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INSIDE VERSUS OUTSIDE MONEY: INDETERMINACY IN GEI MODELS

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Abstract

In this paper I consider the issue of indeterminacy of equilibrium in a general equilibrium model with incomplete markets and nominal assets. First, I present some classic results on nominal and real indeterminacy in those models. I then proceed to analyse a more recent literature that focused on the role of money in eliminating indeterminacy of the price level. I show that determinacy depends crucially on the presence of outside money in the economy. I also point out some of the limitations of this literature and some possible way out. In the last part I present a paper that deals with indeterminacy in an altogether different way, namely introducing non-competitive intermediaries that design assets and price them.

Keywords: GEI, indeterminacy, money

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Inside versus Outside Money: Indeterminacy in GEI models.

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1 Introduction

The General Equilibrium model with Incomplete Markets (henceforth GEI) is concerned with the interaction between commodity markets and financial markets when there aren't enough assets to insure against the realisation of uncertainty. In Debreu's (1954) formulation of the Arrow-Debreu (henceforth AD) model, financial markets are left out of the picture for two main reasons, one connected to the very definition of commodities and one to the market structure as it is assumed in the model. First, consumption goods are defined not only on the basis of their physical characteristics, but also taking into account the time and state of nature -or, in other terms, uncertainty- at which they will become available. Indeed the same physical good is taken to be a different good if it is available at a different time and for a different realization of uncertainty. Second, markets for all goods -in the specific sense given above- are assumed to be open at the start of the economy. Under these assumptions it was shown that an equilibrium exists, is determinate and Pareto Efficient. Moreover markets don't need to reopen, all contracts for all transactions are signed at the beginning of time and delivery of each good will take place at the right time and state of nature.

In the Arrow (1953) formulation, goods are defined only according to their physical characteristics -an apple being after all an apple today and still an apple in one week time-. To perform trades across time and states of uncertainty then, financial markets are needed. This idea was pursued further in Radner (1972). Let me briefly explain how the model works in this case. Assume - adopting a minimalistic approach -that

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there are only two periods, one good and two states of uncertainty. Suppose that there are no markets to buy the good contingent on time and the realization of uncertainty, but there is a financial market on which it is possible to buy financial assets. If there are at least two assets -i.e. there are complete markets[1]-, it is then possible to show that this formulation is completely equivalent to Debreu's one, in the sense of having the same set of equilibria and the same properties, namely existence, determinacy and Pareto Efficiency.

It is moreover easily understood that in such economies money cannot have an essential role neither as an intertemporal carrier of value nor as a medium of exchange. In the Debreu case there is no need for a store of value since all contracts are signed at once and for ever and there is no role for a medium of exchange since all agents are simultaneously present at the date 0 central market in which all trades are conducted netting out expenses and receipts. Everyone being perfectly monitored and completely reliable, no much room is left for a tangible medium of exchange to play a role. The same can also be said of the equivalent Arrow formulation.

A more recent literature on AD with financial markets studied the problem of market incompleteness[2]. In this case the number of assets is less than the number of states of uncertainty. The model with incomplete markets, since the classic paper by Hart (1975), was shown to have very different properties compared to the AD formulation with complete markets. In particular it is much more difficult -and in some models only generically possible- to prove existence, determinacy is not always guaranteed and Pareto Efficiency is almost never obtained.

In this paper I will address mainly the problem of indeterminacy in GEI and the conjecture that has been advanced that indeterminacy could be washed away by the introduction of money.

1.1 Indeterminacy and Money

In Debreu (1970) it has been shown that the AD model has generically -in the space of utilities and endowment (i.e. the fundamentals of the economy)- a finite number of equilibria and equilibria vary in a smooth way with fundamentals. This feature is important when performing comparative statics exercises.

The GEI model on the contrary is not quite so well behaved. In particular in the case in which financial securities are nominal in the sense of paying off in units of account -say, dollars-, the model is characterised by a continuum of equilibria for every

[1] In general as many assets as states of nature are needed to have complete markets, in a two period economy.

[2] For an excellent review of the GEI literature see Geanakoplos (1990).

value of the fundamentals -i.e. equilibria are not locally unique[3]-. This phenomenon often referred to as the indeterminacy problem makes any comparative statics analysis impossible.

Although an interesting phenomenon, since it gives rise to peculiar results such as sunspot equilibria and rational expectations equilibria with partial revelation of information, I consider the presence of indeterminacy as originating from a misspecification of the model, in the sense that some of the relevant equations that define equilibrium have been overlooked[4].

The first solution to the indeterminacy problem was given by Magill and Quinzii (1992). It suggested that determinacy could be obtained by introducing money into the GEI model. The basic idea is that by introducing money in each state the price level is pinned down by a sort of Quantity Theory of Money equation and this is enough to restore determinacy. Beside being very crude in the way it created room for money -namely by forcing people to sell their endowment for money and then using the receipts to shop[5]- the approach seems very sensitive to the way money has been introduced.

To show this point I analyse two recent models -by Dreze and Polemarchakis (2000) and Dubey and Geanakoplos (2000)- in which money is injected through a bank and more traditional cash-in-advance constraints -similar to the ones first introduced by Clower (1967)- are imposed. In the first model equilibria are still indeterminate despite the presence of money, while in the second equilibria are determinate. This is intended to highlight the fact that determinacy of equilibria is sensitive to the way money is modelled. It emerges that two features are crucial for indeterminacy: the presence of outside money alongside inside money and the redistribution of profits by the bank. By outside money it is understood an asset that is no one's liability, as opposed to inside money that is someone's liability. In Dreze and Polemarchakis (2000) -where only inside money is available and profits are redistributed- the market clearing conditions on the money market are redundant and therefore equilibria are indeterminate. In Dubey and Geanakoplos (2000), that can be seen as a more general version of Magill and Quinzii (1992), there are both inside and outside money and profits are not redistributed, giving rise to non redundant equilibrium conditions on the money market that allow to determine the price level.

I will then draw a parallel with a recent debate in macroeconomics about indeterminacy and non-ricardian policies, as set out in Woodford (1994), arguing that the

[3] This result has been proved independently by Balasko and Cass (1989) and Geanakoplos and Mas Colell (1989).

[4] Also from a more practical point of view it seems to me that an interesting model should have clear comparative statics predictions.

[5] Observe that in the model this leads to an extreme way of writing the cash-in-advance constraints.

case of inside money (Dreze and Polemarchakis (2000)) corresponds to a ricardian policy case, whereas the outside money one (Magill and Quinzii (1992) and Dubey and Geanakoplos (2000)) corresponds to non-ricardian policies.

An altogether different way of addressing the determinacy issue has been pursued by Bisin (1998). In that model imperfectly competitive intermediaries create themselves securities and price them. By cleverly specifying the profit functions of the intermediaries Bisin (1998) is able to obtain a result that has the flavour of determinacy, i.e. the fact that equilibria don't depend on the price level. Intuitively, the result holds because intermediaries, following a change in the price level adjust asset prices and returns and effectively restore homogeneity of the budget constraint. I argue informally that this approach could be used to get determinacy also in the Dreze and Polemarchakis (2000) approach[6].

1.2 Structure of the Paper

In section 2 I present the GEI model. Section 3 explains in details the indeterminacy problem. In section 4 I discuss the result by Magill and Quinzii (1992). Section 5 contains the models with a bank by Dreze and Polemarchakis (2000) and Dubey and Geanakoplos (2000). In section 6 I review the model by Bisin (1998). Section 7 concludes. The proofs of the propositions central for my enquiry are in most cases sketched in the main body of the paper to convey the gist of the argument but the details have been often skipped. The interested reader is always referred to the relevant papers.

2 Existence of equilibrium in a GEI model with nominal securities.

2.1 The model

The model is a pure exchange economy with two periods $t = 0, 1$ and uncertainty: at time 0 the goods market and securities market are open, at date $t = 1$ one of the $s = 1, ..., S$ states of uncertainty -often called states of nature- realises and the commodity markets open. There are securities yielding a return contingent on the state of nature. In every state there are $l = 1, ..., L$ commodities. With a slight abuse of notation I will sometimes include the only state of nature at time 0 among the states at time 1: $s = 0, 1, ..., S$. The economy is inhabited by $h = 1, ..., H$ agents who trade commodities and securities to consume in every state of nature.

[6] This is formally done in a companion paper.

I will then define $n = L(S+1)$ as the total number of goods across states. I will label p the n-dimensional vector of prices of commodities. I will also indicate securities with $j = 1, ...J$. A security j is a contract promising to pay at $t = 1$ a return a_{js} in state s , for a price q_j at $t = 0$. The price of the J assets will be the row vector $q = [q_1, ..., q_J]$, and the assets return matrix will be

$$A = \begin{bmatrix} a_{11} & ... & a_{j1}... & a_{J1} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ a_{1S} & ... & a_{jS}... & a_{JS} \end{bmatrix}$$

where each column represents the vector of returns of a particular asset state by state. A portfolio $\theta = [\theta_1, ..., \theta_J]'$ is a column vector with positive quantities of assets bought and negative quantities of assets sold as entries. All assets pay in units of account, i.e. in nominal terms. I will refer to the situation in which the number of assets is equal to the number of states $J = S$ as a complete asset market and as an incomplete asset structure when $J < S$.

In what follows I will be primarily concerned with a situation in which markets are incomplete. I therefore assume:

Assumption 1 *The number of assets is less than the number of states: $J < S$.*

Assumption 2 *The assets return matrix A has full column rank.*

The latter assumption -which can be shown to involve no loss of generality if the returns matrix is exogenously given- is intended to exclude the possibility of redundant assets that may prevent the use of differential methods.

Agents evaluate consumption according to a utility function $u^h : \Re^n_+ \to \Re$ and are endowed initially with different quantities of commodities in different states of nature $\omega^h \in \Re^n_{++}$. These are the fundamentals of the pure exchange economy. The fact that the initial endowment depends on the state of nature means that the uncertainty is intrinsic to the model and not extrinsic as in the case of sunspot economies.

To proceed in the analysis some assumptions on the utility functions will be needed:

Assumption 3 $\forall h$, $u^h : \Re^n_+ \to \Re$ *is twice continuously differentiable and satisfies:*

1. $D_x u^h(\overline{x}^h) \in \Re^n_{++}$, *for every* $\overline{x}^h \gg 0$.

2. $(x^h - \overline{x}^h) D^2_x u^h(\overline{x}^h)(x^h - \overline{x}^h)' < 0$, *for every* $(x^h - \overline{x}^h)$ *s.t.* $D_x u^h(\overline{x}^h)(x^h - \overline{x}^h) = 0$.

3. $\left\{x^h : u^h(x^h) = u^h(\overline{x}^h)\right\} \subset \Re^n_{++}$, *for every* $\overline{x}^h \gg 0$.

This assumption is standard and it serves the purpose of having a maximisation problem that can be solved using the differentiable approach. It says that the utility function should be strictly increasing, strictly concave and the indifference curves shouldn't cross the axis. This last assumption is meant to exclude corner solutions.

Each consumer h solves the following maximisation problem (P1):

$$\underset{x^h \in \Re^n_+, \theta^h \in \Re^J}{Max} u^h(x^h)$$

$$\begin{aligned} s.t. \quad & p_0(x_0^h - \omega_0^h) + q\theta^h = 0 \\ & p_s(x_s^h - \omega_s^h) = A_s\theta^h \quad \forall s \end{aligned}$$

A complication may arise when trying to solve (P1), due to the fact the budget set may not be compact. Suppose that some asset prices and returns are such that there exists the possibility of holding a portfolio that gives positive returns and costs nothing. In this case -i.e. when arbitrage is possible- the maximisation problem will not have a solution since agents will want to hold infinite quantities of such assets, making the budget set unbounded. I want then to make sure that there is no possibility of arbitrage in this economy. This leads to the following assumption.

Assumption 4 *There doesn't exists a $\theta^h, \forall h$ s.t. $q\theta^h \leq 0$ and $A\theta^h \geq 0$.*

This no-arbitrage assumption is necessary and sufficient to ensure that a solution to (P1) exists.

It is also easy to show that when asset prices don't admit arbitrage there always exists (even though it is not always unique when markets are incomplete) a representation of those prices as a linear function of the returns using a vector of implicit prices $\pi = [\pi_0, ..., \pi_S]$. In other terms it can be shown that when the no-arbitrage assumption holds asset prices are equal to the "discounted" value of the returns where the discount factor is given by the implicit prices. In particular I can write $\pi_0 q = \sum_s \pi_s A_s$ or after the normalisation $\pi_0 = 1$, I can write

$$q = \pi A$$

When $J = S$, the complete markets case, moreover, implicit prices are uniquely determined.

2.2 Equilibrium

I will now introduce the equilibrium concept. In an equilibrium, agents will maximise utility subject to the budget constraint and prices will clear the market. Formally

Definition 1 *(p,q,x,θ) is a Financial Equilibrium if :*

1. $\forall h, \quad (x^h,\theta^h)$ solves (P1)

2. $\sum_h x^h = \sum_h \omega^h$ for every commodity in every state of nature

3. $\sum_h \theta^h = 0$ for every asset

Using the no-arbitrage condition, the previous definition can be reduced to one where only commodities and no assets appear. Define as before $\pi = (1, \pi_1)$ the normalised vector of implicit prices. From the no-arbitrage condition I have that $q = \pi A$, and manipulating the period 0 budget constraint,

$$p_0(x_0^h - \omega_0^h) = -q\theta^h = -\pi A\theta^h = -\sum_s \pi_s A_s \theta^h$$

and from period 1 budget constraint:

$$\sum_s \pi_s p_s(x_s^h - \omega_s^h) = \sum_s \pi_s A_s \theta^h$$

Combining the two I get

$$p_0(x_0^h - \omega_0^h) + \sum_s \pi_s p_s(x_s^h - \omega_s^h) = 0$$

or in vector notation simply

$$\pi p(x^h - \omega^h) = 0$$

Define also $P = \pi p$ and $P_1 = [\pi_1 p_1, ..., \pi_s p_s]$.

The new maximisation problem will be (P2):

$$\begin{gathered} \underset{x^h \in \Re^n_+}{Max} \quad u^h(x^h) \\ s.t. \quad P(x^h - \omega^h) = 0 \\ P_1(x_1^h - \omega_1^h) \in span\,[\pi_1 A] \end{gathered}$$

where $span\,[\pi_1 A]$ stands for the smallest space generated by the columns of $\pi_1 A$. I succeeded in transforming the problem into one that is closer to the standard AD formulation with the difference that the value of the excess demand for commodities is constrained to lie in a smaller space since there are not enough assets to complete the markets.

Definition 2 *(P,x) is a No-Arbitrage Equilibrium (NA) if:*

1. $\forall h, \quad (x^h)$solves (P2)

2.$\sum_h x^h = \sum_h \omega^h$or every commodity in every state of nature.

I have to make sure that the two problems are completely equivalent. This is what the next proposition does.

Proposition 1 *A No-Arbitrage Equilibrium exists if and only if a Financial Equilibrium exists.*

Proof. The proof is straightforward. Assume first that a Financial Equilibrium $(\overline{p}, \overline{q}, \overline{x}, \overline{\theta})$ exists. Then we have to show that $(\overline{P}, \overline{x})$ is a NA equilibrium. Observe that by the transformation above the budget constraints of the two problems are equivalent. Agents are then maximising the same utility function on the same budget constraint. Moreover $\overline{x}$ is feasible and $\overline{P} = \overline{p\pi}$.

Assume now that $(\overline{P}, \overline{x})$ is a NA equilibrium, we have to show that there exist $(\overline{q}, \overline{\theta})$ s.t. $(\overline{p}, \overline{q}, \overline{x}, \overline{\theta})$ is a Financial Equilibrium. Let's construct it. $\overline{q} = \overline{\pi} A$, portfolios $\overline{\theta}$ are such as to satisfy $p_s(x_s^h - \omega_s^h) = A_s \theta^h \forall s$ and feasibility. If $\overline{\pi}$ are not unique (as it is usually the case when markets are incomplete) pick without loss of generality $\overline{\pi}^1$ associated with the first agent. Now agents optimise over an equivalent budget constraint and $\overline{x}$ is feasible. ■

2.3 Existence of a No-Arbitrage Equilibrium

The usual tool to prove existence of equilibrium in the AD model are the Theorems by Brower and Kakutani. Here I will use a differentiable version of Brower Theorem, called The Vector Field Theorem.

Theorem 1 *A continuous vector field defined on the positive part of the unit sphere pointing inward at the boundary of the sphere has a zero in the interior of the sphere.*

To apply this theorem to the situation at hand I need to show that the excess demand

$$\sum_h z^h = \sum_h \left(x^h - \omega^h\right)$$

satisfies the following five properties:

1. Walras Law
2. Continuity
3. Homogeneity of degree 0 in prices
4. Boundedness from below
5. Boundary behaviour: if $p^t \in \Re^n_{++}$ is s.t. $p^t \to p^0 \in \partial\Re^n_{++} \setminus \{0\}$, then $\|Z(p^t)\| \to \infty$, when $t \to \infty$, where $Z(p) = \sum_h z^h(p)$.

Homogeneity of degree 0 in fact, allows me to normalise prices to lie in the positive unit sphere:

$$S_{+}^{n-1}=\left\{p:\sum_{n}p_n^2=1\ e\ p_n\geq 0,\forall n\right\}$$

.

Walras law ensures that the excess demand is a vector field on $S_{++}^{n-1}: pZ(p)=0$. Continuity of the excess demand gives immediately the continuity of the vector field, while the boundary behaviour allows me to construct an equivalent vector field that is inward pointing at the boundary of the sphere. Finally the boundedness from below is intended to avoid that the excess demand diverges to $-\infty$.

Contrary to the standard AD model, the present formulation verifies only the first four properties but not the fifth. Looking at the date one budget constraint -in (P1)- it is easily seen that when some price goes to zero the (norm of the) excess demand cannot go to infinity since it is bounded by the returns of the assets. Nevertheless it is possible to use the so called "Cass trick"[7] to prove the existence of equilibrium.

2.3.1 The "Cass trick"

The "Cass trick" states that without loss of generality it is possible to consider the -say- first consumer as unconstrained by the span condition :

$$\underset{x^1\in\Re_{+}^{n}}{Max}\quad u^1(x^1)$$

$$s.t.\quad P(x^1-\omega^1)=0$$

while the others will solve (P2). This is without loss of generality since if $\forall h\geq 2$,

$$P_1(x_1^h-\omega_1^h)\in span\,[\pi_1 A]$$

and market clearing holds

$$\sum x^h=\sum \omega^h$$

it immediately implies that

$$P_1(x_1^1-\omega_1^1)\in span\,[\pi_1 A]$$

is automatically verified for the first agent.

The trick is useful since when one of the prices is going to zero the excess demand of the first consumer which is unconstrained diverges to infinity and this is enough to have the norm of the aggregate excess demand diverge to infinity, thus satisfying property 5. This allows me to construct a vector field $\widetilde{Z}$ equivalent to Z that is inward pointing at the boundary of the unit sphere and existence can be proved.

Proposition 2 *Under assumptions 1, 2 , 3 and 4 a NA Equilibrium exists.*

[7] the trick has been invented by David Cass

3 Indeterminacy

I already said that, contrary to the standard AD model with complete markets, the equilibria of the present model are neither Pareto Efficient nor locally unique. My focus is on the latter issue.

As in AD, here as well there is a problem of nominal indeterminacy: only relative prices in the $S+1$ states of nature can be determined. $S+1$ conditions are then needed to determine prices. Contrary however to AD the normalisation procedure is not neutral with respect to real allocations. In particular $S-1$ of the normalisations have an effect on equilibrium allocations. This is what it is meant by real indeterminacy.

3.1 Walras test and nominal indeterminacy

Historically the first economist to deal with the problem of indeterminacy has been Leon Walras. His procedure to detect indeterminacy was to count equations and unknowns. If the number of the former is higher than the number of the latter there is indeterminacy.

In the GEI model there are n equilibrium conditions for commodities and J for assets, giving a total of $n+J$ equations. Walras law applies $S+1$ times, one for every budget constraint. The total number of linearly independent equations will then sum to

$$W = n + J - S - 1$$

The unknowns are (p, q, ω, A). Counting dimensions I obtain that the number of variables to be determined is

$$K = n + J + Hn + SJ$$

The degree of indeterminacy will then be

$$K - W = Hn + SJ + S + 1$$

Once (ω, A) (dimension equal to $Hn + SJ$) have been fixed, the degree of nominal indeterminacy is

$$S + 1$$

which gives the number of normalisations needed, namely one for every state of nature.

3.2 Real Indeterminacy

Contrary to nominal indeterminacy, real indeterminacy is a phenomenon peculiar to the model with nominal assets. In a model in which assets paid off in real terms -for instance in a basket of commodities- the budget constraints would be homogeneous of degree 0 in prices and this would allow me to normalise prices in all states of nature without any impact on equilibrium allocations, indeterminacy would be purely nominal and would disappear after the normalisation.

In the case under scrutiny however only two normalisations are neutral with respect to allocations and the degree of real indeterminacy is in general $S-1$. More precisely consider the $S+1$ budget constraints in the consumer problem:

$$p_0(x_0^h - \omega_0^h) + q\theta^h = 0$$

$$p_s(x_s^h - \omega_s^h) = A_s\theta^h \quad \forall s$$

and multiply prices by a factor α[8]. There are only two ways in which this operation doesn't affect allocations:

$$\alpha(p_0, q)$$

i.e. rescaling commodities and assets prices at date 0 and

$$(\alpha p_1, \frac{1}{\alpha}q)$$

increasing (if $\alpha > 1$) commodities prices at date 1 and decreasing by the same amount assets prices at date 0. All the other $S-1$ ways of rescaling prices by a factor α, will change real allocations. Indeed for

$$\alpha p_s(x_s^h - \omega_s^h) = A_s\theta^h$$

to be verified,

$$x_s^h - \omega_s^h$$

has to adjust when prices change, since the term on the RHS is not sensitive to normalisations - i.e. the budget constraint is not homogeneous in prices-. This gives the intuition for the result proved by Geanakoplos and Mas-Colell (1989) and Balasko and Cass (1989). The assumptions in the proposition are purely technical.

Proposition 3 *Under assumption 1, 2 and 3 and assuming that A is in general position*[9] *and $H \geq J$, generically there will be $S-1$ degrees of real indeterminacy.*

[8] This amounts to a normalisation.

[9] An $S \times J$ matrix is in general position if every $J \times J$ submatrix obtained deleting rows has full rank. For instance an Arrow Securities matrix - made of assets paying 1 in one state and 0 in the other states- is not in general position.

4 Money and Indeterminacy[10]

4.1 The model

Let me now consider a slightly more complicated model in which there is a Central Market Authority organizing trade and injecting money in the economy.

Each of the two periods is now divided into three subperiods. In the first subperiod of period 0, each agent has to sell his entire initial endowment to the Authority at price p to receive a quantity of money $m^h = p\omega^h$. In the second subperiod agents buy and sell assets and in the third subperiod they buy commodities using money. At date $t = 1$ for every state of nature that may realise, the same sequence of events happens with the exception that in the second subperiod agents receive the returns from the assets they hold. All trades are mediated by the Central Market Authority.

The new maximum problem is (P3):

$$\begin{aligned} &\underset{x^h \in \Re^n_+, \theta^h \in \Re^J}{Max} \quad u^h(x^h) \\ s.t. \quad & p_0 x_0^h = p_0 \omega_0^h - q\theta^h \\ & p_s x_s^h = p_s \omega_s^h + A_s \theta^h \quad \forall s \\ & m_s^h = p_s \omega_s^h \quad s = 0, ..., S \end{aligned}$$

Notice that there is now an additional constraint that resembles a Cash-in-Advance constraint similar to the one first used by Clower (1967). In the present formulation, though, the RHS is represented by the entire initial endowment. This is due to the strong assumption that agents are forced to sell the entire initial endowment to the Central Market Authority. The monetary policy of the Central Market Authority will be given by fixing a nominal amount $M \in \Re^{S+1}_{++}$ s.t.

$$M_s = \sum_h m_s^h, \quad s = 0, ..., S.$$

I am now ready to define equilibrium.

Definition 3 (p, q, x, θ, M) *is a Monetary Equilibrium if:*

1. $\forall h, \quad (x^h, \theta^h)$ *solves (P3)*
2. $\sum_h x^h = \sum_h \omega^h$ *for every commodity in every state of nature*
3. $\sum_h \theta^h = 0$ *for every asset*
4. $p_0 \sum_h x_0^h = M_0$

[10] The model presented here is an adaptation of Magill and Quinzii (1992).

$$p_s \sum_h x_s^h = M_s \quad \forall s$$

The equations at point 4. in the definition represent a particular version of the Quantity Theory of Money, being the velocity of circulation fixed to unity. It is however easy to obtain a more general version with velocity different from 1.

4.2 NA Equilibrium

Exactly as before I can transform the economy into a No-Arbitrage one, turning as before (P3) into (P3')

$$\underset{x^h \in \Re_+^n}{Max} \quad u^h(x^h)$$

$$s.t. \quad P(x^h - \omega^h) = 0$$

$$P\omega^h = \pi m^h$$

$$P_1(x_1^h - \omega_1^h) \in span\,[\pi_1 A]$$

I can then apply the "Cass trick" and solve for the equilibrium.

Definition 4 *(P, x) is a NA equilibrium if:*
1. $\forall h, \quad (x^h)$ solves (P3')
2.$\sum_h x^h = \sum_h \omega^h$ for every good in every state of nature
3. $P \Box \sum_h x^h = \pi \Box M$

where under point 3. I use $\Box$ to indicate the box product:

$$P \Box \sum_h x^h \equiv \begin{bmatrix} P_0 \sum_h x_0^h \\ \dots \\ P_s \sum_h x_s^h \\ \dots \\ P_S \sum_h x_S^h \end{bmatrix} = \begin{bmatrix} \pi_0 M_0 \\ \dots \\ \pi_s M_s \\ \dots \\ \pi_S M_S \end{bmatrix} \equiv \pi \Box M$$

The Vector Field Theorem is then applicable and I can state the following proposition

Proposition 4 *Under assumptions 1, 2, 3 and 4 a Monetary Equilibrium exists.*

4.3 Determinacy

In the present model with money, equilibrium is determinate. The intuitive reason is that the $S+1$ equations determining the monetary policy of the Central Market Authority give exactly the equations needed to fix price levels state by state. To formally show determinacy however I need to introduce some more notation and some theorems that will allow me to handle the problem.

I aim to show that there exist a big enough set of parameters such that the number of equilibria is finite and equilibria vary in a smooth way with parameters. Stated in other terms, the objective is to show that equilibria are generically locally isolated.

For that I need two theorems: the Preimage Theorem and Sard's Theorem.

Theorem 2 *If the function $f : X \to Y$, is smooth, X is compact and $dimX = dimY$ and moreover y is a regular value of f (i.e. $D_x f(x)$ is surjective at every $x : f(x) = y$), then the preimage of y through f, $f^{-1}(y)$ is a finite set of points.*

The former is a generalisation of the inverse function theorem.

Theorem 3 *If the function $f : X \to Y$, is smooth, then the set*

$$\{y : D_x f(x) \text{ is not surjective at } x : f(x) = y.\}$$

has measure zero in Y.

This theorem applied to a simple unidimensional example says that a smooth function from the reals to the reals has only a finite number of stationary values.

Both theorems together imply that -if the assumptions are satisfied- I can take at random[11] a value of a smooth map, invert the map and obtain a finite number of points. Let me apply the two theorems to the model at hand, after introducing some notation. Take $\Pi = \Re^n_{++} \times \Re^s_{++}$ as the space of prices: $(P, \pi) \in \Pi$, Ω the space of initial endowments and M the space of monetary policy.

The aggregate excess demand for commodities is

$$Z : \Pi \times \Omega \times M \to \Re^n, Z(P, \pi, \omega, M) = \sum_h x^h(P, \pi, \omega, M) - \sum_h \omega^h$$

and the excess demand for money is

$$Z^m : \Pi \times \Omega \times M \to \Re^{S+1}_{++}, Z^m(P, \pi, \omega, M) = P \Box \sum_h x^h - \pi \Box M$$

[11] Since by Sard theorem non-regular values have measure zero, I will pick almost always a regular value.

Define also as the equilibrium system

$$H(P,\pi,\omega,M)=0$$

where

$$H(P,\pi,\omega,M)=\begin{bmatrix} Z(P,\pi,\omega,M) \\ Z^m(P,\pi,\omega,M) \end{bmatrix}$$

Let me finally define

$$V=\{P,\pi,\omega,M\in\Pi\times\Omega\times M: H(P,\pi,\omega,M)=0\}$$

as the equilibrium manifold and $\theta : V \to \Omega\times M$ as the natural projection of the equilibrium manifold onto the parameters space. θ will have the role of the function f in the two theorems and it can be shown to be smooth. By Sard's Theorem the set of critical (non regular) values has measure zero. Moreover it can be shown that θ is proper[12] implying that the set is also closed $\Omega\times M$. Finally V is compact and

$$dimV = HN+S+1 = dim\,(\Omega\times M)$$

which, using the preimage theorem implies that $\theta^{-1}(\omega,M)$ is a finite set of points. Moreover θ^{-1} is smooth. This gives a sketch of the proof of the following proposition

Proposition 5 *There exists a set $\Delta\subset\Omega\times M$ open and of full measure[13] s.t.:*

1. Every economy s.t. $(\omega,M)\in\Delta$, has a finite number of equilibria.

2. For every $(\overline{\omega},\overline{M})\in\Delta$, there exists a neighbourhood U s.t. every monetary equilibrium is a smooth function of (ω,M) for $\forall(\omega,M)\in U$.

To summarise I can choose the parameters (ω and M) at random and be almost sure of selecting parameters for which equilibria are locally unique and vary in a smooth way.

Remark 4 *In this model it is also possible to show that monetary policy is non-neutral. This is not surprising since monetary policy fixes $S+1$ normalisations, but only two of them are neutral. There are then $S-1$ ways in which monetary policy can affect allocations, by varying M.*

In order to understand how general is the result that the introduction of money solves the indeterminacy problem, I now turn to alternative ways of modelling money in a GEI framework.

[12] A function is proper if the preimage of a compact set through the function is compact.

[13] open+full measure=generically

5 GEI and Banks: inside and outside money[14]

5.1 Inside money

5.1.1 The model

I will replace the Central Market Authority with a bank that lends money to agents at an interest rate r. In this model only inside money circulates. Here is the sequence of events. At date 0, in the first subperiod the agents borrow money from the bank issuing a promise to return it: $\widehat{m}^h = -b^h$. In the second subperiod they use money to buy commodities and assets: $\widehat{m}^h = pz^{h+} + q\theta^h$. In the third subperiod they sell commodities for money: $m^h = pz^{h-}$, and they receive dividends from the bank v^h. In the forth subperiod they use money and dividends to repay the loan comprehensive of interest: $v^h + m^h = -(1+r)b^h$. The budget constraint at date zero can therefore be formed by observing that $\frac{v^h+m^h}{1+r} = -b^h = \widehat{m}^h = pz^{h+} + q\theta^h$. In turn $pz^{h+} + q\theta^h = pz^h + pz^{h-} + q\theta^h$ and since $pz^{h-} = m^h$ it gives

$$pz^h + m^h + q\theta^h = \frac{v^h + m^h}{1+r}$$

Where I defined net demand and supply of commodities as $z_l^{h+} = \max\left\{z_l^h, 0\right\}$ and $z_l^{h-} = \min\left\{z_l^h, 0\right\}$ and $z_l^h = z_l^{h+} + z_l^{h-} \quad \forall l$.

At date 1 in each state of nature, the same sequence of events repeats itself, with the only difference that in the second subperiod returns accrue to consumers.

The sequence of events gives rise to the usual $S+1$ budget constraints, plus the cash-in-advance constraint which in this case is not any more given by the value of the entire initial endowment, but only by the value of the share of commodities that agents decide to sell.

To avoid some technical difficulties and since the focus is on money as a medium of exchange and not as a store of value, I will assume that there exist a riskless asset. Agents will then use the riskless asset to transfer their wealth to the next date and all money will be returned to the bank at the end of each period.

Assumption 5 *There exists an asset c such that $A_c = [1, ..., 1]$.*

Each agent h solves the following problem (P4):

$$\underset{x^h \in \Re^n_+, \theta^h \in \Re^J, m^h \in \Re_+}{Max} \quad u^h(x^h)$$

[14] I will analyse two recent models, respectively by Dreze and Polemarchakis (2000) and Dubey and Geanakoplos (2000). Both models were originally formulated in a complete market framework. Here I present my version of the models with incomplete markets.

$$s.t. \quad p_0(x_0^h - \omega_0^h) + q\theta^h + \frac{r_0}{1+r_0} m_0^h = \frac{v_0^h}{1+r_0}$$

$$p_s(x_s^h - \omega_s^h) + \frac{r_s}{1+r_s} m_s^h = \frac{v_s^h}{1+r_s} + A_s\theta^h \quad \forall s$$

$$p_s z_s^{h-} = m_s^h \quad s = 0, ..., S$$

Bank's profits are $r_s M_s, \quad s = 0, ..., S$, where $M_s = \sum_h m_s^h$ is the quantity of money injected by the bank in the economy in each state of nature. Let me define dividends as $v_s^h = \delta^h r_s M_s$, where $\sum_h \delta^h = 1$, this is to say that the bank is owned entirely by consumers.

Definition 5 (p, q, r, x, θ, M) *is an equilibrium with inside money if:*

1. $\forall h, \quad (x^h, \theta^h, m^h)$ *solves (P4)*
2. $\sum_h x^h = \sum_h \omega^h$ *for every good in every state of nature*
3. $\sum_h \theta^h = 0$ *for every asset*
4. $p_0 \sum_h z_0^{h+} = M_0$

   $p_s \sum_h z_s^{h+} = M_s \quad \forall s$
5. *bank's profits are given by* $r_s M_s, \quad s = 0, ..., S$

Notice that even in this formulation the equations of the Quantity Theory of Money do appear. However contrary to what happened before they place no restrictions on the equilibrium. This can be easily seen by aggregating the budget constraints:

$$\sum_h \left[p_0(x_0^h - \omega_0^h) + q\theta^h + \frac{r_0}{1+r_0} m_0^h - \frac{v_0^h}{1+r_0} \right] = 0$$

$$\sum_h \left[p_s(x_s^h - \omega_s^h) + \frac{r_s}{1+r_s} m_s^h - \frac{v_s^h}{1+r_s} - A_s\theta^h \right] = 0 \ \forall s$$

and the cash-in-advance constraint:

$$\sum_h \left[p_s z_s^{h-} - m_s^h \right] = 0 \quad s = 0, ..., S.$$

Imposing in the equations above market clearing on commodities and assets markets, I get respectively

$$\sum_h m_0^h = M_0$$

$$\sum_h m_s^h = M_s \forall s$$

and

$$\sum_h \left[p_s z_s^{h-} - m_s^h\right] = 0 \quad s = 0, ..., S$$

Notice also that in equilibrium $\sum_h z_s^{h-} = \sum_h z_s^{h+}$ $s = 0, ..., S$. Putting these equations together I obtain

$$\begin{aligned} p_0 \sum_h z_0^{h+} &= M_0 \\ p_s \sum_h z_s^{h+} &= M_s \quad \forall s \end{aligned}$$

which are exactly the equations under 4. This observation gives the intuition for the indeterminacy result to follow and depends crucially on the fact that the bank is distributing profits in the form of dividends and that there is only inside money, as it can be seen setting $\delta^h = 0, \forall h$ and noticing that the previous reasoning fails to work in this case.

5.1.2 Equilibrium

Here, as in many other models of monetary economies but unlike the model with forced sales, it could happen that the equilibrium exists but money doesn't circulate -or as Hahn (1965) put it money is inessential-. An example is given by a situation in which the initial endowment is Pareto Efficient. In this case the only equilibrium is autarky and obviously money doesn't circulate. To avoid this difficulty I will assume that autarky is not efficient and that the cash-in-advance constraint are always binding. This is enough to prove, with the -by now- familiar methods, the following proposition.

Proposition 6 *Under assumptions 1, 2, 3, 4 and 5 an equilibrium with inside money exists.*

The crucial propositions for our discussion are however the following two.

Proposition 7 *Under the maintained assumptions there are $S+1$ degrees of nominal indeterminacy.*

Proposition 8 *Under the maintained assumptions, if A is in general position and $H \geq J$, generically there are $S-1$ degrees of real indeterminacy.*

These propositions give a precise statement for the intuition given above about the redundancy of the Quantity Theory equations.

5.2 Inside and Outside money.

5.2.1 The model

I will introduce two new features in the previous model. I will assume that banks don't distribute dividends -$\delta^h = 0, \forall h$- and there is outside money in the economy, in the form of an initial endowment $\overline{m}^h$ of an asset called fiat money that is no one's liability. The sequence of events is as follows.

At date 0, in the first subperiod agents receive a loan of money from the bank against a promise of repaying the loan: $\widehat{m}^h = -b^h$. In the second subperiod they use inside and outside money to buy commodities and securities: $\widehat{m}^h + \overline{m}^h = pz^{h+} + q\theta^h$. In the third subperiod they sell commodities for money: $m^h = pz^{h-}$. In the fourth subperiod they use what is left of their money holdings to repay their debt: $m^h = -(1+r)b^h$. Exactly as before net demand and supply are given respectively by $z_l^{h+} = \max\left\{z_l^h, 0\right\}$ and $z_l^{h-} = \min\left\{z_l^h, 0\right\} \quad \forall l$. The budget constraint is formed exactly as before.

At date 1 the same sequence of events takes place with the exception of the second subperiod when agents receive the returns from the assets they have in their portfolio.

The maximisation problem is (P4'):

$$\underset{x^h \in \Re^n_+, \theta^h \in \Re^J, m^h \in \Re_+}{Max} \quad u^h(x^h)$$

$$s.t. \quad p_0(x_0^h - \omega_0^h) + q\theta^h + \frac{r_0}{1+r_0} m_0^h = \overline{m}_0^h$$

$$p_s(x_s^h - \omega_s^h) + \frac{r_s}{1+r_s} m_s^h = \overline{m}_s^h + A_s\theta^h \quad \forall s$$

$$p_s z_s^{h-} = m_s^h \quad s = 0, ..., S$$

Two features are crucial to overturn the indeterminacy result of the previous section, namely the presence of outside money and the fact that dividends are not distributed.

5.2.2 Equilibrium

I already mentioned before that there could be equilibria in which money is not essential. To deal with this issue more formally, let me consider the rate of interest on the loans as a transaction cost incurred by agents in the process of exchange. It's clear that for money to be essential -i.e. for people to want to hold money and trade with it- the benefit of trading with money should be higher than the transaction cost given by r. Call the benefit of trading γ and define with $\gamma(\omega)$ the benefit computed at the initial endowment. From the observation that the bank receives all the (inside and outside) money at the end of each period in an amount equal to $\sum_h \overline{m}_s^h + M_s, \forall s$ I obtain that this amount state by state is equal to the total repayment from consumers

$$\sum_h \overline{m}_s^h + M_s = (1 + r_s)M_s, \forall s$$

Solving the equation for r I obtain, state by state, the interest rate

$$r_s = \sum_h \overline{m}_s^h / M_s, \forall s$$

To have a monetary equilibrium it is then enough to assume:

Assumption 6 *The benefit of trade is higher than the transaction cost* $\gamma_s(\omega) > \sum_h \overline{m}_s^h / M_s$ *in every state.*

I can now formally define the equilibrium.

Definition 6 (p, q, r, x, θ, M) *is an equilibrium with inside and outside money if:*

1. $\forall h, \quad (x^h, \theta^h, m^h)$ *solves (P4')*

2. $\sum_h x^h = \sum_h \omega^h$ *for every good in every state of nature*

3. $\sum_h \theta^h = 0$ *for every asset*

4. $p_0 \sum_h z_0^{h+} = M_0$

$p_s \sum_h z_s^{h+} = M_s \quad \forall s$

5. The bank's profits are $r_s M_s, \quad s = 0, ..., S$

The following proposition guarantees the existence of a monetary equilibrium.

Proposition 9 *Under assumptions 1, 2, 3, 4, 5 and 6 an equilibrium with outside and inside money exists.*

As already seen the interest rate in the equilibrium will be given by $r_s = \sum_h \overline{m}_s^h / M_s, \forall s$

5.2.3 Determinacy

When $\sum_h \overline{m}_s^h > 0$, so that there is outside money in the economy, the equilibrium is determinate. The reason again is that the Quantity Theory equations place $S + 1$ effective restrictions on the equilibrium. This is not however true in the case of no outside money, when $\sum_h \overline{m}_s^h = 0$. In this case the model collapses to a model with incomplete markets and nominal assets and no money at all. In this case of course the equilibrium is indeterminate[15].

[15] This is easily seen by plugging $r_s = 0$ and $\overline{m}_s^h = 0$ for every h and s into the equations above.

Proposition 10 *Assume* $\sum_h \overline{m}_s^h > 0$, *then there exists a set* $\Delta \subset \Omega \times M$ *open and full measure s.t.*

1. Every economy s.t. $(\omega, M) \in \Delta$ *has a finite number of equilibria.*

2. For every $(\overline{\omega}, \overline{M}) \in \Delta$ *there exists a neighbourhood* U *s.t. every equilibrium is a smooth function of* (ω, M) *for* $\forall(\omega, M) \in U$.

As in the case of Magill and Quinzii (1992), increasing the quantity of inside money has real effects through the increase in prices and the decrease in the interest rate.

To sum up, Magill and Quinzii (1992) can be seen as a special case of Dubey and Geanakoplos (2000), where the absence of redistribution of profits and the presence of outside money makes the equilibrium conditions on the money market non-redundant, while in Dreze and Polemarchakis (2000) the equilibrium conditions on the money market are redundant and the model is virtually indistinguishable from a standard GEI model with nominal assets.

Remark 5 *The point raised in this literature has the flavour of a recent debate in macroeconomics concerning non-ricardian policies. Woodford (1994) showed that if the government uses non-ricardian policies -in the sense that transfer payments are independent of equilibrium prices- indeterminacy, which is present in the case of ricardian policies -i.e. when transfer payments depend on equilibrium prices-, disappears. In the present context the distribution of profits combined with the fact that the only form of money is inside money could be broadly interpreted as ricardian policies (they do depend on equilibrium prices and interest rates), giving rise to indeterminacy of equilibrium. On the other hand the outside money world without distribution of dividends is a case in which "transfers" are fixed beforehand without reference to equilibrium prices and interest rates and this, as in Woodford (1994), restores determinacy.*

6 GEI and Intermediation[16]

I will review now an alternative way of dealing with the problem of indeterminacy. One of the crucial implicit assumptions in the GEI model, is that the matrix of returns A, is exogenously given. In other terms it is left unexplained and unmodelled how assets are created. This assumption turns out to be crucial also as faras the indeterminacy of equilibrium is concerned. In particular if the process of creation of assets is explicitly modelled the real indeterminacy result disappears.

[16] In this section I adapt and discuss a recent paper by Bisin (1998). In the original formulation there is no money in the model. The result on real determinacy holds also without money.

6.1 The model

Let me add to the model some imperfectly competitive agents, which I will call financial intermediaries. In particular consider the same model presented in section 3 -with a Central Market Authority and forced sales- in which assets aren't any more exogenously given, but they are designed by financial intermediaries.

Specifically, intermediaries indexed by $k = 1, ..., K$, design the assets that will be exchanged on the financial market (the matrix A is endogenously determined) and impose a spread σ between buying and selling prices[17] to maximise profits, expressed in numeraire. Intermediaries have an initial endowment ω_1^k, only of good 1 and they want to consume good 1 only. There are transaction costs to intermediation : $c(a, p_1)$ are the fixed costs and $\varepsilon(a, p_1)$ the variable costs. Define $p_1 = [p_{10}, p_{11}, ..., p_{1S}]$ the vector of the numeraire price in each state of nature and $a = [a_1, ..., a_S]$ the vector of returns of a security in each state of nature[18]. Observe that since assets are endogenously created by intermediaries it is not any more possible to assume without loss of generality that the matrix A has full row rank. This, together with the bid-ask spread, will in general impair the possibility of using differentiable methods in the proof of existence and determinacy. Costs are expressed in numeraire terms and satisfy the following assumption:

Assumption 7 *1. $c(a, p_1) \geq 0$ and $\varepsilon(a, p_1) \geq 0$; $= 0$ iff $a = 0$.*

2. $c(a, p_1)$ and $\varepsilon(a, p_1)$ are homogeneous of respectively degree 0 and 1 in a for every p_1.

3. $c(a, p_1) = c(a/p_1)$ and $\varepsilon(a, p_1) = \varepsilon(a/p_1)$ for every $a \neq 0$.

4. $c(a, p_1)$ and $\varepsilon(a, p_1)$ are twice continuously differentiable.

The assumption imposes non-negativity on costs, homogeneity, independence from the denomination of returns and differentiability. These assumptions together, imply the possibility of normalising returns and of imposing an upper bound $\overline{c}$ on fixed costs.

Consumers are modelled as before. The sequence of events for them is unchanged.

Intermediaries play a simultaneous game G, choosing the spread σ^k and asset returns A^k. Fixed costs are paid after the choice of spreads and returns. each intermediary can create at most J^k securities, where J^k is s.t. $J^k \overline{c} < \omega_1^k$, to avoid the complication associated with the possibility of default by intermediaries. Define $J = \sum_k J_k$ as the total number of assets intermediaries can create. To formally define a game I need to define the number of players, the strategy spaces and pay-offs.

[17] Only the spread is fixed by the intermediaries, buying and selling prices are still competitively determined.

[18] In what follows I will use the notation a/p_1 to indicate the ratio of the two vectors component by component: $a_1/p_{11}, a_2/p_{12}, ...a_S/p_{1S}$.

The game G is given by:

1. The set of players is $\{1, ..., K\}$

2. Define with Σ^k the space of spreads and A^k the space of returns. These are the strategy spaces.

3. The profit function for each k is given by a function $\Pi^k : \Sigma^k \times A^k \rightarrow \Re$, s.t.

$$\Pi^k = \sum_{j=1}^{J^k} \left[\left(\frac{\sigma^k}{p_1} - \varepsilon(a^k, p_1) \right) \Theta_j^+ - c(a^k, p_1) \right]$$

.

Where demand and supply of securities by consumers are $\theta_j^{h+} = \max \left\{ \theta_j^h, 0 \right\}$ and $\theta_j^{h-} = \min \left\{ \theta_j^h, 0 \right\} \forall j$ and $\Theta_j^+ = \sum_h \theta_j^{h+}$. Sometimes I will also use $\theta^h = \theta^{h+} - \theta^{h-}$.

To avoid negative profits and default, I will assume $\frac{\sigma^k}{p_1} - \varepsilon(a^k, p_1) \geq 0$.

The game is then fully described. Simultaneously and taking the choices of intermediaries and prices[19] as given, consumers solve their usual maximisation problem.

Each consumer h solves (P5):

$$\begin{aligned} & \underset{x^h \in \Re_+^n, (\theta^{h+}, \theta^{h-}) \in \Re^{2J}}{Max} \quad u^h(x^h) \\ s.t. \quad & p_0(x_0^h - \omega_0^h) + (\sigma + q)\theta^{h+} - q\theta^{h-} = 0 \\ & p_s(x_s^h - \omega_s^h) = A_s(\theta^{h+} - \theta^{h-}) \quad \forall s \end{aligned}$$

$$m_s^h = p_s \omega_s^h \quad s = 0, ..., S$$

The difference in this maximisation problem is given by the separation of buying and selling activities on financial markets $(\theta^{h+}, \theta^{h-})$ and the spread σ.

Finally consumption of good 1 by an intermediary is

$$x_1^k = \Pi^k + \omega_1^k$$

6.1.1 Equilibrium

The solution of this model is complicated by the fact that the equilibrium concept is a combination of a standard competitive economy and a non-cooperative game G.

It is as if there were two stages. In the first stage intermediaries choose returns and spreads rationally anticipating the behaviour of consumers and in the second stage consumers demand goods and securities taking as given the choice of intermediaries and prices.

[19] Consumers are price takers in this model.

Definition 7 *(p, q, σ, A, M) is an equilibrium with financial intermediaries and money if:*

1. (p, q) is a Competitive Equilibrium given (σ, A), i.e.:

1.1. $\forall h, \quad (x^h, \theta^h)$ solves (P5)

1.2. $\sum_h x^h = \sum_h \omega^h$ for every good in every state of nature

$\sum_h x_{10}^h + \sum_k x_{10}^k - \sum_j c(a_j, p_1) - \varepsilon(a_j, p_1) \sum_h \theta_j^{h+} = \sum_h \omega_{10}^h + \sum_k \omega_{10}^k$ *for the numeraire at t=0*

1.3. $\sum_h \theta^h = 0$ for every asset

1.4. $p_0 \sum_h x_0^h = M_0$

$p_s \sum_h x_s^h = M_s \quad \forall s$

2. (σ, A) is a Nash Equilibrium of the game G.

3. Intermediaries have rational expectations about (p, q).

The equilibrium condition on the numeraire takes into account the fact that transaction costs are paid in units of numeraire.

6.1.2 Existence of Equilibrium.

To solve for an Equilibrium I proceed "backwards", getting first the competitive equilibrium correspondence $E : (\sigma, A) \rightrightarrows (p, q)$, given (σ, A) and then solve for the Nash equilibrium of the game G. $E(\sigma, A)$ is in general non-empty, compact-valued, and upper hemicontinuous but not convex-valued and the profit functions of the intermediaries inherit the same features. The failure of continuity and convex-valuedness of the payoffs prevents the application of standard existence theorems.[20]. To show existence of equilibrium I will have to apply a theorem due to Simon and Zame (90).

Theorem 6 *If the strategy space is compact and pay-offs are u.h.c., compact valued and convex valued, then a Nash Equilibrium exists.*

Profits are a function of the equilibrium correspondence which is u.h.c. but not convex-valued. To be able to apply the theorem, I will then consider the convex hull

[20] Beside being not convex-valued, the payoff $\Pi_k(E(\gamma, A), \gamma, A)$ is an upperhemicontinuous correspondence, not a continuous function. This is of course inherited from the equilibrium correspondence. Observe that in the version I present here, equilibria in the competitive economy are determinate due to the presence of outside money. It would then be possible to extract a continuous random selection from the equilibrium correspondence (cfr. Allen (1985)) plug it into the profit function and apply standard existence lemmas. In the original Bisin (1998) this is not possible, since a continuous random selection doesn't exists due to the indeterminacy of equilibrium. I will however follow the original Bisin (1998).

of

$$\Pi^k(E(\sigma, A), \sigma, A) = \sum_{j=1}^{J^k} \left[\left(\frac{\sigma^k}{p_1} - \varepsilon(a^k, p_1) \right) \Theta_j^+ (E(\sigma, A), \sigma, A) - c(a^k, p_1) \right]$$

for every k. One way to economically interpret the convexification procedure is to consider the expectations of intermediaries on competitive prices (p, q). I will define the expectations as $\beta^k : \Sigma^k \times A^k \to \Delta(\Re^n_+ \times \Re^J) \quad \forall k$, such that $\beta^k = \beta, \forall k$ and $\text{supp}(\beta(\sigma, A)) \subseteq E(\sigma, A)$. This says respectively that all intermediaries have the same expectations and expectations are rational. The only property left to be verified is that the strategy space $\Sigma \times A$ be compact. Observe that without loss of generality asset returns can be normalised -which ensures compactness of A- and also without loss of generality I can limit myself to consider a compact subset of Σ[21]. This is enough to prove the next proposition.

Proposition 11 *Under assumptions 3, 4 and 7 an equilibrium with financial intermediaries and money exists*[22].

Proof. (Sketch) Fix (σ, A) arbitrarily. Construct a compact cube $K \subset \Re^{n+2J}$ in the space of consumption and demand and supply for assets and demand for money and a

$$\Delta = \{(p, q) : \|q\| \leq 1 \text{ and } 0 < r \leq p_{ls} \leq 1\}$$

Consider the truncated budget set given by the union of the original budget set and the cube. For each fixed (σ, A) the truncated budget set is non-empty, convex valued, compact valued and continuos in prices. The resulting truncated demand correspondences for commodities and assets are non-empty, compact valued convex valud and u.h.c. by the Maximum Theorem. In turn applying Kakutani Theorem there exists an equilibrium $(\overline{x}, \overline{\theta}, \overline{p}, \overline{q}, \overline{M})$ of the truncated economy. I need to show that this is an equilibrium of the original economy. Observe that by adding budget constraints over individuals and feasibility for assets it must be that $\overline{p}_s \sum_h (\overline{x}_s^h - \omega_s^h) = 0$. Take a sequence $\{r_n\}$ converging to 0, and consider the corresponding sequence of equilibria of the truncated economy E_n. Since the sequence is bounded in $K \times \Delta$ there exists a subsequence converging to $E^* = (x^*, \theta^*, p^*, q^*, M^*)$. By the same reasoning as before $p_s^* \sum_h (x_s^{h*} - \omega_s^h) = 0 \forall s$. By strict monotonicity of the utility function there exists a $\widehat{r} > 0$ s.t. $\min \overline{p}_n > \widehat{r}$ implying that $\min p^* > 0$. This in turn gives $\sum_h (x_s^{h*} - \omega_s^h) = 0 \forall s$. The limit allocation solves the truncated problem at prices

[21] These facts are easy to prove. The proof can be found in Bisin (1998).

[22] The proposition only proves existence of a mixed strategies equilibrium. An equilibrium in pure strategies may not exist.

(p^*, q^*), is in the interior of the cube K and is feasible. This is enough to show that the equilibrium set of the untruncated economy is non-empty for every (σ, A). The second step of the proof involves showing that the profit correspondence is non-empty, convex and compact valued and u.h.c. Non-emptiness follows from non-emptiness of the equilibrium correspondence. To show u.h.c., take a sequence $(\sigma, A)_n$ and a corresponding sequence $(x, \theta, p, q, M)_n$ in the equilibrium correspondence. $(x, \theta, p, q, M)_n$ lies in a compact set so there is a subsequence converging to $(x^*, \theta^*, p^*, q^*, M^*)$. In turn by the continuity properties of the utility function and of the (truncated) budget set, $(x^*, \theta^*, p^*, q^*, M^*)$ is an equilibrium associated to (σ^*, A^*). The profit correspondence inherits immediately this property. Boundedness is also immediate. Taking the convex hull of $\Pi^k(E(\sigma, A), \sigma, A)$ I get a correspondence that is non-empty, closed, bounded, convex valued and u.h.c. The strategy space is compact by construction and Simon and Zame Theorem can be applied. This shows that a mixed strategy equilibrium exists. ■

Remark 7 *One of the interesting features of the analysis is that markets may be endogenously incomplete in equilibrium. Since there are fixed costs to intermediation, each intermediary will design only a limited number of assets. Depending on the level of costs, this may give rise to a total number of assets designed in equilibrium inferior to the number of states of nature $J < S$.*

6.1.3 Determinacy

The reason why in equilibrium there is no real indeterminacy -in the sense that the price level doesn't affect equilibrium allocations- is twofold. First the fact that there is outside money, imposes $S + 1$ extra restrictions on the equilibrium, thus fixing the price level state by state. The second reason, which is present also in the same model without money, is that the return matrix is endogenously determined. Since profits are expressed in numeraire terms and costs depend only on real returns, at the equilibrium intermediaries effectively index the returns of the assets they design to the price of the numeraire p_1. This implicitly transform the model into one in which assets pay off in real terms (in terms of numeraire). It is well known that in those economies the $S + 1$ degrees of indeterminacy are purely nominal, since the budget constraint is homogeneous in prices. The two effects together give not only the result that indeterminacy (nominal and real) disappears, but also -contrary to what happened in Magill and Quinzii (1992)- that monetary policy is completely neutral: a change in M will change the price level, intermediaries will adjust the returns to the new level and real allocations will remain the same.

Proposition 12 *The equilibrium allocations don't depend on the price level p_1*

Proof. The proof consists simply in showing that demand, profit functions and the strategy space of intermediaries are dependent on p_1 only through $\left(\frac{\sigma}{p_1}, \frac{A}{p_1}\right)$. The strategy space depends only on $\left(\frac{\sigma}{p_1}, \frac{A}{p_1}\right)$ by definition. The profit function by the assumption on costs. As for demands of commodities and assets, rewrite the budget constraint as follows:

$$s.t. \quad p_0(x_0^h - \omega_0^h) + (p_1 \frac{\sigma}{p_1} + q)\theta^{h+} - q\theta^{h-} = 0$$

$$p_s(x_s^h - \omega_s^h) = p_1 \frac{A_s}{p_1}(\theta^{h+} - \theta^{h-}) \quad \forall s$$

This is now a numeraire economy, observing that intermediaries in equilibrium effectively change (σ, A) every time p_1 changes. The proof is complete. ■

Remark 8 *Observe that the proposition is different from the ones encountered before on the determinacy of equilibrium. In particular it doesn't say that equilibria are locally unique. This depends on the fact that intermediaries expectations are not pinned down by the equilibrium. Each different belief about prices will generate a different equilibrium that in principle could be arbitrarily close to the former one.*

Remark 9 *I see the model just described as potentially useful to get rid of the real indeterminacy issue in the model with inside money. This is explored in a companion paper.*

7 Conclusion

In this paper I reviewed some recent results concerning the relationship between indeterminacy of equilibria and the presence of money as a medium of exchange. I have stressed throughout the paper that introducing money per se doesn't allow to solve the problem. In particular the crucial features turned out to be the presence of outside money and the redistribution of profits by the bank. In the model with only inside money and no redistribution of profits, the restrictions imposed by the quantity theory equations on the equilibrium are vacuous and indeterminacy persists. The fact of making explicit the design of assets, through non-competitive agents that maximise profits in a way that is free from "money illusion" could help in this sense to eliminate at least real indeterminacy from that model. This is the subject of a companion paper.

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