Applications of Quantified Constraint Solving over the Reals
Bibliography

Stefan Ratschan*

April 8, 2003

Quantified constraints over the reals appear in numerous contexts. Usually existential quantification occurs when some parameter can be chosen by the user of a system, and univeral quantification when the exact value of a parameter is either unknown, or when it occurs in infinitely many, similar versions.

The following is a list of application areas and publications that contain applications for solving quantified constraints over the reals. The list is certainly not complete, but grows as the author encounters new items. Contributions are very welcome!

Electrical Engineering/Electronics: [68, 64]

Stability: [55, 66, 56, 67, 38, 37]

Control: [1, 46, 45, 25, 44, 60, 7, 5, 33, 4, 17, 62, 65, 63], survey [24], reachability [52, 53, 51, 6], embedded control systems [71], hybrid systems [28, 30, 32, 70, 64]

Computational Geometry/Motion Planning/Collision Detection: [69, 44, 35, 73]

Constraint Databases: [50, 14, 61, 9, 2, 10, 29, 49]

Theorem Proving in Real Geometry: [22, 21]

Several Different: [23, 72, 57, 44]

Use of Predicate Language for Modeling Engineering Problems: [26, 13]

*This work was supported by a Marie-Curie Fellowship of the European Union

Other: Camera motion: [11], Constraint Logic Programming: [36], Mechanical Engineering: [40, 39], Mathematics: [54] (from [47]), [15, 48], Biology: [18, 32], Interpolation: [34], Scheduling [31], Automated Theorem Proving [12, 42, 41], Optimization: [3], Termination of Rewrite Systems [19, 20]

1 Acknowledgments

This work has been supported by a Marie Curie fellowship of the European Union under contract number HPMF-CT-2001-01255.

Thanks to Hirokazu Anai for contributing references.

References

[1] C. Abdallah, P. Dorato, W. Yang, R. Liska, and S. Steinberg. Applications of quantifier elimination theory to control system design. In *4th IEEE Mediterranean Symposium on Control and Automation*, Crete, Greece, 1996.

[2] F. Afrati, S. S. Cosmadakis, S. Grumbach, and G. M. Kuper. Linear vs. polynomial constraints in database query languages. In *Proc. PPCP'94*, 1994.

[3] H. Anai. On solving semidefinite programming by quantifier elimination. *In Proc. of American Control Conference, Philadelphia*, pages 2814–2818, 1998.

[4] H. Anai. Algebraic approach to analysis of discrete-time polynomial systems. In *Proc. of European Control Conference*, 1999.

[5] H. Anai and S. Hara. Fixed-structure robust controller synthesis based on sign definite condition by a special quantifier elimination. In *Proceedings of American Control Conference 2000*, pages 1312–1316, 2000.

[6] H. Anai and V. Weispfenning. Reach set computation using real quantifier elimination. In *Proceedings of International Workshop on Hybrid Systems: Computation and Control (HSCC2001)*, volume 2034 of *LNCS*, pages 63–76. Springer, 2001.

[7] B. D. O. Anderson, N. K. Bose, and E. I. Jury. Output feedback stabilization and related problems — solution via decision methods. *IEEE Trans. Automatic Control*, AC-20:53–66, 1975.

[8] B. Barmish and Z. Shi. Robust stability of a class of polynomials with coefficients depending multilinearly on perturbations. *IEEE Trans. Autom. Control*, AC–35:1040–1043, 1990.

[9] S. Basu. New results on quantifier elimination over real closed fields and applications to constraint databases. *Journal of the ACM*, 46(4):537–555, 1999.

[10] M. Baudinet, J. Chomicki, and P. Wolper. Constraint- generating dependencies. In G. Gottlob and M. Vardi, editors, *Database Theory – ICDT'95, 5th International Conference*, volume 893 of *LNCS*, Prague, Czech Republic, 1995. Springer- Verlag.

[11] F. Benhamou and F. Goualard. Universally quantified interval constraints. In *Proc. of the Sixth Intl. Conf. on Principles and Practice of Constraint Programming (CP'2000)*, number 1894 in LNCS, Singapore, 2000. Springer Verlag.

[12] N. S. Bjørner, M. E. Stickel, and T. E. Uribe. A practical integration of first-order reasoning and decision procedures. In *CADE-97*, volume 1249, pages 101–115, 1997.

[13] J. Bowen and D. Bahler. Full first- order logic in knowledge representation: A constraint- based approach. Technical Report TR-93-03, Department of Computer Science, University College, Cork, Ireland, 1993.

[14] A. Brodsky. Constraint databases: Promising technology or just intellectual exercise? *ACM Computing Surveys*, 28(4), 1996.

[15] C. Brown. Simple CAD construction and its applications. *Journal of Symbolic Computation*, 31(5):521–547, 2001.

[16] B. F. Caviness and J. R. Johnson, editors. *Quantifier Elimination and Cylindrical Algebraic Decomposition*. Springer, Wien, 1998.

[17] C. Champetier and J. F. Magni. Pole assignment by output feedback using cylindrical algebraic decomposition. *Internation Journal of Control*, 48(5):2107–2120, 1988.

[18] C. Chauvin, M. Müller, and A. Weber. An application of quantifier elimination to mathematical biology. In *Computer Algebra in Science and Engineering*, pages 287–296. World Scientific, 1994.

[19] G. E. Collins and H. Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12:299–328, 1991. Also in [16].

[20] N. Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3:69–116, 1987.

[21] A. Dolzmann. Solving geometric problems with real quantifier elimination. In X.-S. Gao, D. Wang, and L. Yang, editors, *Automated Deduction in Geometry*, pages 14–29. Springer, 1999.

[22] A. Dolzmann, T. Sturm, and V. Weispfenning. A new approach for automatic theorem proving in real geometry. *Journal of Automated Reasoning*, 21(3):357–380, 1998.

[23] A. Dolzmann, T. Sturm, and V. Weispfenning. Real quantifier elimination in practice. In B. H. Matzat, G.-M. Greuel, and G. Hiss, editors, *Algorithmic Algebra and Number Theory*, pages 221–248. Springer, 1998.

[24] P. Dorato. Quantified multivariate polynomial inequalities. *IEEE Control Systems Magazine*, 20(5):48–58, October 2000.

[25] P. Dorato, W. Yang, and C. Abdallah. Robust multi-objective feedback design by quantifier elimination. *Journal of Symbolic Computation*, 24:153–159, 1997.

[26] B. Faltings and K. Sun. FAMING: Supporting innovative mechanism shape design. *Computer-Aided Design*, 28(3):207–216, 1996.

[27] G. Fiorio, S. Malan, M. Milanese, and M. Taragna. Robust performance design of fixed structure controllers with uncertain parameters. In *Proceedings of the 32nd IEEE Conf. Decision and Control*, pages 3029–3031, 1993.

[28] M. Fränzle. Analysis of hybrid systems: An ounce of realism can save an infinity of states. In J. Flum and M. Rodriguez-Artalejo, editors, *Computer Science Logic (CSL'99)*, number 1683 in LNCS. Springer, 1999.

[29] V. Gaede and O. Günther. Constraint-based query optimization and processing. In *Proc. 1st Intl. Database Workshop on Constraint Database Systems (CDB'95)*, 1995.

[30] Y. Gao, J. Lygeros, M. Quincampoix, and N. Seube. Approximate stabilisation of uncertain hybrid systems. In Maler and Pnueli [59], pages 203–215.

[31] R. Gerber, W. Pugh, and M. Saksena. Parametric dispatching of hard real-time tasks. *IEEE Transactions on Computers*, 44(3):471–479, 1995.

[32] R. Ghosh. Automated symbolic reachability analysis; with application to delta-notch signaling automata. In Maler and Pnueli [59].

[33] S. T. Glad. An algebraic approach to bang-bang control. In *Proc. 3rd European Contr. Conf.*, pages 2892–2895, Rome, Italy, 1995.

[34] L. Gonzalez-Vega. Applying quantifier elimination to the Birkhoff interpolation problem. *Journal of Symbolic Computation*, 22(1):83–103, 1996.

[35] H. Hong. Collision problems by an improved CAD- based quantifier elimination algorithm. Technical Report 91-05, RISC- Linz, 1991.

[36] H. Hong. Non-linear constraints solving over real numbers in constraint logic programming. Technical Report 92-08, RISC-Linz, 1992.

[37] H. Hong, R. Liska, and S. Steinberg. Logic, quantifiers, computer algebra and stability. *SIAM News*, 30(6):10–,13, 1997.

[38] H. Hong, R. Liska, and S. Steinberg. Testing stability by quantifier elimination. *Journal of Symbolic Computation*, 24(2):161–187, 1997.

[39] N. I. Ioakimidis. Quantifier elimination in applied mechanics problems with cylindrical algebraic decomposition. *International Journal of Solids and Structures*, 34(30):4037–4070, 1997.

[40] N. I. Ioakimidis. REDLOG-aided derivation of feasibility conditions in applied mechanics and engineering problems under simple inequality constraints. *Journal of Mechanical Engineering (Strojnícky Časopis)*, 50(1):58–69, 1999.

[41] P. Janičić and A. Bundy. A general setting for combining and integrating decision procedures into theorem provers. *Journal of Automated Reasoning*, 2001.

[42] P. Janičić, A. Bundy, and I. Green. A framework for the flexible integration of a class of decision procedures in theorem provers. In *CADE-16*, number 1632 in LNAI, pages 127–141. Springer, 1999.

[43] L. Jaulin, I. Braems, and E. Walter. Interval methods for nonlinear identification and robust control. In *IEEE 2002 Conference on Decision and Control*, 2002. submitted.

[44] L. Jaulin and É. Walter. Guaranteed tuning, with application to robust control and motion planning. *Automatica*, 32(8):1217–1221, 1996.

[45] M. Jirstrand. Nonlinear control system design by quantifier elimination. *Journal of Symbolic Computation*, 24(2):137–152, 1997.

[46] M. Jirstrand. *Algebraic Methods for Inequality Constraints in Control*. PhD thesis, Linköping University, Department of Electrical Engineering, Division of Automatic Control, 1998. No. 527.

[47] W. Kahan. Problem no. 9: An ellipse problem. *SIGSAM Bulletin of the ACM*, 9(35):11, 1975.

[48] M. E. Kahoui and A. Weber. Deciding Hopf bifurcations by quantifier elimination in a software-component architecture. *Journal of Symbolic Computation*, 30(2):161–179, 2000.

[49] M. Koubarakis. Foundations of indefinite constraint databases. In *Proc. PPCP'94*, 1995.

[50] G. Kuper, L. Libkin, and J. Paredaens, editors. *Constraint Databases*. Springer, 2000.

[51] G. Lafferriere, G. J. Pappas, and S. Yovine. Symbolic reachability computation for families of linear vector fields. *Journal of Symbolic Computation*, 32(3):231–253, 2001.

[52] G. Lafferriere, G. S. Pappas, and S. Yovine. A new class of decidable hybrid systems. In *Hybrid Systems: Computation and Control*, volume 1569 of *LNCS*, pages 137–151. Springer, 1999.

[53] G. Lafferriere, G. S. Pappas, and S. Yovine. Reachability computation for linear hybrid systems. In *Proceedings of the 14th IFAC World Congress*, volume E, pages 7–12, Beijin, China, 1999.

[54] D. Lazard. Quantifier elimination: Optimal solutions for two classical examples. *Journal of Symbolic Computation*, 5(1,2):261–266, 1988.

[55] R. Liska and S. Steinberg. Applying quantifier elimination to stability analysis of difference schemes. *Computer Journal*, 36(5):497–593, 1993.

[56] R. Liska and S. Steinberg. Solving stability problems using quantifier elimination. In R. Jeltsch and M. Mansour, editors, *Stability Theory, Proceedings of Conference on Centennial Hurwitz on Stability Theory*, pages 205–210, Basel, 1996. Birkhäuser Verlag. International Series of Numerical Mathematics (ISNM) Vol. 121.

[57] R. Loos and V. Weispfenning. Applying linear quantifier elimination. *The Computer Journal*, 36(5):450–462, 1993.

[58] S. Malan, M. Milanese, and M. Taragna. Robust analysis and design of control systems using interval arithmetic. *Automatica*, 33(7):1363–1372, 1997.

[59] O. Maler and A. Pnueli, editors. *Hybrid Systems: Computation and Control (HSCC)*, number 2623 in LNCS. Springer, 2003.

[60] D. Nešić and I. M. Y. Mareels. Dead beat controllability of polynomial systems: Symbolic computation approaches. *IEEE Transactions on Automatic Control*, 43(2):162–175, 1998.

[61] J. Paredaens, J. van den Bussche, and D. van Gucht. First-order queries on finite structures over the reals. *SIAM Journal on Computing*, 27(6):1747–1763, 1998.

[62] S. Ratschan and L. Jaulin. Solving composed first-order constraints from discrete-time robust control. In *Proceedings of the Sixth Annual Workshop of the ERCIM Working Group on Constraints*, 2001. `http://arXiv.org/html/cs/0110012`.

[63] S. Ratschan and J. Vehí. Robust pole clustering of parametric uncertain systems using interval methods. In *4th IFAC Symposium on Robust Control Design*, 2003. To appear.

[64] M. Senesky, G. Eirea, and T. J. Koo. Hybrid modelling and control of power electronics. In Maler and Pnueli [59].

[65] Y. Smagina. Robust modal P and PI regulator synthesis for a plant with interval parameters in the state space. In *Proc. of the American Control Conference*, 2000.

[66] S. Steinberg and R. Liska. Stability analysis by quantifier elimination. In G. Jacob, N. Ousous, and S. Steinberg, editors, *Proceedings SC 93, International IMACS Symposium on Symbolic Computation, New Trends and Developments*, pages 62–67, 1993. Lille, France, June 14 - 17, 1993.

[67] S. Steinberg and R. Liska. Stability analysis by quantifier elimination. *Mathematics and Computers in Simulation*, 42(4-6):629–638, 1996.

[68] T. Sturm. Reasoning over networks by symbolic methods. *Applicable Algebra in Engineering Communication and Computing*, 10(1):79–96, September 1999.

[69] T. Sturm and V. Weispfenning. Computational geometry problems in REDLOG. In D. Wang, editor, *Automated Deduction in Geometry*, volume 1360 of *LNAI*, pages 58–86, Springer-Verlag, 1998.

[70] A. Tiwari and G. Khanna. Series of abstractions for hybrid automata. In C. J. Tomlin and M. R. Greenstreet, editors, *Hybrid Systems: Computation and Control HSCC*, volume 2289 of *LNCS*, pages 465–478. Springer, Mar. 2002.

[71] A. Tiwari, N. Shankar, and J. Rushby. Invisible formal methods for embedded control systems. *Proc. of the IEEE*, 2002. to appear.

[72] V. Weispfenning. Simulation and optimization by quantifier elimination. *Journal of Symbolic Computation*, 24(2), 1997.

[73] V. Weispfenning. Semilinear motion planning with REDLOG. *Applicable Algebra in Engineering, Communication, and Computing*, 2001. To appear.

2 Examples

A collection of examples from the above papers. I am thankful for any contribution!

- Robust-1 [24]:

```
FORALL([p], [[0, 1]],
  9 + 48 p + 48 q + 32 p q > 0 /\
  1 + p + q > 0 /\
  -16 p - 16 q + 16 p^2 + 16 q^2 + 7 > 0
);
[q];
[[-2, 2]];
```

  Solution: [24]: $4q-1 < 0 /\backslash 16q+3 > 0 /\backslash 4q-3 > 0$, that is $(-0.1875, 0.25)$ union $(0.75, \infty)$.

- Robust-2 [27, 58]:

```
FORALL([q1, q2, ww], [[0.8, 1.25], [0.8, 1.25], [0.0, infty]],
 -k2 - k1 q1 > 0 /\
 - q1 k1 - 50 > 0 /\
 2 k2^2 ww^2  + 4 k2^2 ww k1 q1 q2 + k2^2 k1^2 q1^2 q2^2 + 2 k2^2 ww q2^2 + 4 k2 w
 400 k2^2 ww^2 + 800 k2^2 ww k1 q1 q2 + 400 k2^2 k1^2 q1^2 q2^2 + 400 k2^2 ww q2^2
);
[k1, k2];
[[-200, 0], [0, 10]];
```

- Robust-3 [25]:

```
FORALL([ p1, p2 ], [ [0.8, 1.25], [0.8, 1.25]],
  p2*(1+p1*q1) < 0 /\
  FORALL([w1], [[0,10]], 99*w1^2 + p2^2 * (100*(1+p1*q1)^2 - 1) > 0) /\
  FORALL([w2], [[-infty,infty]], (400-q1^2)*w2^2 + p2^2*(400*(1+p1*q1)^2-q1^2) > 0
);
[q1];
[ [-50, 10] ];
```

  Solution [25]: $-20 \le q_1 < -1.375$

- Robust-4 [58, 8]

```
FORALL([p1, p2, p3, p4], [[2.5, 3.5], [1.5, 2.5], [2.5, 3.5], [9.5, 10.5]],
  p2 q2 + p4 q4 > 0 /\
  p4 + q4 + p1 q1 - p3 q3 > 0 /\
  p1 p3 q2 q4 - p1 p3 q2 p4 - p2 q1 p3 p4 - 2 p1 p2 q1 q2 - p1 q2 p4 q3 + p2 q1 p3
  q3 - p3 > 0
);
[q1, q2, q3, q4];
[[19, 21], [22, 24], [9, 11], [4, 5]];
```

- Robust-5 [44]

```
FORALL([z, T, w0, K], [[0.95, 1.05], [-1.05, -0.95], [0.95, 1.05], [0.95, 1.05]],
  K c1 w0^2 < 0 /\
  2 z T w0 + 1 < 0 /\
  (2 z w0 + w0^2 T + w0^2 T c3)*(2 z T w0 + 1) - w0^2 (K*c2+1) < 0 /\
  w0^2*(K*c2 + 1)*(2*T*z*w0 + 1)*(2*z*w0 + w0^2*(T + K*c3)) - K*c1*w0^2*(2*T*z*w0
);
[ c1, c2, c3 ];
[[ -10, 10], [-10, 10], [-10, 10]];
```

- Robust-6 [43]

```
FORALL([p1, p2, p3], [[0.9, 1.1], [0.9, 1.1], [0.9, 1.1]],
  (1 + c2*p1 )*( (p2*p3^2 +p3)*(p2*p3 +1) - p2*(p3^2 +c2*p1*p3^2)) - (p2*p3 + 1)^2
  (p2*p3 + 1)*(p2*p3 + 1) - p2*( p3 + c2*p1*p3) > 0 /\
  c1*p1*p3^2 > 0
);
[c1, c2];
[[0, 1], [0, 1]];
```

- Control-Stabilization-1 [1]

```
EXISTS([A, B, D, P1, P2, P3, P4, P5, P6, P7, P8, P9],
  P1 > 0 /\ P2 > 0 /\ P3 > 0 /\ P4 P5 > 0 /\ A > 0 /\ B P6 P7 > 0 /\ P8 P9 > 0 /\
  P1 = A B^2 - D^2 /\
  P2 = - A B + A + D^2 - D - 1 /\
  P3 = A B - A D - 2 A + D^3 + 4 D^2 + 4 D /\
  P4 = A B - 2 A - B D^2 - 4 B D - 4 B + 2 D^2 + 5 D + 2 /\
  P5 = A B^3 - A B^2 D - 4 A B^2 + 2 A B D + 4 A B + 2 B D^3 + 5 B D^2 + 2 B D - D
  P6 = A B - 2 A - B D^2 - 4 B D - 4 B + 2 D^2 + 4 D /\
  P7 = A B^2 - A B D - 4 A B + 2 A D + 4 A + 2 D^3 + 4 D^2 /\
```

```
  P8 = A B - 2 A - B D^2 - 4 B D - 4 B + 2 D^2 + 3 D - 2 /\
  P9 = A B^3 - A B^2 D - 4 A B^2 + 2 A B D + 4 A B + 2 B D^3 + 3 B D^2 - 2 B D + D
);
```

Solution: True, witness: A=110, B=3/2, D=15

- Biology-1 [18]

```
FORALL([c, t, r, h, d, v],
  (r>0 /\ t>0 /\ c<1 /\ d>0 /\ v>0) ==>
     ((d+1)(v+t+1) = r h (c t + d + 1) ==> -(d+1)(v d + v + t d + t + d + 1 - h r
```

Solution [18]: True

- Biology-2 [18]

```
FORALL([r2, p1],
  (r2>1/2 ==>
    (4 (180 + 180 r2^2) = 75 (12+6 r2) /\
      (8 + 24 p1 + 20 r2 - 16 r2^2 p1 + 198 r2^2 p1^2 + 95 p1 r2 + 185 r2 p1^2 - 3
```

Solution [18]: True

- Biology-3 [18]

```
FORALL([r1, r2, p1],
  (r2>0 /\ r1>0 /\ r2>r1) ==>
    (( 4 (720 r1^2 + 180 r2^2)= 75 (24 r1 + 6 r2) /\
         (-88 r1 r2^2 p1^2 + 56 r1 r2^2 p1 - 480 r1^2 p1^2 r2 - 335 r1 p1 r2 + 55
  p1^2 + 480 r1^2 p1 r2 - 80 r1^2 + 128 r1^3 + 80 r1^2 p1 - 20 r2^2 p1 - 20 r2 p1
  r2^2 p1^2 - 256 r1^3 p1 + 128 r1^3 p1^2 + 32 r1 r2^2) p1 = 0) ==> p1 <= 0)
```

Solution [18]: True

- Biology-4 [18]

```
FORALL([g2, p1],
  (r2>0 /\ r1>0 /\ r2>r1 /\ 0<g2 /\ g2 < 1/2) ==>
    (( 4 (900 r1^2 (1-g2) + 900 r2^2 g2 ) = 75 (30 r1 (1-g2) + 30 r2 g2) /\
        (55 r2^2 g2 p1^2 + 20 r2^2 g2 p1 + 20 r2 g2 r1 - 40 r2 g2 r1 p1 + 20 r1^2
  r1^2 g2 - 32 r1^3 + 32 r1^3 g2 + 75 r1 p1 r2 - 55 r2 g2 r1 p1^2 - 120 r1^2 p1 r2
  r1^2 p1 r2 g2 + 120 r1^2 p1^2 r2 - 120 r1^2 p1^2 r2 g2 - 56 r1 r2^2 g2 p1 + 88 r
  g2 p1^2 - 32 r1^3 p1^2 + 32 r1^3 p1^2 g2 - 20 r1^2 p1 + 20 r1^2 p1 g2 + 64 r1^3
  64 r1^3 p1 g2 - 32 r1 r2^2 g2) p1 = 0 ) ==> p1 <= 0))
```

Solution [18]: True

- Biology-5 [18]

```
FORALL([p1], (t > 0) ==> (( 2 (48 t + 7) = 7 (1 + 8 t) /\
  -3472875 p1 - 26162325 p1^2 + 10584000 t - 69325200 t p1 - 327499200 t^2 p1 - 57
t p1^2 - 4222108800 t^2 p1^2 - 10163232000 t^3 p1^2 + 1852200 = 0) ==> (p1 <= 0)))
```

Solution [18]: True

- Biology-6 [18]

```
FORALL([p1], (t > 0 /\ 0 < c /\ c < 1) ==> (( 2 (60 c t + 7) = 7 (1 + 8 t) /\
  139179600 t p1 - 87318000 c t p1 - 662256000 c t^2 p1 + 357210000 t p1^2 c + 576
  t^2 p1^2 c + 3472875 p1 + 14817600 t + 26162325 p1^2 + 23250240000 c t^3 p1^2 +
  292515300 t p1^2 + 857304000 t^2 p1 - 390096000 t^2 p1^2 - 31752000 c t - 843696
  p1^2 - 1852200 = 0) ==> (p1 <= 0))
```

Solution [18]: True

- Biology-7 [18]

```
FORALL([p1], (t > 0 /\ 2 r2 > 1) ==> (( 8 (1+r2^2) (1+ 6 t) = (2 + r2) 5 (1 + 8 t)
  -370440 - 1111320 p1 + 8678880 r2^2 t p1 - 926100 r2 + 25401600 r2^2 t^2 p1 - 35
r2^2 t^3 p1^2 + 1481760 p1^2 + 29529360 t p1^2 - 91551600 t p1 r2 - 153071100 t p1
791985600 t^2 p1^2 r2 - 202127940 r2^2 t p1^2 - 1472385600 r2^2 t^2 p1^2 - 5927040
740880 r2^2 p1 - 4398975 p1 r2 - 7726320 t p1 + 182347200 t^2 p1^2 - 9168390 r2^2
8566425 r2 p1^2 + 8890560 r2^2 t + 1481760 r2^2 - 450878400 t^2 p1 r2 - 925344000
r2 - 7408800 t r2 + 12700800 t^2 p1 + 326592000 t^3 p1^2 = 0) ==> (p1 <= 0)))
```

Solution [18]: True

- Control-1 [45], Example 2.2

```
EXISTS([u], [[-0.5, 0.5]],
  -x1 + x2 u = 0 /\ -x2 + (1+x1^2) u + u^3 = 0
)
```

Solution [45]:

```
x2^4 - x1^3 x2^2 - x1 x2^2 - x1^3 = 0 /\ (x2 + 2 x1 <= 0 \/ x2 - 2 x1 >= 0)
```

- Control-2 [45], Example 4.1

```
FORALL([t], [[0, 1]]
  EXISTS([u, l], [[-1,1], (0, \infty]],
    -t + 2 = l /\ -(3 t^2 - 2 t^3) - t^2 + 4 u = l ( 6 t - 6 t^2 ) ))
```

- Reachability-1 [51], Example 3.4

```
EXISTS([z], [[ 1, \infty ]],
  x2 >= 3 /\
  x1 z^2 = 4 /\
  (2 x2 - 1) + z^2 = 6 z
)
```

Solution: $x_2 \geq 3 \wedge 4x1^2x2^2 - 4x1^2x2 + 16x1x2 + x1^2 - 152x1 + 16 = 0$

- Reachability-2 [51], Example 3.5

```
EXISTS([a, z], [[ 0, 1 ], [ 1, \infty ]],
  y1 = 2/3 a (-z^4 + z) /\
  y2 z^2 = 1/2 a (z^4 - 1)
)

EXISTS([z], [[ 1, \infty ]],
  3 y1 (z^3 + z^2 + z + 1) + 4 y2 (z^5 + z^4 + z^3) = 0
)
```

Solution: $(y_2 > 0 \wedge y_1 + y_2 \leq 0) \vee (y_2 < 0 \wedge y_1 + y_2 \geq 0) \vee 4y_2 + 3y_1 = 0$

- Reachability-3 [51], Example 3.6

```
EXISTS([w, z], [[-2,2], [-2,2]],
  w^2+z^2 = 1 /\
  0 = (z^2-w^2)-2/3 w /\
  y2 = - 4 z w - 5/3 z
);
[y2];
[[0, 4]];
```

Solution: $y2 = \frac{\sqrt{-181\sqrt{19}+908}}{9\sqrt{2}} \vee y2 = \frac{\sqrt{181\sqrt{19}+908}}{9\sqrt{2}}$

- Reachability-4 [51], Example 3.7

```
EXISTS([w, z, a], [[-2,2], [-2,2], [0, \infty]],
   a>0 /\
   w^2+z^2 = 1 /\
   -3a = w ((4 a^2 - 2) z + 2 - a^2) /\
   3a = (a^2-2)(w^2-z^2+z)
);
[];
[];
```