Cost minimisation interpretation of fourth power phase estimator and links to multimodulus algorithm

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It is shown that the fourth power phase estimator minimises or maximises (depending on a condition on the transmitted constellation) the cost function associated to the recently proposed multimodulus algorithm (MMA) for blind equalisation. Implications for operation of MMA are discussed.

Introduction: In burst-mode digital transmission systems adopting coherent demodulation, phase recovery within each burst becomes a crucial issue. Among the blind (or non-data-aided) carrier phase estimators available from the literature, perhaps the fourth power estimate appears to be the most popular. It is known to yield an approximate maximum likelihood estimator in the limit of small SNR [1]. Serpedin *et al.* have shown in [2] that several seemingly different phase estimators are in fact equivalent to the fourth power estimator.

In this Letter we present a reinterpretation of the fourth power estimator in terms of the minimisation of a dispersion-based cost function. This cost has been recently proposed in the context of blind equalisation [3, 4] as the basis of the so-called multimodulus algorithm (MMA). One advantage of MMA over the standard constant modulus algorithm (CMA) [5] is precisely its ability to perform blind phase acquisition. Thus, this inherent phase recovery property of MMA can be seen as a 'built-in' fourth power estimator; consequently one cannot expect MMA to synchronise the phase with constellations for which the fourth power estimator is not well suited, such as M-PSK ($M>4$).

Problem statement and derivation: In the absence of channel impairments other than phase offset and noise, the received samples can be written as

$$Y(k) = X(k)e^{j\theta} + N(k) = Y_r(k) + jY_i(k), \quad 1 \le k \le K \tag{1}$$

where $X(k)$ is the complex-valued transmitted symbol, $N(k)$ is complex-valued additive noise, assumed independent of the symbols and circular (i.e. $E\{N^p(k)\} = 0$ for all positive integers p), and θ is the phase angle to be determined from the observed values $\{Y(k)\}$.

Let ϕ be a candidate estimate, and define the derotated samples

$$\hat{X}(k) = Y(k)e^{-i\phi} = \hat{X}_r(k) + j\hat{X}_i(k) \tag{2}$$

The multimodulus cost function [3, 4] is defined as the sum of the dispersions of the real and imaginary parts of the derotated signal:

$$J(\phi) = E\{(\hat{X}_r^2(k) - \gamma_r)^2 + (\hat{X}_i^2(k) - \gamma_i)^2\} \tag{3}$$

for some constants γ_r, γ_i. We will assume that $\gamma_r = \gamma_i$; in that case, it is readily seen by expanding (3) that this constant merely adds a constant term to J, so that we can take $\gamma_r = \gamma_i = 0$ and $J(\phi) = E\{\hat{X}_r^4(k) + \hat{X}_i^4(k)\}$. By noting that

$$\frac{\partial \hat{X}_r(k)}{\partial \phi} = \hat{X}_i(k), \quad \frac{\partial \hat{X}_i(k)}{\partial \phi} = -\hat{X}_r(k) \tag{4}$$

the derivative of the cost is seen to be

$$\begin{aligned}\frac{\partial J(\phi)}{\partial \phi} &= 4E\{\hat{X}_r(k)\hat{X}_i(k)[\hat{X}_r^2(k) - \hat{X}_r^2(k)]\} \\ &= \text{Im } E\{\hat{X}^4(k)\}\end{aligned} \tag{5}$$

After some algebra, this can be written explicitly in terms of ϕ as

$$\begin{aligned}\frac{\partial J(\phi)}{\partial \phi} &= 4E\{Y_r(k)Y_i(k)[Y_r^2(k) - Y_i^2(k)]\}\cos 4\phi \\ &\quad - E\{Y_r^4(k) + Y_i^4(k) - 6Y_r^2(k)Y_i^2(k)\}\sin 4\phi \\ &= \text{Im } E\{Y^4(k)\}\cos 4\phi - \text{Re } E\{Y^4(k)\}\sin 4\phi\end{aligned} \tag{6}$$

Therefore the stationary points of the cost, at which (6) vanishes, are given by

$$\tan 4\phi = \frac{\text{Im } E\{Y^4(k)\}}{\text{Re } E\{Y^4(k)\}}$$

or equivalently

$$\phi = \frac{1}{4}\text{angle } E\{Y^4(k)\} \tag{7}$$

Differentiating (5) and using (4), the second derivative of the cost is found:

$$\frac{\partial^2 J(\phi)}{\partial \phi^2} = -4\text{Re } E\{\hat{X}^4(k)\} \tag{8}$$

Hence we see that $E\{\hat{X}^4(k)\}$ is real at the stationary points of J, becoming negative at the minima and positive at the maxima.

For circular noise independent of the data, it follows that $E\{\hat{X}^4(k)\} = E\{\hat{X}^4(k)\}e^{j(\theta-\phi)}$. From (5) and (8), it is seen that the noise does not alter the location or character (minimum/maximum) of the stationary points under these conditions.

If the constellation to which $\{X(k)\}$ belong has quadrant symmetry, and if the symbols are drawn equiprobably, then $E\{X^4(k)\}$ is real. In that case, the stationary points of J are given by

$$\text{Im } E\{\hat{X}^4(k)\} = E\{X^4(k)\}\sin 4(\theta - \phi) = 0 \Rightarrow \phi - \theta + n\pi/4$$

and

$$\left.\frac{\partial^2 J(\phi)}{\partial \phi^2}\right|_{\phi=\theta+n\pi/4} = (-1)^{n+1}4E\{X^4(k)\}$$

Hence consecutive minima (or maxima) of $J(\phi)$ are $\pi/2$ rad apart, which reflects the inherent ambiguity due to the quadrant symmetry of the constellation.

The fourth power estimator is defined as [1, 2]

$$\hat{\theta} = \frac{1}{4}\text{angle}[E\{X^{*4}(k)\}E\{Y^4(k)\}] \tag{9}$$

Note that $E\{X^{*4}(k)\} = E\{X^4(k)\}$ since this quantity is real. Then in view of (7) and (8), it is seen that the fourth power estimate minimises or maximises the cost J according to the sign of $E\{X^4(k)\}$.

Effect in multimodulus algorithm: As MMA is designed to minimise (3), phase correction (modulo $\pi/2$) will be accomplished provided that $E\{X^4(k)\} < 0$. Most rectangular and cross QAM constellations satisfy this property. Other constellations, such as that of the CCITT V.29 standard, do not, so that if the MMA is employed in those systems the equaliser output should be rotated by $\pi/4$ rad. For QPSK, the constellation $\{(\pm 1 \pm j)/\sqrt{2}\}$ satisfies $E\{X^4(k)\} < 0$, while $\{\pm 1, \pm j\}$ does not. Finally, for M-PSK constellations with $M > 4$, $E\{X^4(k)\} = 0$ (and in fact $J(\phi)$ becomes flat), so that MMA cannot compensate for the phase in these systems.

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From Chapman-Robbins bound towards Barankin bound in threshold behaviour prediction

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The evolution of the Barankin bound and the threshold behaviour occurrence with the number of test points for which the bias is constrained to be null is studied. It is first shown that the Barankin bound is an increasing function of the number of test points used in its computation. This result is further used to characterise the threshold occurrence evolution with the number of test points.

Introduction: In signal processing, the knowledge of a lower bound on the estimation variance and of the so-called 'SNR threshold' [1, 2] is important to evaluate relevance of the algorithms and to choose the operating conditions (transmission power, number of measurements, etc.) necessary to obtain a desired performance. The Cramer-Rao bound (CRB) [3] is the most commonly used. It is achieved asymptotically in the number of snapshots by a maximum likelihood estimator. At low signal-to-noise ratio (SNR) or if the number of snapshots N is small, a threshold behaviour characterised by the sudden increase of the estimation mean square error (MSE) over the CRB is observed and has been reported in several works [1, 2, 4]. This incited development of lower bounds that are closer to the best attainable performance than is the CRB. The sudden departure of such bounds from the CRB was used by some authors to predict the SNR threshold [2].

In [5] Barankin derived the greater lower bound of the MSE of an estimator unbiased for all the admissible values of the unknown parameter. It is referred to as the Barankin bound (BB). The BB and the CRB coincide asymptotically for high SNR or large number of snapshots. The BB provides a good means to predict the threshold region.

It is shown in [6] that the BB can be obtained through the optimisation of a constrained quadratic problem where each constraint expresses the bias nullity for a discrete value of the parameter, called the test point. The BB analytical computation can be obtained in some specific cases [7] but is usually computed numerically. Because of its prohibitive computational complexity, an approximated version of the BB was proposed by Chapman and Robbins [8] where a single test point is used; it is referenced by the Chapman-Robbins bound (ChRB).

In this Letter, we show that the threshold occurrence value predicted by the Chapman-Robbins bound is optimistic. For this, we first show that the Barankin bound is an increasing function of the number n of the test points. This result is then used to show that the threshold occurrence predicted by the BB is also an increasing function of n.

Barankin bound: The Barankin bound is expressed as [6]

$$\mathrm{BB} = \lim_{n\to\infty} \sup_{h_1\cdots h_n} [h_1 \cdots h_n](\mathbf{D}_n - \mathbf{1}\mathbf{1}^T)^{-1}[h_1 \cdots h_n]^T \tag{1}$$

where Φ_k ($k = 0, \ldots, n$) denote the set of test points at which the estimator is constrained to be unbiased and $h_k = \Phi_k - \Phi_0$. The generic element of $\mathbf{D}_n$ is given by

$$\mathbf{D}_n(k, l) = d_{k,l} = \int \frac{p(x, \Phi_k)p(x, \Phi_l)}{p(x, \Phi_0)} dx \tag{2}$$

where $p(x, \Phi)$ is the probability density of the observation x. If the nth-order Barankin bound denoted by BB_n is defined as the lowest MSE that can be obtained by an estimator the bias of which is constrained to be null for $n+1$ discrete values $\Phi_0, \ldots, \Phi_n$ of the parameter Φ to estimate, then

$$\mathrm{BB}_n = \sup_{h_1,\ldots,h_n} \underbrace{[h_1 \cdots h_n](\mathbf{D}_n - \mathbf{1}\mathbf{1}^T)^{-1}[h_1 \cdots h_n]^T}_{\mathrm{MSE}_n} \tag{3}$$

and $\mathrm{BB} = \lim_{n\to\infty}\mathrm{BB}_n$. The Chapman-Robbins bound uses a single test point and is given by

$$\mathrm{ChRB} = \sup_h \frac{h^2}{\int p(x, \Phi + h)^2/p(x, \Phi)dx - 1} = \mathrm{BB}_1 \tag{4}$$

Theorem 1: The Barankin bound is an increasing function of the number of test points:

$$\mathrm{BB}_n \le \mathrm{BB}_{n+1} \tag{5}$$

Proof: Using (2), it is straightforward that

$$\mathbf{D}_{n+1} = \begin{pmatrix} \mathbf{D}_n & & & d_{n+1,1} \\ & & & \vdots \\ & & & d_{n+1,n} \\ d_{n+1,1} & \cdots & d_{n+1,n} & d_{n+1,n+1} \end{pmatrix} \tag{6}$$

and thus

$$\mathbf{D}_{n+1} - \mathbf{1}\mathbf{1}^T = \begin{pmatrix} \mathbf{D}_n - \mathbf{1}\mathbf{1}^T & & & d_{n+1,1} - 1 \\ & & & \vdots \\ & & & d_{n+1,n} - 1 \\ d_{n+1,1} - 1 & \cdots & d_{n+1,n} - 1 & d_{n+1,n+1} - 1 \end{pmatrix} \tag{7}$$

where $\mathbf{1}$ is a column vector of ones of convenient size. Let

$$\mathbf{d}_{n+1} = \begin{pmatrix} d_{n+1,1} - 1 \\ \vdots \\ d_{n+1,n} - 1 \end{pmatrix} \tag{8}$$

$$c_{n+1} = d_{n+1,n+1} - 1 \tag{9}$$

Then, the block inverse of the matrix given by (7) is

$$\mathbf{M}_{n+1} \equiv (\mathbf{D}_{n+1} - \mathbf{1}\mathbf{1}^T)^{-1} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} \tag{10}$$

where

$$\mathbf{B}_{11} = \mathbf{M}_n + \mathbf{M}_n\mathbf{d}_{n+1}(c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1})^{-1}\mathbf{d}_{n+1}^T\mathbf{M}_n$$
$$\mathbf{B}_{12} = -\mathbf{M}_n\mathbf{d}_{n+1}(c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1})^{-1}$$
$$\mathbf{B}_{21} = -\frac{1}{c_{n+1}}\mathbf{d}_{n+1}^T[\mathbf{M}_n + \mathbf{M}_n\mathbf{d}_{n+1}(c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1})^{-1}\mathbf{d}_{n+1}^T\mathbf{M}_n]$$
$$\mathbf{B}_{22} = (c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1})^{-1}$$

Based on (3) and (10) and using $\mathbf{h}_n = [h_1 \cdots h_n]^T$, the $n+1$th-order Barankin bound is expressed as

$$\mathrm{BB}_{n+1} = \sup_{h_n, h_{n+1}} \underbrace{[\mathbf{h}_n^T h_{n+1}]\mathbf{M}_{n+1}\begin{bmatrix} \mathbf{h}_n \\ h_{n+1} \end{bmatrix}}_{\mathrm{MSE}_{n+1}} \tag{11}$$

Some computations lead to the following expression

$$\mathrm{MSE}_{n+1} = \mathrm{MSE}_n + (c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1})^{-1}(\mathbf{h}_n^T\mathbf{M}_n\mathbf{d}_{n+1} - h_{n+1})^2 \tag{12}$$

Using (7),

$$det(\mathbf{D}_{n+1} - \mathbf{1}\mathbf{1})^T = (c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1})det(\mathbf{D}_n - \mathbf{1}\mathbf{1}^T) \tag{13}$$

Note that $\mathbf{M}_p$ (for $p \ge 1$) is positive definite, thus $det(\mathbf{M}_p) = 1/det(\mathbf{D}_p - \mathbf{1}\mathbf{1}^T) \ge 0$. Then (13) leads to

$$c_{n+1} - \mathbf{d}_{n+1}^T\mathbf{M}_n\mathbf{d}_{n+1} \ge 0 \tag{14}$$

Returning to (12), since $(\mathbf{h}_n^T\mathbf{M}_n\mathbf{d}_{n+1} - h_{n+1})^2$ is a real and quadratic form, then (14) implies

$$\forall h_1, \ldots, h_n, h_{n+1}, \quad \mathrm{MSE}_{n+1} \ge \mathrm{MSE}_n \tag{15}$$

Therefore,

$$\sup_{\mathbf{h}_n, h_{n+1}} \mathrm{MSE}_{n+1} \ge \sup_{\mathbf{h}_n} \mathrm{MSE}_n \tag{16}$$

which is equivalent to

$$\mathrm{BB}_{n+1} \ge \mathrm{BB}_n \tag{17}$$

Theorem 2: The threshold occurrence ρ is an increasing function of the number of test points n,