -F(x)$ for all $x \in x$. (ii) An odd superadditive fuzzy multivalued function between groups is additive.

A fuzzy multivalued function F between vector spaces X and Y (over Γ) is called *homogeneous* if $F(\lambda x) = \lambda F(x)$ for all nonzero $\lambda \in \Gamma$ and $x \in X$. The fuzzy multivalued function F is homogeneous if and only if $\lambda F(x) \subset F(\lambda x)$ (or equivalently $F(\lambda x) \subset \lambda F(x)$) for all nonzero $\lambda \in \Gamma$ and $x \in X$. A homogeneous fuzzy multivalued function F is called *linear* if it is additive. Homogeneous fuzzy multivalued functions are odd.

For more details we refer to [5,6,15,16].

Definition 6.2. Let X and Y be two groups and Z be a subgroup of X. If $F : X \rightsquigarrow Y$ is a fuzzy multivalued function and $T : Z \rightsquigarrow Y$ then the fuzzy multivalued function $F * T$ defined by,

$$(F * T)(x) = \underset{z \in Z}{\cap} [F(x - z) + T(z)],$$

for all $x \in X$, is called the *intersection convolution* of F and T.

The fuzzy multivalued function $[F + T(0)] : X \rightsquigarrow Y$ is defined by,

$$(F + T(0))(x) = F(x) + T(0).$$

Similarly, $[F(0) + T] : Z \rightsquigarrow Y$ is defined by

$$(F(0) + T)(z) = F(0) + T(z).$$

Theorem 6.3. (i) $F * T \subset F + T(0)$ and (ii). $(F * T)|_Z \subset F(0) + T$.

Proof. By definition 6.2,

$$(F * T)(x) \subset F(x) + T(0) = (F + T(0))(x),$$

for all $x \in X$.

And,

$$(F * T)(z) \subset F(z - z) + T(z) = (F(0) + T)(z),$$

for all $z \in Z$.

Corollary 6.4. (i). If

$$[T(0)](y) = \{ \begin{matrix} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{matrix} \quad \text{where } r \in (0, 1],$$

then $(F * T) \subset T$, and

(ii). If

$$[F(0)](y) = \{ \begin{matrix} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{matrix} \quad \text{where } r \in (0, 1],$$

then $(F * T)|_z \subset T$.

Theorem 6.5. Let X and Y be two groups and Z be a subgroup of X. If $F : X \rightsquigarrow Y$ is a fuzzy multivalued function and T is a subadditive selector fuzzy multivalued function of $F|_Z$, then $T \subset (F * T)|_z$.

Moreover, if

$$[F(0)](y) = \{ \begin{matrix} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{matrix} \quad \text{where } r \in (0, 1],$$

then $T = (F * T)|_z$.

Proof. If $z \in Z$, then

$$T(z) = T(z - v + v) \subset T(z - v) + T(z)$$

$$\subset F(z - v) + T(z)$$

for all $v \in Z$. Hence $T \subset (F * T)|_Z$.

If

$$[F(0)](y) = \{ \begin{array}{ll} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{array} \text{ where } r \in (0,1],$$

then by corollary 6.4 the converse inclusion also hold. Hence $T = (F * T)|_z$.

Corollary 6.6. If F is a fuzzy multivalued function such that,

$$[F(0)](y) = \{ \begin{array}{ll} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{array} \text{ where } r \in (0,1],$$

and T is a subadditive selector fuzzy multivalued function of F then $T = F * T$.

Corollary 6.7. If F is a subadditive fuzzy multivalued function such that

$$[F(0)](y) = \{ \begin{array}{ll} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{array} \text{ where } r \in (0,1],$$

then $F = F * F$.

Theorem 6.8. Let X and Y be two groups and Z be a subgroup of X. If $F : X \rightsquigarrow Y$ is a fuzzy multivalued function and T is an additive selector fuzzy multivalued function of $F|_Z$, then T is also a selector fuzzy multivalued function of $(F + T(0))|_Z$ and

$$F * T = (F + T(0)) * T.$$

Proof. For $z \in Z$,

$$T(z) = T(z) + T(0) \subset F(z) + T(0) = (F + T(0))(z).$$

Therefore T is a selector fuzzy multivalued function of $(F + T(0))|_Z$. For $x \in X$,

$$\begin{aligned} (F * T)(x) &= \bigcap_{z \in Z} (F(x - z) + T(z)) \\ &= \bigcap_{z \in Z} (F(x - z) + T(0) + T(z)) \\ &= \bigcap_{z \in Z} ((F + T(0))(x - z) + T(z)) \\ &= ((F + T(0)) * T)(x) \end{aligned}$$

Hence $F * T = (F + T(0)) * T$.

Theorem 6.9. Let X and Y be two groups and Z be a subgroup of X. Let $F : X \rightsquigarrow Y$ be a fuzzy multivalued function, T a subadditive selector fuzzy multivalued function of $F|_Z$ and S is a subadditive selector fuzzy multivalued function of F such that $T = S|_Z$, then $S \subset F * T$.

Proof. Theorem 6.5 and definition 6.2 imply

$$S(x) \subset (F * S)(x) = \underset{z \in X}{\cap}(F(x - z) + S(z) \subset \underset{z \in Z}{\cap}(F(x - z) + S(z))$$

$$= \underset{z \in Z}{\cap}(F(x - z) + T(z)) = (F * T)(x)$$

for all $x \in X$. Hence $S \subset F * T$.

Corollary 6.10. If F and T are as in theorem 6.9 and T can be extended to a subadditive selector fuzzy multivalued function of F then $(F * T)(x)$ is nonempty for all $x \in X$.

Proof. Assume that T can be extended to S, then $S(x) \subset (F * T)(x)$. Hence $(F * T)(x)$ is nonempty for all x in X.

Corollary 6.11. If F is subadditive fuzzy multivalued function with

$$[F(0)](y) = \{ \begin{matrix} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{matrix} \quad \text{where } r \in (0, 1],$$

then $F = F * (F|_Z)$.

Theorem 6.12. Let X and Y be two groups, Z be a subgroup of X and $F : X \rightsquigarrow Y$ a fuzzy multivalued function. If T is an additive selector fuzzy multivalued function of $F|_Z$, which can be extended to a subadditive selector fuzzy multivalued function of $F + T(0)$ then $(F * T)(x)$ is nonempty for all $x \in X$.

Proof. Theorem 6.8 implies that T is a selector fuzzy multivalued function of $(F + T(0))|_Z$. Theorem 6.8 and corollary 6.10 further imply that $(F * T)(x) = ((F + T(0)) * T)(x) \neq \phi$, for all $x \in X$.

Theorem 6.13. Let X and Y be two groups Z be a subgroup of X and $F : X \rightsquigarrow Y$ be a superadditive fuzzy multivalued function. If T is a selector fuzzy multivalued function of $F|_Z$ with $0 \in T(0)$, then $F = F + T(0)$.

Proof. For $x \in X$,

$$F(x) \subset F(x) + T(0) = (F + T(0))(x),$$

and

$$(F + T(0))(x) = F(x) + T(0) \subset F(x) + F(0) \subset F(x).$$

It implies that, $F = F + T(0)$.

Theorem 6.14. Let X and Y be two group, Z be a subgroup of X and $F : X \rightsquigarrow Y$ be a superadditive fuzzy multivalued function. If T is a subadditive selector fuzzy multivalued function of $F|_Z$ then $F * T = F + T(0)$.

Proof. For $x \in X$,

$$(F + T(0))(x) = F(x) + T(0) \subset F(x) + T(-z) + T(z)$$

$$\subset F(x) + F(-z) + T(z) \subset F(x - z) + T(z),$$

for all $z \in Z$. It implies that,

$$(F + T(0))(x) \subset (F * T)(x).$$

Theorem 6.3 further implies $(F * T)(x) \subset (F + T(0))(x)$. Hence $F * T = F + T(0)$.

Remark 6.15.(i). If F and T are as in theorem 6.14, and $0 \in T(0)$, then $F = F * T$

(ii) If F is an additive fuzzy multivalued function from a group X into another group Y with $0 \in F(0)$ and Z is a subgroup of X, then $F = F * (F|_Z)$.

(iii) If F is an additive fuzzy multivalued function from a group X into another group Y with $0 \in F(0)$, then $F = F * F$.

Theorem 6.16. Let X and Y be two groups and Z be a subgroup of X. Let $F : X \rightsquigarrow Y$ be a fuzzy multivalued function and $T : Z \rightsquigarrow Y$ be an odd additive fuzzy multivalued function then

$$(F * T)(x + z) = (F * T)(x) + T(z),$$

for all $x \in X$ and $z \in Z$.

Proof. Let $x \in X$ and $z \in Z$, then for $u \in Z$,

$$(F * T)(x) + T(z) \subset F(x - (u - z)) + T(u - z) + T(z)$$

$$= F((x + z) - u) + T(u) = (F * T)(x + z).$$

It further implies that

$$(F * T)(x + z) + T(-z) \subset (F * T)(x).$$

Also

$$(F * T)(x + z) \subset (F * T)(x) - T(-z) = (F * T)(x) + T(z).$$

Hence

$$(F * T)(x + z) = (F * T)(x) + T(z).$$

Corollary 6.17. Let X and Y be two groups, Z be a subgroup of X, and $F : X \rightsquigarrow Y$ be a fuzzy multivalued function with

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0 \ , \\ r, & \text{if } y = 0 \ , \end{cases} \quad \text{where } r \in (0, 1].$$

If T is an additive selector fuzzy multivalued function of $F|_Z$, then

$$(F * T)(x + z) = (F * T)(x) + (F * T)(z)$$

for all $x \in X$ and $z \in Z$.

Proof. Using theorem 6.16 and theorem 6.5, we have

$$\begin{aligned}(F * T)(x+z) &= (F * T)(x) + T(z) \\ &= (F * T)(x) + (F * T)(z)\end{aligned}$$

for all $x \in X$ and $z \in Z$.

Theorem 6.18. Let X and Y be two vector spaces (over Γ) and Z be a subspace of X. If $F : X \rightsquigarrow Y$ is a homogeneous fuzzy multivalued function and $T : Z \rightsquigarrow Y$ is a homogeneous fuzzy multivalued function then $F * T$ is also homogeneous.

Proof. Let λ be any nonzero element of Γ and $x \in X$, then for all $z \in Z$,

$$\begin{aligned}\lambda(F * T)(x) &\subset \lambda\left(F(x - \frac{1}{\lambda}z) + T(\frac{1}{\lambda}z)\right) \\ &= \lambda F(x - \frac{1}{\lambda}z) + \lambda T(\frac{1}{\lambda}z) \\ &= F(\lambda x - z) + T(z) \\ &= (F * T)(\lambda x).\end{aligned}$$

Therefore $\lambda(F * T)(x) \subset (F * T)(\lambda x)$. Hence $F * T$ is homogeneous.

Theorem 6.19. Let X and Y be two vector spaces (over Γ) and Z a vector subspace of X such that codim $(Z) = 1$. Let $F : X \rightsquigarrow Y$ be a homogeneous fuzzy multivalued function and T be a linear selector fuzzy multivalued function of $F|_Z$, then: there exists a linear selector fuzzy multivalued function S of $F + T(0)$ such that $S|_Z = T$ if and only if $(F * T)(x)$ is nonempty for some $x \in X \backslash Z$.

Proof. Assume that there exists a linear selector fuzzy multivalued function S of $F + T(0)$ such that $S|_Z = T$. Theorem 6.12 implies $(F * T)(x)$ is nonempty for all $x \in X$.

Conversely, let there exists $x_0 \in X \backslash Z$ such that $(F * T)(x_0)$ is nonempty. It implies that there exists some $y_0 \in Y$ such that $y_0 \in (F * T)(x_0)$. Since

codim $(Z) = 1$, therefore $span\{x_0\} \oplus Z = X$. Define $S : X \rightsquigarrow Y$ by

$$S(\lambda x_0 + z) = \lambda y_0 + T(z)$$

for all $\lambda \in \Gamma$ and $z \in Z$.

Obviously, S is well defined linear fuzzy multivalued function from X into Y and $S|_Z = T$. For $z \in Z$, we have

$$S(\lambda x_0 + z) = \lambda y_0 + T(z)$$

$$\subset \lambda(F * T)(x_0) + T(z)$$

(By Theorem 6.18)

$$\subset (F * T)(\lambda x_0) + T(z)$$

(By Theorem 6.16)

$$= (F * T)(\lambda x_0 + z)$$

(By Theorem 6.3)

$$\subset F(\lambda x_0 + z) + T(0)$$

$$= (F + T(0))(\lambda x_0 + z)$$

for all non-zero λ in Γ. Also for $z \in Z$, using theorem 6.5 and theorem 6.3 we obtain,

$$S(z) = T(z) \subset (F * T)(z) \subset F(z) + T(0) = (F + T(0))(z).$$

It further implies that S is a linear selector fuzzy multivalued function of $F + T(0)$ with $S|_Z = T$.

Theorem 6.20. Let X and Y be two vector spaces (over Γ) and Z be a subspace of X. Let $F : X \rightsquigarrow Y$ be a homogeneous fuzzy multivalued function, T be a linear selector fuzzy multivalued function of $F|_Z$ and $G \subset F + T(0)$ be a linear fuzzy multivalued function with $T = G|_Z$. If $(F * G)(x)$ is nonempty for all $x \notin D_G$ then there exists a linear selector fuzzy multivalued function S of $F + T(0)$ with $T = S|_Z$.

Proof. Let $\mathcal{L}$ be the family of all linear fuzzy multivalued function $G \subset F+T(0)$ with $T = G|_Z$. Since T is a linear selector fuzzy multivalued function of $F + T(0)$ (see theorem 6.8) therefore $T \in \mathcal{L}$. The family $\mathcal{L}$ is partially ordered by inclusion. By the Hausdorff maximality principle, there exists a maximal totally ordered subset $\mathcal{B}$ of $\mathcal{L}$.

Define $S = \cup\mathcal{B}$. If $G_1, G_2 \in \mathcal{B}$, then either $G_1 \subset G_2$ or $G_2 \subset G_1$. It further implies S is a linear fuzzy multivalued function from some subspace W of X into Y. Since for any $G \in \mathcal{B},\ T = G|_Z$ and $G \subset F + T(0)$, therefore $T = S|_Z$ and $S \subset F + T(0)$.

We claim $W = X$. Assume that, on the contrary $W \neq X$, then there exists an $x_0 \in X$ such that $x_0 \notin W$.

For $x_0 \in X \setminus W$,

$$((F + T(0)) * S)(x_0) = ((F + S(0)) * S)(x_0)$$

(By Theorem 6.8)

$$= (F * S)(x_0)$$

(By hypothesis)

$$\neq \phi.$$

Theorem 6.19 further implies that there exists a linear selector fuzzy multivalued function P of $(F + T(0)) + S(0)$ to the subspace $(span\{x_0\}) \oplus W$ of X with $S = P|_W$ and thus $T = P|_Z$.

But $\mathcal{B}\cup\{P\}$ is a totally ordered subset of $\mathcal{L}$, contradiction to the maximality of $\mathcal{B}$, because $P \notin B$. Hence $W = X$.

7. A GENERAL THEOREM ON SELECTORS OF FUZZY MULTIVALUED FUNCTIONS

One of the most interesting and important problems in ordered set theory is the extension problem. Two ordered sets X and Y are given, together with a subset $A \subset X$, we would like to know whether every order preserving function $g : A \to Y$ can be extended to an order preserving function $f : X \to Y$. Sometimes there are additional requirements on f e.g., for every $x \in X,\ f(x)$ must be an element of a pre-assigned subset of Y. This new problem is clearly more general than the extension problem and is called a selection problem. Even though there is a lot of work in the classical set theory on selection problems (see; Knaster [kna], Tarski [tar], Davis [dav], Birkhoff [bir] and Smithson [smi]). The aim of this section is to prove the existence of a fuzzy order preserving selectors for fuzzy multivalued functions under suitable conditions. A fixed point theorem for fuzzy order preserving fuzzy multivalued functions is also proved.

Definition 7.1. Let X be a crisp set. A *fuzzy ordering relation* on X is a fuzzy subset R of $X \times X$ with the following properties:

(i). For all $x \in X,\ r(x,x) \in (0,1]$.

(ii). For all $x, y \in X,\ r(x,y) + r(y,x) > 1$ implies $x = y$.

(iii). For all $(x,y,z) \in X^3,\ [r(x,y) \geq r(y,x)$ and $r(y,z) \geq r(z,y)]$ imply $r(x,z) \geq r(z,x)$.

A set with a fuzzy ordering relation defined on it, is called a *fuzzy ordered set.*

Definition 7.2. Let X be a set with a fuzzy ordered relation R. Then:

(a). The fuzzy order R is said to be *total* if for all $x \neq y$ we have $r(x,y) \neq r(y,x)$.

(b). A fuzzy ordered fuzzy subset of X on which the fuzzy order R is total is called a *fuzzy chain*.

(c). For a fuzzy subset $A \subset X$, the fuzzy set $U(A)$ of *upper bounds* is defined by $U(A)(x) = \sup\{\inf\{r(y,x) - r(x,y) : y \in A\}, 0\}$. Thus an upper bound (strict upper bound) of A is an element $x \in X$ satisfying $r(y,x) \geq r(x,y)$ $(r(y,x) > r(x,y))$ for all $y \in A$.

(d). An element x is called a *maximal element* of A if there is no $y \neq x$ in A for which $r(x,y) \geq r(y,x)$.

(e). An $x \in A$ satisfying $r(y,x) \geq r(x,y)$ for all $y \in A$ is called the *greatest element* of A.

Similarly, we can define a lower bound, a minimal and a least element of A. In a fuzzy chain C, the least and greatest elements are unique. We define, $\sup A$ = the least element of upper bounds of A and $\inf A$ = the greatest element of lower bounds of A, if it exists.

Moreover using Fuzzy Axiom of Choice (Chapin [cha1-2], Ax. 14), we choose a function f that assigns to every bounded fuzzy chain C a strict upper bound $f(C)$.

Theorem 7.3. ([beg5]) (Fuzzy Zorn's lemma). Let X be a fuzzy ordered set. If every fuzzy chain in X has an upper bound then X has a maximal element.

Let X be a fuzzy ordered set with a fuzzy order R and $F : X \to [0,1]^X \backslash \{\phi\}$ be a *fuzzy multivalued function*, that is, for $x \in X$, $F(x)$ is a nonempty fuzzy subset of X. If F maps the points of its domain to singletons, then F is said to be a *single valued fuzzy function*. In this section no distinction will be made between a single valued fuzzy function and a fuzzy multivalued function. The fuzzy multivalued function F is said to be *fuzzy order preserving* if and only if $x_1, x_2 \in X$ and $y_1 \in F(x_1)$, $r(x_1,x_2) \geq r(x_2,x_1)$ implies that there is a $y_2 \in F(x_2)$ such that $r(y_1,y_2) \geq r(y_2,y_1)$. A *selector* for F is a fuzzy function $f : X \longrightarrow X$ such that $\{f(x)\} \subseteq F(x)$ for each $x \in X$. A point $x \in X$ is a

fixed point of F, if $\{x\} \subseteq F(x)$.

Theorem 7.4. Let X be a fuzzy ordered set and let F be a fuzzy order preserving fuzzy multivalued function on X. If $\sup F(x) \subseteq F(x)$ for all $x \in X$, then there is a fuzzy order preserving selector f for F.

Proof. Let $f(x) = \sup F(x)$ for each $x \in X$. Then f is a fuzzy order preserving selector for F. Indeed, let $r(x_1, x_2) \geq r(x_2, x_1)$. Since $\{f(x_1)\} \subseteq F(x_1)$ there is a $z \in F(x_2)$ such that $r(f(x_1), z) \geq r(z, f(x_1))$. But $r(z, f(x_2)) \geq r(f(x_2), z)$. Hence $r(f(x_1), f(x_2)) \geq r(f(x_2), f(x_1))$.

Theorem 7.5. Let X be a fuzzy ordered set in which each nonempty fuzzy chain C has a supremum and X contains a least element u. If $F : X \to [0,1]^X \backslash \{\phi\}$ is a fuzzy multivalued function which satisfies (I)-(III) as follows:

(I). Let there be a fuzzy order preserving fuzzy function $g : C \longrightarrow X$ such that $\{g(x)\} \subseteq F(x)$ for all $x \in C$. Then there exists $y_0 \in F(\sup C)$ such that $r(g(x), y_0) \geq r(y_0, g(x))$ for all $x \in C$.

(II). Let $r(x_1, x_2) \geq r(x_2, x_1)$ and let $y_1 \in F(x_1), y_2 \in F(x_2)$ with $r(y_1, y_2) \geq r(y_2, y_1)$. If $r(x_1, x) \geq r(x, x_1)$ and $r(x, x_2) \geq r(x_2, x)$ then

$$F(x) \cap \{z : r(y_1, z) \geq r(z, y_1) \text{ and } r(z, y_2) \geq r(y_2, z)\} \neq \phi.$$

(III). Let $D = \{z : r(x_1, z) \geq r(z, x_1) \text{ and } r(z, x_2) \geq r(x_2, z)\}$for $r(x_1, x_2) \geq r(x_2, x_1)$. If $F(x) \cap D \neq \phi$ then $\sup(D \cap F(x)) \in D \cap F(x)$.
Then there exists a fuzzy order preserving selector f for F on X.

Proof. Let P be the collection of fuzzy subsets Y of X with properties:

(1). $u \in Y$.

(2). If $x \in Y$ and $r(z, x) \geq r(x, z)$ then $z \in Y$, and

(3). There is an order preserving fuzzy function $g : Y \longrightarrow X$ such that $g(x) \in F(x)$ for all $x \in Y$.

Let $(P, g) = \{(Y, g) : \ Y \in P \text{ and } g \text{ is a fixed fuzzy function from (3) }\}$. Define a fuzzy order on (P, g) as follows: $(Y_1, g_1) < (Y_2, g_2)$ if and only if $Y_1 \subseteq$

Y_2 and $g_1 = g_2|_{Y_1}$. Then by fuzzy Zorn's lemma there is a maximal element (X_0, f_0) of (P, g). If $X = X_0$, we are done. Otherwise, suppose $x \in X \backslash X_0$, and let C be a maximal fuzzy chain containing u and x. Then $C \cap X_0 = C_1$ is a fuzzy chain in X_0. Let $x_0 = \sup C_1$. Condition (I) and maximality of X_0 imply that $x_0 \in X_0$. Now pick a $y' \in F(x)$ such that $r(f(x_0), y') \geq r(y', f(x_0))$. Let $Y = X_0 \cup \{z : r(x_0, z) \geq r(z, x_0) \text{ and } r(z, x) \geq r(x, z)\}$. Define $f : Y \longrightarrow Y$ as follows:

If $z \in X_0$, then $f(z) = f_0(z)$ and if $z \in \{z : r(x_0, z) \geq r(z, x_0)$ and $r(z, x) \geq r(x, z)\}$ then

$$f(z) = \sup(F(z) \cap \{y : r(f(x_0), y) \geq r(y, f(x_0)) \text{ and } r(y, y') \geq r(y', y)\}).$$

Set $f(x) = y'$. Conditions (II) and (III) show that f is well defined order preserving fuzzy function. This contradicts to the maximality of X_0. Hence $X = X_0$.

Theorem 7.6. Let X be a fuzzy ordered set in which each nonempty fuzzy chain C has a supremum. Let $F : X \longrightarrow [0, 1]^X \backslash \{\phi\}$ be a fuzzy order preserving fuzzy multivalued function such that, given a chain C in X and a single valued order preserving fuzzy function $g : C \longrightarrow X$ satisfying $\{g(x)\} \subseteq F(x)$ for all $x \in C$, there exists a $y_0 \in F(\sup C)$ such that $r(g(x), y_0) \geq r(y_0, g(x))$ for all $x \in C$. If there is a point $p \in X$ and $y \in F(p)$ such that $r(p, y) \geq r(y, p)$, then F has a fixed point.

Proof. Let $p \in X$ and let $y \in F(p)$ with $r(p, y) \geq r(y, p)$. Define a collection P of fuzzy subsets Y of X by:

(i). $p \in Y$.

(ii). If $r(p, z) \geq r(z, p), r(z, x) \geq r(x, z)$ and $x \in Y$ then $z \in Y$; and

(iii). If $x \in Y$, then there is a $z \in F(x)$ such that $r(x, z) \geq r(z, x)$.

Fuzzy order P by inclusion. Since $\{p\} \in P$. Therefore by fuzzy Zorn's lemma there is a maximal element $X_0 \in P$. Let C be a maximal chain in X_0 (existence of C is implied by fuzzy Zorn's lemma) and let $x_0 = \sup C$.

Element $x_0 \in X_0$. Indeed; let there be a fuzzy subset $C_0 \subseteq C$ such that :

(iv). $x_0 = \sup C_0$, and

(v). There is an order preserving fuzzy function $g : C_0 \longrightarrow X$ such that $\{g(x)\} \subseteq F(x)$ and $r(x, g(x)) \geq r(g(x), x)$ for each $x \in C_0$.

Let Q be the collection of fuzzy subsets of C which satisfies (v). If $C_1, C_2 \in Q$, then fuzzy order the pairs $(C_1, g_1), (C_2, g_2)$, where g_1 and g_2 are fixed fuzzy functions from condition (v), by $(C_1, g_1) < (C_2, g_2)$, if and only if $C_1 \subseteq C_2$ and $g_1 = g_2|_{C_1}$. By fuzzy Zorn's lemma there is a maximal set C_0 with function g_0 in Q. Let $x' = \sup C_0$. If $x' \neq x_0$ then there is an $x \in C$ such that $r(x', x) \geq r(x, x')$. By hypothesis, we can extend g_0 to the set $C_0 \cup \{x\}$ which contradicts the maximality of C_0. Thus $\sup C_0 = x_0$. By hypothesis there is a $y \in F(x_0)$ such that $r(g(x), y) \geq r(y, g(x))$ for all $x \in C_0$. But $r(x, g(x)) \geq r(g(x), x)$ for $x \in C_0$ and so y is an upper bound for C_0. Thus $r(x_0, y) \geq r(y, x_0)$. Hence, $x_0 \in X_0$.

Since $x_0 \in X_0$, there exists a $y_0 \in F(x_0)$ such that $r(x_0, y_0) \geq r(y_0, x_0)$. If $x_0 = y_0$, we are done. Otherwise suppose $r(x_0, y_0) > r(y_0, x_0)$. Then put $X_1 = X_0 \cup Z$, where $Z = \{z : r(x_0, z) \geq r(z, x_0)$ and $r(z, y_0) \geq r(y_0, z)\}$.

Since F is fuzzy order preserving, therefore for each $z \in Z$, there is a $w \in F(z)$ such that $r(y_0, w) \geq r(w, y_0)$. But then, $r(z, w) \geq r(w, z)$. Thus (i), (ii) and (iii) are satisfied by X_1, which contradicts the maximality of X_0. Hence, $x_0 = y_0$ and thus $x_0 \in F(x_0)$.

Remark 7.7. Theorem 7.6 generalizes/extends several known results including among them are Knaster [kna], Tarski [tar], Abian and Brown [abi, Theorem 2], Beg [beg2, Theorem 2.4] and Beg [beg4]. Also Theorem 7.4 is a fuzzy analogue of Smithson [smi, Theorem 1.1].

8. LINEAR FUZZY MULTIVALUED OPERATORS

Linear multivalued operators made their first appearance in functional

analysis in von Neumann [neu], which was subsequently studied by Arens [are], Rockafellar [roc], Cech [cec], De Wilde [wil], Szaz and Szaz [sza] and many other authors. Theory of linear multivalued operators is one of the most exciting and influential field of research in modern mathematics. Applications of this theory can be found in economic theory, noncooperative games, artificial intelligence, medicine, and existence of solutions for differential inclusions (see Klein and Thompson [kle], Aubin and Frankowska [aub2] and references therein). In this section the notion of linear fuzzy multivalued operators is given and we studied the operations of addition and scalar multiplication in the class of linear fuzzy multivalued operators. We also obtained the laws governing these operations combined with the operations of composition and inversion.

A *fuzzy multivalued function* (or fuzzy relation) f from a set X into a set Y assigns to each x in domain $D(f)$ of f a nonempty fuzzy subset $f(x)$ of Y. If f maps the points of its domain to singletons, then f is said to be a *single valued fuzzy function.* For $x \in X$, $x \notin D(f)$ we define $f(x) = \phi$. With this convention, we have

$$D(f) = \{x \in X : f(x) \neq \phi\}. \tag{1}$$

with membership function given by,

$$D(f)(x) = \sup_{y \in Y} f(x)(y)$$

We can identify f with a fuzzy subset G_f of $X \times Y$ and $f(x)(y) = G_f(x, y)$.

If A is a fuzzy subset of X, then the fuzzy subset $f(A)$ in Y is defined by

$$f(A)(y) = \sup_{x \in X} \left[G_f(x, y) \wedge A(x) \right]. \tag{2}$$

The fuzzy subset $f(A)$ is called *image* of fuzzy set A. The graph G_f of f is the fuzzy subset of $X \times Y$ associated with f,

$$G_f = \{(x, y) \in X \times Y : [f(x)](y) \neq 0\}. \tag{3}$$

The class of all fuzzy multivalued functions from X into Y is denoted by $FR(X,Y)$.

The *inverse* f^{-1} of a fuzzy multivalued function $f : X \rightsquigarrow Y$ is defined by

$$f^{-1}(B)(x) = \sup_{y \in Y} \left[G_f(x,y) \wedge B(y) \right]. \tag{4}$$

where B is a nonempty fuzzy subset of Y. For the inverse of a singleton, we introduce

$$f^{-1}(y) = \{x \in X : G_f(x,y) \neq 0\}. \tag{5}$$

This type of inverse is usually called lower inverse (see section 2). However, in this section we only use this type of inverse of a fuzzy multivalued function, thus we omit the term lower for simplicity. Fuzzy set $R(f) = f(X)(= f(D(f)))$ is called the *range* of f. If $R(f) = Y$, then f is called *surjective*. If f^{-1} is single valued, then f is called *injective*. If f is injective then $f(x_1) = f(x_2)$ implies $x_1 = x_2$, $(x_1, x_2 \in D(f))$. Also, $D(f^{-1}) = R(f)$ and $R(f^{-1}) = D(f)$.We shall denote $f(\{x\}) = f(x)$; no distinction will be made between a single valued fuzzy map and a fuzzy map into Y.

$$G_{f^{-1}} = \{(y,x) \in Y \times X : G_f(x,y) \neq 0\} \tag{6}$$

and

$$G_{f^{-1}}(y,x) = G_f(x,y). \tag{7}$$

The identity relation defined on a nonempty fuzzy subset E of X is denoted by I_E or simply I when E is understood; $I_E \in FR(X,X)$ whose graph is $G_{I_E} = \{(e,e) : E(e,e) \neq 0\}$.

Let $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ be two fuzzy multivalued functions where $R(f) \cap D(g) \neq \phi$. The *composition* $gf : X \rightsquigarrow Z$ is defined by

$$(gf)(x) = g(f(x)) = \bigcup_{y \in f(x)} g(y) \tag{8}$$

for $x \in X$. The domain of gf is computed by

$$\begin{aligned} D(gf) &= \{x \in X : g(f(x)) \neq \phi\} \\ &= \{x \in X : f(x) \cap D(g) \neq \phi\}. \end{aligned}$$

Using (1), we obtain,

$$D(gf) = f^{-1}(D(g)) \tag{9}$$

Also from the definition of gf, we have

$$\begin{aligned} G_{gf} &= \{(x,z) \in X \times Z : [(gf)(x)](z) \neq 0\} = \{(x,z) \\ &\in X \times Z : G_f(x,y) \neq 0 \text{ and } G_g(y,z) \neq 0 \text{ for some } y \in Y\} \end{aligned} \tag{10}$$

Using above defined notions, we can easily prove the following proposition.

Proposition 8.1. [*Elementary Properties of Composition*]. Let $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$ be two fuzzy multivalued functions. Then:

(i). $f = fI_X = I_Y f$.

(ii). $G_{I_{D(f)}} \subset G_{f^{-1}f}$

(ii). $(gf)^{-1} = f^{-1}g^{-1}$.

(iv). $D(gf) = f^{-1}g^{-1}(R(g))$.

(v). If $h \in FR(Z,W)$, then $h(gf) = (hg)f$.

(vi). The fuzzy multivalued function f is injective if and only if $ff^{-1} = I_{D(f)}$.

(vii). f is single valued fuzzy function if and only if $ff^{-1} = I_{R(f)}$.

Let A be a fuzzy subset of X and $A \cap D(f) \neq \phi$. The *restriction* of f to A is denoted by $f \mid_A$, is defined by:

$$\begin{aligned} f \mid_A &\in FR(X,Y), \\ D(f \mid_A) &= D(f) \cap A \\ (f \mid_A)(a) &= f(a) \text{ for } a \in A. \end{aligned}$$

Also

$$\begin{aligned} G_{f|_A} &= G_f \cap (A \times Y) \\ &= \{(x, y) \in G_f : x \in A\}. \end{aligned}$$

Thus

$$f \mid_A = f \mid_{A \cap D(f)} \tag{11}$$

Let f and g be in $FR(X, Y)$, we say that g is an *extension* of f if

$$g \mid_{D(f)} = f.$$

If g is an extension of f then $G_f \subset G_g$.

Let E be a vector space over K, where K denotes either the real or the complex numbers. Let $A_1, A_2, \cdots, A_n$ be fuzzy subsets of E and denote by $A_1 \times A_2 \times A_3 \times \cdots \times A_n$ the fuzzy subset A in E^n defined by

$$A(x_1, x_2, \cdots, x_n) = min\{A_1(x_1), A_2(x_2), \cdots, A_n(x_n)\} \tag{12}$$

If $f : E^n \to E$ is defined by

$$f(x_1, x_2; \cdots x_n) = x_1 + x_2 + \cdots + x_n,$$

then the fuzzy set $f(A)$ in E is called the *sum of the fuzzy sets* $A_1, A_2, \cdots, A_n$ and it is denoted by $A_1 + A_2 + \cdots + A_n$. For a fuzzy subset A of E and t a scalar, we denote tA the image of A under the map $g : E \to E,\ g(x) = tx$. If $\boldsymbol{\alpha}$ is a fuzzy set in K and A a fuzzy set in E, then the image in E of the fuzzy set $\boldsymbol{\alpha} \times A$, a fuzzy subset of $K \times E\,[(\boldsymbol{\alpha} \times A)(t, x) = min\{\boldsymbol{\alpha}(t),\ A(x)\}]$ under the map $h : K \times E \to E,\ h(t, x) = tx$, is denoted by $\boldsymbol{\alpha}A$.

A fuzzy subset F of vector space E is called a *fuzzy linear space* (subspace) of E if (i). $F + F \subset F$; (ii). $\lambda F \subset F$, for every scalar λ. For further details we refer to Katsaras and Liu [kat3] and Beg, Arshad and Shabir [beg13].

In the sequel X, Y, Z will denote vector spaces over K ($= \mathcal{R}$ or $\mathcal{C}$). A fuzzy multivalued function $f : X \rightsquigarrow Y$ is called *linear fuzzy multivalued operator*, if for all $x, y \in D(f)$, and nonzero scalar α, we have

$$f(x) + f(y) = f(x + y) \tag{13}$$

and

$$\alpha f(x) = f(\alpha x) \tag{14}$$

where $[f(\alpha x)](z) = \frac{1}{\alpha}[f(x)](z)$ for z in Y. Clearly, $D(f)$ is a fuzzy linear subspace of X. The class of all linear fuzzy multivalued operators from X into Y will be denoted by $LFR(X, Y)$. We write $LFR(X, X) = LFR(X)$.

If E is a fuzzy linear subspace of X and $f \in LFR(X, Y)$, then

$$f \mid_E \in LFR(X, Y)$$

and

$$fJ_E \in LFR(X, Y)$$

where $J_E : E \to X$ is defined by $J_E(e) = \{e\}$ for all $e \in E$.

Theorem 8.2. Let $f \in FR(X, Y)$. Then f is a linear fuzzy multivalued opeator if and only if G_f is a fuzzy linear subspace of $X \times Y$.

Proof. Let $f \in LFR(X, Y)$ and $G_f(x_1, y_1) \neq 0, G_f(x_2, y_2) \neq 0$, $0 \neq \alpha \in K$. Then $f(x_1)(y_1) \neq 0$ and $f(x_2)y_2 \neq 0$ and so

$$[f(x_1 + x_2)](y_1 + y_2) = [f(x_1) + f(x_2)](y_1 + y_2) = f(x_1)y_1 + f(x_2)y_2.$$

Therefore

$$0 \neq f(x_1)y_1 + f(x_2)y_2 \leq [f(x_1 + x_2)](y_1 + y_2). \tag{15}$$

Thus $G_f + G_f \subset G_f$. Since $f(x_1)(y_1) \neq 0$, so $[f(\alpha x_1)](\alpha y_1) \neq 0$. Thus

$$G_f(\alpha(x_1, y_1)) = G_f((\alpha x_1, \alpha y_1)) \neq 0. \tag{16}$$

From (15), we have $G_f(0,0) \neq 0$. So (16) also holds for $\alpha = 0$. Hence G_f is a fuzzy linear space.

Conversely, suppose that G_f is a fuzzy linear subspace of $X \times Y$. Then $D(f)$ is a fuzzy linear subspace of X. Let $D(f)(x_1) \neq 0, D(f)(x_2) \neq 0$. Select y_1, y_2 such that $f(x_1)(y_1) \neq 0$ and $f(x_2)(y_2) \neq 0$. Then

$$G_f(x_1 + x_2, y_1 + y_2) = G_f(x_1, y_1) + G_f(x_2, y_2) \neq 0.$$

Therefore

$$[f(x_1 + x_2)](y_1 + y_2) = f(x_1)(y_1) + f(x_2)(y_2) \neq 0,$$

which further implies that

$$f(x_1) + f(x_2) \subset f(x_1 + x_2). \tag{17}$$

Let $f(x_1 + x_2)(y) \neq 0$ and select arbitrarily y_1 such that $f(x_1)(y_1) \neq 0$. Then

$$G_f(x_1 + x_2, y) = G_f(x_1, y_1) + G_f(x_2, y - y_1) \neq 0.$$

It implies that $G_f(x_2, y - y_1) \neq 0$, whence $f(x_2)(y - y_1) \neq 0$. Thus y is in $y_1 + f(x_2)$ and $y_1 + f(x_2) \subset f(x_1) + f(x_2)$. It further implies that

$$f(x_1 + x_2) \subset f(x_1) + f(x_2). \tag{18}$$

The inclusions (17) and (18) imply that

$$f(x_1 + x_2) = f(x_1) + f(x_2).$$

Next, suppose $0 \neq \alpha \in K$ and x is in $D(f)$. If y is in $f(x)$ then

$$G_f(\alpha(x, y)) = G_f(\alpha x, \alpha y) \neq 0.$$

Therefore $\alpha f(x) \subset f(\alpha x)$. On the other hand if y is in $f(\alpha x)$, then

$$G_{\alpha f}(x, y) = G_f(x, \frac{1}{\alpha} y) = G_f(\alpha x, y) \neq 0.$$

Thus $f(\alpha x) \subset \alpha f(x)$. It further implies $\alpha f(x) = f(\alpha x)$.

Hence f is a linear fuzzy multivalued operator from X into Y.

Theorem 8.3. Let $f \in FR(X, Y)$. Then f is a linear fuzzy multivalued operator if and only if f^{-1} is a linear fuzzy multivalued operator.

Proof. Let $f \in LFR(X, Y)$ and let y_1, y_2 be in $D(f^{-1})$. Let x_1 be in $f^{-1}(y_1)$ and x_2 be in $f^{-1}(y_2)$. Then y_1 is in $f(x_1)$ and y_2 is in $f(x_2)$ and so

$$f(x_1 + x_2)(y_1 + y_2) = [f(x_1) + f(x_2)](y_1 + y_2) \neq 0.$$

Therefore $x_1 + x_2$ is in $f^{-1}(y_1 + y_2)$ and

$$f^{-1}(y_1)(x_1) + f^{-1}(y_2)(x_2) \leq [f^{-1}(y_1 + y_2)](x_1 + x_2).$$

Thus

$$f^{-1}(y_1) + f^{-1}(y_2) \subset f^{-1}(y_1 + y_2). \tag{19}$$

Next let x be in $f^{-1}(y_1 + y_2)$ and select x_1 arbitrarily in $f^{-1}(y_1)$. Then

$$G_f(x, y_1 + y_2) = G_f(x_1, y_1) + G_f(x - x_1, y_2) \neq 0.$$

Therefore $G_f(x - x_1, y_2) \neq 0$, whence $x - x_1$ is in $f^{-1}(y_2)$. Thus x is in $x_1 + f^{-1}(y_2)$ and

$$x_1 + f^{-1}(y_2) \subset f^{-1}(y_1) + f^{-1}(y_2).$$

It further implies

$$f^{-1}(y_1 + y_2) \subset f^{-1}(y_1) + f^{-1}(y_2). \tag{20}$$

Now (19) and (20) imply,

$$f^{-1}(y_1 + y_2) = f^{-1}(y_1) + f^{-1}(y_2). \tag{21}$$

Let $0 \neq \alpha \in K$ and y be in $D(f^{-1})$. If x is in $f^{-1}(y)$, then

$$G_f(\alpha(x, y)) = G_f(\alpha x, \alpha y) \neq 0.$$

So

$$0 \neq \alpha f^{-1}(y)(\alpha x) \leq f^{-1}(\alpha y)(x).$$

Hence $\alpha f^{-1}(y) \subset f^{-1}(\alpha y)$.

If x is in $f^{-1}(\alpha y)$ then

$$0 \neq G_{f^{-1}}(\alpha y, x) = G_f(x, \alpha y) \leq G_{\alpha^{-1} f}(x, y) = G_{\alpha f^{-1}}(y, x).$$

It further implies that

$$0 \neq f^{-1}(\alpha y)(x) \leq \alpha f^{-1}(y)(x).$$

Thus $f^{-1}(\alpha y) \subset \alpha f^{-1}(y)$. Therefore

$$\alpha f^{-1}(y) = f^{-1}(\alpha y). \tag{22}$$

Equalities (21) and (22) imply that f^{-1} is a linear fuzzy multivalued operator from Y into X.

Similarly, we can prove that if $f^{-1} \in LFR(Y, X)$ then $f \in LFR(X, Y)$.

Corollary 8.4. If $f \in LFR(X, Y)$ then $f(0)$ and $f^{-1}(0)$ are fuzzy linear subspaces.

Definition 8.5. The fuzzy linear subspace $f^{-1}(0)$ is called the *null space* (or *kernel*) of f and is denoted by $N(f)$.

Corollary 8.6. Let $f \in FR(X, Y)$ then f is a linear fuzzy multivalued operator if and only if

$$\alpha f(x_1) + \beta f(x_2) = f(\alpha x_1 + \beta x_2), \tag{23}$$

for all x_1, x_2 in $D(f)$ and nonzero α, β in K.

Theorem 8.7. Let $f : X \rightsquigarrow Y$ be a linear fuzzy multivalued operator. If x is in $D(f)$ then: y is in $f(x)$ if and only if $f(x) = y + f(0)$.

Proof. Let y be in $f(x)$ then

$$y + f(0) \subset f(x) + f(0) = f(x + 0) = f(x).$$

Also for any y_1 in $f(x)$, we have $y_1 = y + (y_1 - y)$ and $y + (y_1 - y)$ is in $y + (f(x) - f(x))$. It further implies that

$$\begin{aligned} [y + f(0)](y_1) &= [y + f(0)][y + (y_1 - y)] \\ &= [y + (f(x) - f(x))][y + (y_1 - y)] \neq 0. \end{aligned}$$

Thus $f(x) \subset y + f(0)$. Hence $f(x) = y + f(0)$.

Conversely, let $f(x) = y + f(0)$ then y is in $f(x) + f(0)$ and $f(x) + f(0) = f(x + 0) = f(x)$.

Corollary 8.8. Let $f : X \rightsquigarrow Y$ be a linear fuzzy multivalued operator. If x_1, x_2 are in $D(f)$ then: $f(x_1) \cap f(x_2) \neq \phi$ if and only if $f(x_1) = f(x_2)$.

Proof. Let y be in $f(x_1) \cap f(x_2)$. Then by the Theorem 8.7, we have $f(x_1) = y + f(0)$ and $f(x_2) = y + f(0)$. Hence $f(x_1) = f(x_2)$.

The converse is trivial.

Now we state several simple consequences of above results whose proofs are obvious and thus omitted.

Corllary 8.9. Let $f \in LFR(X, Y)$. Then f is single valued if and only if $f(0) = \{0\}$.

Corollary 8.10. Let $f \in LFR(X, Y)$. Then f is injective if and only if $N(f) = \{0\}$.

Corollary 8.11. Let $f \in LFR(X, Y)$. Then

(i). $ff^{-1}(0) = f(0)$, and

(ii). $f^{-1}f(0) = f^{-1}(0)$.

Corollary 8.12. Let $f \in LFR(X, Y)$. Then

(i). $ff^{-1}(y) = y + f(0)$, (y is in $R(f)$), and

(ii). $f^{-1}f(x) = x + f^{-1}(0)$, ($x$ is in $D(f)$).

Corollary 8.13. Let $f, g \in LFR(X, Y)$. Let $G_g \subset G_f$ then f is an extension of g if and only if $g(0) = f(0)$.

Theorem 8.14. Let $f \in FR(X, Y)$. Then f is a linear fuzzy multivalued operator if and only if for all $x_1, x_2 \in D(f)$, and for all $\alpha, \beta \in K$,

$$\alpha f(x_1) + \beta f(x_2) \subset f(\alpha x_1 + \beta x_2). \tag{24}$$

Proof. Let $f \in LFR(X, Y)$. Let $\alpha, \beta \in K$ and x_1, x_2 be in $D(f)$. Select y_1 in $f(x_1)$ and y_2 in $f(x_2)$. Since G_f is a fuzzy linear subspace, therefore $G_f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \neq 0$ and

$$\begin{aligned} 0 &\neq \alpha f(x_1)(y_1) + \beta f(x_2)(y_2) \\ &\leq G_f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = [f(\alpha x_1 + \beta x_2)](\alpha y_1 + \beta y_2). \end{aligned}$$

Hence (24) follows.

Conversely, assume that (24) is satisfied and let $G_f(x_1, y_1) \neq 0, G_f(x_2, y_2) \neq 0$. Then for $\alpha, \beta \in K$, we obtain

$$0 \neq [\alpha f(x_1) + \beta f(x_2)](\alpha y_1 + \beta y_2) \leq f(\alpha x_1 + \beta x_2)(\alpha y_1 + \beta y_2).$$

It further implies that $G_f[\alpha(x_1, y_1) + \beta(x_2, y_2)] \neq 0$. Therefore G_f is a fuzzy linear subspace of $X \times Y$. Theorem 8.2 further implies that f is a linear fuzzy multivalued operator.

Definition 8.15. [*Addition and scalar multiplication in* $LFR(X, Y)$]. Let $f, g : X \rightsquigarrow Y$ be two linear fuzzy multivalued operators and α be a scalar, then we define the *sum linear fuzzy multivalued operator* $f + g$ and *scalar multiplication* αf by

$$(f + g)(x) = f(x) + g(x) = \{y + z : y \in f(x) \text{ and } z \in g(x)\}, \tag{25}$$

and

$$(\alpha f)(x) = \alpha[f(x)]. \tag{26}$$

Theorem 8.16. Let $f, g \in LFR(X, Y)$. Then

(i). $D(f + g) = D(f) \cap D(g)$, and

(ii). $D(\alpha f) = D(f)$.

Proof. (i). Let x is in $D(g) \cap D(f)$. Then $g(x) + f(x) = (g + f)(x) \neq \phi$ and so $[D(f) \cap D(g)](x) \leq D(g + f)(x)$.

Conversely, assume that x is in $D(g + f)$, then since $(g + f)(x) = g(x) + f(x) \neq \phi$, therefore $g(x) \neq \phi$ and $f(x) \neq \phi$. Thus $D(f + g)(x) \leq [D(g) \cap D(f)](x)$.

(ii). By definition.

Remark 8.17. Theorem 8.16 implies that if $f, g \in LFR(X, Y)$ then $f + g$ and αf are linear fuzzy multivalued operators.

Remark 8.18. Let $f, g \in LFR(X, Y)$. Then

(i). $G_{f+g} = \{(x, y) \in X \times Y : y = a + b$, where $G_f(x, a) \neq 0$ and $G_g(x, b) \neq 0\}$.

(ii). $G_{\alpha f} = \{(x, \alpha y)$ in $X \times Y : G_{\alpha f}(x, \alpha y) = G_f(x, y) \neq 0\}$.

(iii). $\alpha(\beta f) = (\alpha\beta)f$, $(\alpha, \beta \in K)$.

(iv). $f + g = g + f$

(v). If $h \in LFR(X, Y)$, then $h + (g + f) = (h + g) + f$.

Theorem 8.19. Let $f \in LFR(X, Y)$, $g \in LFR(Y, Z)$. Then

(i). $ff^{-1} = I_{R(f)} + (ff^{-1} - ff^{-1})$, and

(ii). $\lambda(gf) = (\lambda g)f = g(\lambda f)$, $(0 \neq \lambda \in K)$.

Proof. (i). Let y be in $R(f)$. Theorem 8.7 and Corollary 8.4 imply that $ff^{-1}(y) = y + f(0)$ and $ff^{-1}(y) - ff^{-1}(y) = f(0)$. Therefore $ff^{-1}(y) = y + (ff^{-1}(y) - ff^{-1}(y))$. Hence $ff^{-1} = I_{R(f)} + (ff^{-1} - ff^{-1})$.

(ii). Let $0 \neq \lambda \in K$. If now $G_{(\lambda g)f}(x,z) \neq 0$ then there exists y, such that $G_f(x,y) \neq 0$, $G_{\lambda g}(y,z) \neq 0$. It implies that, $G_f(x,y) \neq 0$, $G_g(y,\lambda^{-1}z) \neq 0$. Therefore $G_{\lambda(gf)}(x,z) \geq G_{(\lambda g)f}(x,z) \neq 0$. Thus

$$G_{(\lambda g)f} \subset G_{\lambda(gf)}. \tag{27}$$

Next assume that (x,z) is in $G_{\lambda(gf)}$ then there exists y, such that $G_f(x,y) \neq 0$, $G_g(y,\lambda^{-1}z) \neq 0$. Therefore $G_f(x,y) \neq 0$, $G_{\lambda g}(y,z) \neq 0$. Thus $G_{(\lambda g)(f)}(x,z) \geq G_{\lambda(gf)}(x,z) \neq 0$. It implies that

$$G_{\lambda(gf)} \subset G_{(\lambda g)f}. \tag{28}$$

Inclusions (27) and (28) further imply that

$$G_{\lambda(gf)} = G_{(\lambda g)f}. \tag{29}$$

Similarly, we can prove

$$G_{\lambda(gf)} = G_{g(\lambda f)}. \tag{30}$$

Equalities (29) and (30) imply

$$(\lambda g)f = \lambda(gf) = g(\lambda f).$$

Theorem 8.20. Let $h \in LFR(X,Y)$ and $f,g \in LFR(Y,Z)$. If $G_g \subset G_f$ then $G_{gh} \subset G_{fh}$.

Proof. Let $G_{gh}(x,z) \neq 0$. Then $G_h(x,y) \neq 0$ and $G_g(y,z) \neq 0$ for some x in $D(g)$. Hence $G_{fh}(x,z) \geq G_{gh}(x,z) \neq 0$ and $G_{gh} \subset G_{fh}$.

Theorem 8.21. Let $f \in LFR(X,Y)$ and $g,h \in LFR(Y,Z)$ then

$$[(g+h)f](x) \subset (gf+hf)(x) \tag{31}$$

Proof. Since

$$\begin{aligned}
G_{(g+h)f} &= \{(x,z) : G_f(x,y) \neq 0 \text{ and } G_{g+h}(y,z) \neq 0\} \\
&= \{(x,z) : G_f(x,y) \neq 0, G_g(y,z_1) \neq 0, G_h(y,z_2) \neq 0, z_1+z_2 = z\} \\
&\subset \{(x,z) : G_{gf}(x,z_1) \neq 0, G_{hf}(x,z_2) \neq 0, z_1+z_2 = z\} \\
&= G_{gf+hf}.
\end{aligned}$$

Hence

$$[(g+h)f](x) \subset (gf+hf)(x).$$

Remark 8.22. If f is single valued in the Theorem 8.21 then $(g+h)f = gf+hf$.

Theorem 8.23. Let $f \in LFR(X,Y)$ and $g,h \in LFR(Z,X)$. Then $f(h+g)$ is an extension of $fh+fg$.

Proof. Since

$$\begin{aligned}
G_{(fh+fg)} &= \{(z,y) : y = a+b \text{ where } G_{fh}(z,a) \neq 0 \text{ and } G_{fg}(z,b) \neq 0\} \\
&= \{(z,y) : y = a+b \text{ where } G_h(z,x_1) \neq 0 \text{ and } G_f(x_1,a) \\
&\neq 0, G_g(z,x_2) \neq 0, G_f(x_2,b) \neq 0\} \\
&\subset \{(z,y) : y = a+b \text{ where } G_{(h+g)}(z,x_1+x_2) \\
&\neq 0, G_f(x_1+x_2,a+b) \neq 0\} \\
&= \{(z,y) : G_{(h+g)}(z,x) \neq 0, G_f(x,y) \neq 0\} = G_{f(h+g)}. \qquad (32)
\end{aligned}$$

Hence $f(h+g)$ is an extension of $fh+fg$.

Theorem 8.24. If $D(f) = X$ in the Theorem 8.23. then $f(h+g) = fh+fg$.

Proof. Let z be in $D(fh+fg)$. Then by inclusion (32), z is in $D[f(h+g)]$. Let y be in $f(h+g)(z)$. Then y is in $f(x)$ where x is in $(h+g)(z)$. Write $x = x_1+x_2$ where x_1 is in $h(z)$, x_2 is in $g(z)$. Then

$$0 \neq f(x_1+x_2)(y) = [f(x_1)+f(x_2)](y) \leq [fh(z)+fg(x_2)](y).$$

Therefore $f(h+g)(z) \subset fh(z)+fg(z)$. Inclusion (32) further implies,

$$f(h+g)(z) = fh(z)+fg(z),$$

for all z in Z. Hence $f(h+g) = fh+fg$.

Remark 8.25. Most of the above results proved in this section are fuzzy analogue of Arens [are] and Mennicken and Sagraloff [men].

The rules governing the operations of single valued linear fuzzy operator $F : X \to Y$ on fuzzy subsets M, N of X, $0 \neq \alpha \in K$ are stated below:

(i). $F(\alpha M) = \alpha F(M)$.

(ii). $F(M + N) \supset F(M) + F(N)$.

(iii). If $D(F) = X$, then $F(M + N) = F(M) + F(N)$.

(iv). If B is a fuzzy subset of Y, then $FF^{-1}(B) = B \cap R(F) + F(0)$.

(v). If A is a fuzzy subset of X then $F^{-1}F(A) = A \cap D(F) + F^{-1}(0)$.

(vi). $F^{-1}(0) \times \{0\} = G_F \cap (X \times \{0\}) = G_F \cap G_{(0_X)}$, where 0_X denotes the zero operator on X.

(vii). $\{0\} \times F(0) = G_F \cap (\{0\} \times Y)$.

(viii). $X \times R(F) = G_F + (X \times \{0\}) = G_F + G_{(0_X)}$.

(ix). $D(F) \times Y = G_F + (\{0\} \times Y)$.

9. FIXED POINTS OF FUZZY MULTIVALUED MAPPINGS WITH VALUES IN FUZZY ORDERED SETS

We review briefly some definitions and terminologies [cha1-2, hei, lee, zad1-2].

Let X be a space of points , with a generic element of X denoted by x. A fuzzy subset of X is characterized by a membership function"θ" which associates with each element in X a real number in the interval $[0, 1]$.

In addition to first order fuzzy theory axioms (Za) (for details see Chapin [cha1-2]) we also assume Fuzzy Hausdorff Maximal Principle: if every fuzzy chain in a fuzzy ordered set X has an upper bound , then X has a maximal element.

Let (X, d) be a metric linear space (i.e., a complex or real vector space) and A be a fuzzy set in X characterized by a membership function α . The $\lambda-$level set of A , denoted by A_λ, is defined by

$A_\lambda = \{x : \alpha(x) \geq \lambda\}$ if $\lambda \in (0, 1]$,

$A_0 = \overline{\{x : \alpha(x) > 0\}}$,

where $\overline{B}$ denotes the closure of the (nonfuzzy) set B.

Definition 9.1. A fuzzy set A in a metric linear space X is said to be an approximate quantity if and only if A_λ is compact and convex in X for each $\lambda \in [0, 1]$ and $\sup_{x \in X} \alpha(x) = 1$.

Let $W(X)$ be a collection of all approximate quantities in X.

Definition 9.2. Let $A, B \in W(X)$. Then A is said to be more accurate than B (or B includes A) denoted by $A \subset B$ if and only if $\alpha(x) \leq \beta(x)$ for each $x \in X$, where β is the membership function of B.The relation $\subset$ induces an order on $W(X)$.

Let X be a nonempty fuzzy ordered set and T be a fuzzzy multivalued mapping with domain in X and range in the family of all fuzzy subsets of X i.e., $T : X \to [0, 1]^X$. We assume that T is nonempty valued, meaning that $T(x) \neq \phi$ for all $x \in X$. Also $T(a) \geq a$ if and only if for all $x \in T(a)$, $\theta(a, x) \geq \theta(x, a)$. A point $x \in X$ is called a fixed point of T if $\{x\} \subset T(x)$.

Lemma 9.3. Let X be a fuzzy ordered set . Let $T : X \to [0, 1]^X$ be a nonempty fuzzy multivalued mapping and there exists some $a \in X$ such that $T(a) \geq a$. Let $P = \{x \in X : T(x) \geq x\}$. Assume that every nonempty T-invariant subset A of P has a maximal point. Then T has a fixed point.

Proof. Let $a \in A$ be a maximal point of A. Since $a \in A \subset P$ .Therefore for all $b \in T(a)$, $\theta(a, b) \geq \theta(b, a)$. As $T(a) \subset T(A) \subset A$, maximality of a in A ensures that for all $b \in T(a)$, $a = b$, proving that $\{a\} = T(a)$.

Definition 9.4. Let X be a fuzzy ordered set and $T : X \to [0, 1]^X$.The fuzzy multivalued mapping T is called nondecreasing if $\theta(a, b) \geq \theta(b, a)$ and $a \neq b$

imply that for all $x \in T(a)$, for all $y \in T(b)$, $\theta(x, y) \geq \theta(y, x)$

Lemma 9.5. Let X be a fuzzy ordered set. Let $T : X \to [0, 1]^X$ be a nonempty nondecreasing fuzzy multivalued mapping and there exists some $a \in X$ such that $T(a) \geq a$. Let $P = \{x \in X : T(x) \geq x\}$. Then $T(P) \subset P$.

Proof. Let $y \in T(P)$. Then there exists $x \in P$, such that $y \in T(x)$. As $x \in P$, therefore for all $z \in T(x)$, $\theta(x, z) \geq \theta(z, x)$. In particular , $\theta(x, y) \geq \theta(y, x)$. If $y = x$, the result follows. If $y \neq x$, then as T is nondecreasing , therefore for all $z_1 \in T(x)$ and $z_2 \in T(y)$, $\theta(z_1, z_2) \geq \theta(z_2, z_1)$. As $y \in T(x)$, we can conclude that for all $z_2 \in T(y)$, $\theta(y, z_2) \geq \theta(z_2, y)$. Hence $y \in H$.

Theorem 9.6. Let X be a fuzzy ordered set in which every nonempty fuzzy chain has a supremum. Let $T : X \to [0, 1]^X$ be a nonempty nondecreasing multivalued fuzzy mappings and there exits some $a \in X$ such that $T(a) \geq a$.Then there exits $x_0 \in X$ such that $\{x_0\} = T(x_0)$.

Proof. Consider $P = \{x \in X : T(x) \geq x\}$. Since $T(a) \geq a$, P is nonempty. Take a nonempty fuzzy chain C in P. By hypothesis C has a supremum b in X , i.e., $\theta(c, b) \geq \theta(b, c)$ for all $c \in C$. If $b = c$ for some $c \in C$, then $b \in H$. Suppose now that $b \neq c$ for all $c \in C$. Since T is nondecreasing, thus for all $y \in T(c)$ and for all $z \in T(b)$, $\theta(y, z) \geq \theta(z, y)$. As $c \in C \subset P$, therefore $\theta(c, y) \geq \theta(y, c)$ for all $y \in T(c)$. It further implies that $\theta(c, z) \geq \theta(z, c)$ for all $z \in T(b)$. As c is an arbitrary element of C, therefore every z in $T(b)$ is a majorant of C and b is the smallest element of all majorants i.e., $\theta(b, z) \geq \theta(z, b)$ for all $z \in T(b)$.Thus $b \in P$. By Fuzzy Hausdorff Maximal Principle, P has a maximal element x_0. Lemma 9.5 implies that $T(P) \subset P$. Lemma 9.3 further implies that $\{x_0\} = T(x_0)$.

Remark 9.7. Theorem 9.6 is fuzzy multivalued version of the results of Knaster [kna] and Tarski [tar].

Next we prove a general principle on fuzzy ordered sets for use to prove a

Caristi [car] type fixed point theorem for fuzzy multivalued mappings.

Let X be a fuzzy ordered set. For $x \in X$, we denote $S(x) = \{y \in X : \theta(x,y) \geq \theta(y,x)\}$.

Definition 9.8. A sequence $\{x_n\}$ in a fuzzy ordered set X is said to be *increasing*, provided $\theta(x_n, x_{n+1}) \geq \theta(x_{n+1}, x_n)$ for all n.

Theorem 9.9. Let $\psi : X \to R$ be a function which satisfies :

i) $\theta(x,y) \geq \theta(y,x)$ and $x \neq y$ implies $\psi(x) \leq \psi(y)$,

ii) for any increasing sequence $\{x_n\}$ in X such that $\psi(x_n) < c < \infty$ for all n , there exists some $y \in X$ such that $\theta(x_n, y) \geq \theta(y, x_n)$ for all n,

iii) for each $x \in X$, $\psi(S(x))$ is bounded.

Then for each $x \in X$, there exists $x^{/} \in S(x)$ such that $x^{/}$ is maximal i.e., $\{x^{/}\} = S(x^{/})$.

Proof. For $a \in X$, let $p(a) = sup\{\psi(b) : b \in S(a)\}$. Assume the conclusion of the theorem fails for some $x \in X$ and define by induction a sequence $\{x_n\}$ such that $x_1 = x$ and $x_{n+1} \in S(x_n)$ satisfies $p(x_n) \leq \psi(x_{n+1}) + \frac{1}{n}$ for all n.

Since $\psi(x_{n+1}) \leq p(x_n)$ and $p(x_n) \neq +\infty$, it follows from (ii) that there exists some $y \in X$ such that $\theta(x_n, y) \geq \theta(y, x_n)$ for all n. Also by assumption y is not maximal in $S(x)$, so there exists $u \in X$ such that $\theta(y,u) \geq \theta(u,y)$ and $\psi(y) < \psi(u)$. Since $\theta(x_n, u) \geq \theta(u, x_n)$, $\psi(u) \leq p(x_n)$ for all n. Also $\theta(x_{n+1}, y) \geq \theta(y, x_{n+1})$ thus $\psi(x_{n+1}) \leq \psi(y)$. Therefore,

$$\psi(u) \leq p(x_n) \leq \psi(x_{n+1}) + \frac{1}{n} \leq \psi(y) + \frac{1}{n}$$

for all n.Hence $\psi(u) \leq \psi(y)$, a contradiction.

Theorem 9.10. Let (X,d) be a complete metric linear space and $\varphi : X \to R$ be a bounded below lower semicontinuous function. Suppose $T : X \to W(X)$ is a fuzzy multivalued mapping such that for each $x \in X$, there exists $\{y\} \subset T(x)$ satisfying $d(x,y) \leq \varphi(x) - \varphi(y)$. Then there exists $x^{/} \in X$ such that

$\{x^{/}\} \subset T(x^{/})$

Proof. Set $\psi = -\varphi$ and for $x, y \in X$, say $\theta(x, y) \geq \theta(y, x)$ provided $d(x, y) \leq \varphi(x) - \varphi(y)$. Note that by assumption for every $u \in X$ there exists $v \in T(u)$ such that $\theta(u, v) \geq \theta(v, u)$. We must verify conditions (i),(ii), and (iii) of theorem 9.9. Condition (i) obviously holds. To see that (ii) holds, observe that if $\{x_n\}$ is any increasing sequence then $\{\varphi(x_n)\}$ is decreasing and bounded below, hence $\{\varphi(x_n)\}$ converges, say to $r \in R$. It further implies that $\{x_n\}$ is a Cauchy sequence. Thus $\{x_n\}$ converges to a point $y \in X$ and since φ is lower semicontinuous it follows that,

$$d(x_n, y) \leq \varphi(x_n) - r \leq \varphi(x_n) - \varphi(y).$$

Thus $\theta(x_n, y) \geq \theta(y, x_n)$ for all n. Since (iii) follows from the fact that φ is bounded below, we conclude [for each $u \in X$, put $f(u) = v$, then $x^{/} = f(x^{/})$] that for each $x \in X$ there exists $x^{/}$ such that $\theta(x, x^{/}) \geq \theta(x^{/}, x)$ and $\{x^{/}\} \subset T(x^{/})$.

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